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On the solution of some fractional differential equations

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A B S T R A C T

There has been a great deal of interest in fractional differential equations. These equations arise in mathematical physics and engineering sciences. An attempt is made to solve some fractional differential Equation using the method of Laplace transforms. Some special cases with graphs have been discussed.

Keywords: Fractional Derivatives, Laplace transforms, Fractional differential equations.

MSC 2010 codes: 26A33, 44A10

1 Introduction

The methods of Integral transforms have their genesis in nineteenth century work of Joseph Fourier and Oliver Heaviside. An integral transformation (Sneddon 1972) simply means a unique mathematical operation through which a real or complex-valued function f is transformed into another new function F, or into a set of data that can be measured (or observed) experimentally. The solution is then mapped back to the original function with the inverse of the integral transform.

The Laplace Transform is defined as follows:

If $f(t)$ is of exponential order α and is a piece-wise continuous function on Real line, then Laplace transform of $f(t)$ for $s>\alpha$ is defined by(Sneddon 1972):

$$F(s)=L\left[f(t)\right]=\int_0^\infty e^{-st}f(t)\,dt \tag{1.1}$$

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And the inverse Laplace transform of $F(s)$ is defined by (Sneddon 1972):

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds \tag{1.2}$$

The convolution theorem for Laplace transform is given by (Sneddon 1972):

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(t-u)g(u)\,du$$

There has been a great deal of interest in fractional differential equations (Miller et al. 1993; Oldham et al. 1974). These equations arise in mathematical physics and engineering sciences. There are many definitions of fractional calculus are given by many different mathematicians and scientists. Here, we formulate the problem in terms of the Caputo fractional derivative (Caputo 1967,1969), which is defined as:

If α is a positive number and n is the smallest integer greater than α such that $n-1<\alpha<n$, the nth fractional derivative of a function $f(t)$ is defined by (Podlubny 2005):

$$C^{\alpha}\left[f(t)\right] = \frac{d^{\alpha}f}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{n}(x)}{(t-x)^{\alpha+n-1}}dx \tag{1.3}$$

Further we used the result due to (Caputo 1967, 1969):

$$L\left\{\frac{d^{\alpha}f}{dt^{\alpha}}\right\} = s^{\alpha}F(s) - \sum_{r=0}^{n-1} s^{\alpha-r-1}f^{(r)}(0) \tag{1.4}$$

where n is the smallest integer greater than α.

2 Fractional differential equations

In this section, we obtain the solution of some special fractional differential equations using Laplace transform.

(A) Consider the fractional Differential Equations is of the form

$$\frac{d^{\alpha}y}{dt^{\alpha}} = f(t) \text{ with initial condition } y^{(r)}(0) = c_r;\ r = 0,1,2...,n-1 \tag{1.5}$$

and n is the smallest integer greater than α.

Solution: Suppose that $f(t)$ is a sufficiently good function i.e. Laplace transform of $f(t)$ exists.

Applying the Laplace transform on both the sides of equation (1.5), we have

$$s^{\alpha}Y(s)-\sum_{r=0}^{n-1}s^{\alpha-r-1}y^{(r)}(0)=F(s) \tag{1.6}$$

Further simplification yields,

$$Y(s)=\frac{F(s)+\sum_{r=0}^{n-1}c_r s^{\alpha-r-1}}{s^{\alpha}} \tag{1.7}$$

Taking inverse Laplace transform of (1.7), we have

$$y(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(s)+\sum_{r=0}^{n-1}c_r s^{\alpha-r-1}}{s^{\alpha}}e^{st}ds \tag{1.8}$$

OR

Using convolution theorem for Laplace transform,

$$y(t)=\sum_{r=0}^{n-1}c_r\frac{t^r}{\Gamma(r+1)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}f(u)du \tag{1.9}$$

Hence, (1.8) and (1.9) are the solutions of (1.5).

Some special cases:

Case 1: If $\alpha = \frac{3}{2}, c_0 = 0, c_1 = 0$ and $f(t) = c$ (c is constant) then $y(t) = -\frac{4ct}{3}\sqrt{\frac{t}{\pi}}$

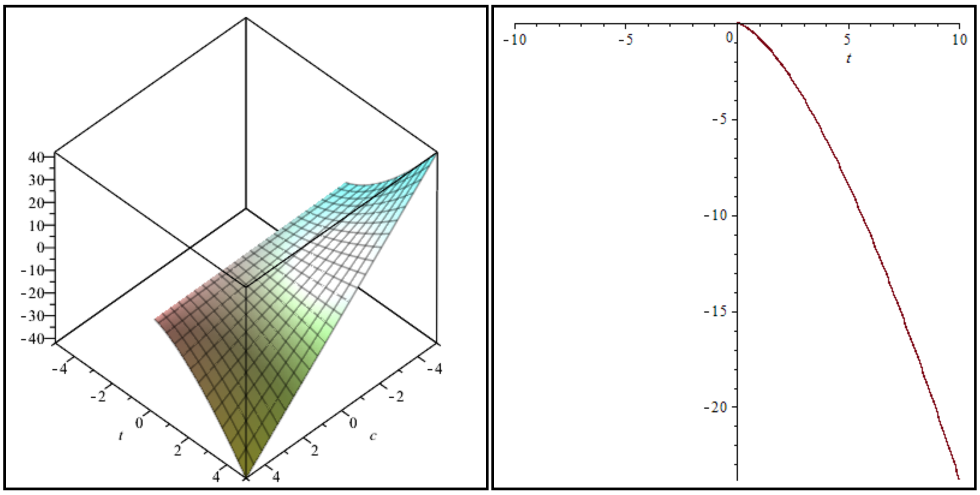

Fig. 1: **(A)** $y(t) = -\frac{4ct}{3}\sqrt{\frac{t}{\pi}}$ **(B)** $y(t) = -\frac{4t}{3}\sqrt{\frac{t}{\pi}}$

Case 2: If $\alpha = \frac{3}{2}, c_0 = 0, c_1 = c_1$ and $f(t) = 0$ then $y(t) = c_1$

Case 3: If $\alpha = \frac{1}{2}, c_0 = c_0$ and $f(t) = c$ (c is constant) then $y(t) = c_0 + \frac{c}{\sqrt{\pi t}}$

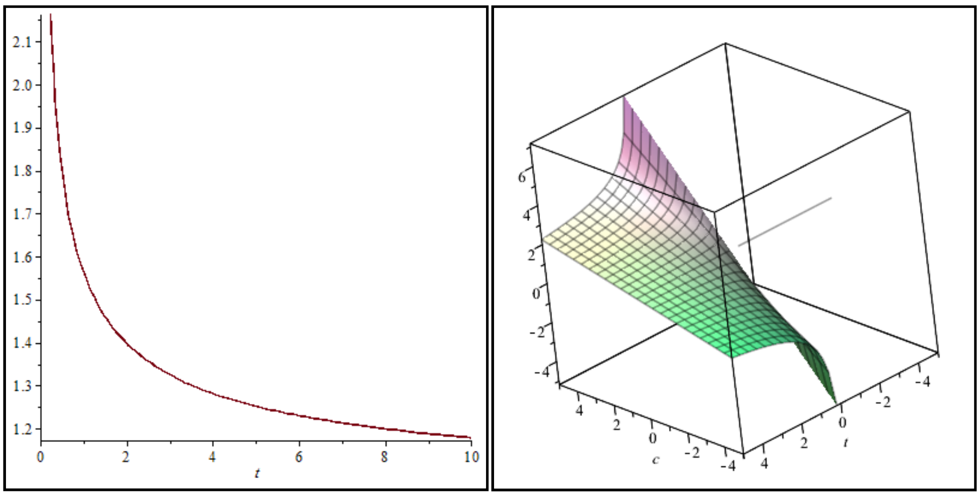

Fig. 2: **(A)** $y(t) = 1 + \frac{1}{\sqrt{\pi t}}$ **(B)** $y(t) = 1 + \frac{c}{\sqrt{\pi t}}$

(B) Consider the fractional Differential Equations is of the form

$$\frac{d^{\alpha}y}{dt^{\alpha}}+\frac{d^{\beta}y}{dt^{\beta}}=f(t) \text{ with initial condition } y^{(r)}(0)=c_r;\, r=0,1,2...,n-1 \tag{1.10}$$

where α and β are positive numbers with $\alpha>\beta$

and n is the smallest integer greater than α.

Solution: Suppose that $f(t)$ is a sufficiently good function i.e. Laplace transforms of $f(t)$ exists.

Applying the Laplace transform on both the sides of equation (1.10), we have

$$\left(s^{\alpha}+s^{\beta}\right)Y(s)-\sum_{r=0}^{n-1}s^{\alpha-r-1}y^{(r)}(0)-\sum_{k=0}^{m-1}s^{\beta-k-1}y^{(k)}(0)=F(s) \tag{1.11}$$

Here m is the smallest integer greater than β.

Further simplification yields,

$$Y(s)=\frac{F(s)+\sum_{r=0}^{n-1}c_r s^{\alpha-r-1}+\sum_{k=0}^{m-1}c_k s^{\beta-k-1}}{s^{\alpha}+s^{\beta}} \tag{1.12}$$

Taking inverse Laplace transform of (1.7), we have

$$y(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(s)+\sum_{r=0}^{n-1}c_r s^{\alpha-r-1}+\sum_{k=0}^{m-1}c_k s^{\beta-k-1}}{s^{\alpha}+s^{\beta}}e^{st}ds \tag{1.13}$$

Some Special Cases:

Case 1: If $\alpha=\frac{3}{2},\beta=\frac{1}{2}\ c_0=0,c_1=0$ and $f(t)=t$

then $y(t)=\dfrac{4t^{3/2}-6\sqrt{t}-3ie^{-t}\sqrt{\pi}\,\mathrm{erf}\left(i\sqrt{t}\right)}{3\sqrt{\pi}}$

where $\mathrm{erf}(t)$ is the well-known error function.

Case 2: If $\alpha=\frac{3}{2},\beta=\frac{1}{2}\ c_0=0,c_1=0$ and $f(t)=\sin t$ then $y(t)=-i\int_0^t e^{-u}\mathrm{erf}\left(i\sqrt{u}\right)\sin(t-u)\,du$

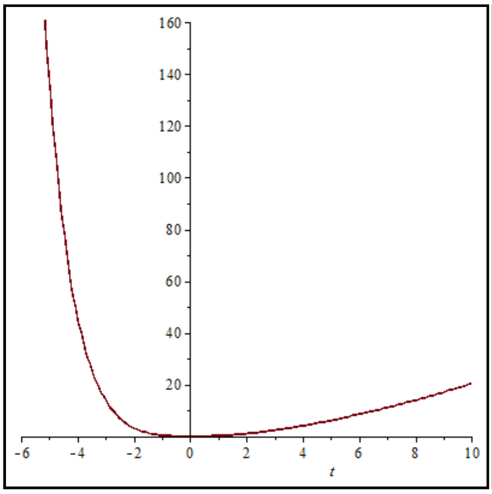

Fig. 3 (A)

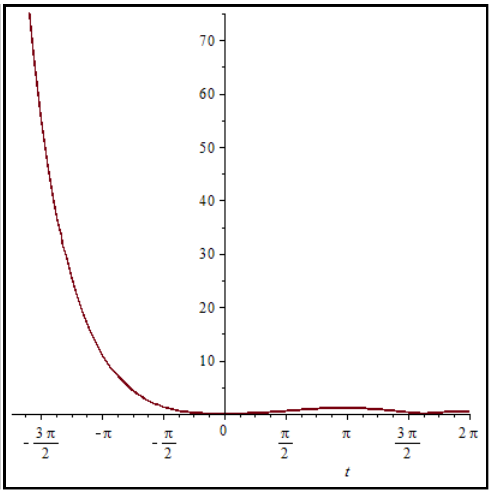

Fig. (B)

$$y(t)=\left|\frac{4t^{3/2}-6\sqrt{t}-3ie^{-t}\sqrt{\pi}\,\mathrm{erf}\left(i\sqrt{t}\right)}{3\sqrt{\pi}}\right| \qquad y(t)=\left|-i\int_0^t e^{-u}\,\mathrm{erf}\left(i\sqrt{u}\right)\sin(t-u)\,du\right|$$

Case 3: If $\alpha=\frac{3}{2},\beta=\frac{1}{2}\ c_0=1,c_1=2$ and $f(t)=0$ then $y(t)=4-4e^{-t}$

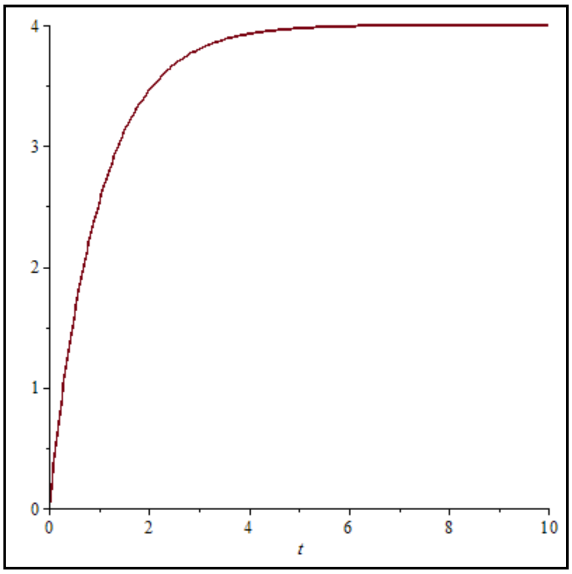

Fig. 4 : $y(t)=4-4e^{-t}$

Case 4: If $\alpha=\frac{3}{2},\beta=\frac{1}{2}\ c_0=1,c_1=2$ and $f(t)=e^t$ then

$$y(t)=4+1/2\,\mathrm{erf}\left(\sqrt{t}\right)e^{t}+1/2\left(i\,\mathrm{erf}\left(i\sqrt{t}\right)-8\right)e^{-t}$$

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