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Physicists Hermite wavelet method for singular differential equations

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ABSTRACT

This manuscript witnesses a modification in the Legendre Wavelet Method (LWM) by inserting Physicists Hermite Polynomials instead of the traditional Legendre's Polynomials. The modified version which is called the Physicists Hermite Wavelet Method (PHWM) is highly accurate and is tested on Singular Differential Equations (SDEs). Five examples are given to elucidate the solution procedure. Comparison of numerical results explicitly reflects the very high level of accuracy.

Keywords: Physicists Hermite polynomials, wavelets, Legendre wavelets method, Singular Differential Equations (SDEs), MAPLE 13.

MSC 2010 codes: 35K99, 35P99, 35P05.

1 Introduction

Singular Differential Equations (SDEs) are of extreme importance in physical, engineering and applied sciences. It is an established fact (Hasan and Zhu, 2008) that wide range of real time problems is modeled by such equations. The general form of Singular Differential Equations (SDEs) (Hasan and Zhu, 2008) is given as:

$$y''(x) + \frac{r}{x} y'(x) + p(x)y(x) = q(x), \qquad a \le x \le b, \tag{1}$$

with the physical boundary conditions

$$y(0) = \alpha, \quad y(A) = \beta, \tag{2}$$

where r, α, β are the real constants. The through study of literature reveals that many researchers have investigated these equations. Hasan and Zhu (2008) applied Modified Adomian's decomposition method for singular initial value problems in the second-order ordinary differential equations. Moreover, various numerical studies have been reported based on the Legendre Wavelet (Jafari et al., 2011; Hesameddini and Shekarpaz, 2011; Razzaghi

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and Yousefi, 2000, 2010). Finite Difference (Meerschaert and Tadjeran, 2003, Meerschaert, et. al., 2006), Shooting (Kwong, 2006), Chebyshev Wavelet (Zhu, et ai., 2011), and Galerkin Methods (Xiao, et. al, 2006). In addition, Chebyshev, Legendre and Shannon Wavelet Methods have been used for obtaining the numerical solutions of the Fredholm integral equation in Paryab, M. Rostami (2008). Kajani and Mahdavi (2011) solved the nonlinear integral equations by using Galerkin method with hybrid Block-Pulse function. Mohammadi, Hosseini and Mohyud-Din (2011) implemented Legendre Wavelet Galerkin Method for solving ordinary differential equations. Hosseini, Mohyud-Din and Nakhaeei (2011) solved the telegraph equations with a new Rothe-wavelet method. Venkatesh et al. (2012) used Legendre Wavelets Based Approximation Method for Cauchy Problems. It is to be highlighted that Wavelets theory is a relatively new and emerging area in mathematical research. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis and many other areas. Wavelets allow the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms (Razzaghi and Yousefi, 2001a, 2001b). Inspired and motivated by the ongoing research in this area, we replace Physicists Hermite Polynomials (Koci´c et al., 2012; Schleicher, 2008) with Legendre's Polynomials in the traditional Legendre Wavelet Method (LWM). The re-formulated scheme is called Physicists Hermite Wavelet Method (PHWM) and is applied on Singular Differential Equations (SDEs). It is worthmentioning that applied scheme is fully compatible with the complexity of such problems and obtained results are highly accurate. Moreover, is also observed that PHWM is very reliable and may be implemented on other physical problems also.

2 Preliminaries

2.1 Physicists Hermite polynomials

Physicists Hermite polynomial, named is given by

$$PH_n(x) = (-1)^n \mathrm{e}^{\frac{x^2}{2}}. \frac{d^n}{dx^n} \mathrm{e}^{-\frac{x^2}{2}} \tag{3}$$

few Physicists Hermite polynomials given below

$PH_0 = 1,$

$PH_1 = 2x,$

$PH_2 = -2 + 4x^2,$

$PH_3 = 4x(-3 + 2x^2),$

$PH_4 = 12 - 48x^2 + 16x^4,$

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2.2 Wavelets and Physicists Hermite Wavelets

In recent years, wavelets have found their way into many different fields of science and engineering (Mohammadi and Hosseini, 2010, 2011a, 2011b; Yousefi, 2006, 2007). Wavelets constitute a family of functions constructed from the dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right) \quad a, b \in R, a \neq 0. \tag{4}$$

Physicists Hermite wavelets $\psi_{nm}(t) = \psi(k, n, m, t)$ have four arguments; n, k can assume any positive integer, m is the order for Physicists Hermite polynomials and t is the normalized time. They are defined on the interval [0, 1) by;

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}}\, PH_m\left(2^k t-(2n+1)\right) & \frac{n}{2^k} \leq t \leq \frac{n+1}{2^k} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where $m = 0,1,2,\dots,M-1$ and $n = 0,1,\dots,2^{k-1}$ The coefficient $\left(m+\frac{1}{2}\right)$ is for orthonormality. Here $PH_m(t)$ are the well-known Physicists Hermite polynomials of order m, which have been previously described.

2.3 Functions approximation

A function $f(t)$ define over [0,1) may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{6}$$

where $c_{nm} = \langle f(t), \psi_{nm}(t)\rangle$, in which $\langle\cdot,\cdot\rangle$ denotes the inner product. If the infinite series in Eq. (6) is truncated, then Eq. (6) can be written as

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t). \tag{7}$$

It can be written as

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\,\psi(t). \tag{8}$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given as

$$C = \begin{bmatrix} c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, \\ c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1} \end{bmatrix}^T, \tag{9}$$

and

$$\psi(t) = \begin{bmatrix} \psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \dots, \\ \psi_{2M-1}, \dots, \psi_{2^{k-1}0}, \dots, \psi_{2^{k-1}M-1} \end{bmatrix}^T. \tag{10}$$

3 Methodology

Consider the singular differential equation (1-2) as

$$y''(x) + \frac{r}{x} y'(x) + p(x)y(x) = q(x), \qquad a \le x \le b,$$
$$y(0) = \alpha, \quad y(A) = \beta,$$

where r is nonzero constant and $p(x), q(x)$ are real valued functions, α, β, a, b and A are constants. In order to solve singular differential equation, we suppose that $[a, b] = [0, 1]$, otherwise the problem can be mapped from $[a, b]$ to [0, 1] easily. For solving singular differential equation, it is sufficient to suppose that the approximate solution is as

$$y(x) = \mathbf{C}^T \boldsymbol{\psi}(x),$$

where $\mathbf{C}$ and $\boldsymbol{\psi}(x)$ are given in above. Therefore, we obtain

$$\mathbf{C}^T \boldsymbol{\psi}''(x) + \frac{r}{x} \mathbf{C}^T \boldsymbol{\psi}'(x) + p(x)\mathbf{C}^T \boldsymbol{\psi}(x) = q(x), \qquad a \le x \le b,$$
$$\mathbf{C}^T \boldsymbol{\psi}(0) = \alpha, \quad \mathbf{C}^T \boldsymbol{\psi}(A) = \beta.$$

Therefore, in order to apply the Physicists Hermite Wavelet Method (PHWM); we require $2^{k-1}M - 1$ collocating points. Suitable collocating points are as

$$x_i = \cos\left(\frac{(2i+1)\pi}{2^k M}\right), \qquad i = 2, \dots, 2^{k-1}M - 1.$$

Implementing the collocating points and imposing the initial value to the singular differential equation, we obtain

$$\mathbf{C}^T \boldsymbol{\psi}''(x_i) + \frac{r}{x_i} \mathbf{C}^T \boldsymbol{\psi}'(x_i) + p(x_i)\mathbf{C}^T \boldsymbol{\psi}(x_i) = q(x_i), a \le x \le b,$$
$$\mathbf{C}^T \boldsymbol{\psi}(0) = \alpha, \quad \mathbf{C}^T \boldsymbol{\psi}(A) = \beta.$$

The differential equation yields $2^{k-1}M - 3$ equations and initial condition produces 2 number of equations. Therefore, the obtained system has $2^{k-1}M - 1$ equations and $2^{k-1}M - 1$ unknowns. Solving this system gives the unknown coefficients $\mathbf{C}$.

4 Numerical applications

In this section, we apply proposed technique (PHWM) to construct numerical solutions of Singular differential equations SDEs). Numerical results are very encouraging.

4.1 Example

Consider the following singular BVP as

$$-y''(x) - \frac{2}{x} y'(x) + (1 - x^2) y(x) = x^4 - 2x^2 + 7, \tag{11}$$

subject to the boundary conditions

$$y(0) = 1, \qquad y(1) = 0. \tag{12}$$

The exact solution of Eq. (11-12) is

$$y(x) = 1 - x^2. \tag{13}$$

According to the Physicists Hermite Wavelets Method (PHWM), we assume the trial solution

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = \mathbf{C}^{\mathbf{T}}\, \boldsymbol{\psi}(x), \tag{14}$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given as

$$\mathbf{C} = \left[c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}\right]^T,$$

and $\boldsymbol{\psi}(x) = \left[\psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \dots, \psi_{2M-1}, \dots, \psi_{2^{k-1}0}, \dots, \psi_{2^{k-1}M-1}\right]^T.$

We apply the proposed technique to solve Eq. (11-12) with $K = 1,$ and $M = 3.$ We have Eq. (14) is

$$y(x) = \sum_{n=1}^{1} \sum_{m=0}^{2} c_{nm} \psi_{nm}(x) = c_{10}\psi_{10}(x) + c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) \\ = \mathbf{C}^{\mathbf{T}} \boldsymbol{\psi}(x), \tag{15}$$

where

$$\psi_{nm}(x) = \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}}\, PH_m\left(2^k t - (2n+1)\right). \tag{16}$$

In the above expression, PH_m are the Physicists Hermite polynomials. Therefore, we have the trial solution is

$$y(x) = c_{1,0} + \sqrt{3}(2x-1)c_{1,1} + 4\sqrt{5}x(x-1)c_{1,2} = \mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(x),$$

where $\mathbf{C} = [c_{1,0}, c_{1,1}, c_{1,2}]^T$ and $\boldsymbol{\psi}(x) = [1, \sqrt{3}(2x-1), 4\sqrt{5}x(x-1)]^T$.

Substituting into the given problem we get

$$-\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}''(x) - \frac{2}{x}\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}'(x) + (1+x^2)\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(x) = x^4 - 2x^2 + 7, \tag{17}$$

subject to the boundary conditions

$$\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(0) = 1, \qquad \mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(1) = 0. \tag{18}$$

Substitute the collocating points are in Eq. (17), we have the system of equations

$$x_i = \cos\left(\frac{(2i+1)\pi}{2^k M}\right), \qquad i = 3. \tag{19}$$

Implementing the collocating points and imposing the initial value of the system, the matrix form is given as

$$\mathbf{A}_{3\times3}\mathbf{C}_{3\times1} = \mathbf{b}_{3\times1}.$$

After solving we get the following exact solution

$$y(x) = 1 - x^2.$$

4.2 Example

Consider the following singular IVP as

$$y''(x) + \frac{4}{x}y'(x) + \frac{2}{x^2}y(x) = 12, \tag{20}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y'(0) = 0. \tag{21}$$

The exact solution of Eq. (20-21) is

$$y(x) = x^2. \tag{22}$$

According to the Physicists Hermite Wavelets Method (PHWM), we assume the trial solution

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = \mathbf{C}^{\mathbf{T}}\,\boldsymbol{\psi}(x). \tag{23}$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given as

$$\mathbf{C} = \left[c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}\right]^T,$$

and $$\boldsymbol{\psi}(x) = \left[\psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \dots, \psi_{2M-1}, \dots, \psi_{2^{k-1}0}, \dots, \psi_{2^{k-1}M-1}\right]^T.$$

We apply the proposed technique to solve Eq. (20-21) with $K = 1$, and $M = 3$. We have Eq. (23) is

$$y(x) = \sum_{n=1}^{1}\sum_{m=0}^{2} c_{nm}\psi_{nm}(x) = c_{10}\psi_{10}(x) + c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) \\ = \mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(x), \tag{24}$$

where

$$\psi_{nm}(x) = \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}}\, PH_m\left(2^k t - (2n+1)\right). \tag{25}$$

In the above expression, PH_m are the Physicists Hermite polynomials. Therefore, we have the trial solution is

$$y(x) = c_{1,0} + \sqrt{3}(2x-1)c_{1,1} + 4\sqrt{5}x(x-1)c_{1,2} \ = \mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(x),$$

where $\mathbf{C} = \left[c_{1,0}, c_{1,1}, c_{1,2}\right]^T$ and $\boldsymbol{\psi}(x) = \left[1, \sqrt{3}(2x-1), 4\sqrt{5}x(x-1)\right]^T$.

Substituting into the given problem we get

$$\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}''(x) + \frac{4}{x}\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}'(x) + \frac{2}{x^2}\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(x) = 12, \tag{26}$$

subject to the boundary conditions

$$\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(0) = 1, \qquad \mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}'(0) = 0. \tag{27}$$

Substitute the collocating points are in Eq. (26), we have the system of equations

$$x_i = \cos\left(\frac{(2i+1)\pi}{2^k M}\right), \qquad i = 2. \tag{28}$$

Implementing the collocating points and imposing the initial value of the system, the matrix form is given as

$$\mathbf{A}_{3\times3}\mathbf{C}_{3\times1} = \mathbf{b}_{3\times1}.$$

After solving we get the following exact solution

$$y(x) = x^2.$$

4.3 Example

Consider the following singular IVP as

$$y''(x) + \frac{2}{x} y'(x) + y(x) = 6 + 12x + x^2 + x^3, \tag{29}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y'(0) = 0. \tag{30}$$

The exact solution of Eq. (29-30) is

$$y(x) = x^2 + x^3. \tag{31}$$

According to the Physicists Hermite Wavelets Method (PHWM), we assume the trial solution

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = \mathbf{C}^{\mathbf{T}}\, \boldsymbol{\psi}(x). \tag{32}$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given as

$$\mathbf{C} = \left[c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}\right]^T,$$

and $$\boldsymbol{\psi}(x) = \left[\psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \dots, \psi_{2M-1}, \dots, \psi_{2^{k-1}0}, \dots, \psi_{2^{k-1}M-1}\right]^T.$$

We apply the proposed technique to solve Eq. (29-30) with $K = 1$, and $M = 4$. We have Eq. (32) is

$$\begin{aligned} y(x) &= \sum_{n=1}^{1} \sum_{m=0}^{4} c_{nm} \psi_{nm}(x) \\ &= c_{10}\psi_{10}(x) + c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) + c_{13}\psi_{13}(x) \\ &= \mathbf{C}^{\mathbf{T}} \boldsymbol{\psi}(x), \end{aligned} \tag{33}$$

where

$$\psi_{nm}(x) = \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}}\, PH_m\left(2^k t - (2n + 1)\right). \tag{34}$$

In the above expression, PH_m are the Physicists Hermite polynomials. Therefore, we have the trial solution is

$$y(x) = c_{1,0} + \sqrt{3}(2x-1)c_{1,1} + 4\sqrt{5}x(x-1)c_{1,2} + 2\sqrt{7}(2x-1)(2x^2-2x-1)c_{1,3} = \mathbf{C^T\psi}(x),$$

where

$$\mathbf{C} = \left[c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}\right]^T \text{ and } \boldsymbol{\psi}(x) = \begin{bmatrix} 1, \sqrt{3}(2x-1), 4\sqrt{5}x(x-1), \\ 2\sqrt{7}(2x-1)(2x^2-2x-1) \end{bmatrix}^T.$$

Substituting into the given problem we get

$$\mathbf{C^T\psi}''(x) + \frac{2}{x}\mathbf{C^T\psi}'(x) + \mathbf{C^T\psi}(x) = 6 + 12x + x^2 + x^3, \tag{35}$$

subject to the boundary conditions

$$\mathbf{C^T\psi}(0) = 1, \qquad \mathbf{C^T\psi}'(0) = 0. \tag{36}$$

Substitute the collocating points are in Eq. (35), we have the system of equations

$$x_i = \cos\left(\frac{(2i+1)\pi}{2^k M}\right), \qquad i = 2,3. \tag{37}$$

Implementing the collocating points and imposing the initial value of the system, the matrix form is given as

$$\mathbf{A}_{4\times4}\mathbf{C}_{4\times1} = \mathbf{b}_{4\times1}.$$

After solving we get the following exact solution

$$y(x) = x^2 + x^3.$$

4.4 Example

Consider the following singular IVP as

$$y''(x) + \frac{8}{x}y'(x) + xy(x) = -30x + 44x^2 - x^4 + x^5, \tag{38}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y'(0) = 0. \tag{39}$$

The exact solution of Eq. (38-39) is

$$y(x) = x^4 - x^3. \tag{40}$$

According to the Physicists Hermite Wavelets Method (PHWM), we assume the trial solution

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = \mathbf{C^T}\,\boldsymbol{\psi}(x). \tag{41}$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given as

$$\mathbf{C} = \left[c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}\right]^T,$$

and $$\boldsymbol{\psi}(x) = \left[\psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \dots, \psi_{2M-1}, \dots, \psi_{2^{k-1}0}, \dots, \psi_{2^{k-1}M-1}\right]^T.$$

We apply the proposed technique to solve Eq. (38-39) with $K = 1$, and $M = 5$. We have Eq. (41) is

$$\begin{aligned} y(x) &= \sum_{n=1}^{1} \sum_{m=0}^{4} c_{nm}\psi_{nm}(x) \\ &= c_{10}\psi_{10}(x) + c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) + c_{13}\psi_{13}(x) + \cdots \\ &= \mathbf{C^T}\boldsymbol{\psi}(x), \end{aligned} \tag{42}$$

where

$$\psi_{nm}(x) = \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}}\, PH_m\left(2^k t - (2n+1)\right). \tag{43}$$

In the above expression, PH_m are the Physicists Hermite polynomials. Therefore, we have the trial solution is

$$y(x) = c_{1,0} + \sqrt{3}(2x-1)c_{1,1} + 4\sqrt{5}x(x-1)c_{1,2} + 2\sqrt{7}(2x-1)$$
$$(2x^2 - 2x - 1)c_{1,3} + \cdots = \mathbf{C^T}\boldsymbol{\psi}(x),$$

where $\mathbf{C} = \left[c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, \dots\right]^T$ and $\boldsymbol{\psi}(x) = \begin{bmatrix} 1, \sqrt{3}(2x-1), 4\sqrt{5}x(x-1), \\ 2\sqrt{7}(2x-1)(2x^2-2x-1), \dots \end{bmatrix}^T.$

Substituting into the given problem we get

$$\mathbf{C^T}\boldsymbol{\psi}''(x) + \frac{8}{x}\mathbf{C^T}\boldsymbol{\psi}'(x) + x\mathbf{C^T}\boldsymbol{\psi}(x) = -30x + 44x^2 - x^4 + x^5, \tag{44}$$

subject to the boundary conditions

$$\mathbf{C^T}\boldsymbol{\psi}(0) = 0, \qquad \mathbf{C^T}\boldsymbol{\psi}'(0) = 0. \tag{45}$$

Substitute the collocating points are in Eq. (44), we have the system of equations

$$x_i = \cos\left(\frac{(2i+1)\pi}{2^k M}\right), \qquad i = 2,3,4. \tag{46}$$

Implementing the collocating points and imposing the initial value of the system, the matrix form is given as

$$\mathbf{A}_{5\times 4}\mathbf{C}_{5\times 1} = \mathbf{b}_{5\times 1}.$$

After solving we get the following exact solution

$$y(x) = x^4 - x^3.$$

4.5 Example

Consider the following singular BVP as

$$y''(x) + \frac{2}{x}y'(x) - 10y(x) = 12 - 10x^4, \tag{47}$$

subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = 3. \tag{48}$$

The exact solution of Eq. (47-48) is

$$y(x) = 2x^2 + x^4. \tag{49}$$

According to the Physicists Hermite Wavelets Method (PHWM), we assume the trial solution

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = \mathbf{C}^{\mathbf{T}}\, \boldsymbol{\psi}(x). \tag{50}$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given as

$$\mathbf{C} = \left[c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}\right]^T,$$

and $$\boldsymbol{\psi}(x) = \left[\psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \dots, \psi_{2M-1}, \dots, \psi_{2^{k-1}0}, \dots, \psi_{2^{k-1}M-1}\right]^T.$$

We apply the proposed technique to solve Eq. (47-48) with $K = 1,$ and $M = 5$. We have Eq. (50) is

$$\begin{aligned} y(x) &= \sum_{n=1}^{1} \sum_{m=0}^{4} c_{nm}\psi_{nm}(x) \\ &= c_{10}\psi_{10}(x) + c_{11}\psi_{11}(x) + c_{12}\psi_{12}(x) + c_{13}\psi_{13}(x) + \cdots \\ &= \mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(x), \end{aligned} \tag{51}$$

where

$$\psi_{nm}(x) = \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}}\, PH_m\left(2^k t - (2n+1)\right). \tag{52}$$

In the above expression, PH_m are the Physicists Hermite polynomials. Therefore, we have the trial solution is

$$y(x) = c_{1,0} + \sqrt{3}(2x-1)c_{1,1} + 4\sqrt{5}x(x-1)c_{1,2} + 2\sqrt{7}(2x-1)(2x^2-2x-1)c_{1,3} + \cdots = \mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(x),$$

where $\mathbf{C} = [c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, \ldots]^T$ and $\boldsymbol{\psi}(x) = \begin{bmatrix} 1, \sqrt{3}(2x-1), 4\sqrt{5}x(x-1), \\ 2\sqrt{7}(2x-1)(2x^2-2x-1), \ldots \end{bmatrix}^T$.

Substituting into the given problem we get

$$\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}''(x) + \frac{2}{x}\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}'(x) - 10\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(x) = 12 + 10x^4, \tag{53}$$

subject to the boundary conditions

$$\mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(0) = 0, \qquad \mathbf{C}^{\mathbf{T}}\boldsymbol{\psi}(1) = 3. \tag{54}$$

Substitute the collocating points are in Eq. (53), we have the system of equations

$$x_i = \cos\left(\frac{(2i+1)\pi}{2^k M}\right), \qquad i = 2,3,4. \tag{55}$$

Implementing the collocating points and imposing the initial value of the system, the matrix form is given as

$$\mathbf{A}_{5\times4}\mathbf{C}_{5\times1} = \mathbf{b}_{5\times1}.$$

After solving we get the following exact solution

$$y(x) = 2x^2 + x^4.$$

5 Conclusions

Physicists Hermite Wavelet Method is applied to obtain numerical solutions of linear Singular Differential Equations (SDEs). The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. With the aid of MAPLE 13, the correctness of solutions is verified by reverse substitution. It is concluded that proposed algorithm can be extended to other linear and nonlinear problems of physical nature also.

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