International Journal of Advances in Applied Mathematics and Mechanics
Volume 1, Issue 2 : (2013) pp. 30-38
Available online at www.ijaamm.com
ISSN: 2347-2529

Thermal instability of rotating Maxwell visco-elastic fluid with variable gravity in porous medium

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Received 20 September 2013; Accepted (in revised version) 20 October 2013

ABSTRACT

The effect of variable gravity and rotation is investigated on the thermal instability of Maxwell visco-elastic fluid with variable gravity in porous medium. It is found that, in case of stationary convection, variable gravity has destabilizing effect while rotation has stabilizing effect on the system. Medium permeability has stabilizing/destabilizing effect on system depending upon certain condition. The mode may be non oscillatory or oscillatory.

Keywords: Thermal instability, Maxwell Visco -elsatic fluid, porous medium, variable gravity, rotation.

MSC 2010 codes: 76S99, 74B20.

1 Introduction

Many common materials such as paints, polymers, plastics, and more exotic one such as magma, saturated soils and Earth's lithosphere behaves as Visco-elastic fluids. Visco-elastic fluids have range of unlikely behaviors. In condition of shear flow, they develop transverse normal stresses that cause many phenomenons as shear thinning or thickening, not climbing and extrudali extensively swell. The subject of thermal or thermal instability in porous medium has been studied extensively in recent years. There are many important reasons for development of the subject. On the practical side, the need of model-geothermal system, packed-bed catalytic reactions. On the other hand, there also a need to comprehend fully the implication of recent model of fluid flows of Maxwellian Visco-elastic fluid in porous medium. Theoretical and experimental results on the stability of cellular convection of a fluid layer in nonporous medium, in the presence of rotation and magnetic field, have been given by Chandrasekhar (1961). The problem of convective instability of Visco-elastic fluid heated from below, was first studied by Green (1968). Vest and Arpaci (1969) have investigated the problem of overstability in a horizontal layer of a Visco-elastic fluid heated from below. The idealization of uniform gravity assumed in theoretical investigations, although valid for laboratory purposes, can scarcely be justified for large-scale convection phenomena occurring

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in atmosphere, the ocean or mantle of the Earth. It then becomes imperative to consider gravity as variable quantity varying with distance from surface or reference point. Pradhan et al. (1989) studied the thermal instability of a fluid layer in a variable gravitational field and found that variable gravity has destabilizing effect on the fluid layer. Rotation too has profound effect on the onset of instability. The problem of thermal convection in fluids in porous medium is of considerable importance in geophysics, soil sciences, found water hydrology and astrophysics. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical context (McDonnel, 1978). The physics of flow through porous medium has been given in a treatise by Scheidegger (1960). The Rayleigh instability of a thermal boundary layer in flow in porous medium is studied by Wooding (1960). Such problem arises in oceanography, limnology and engineering. Such problem arises in oceanography, limnology and engineering. It induces the number of new elements into the problem and some of its consequence at the first sight unexpected. In the present problem an attempt has been made to study the thermal instability of Maxwell Visco-elastic fluid in the presence variable gravity and rotation in porous medium.

2 Formulation of problem and perturbation equations

Consider an infinite horizontal layer of Maxwellian Visco- elastic fluid of thickness 'd' bounded by plane z = 0 and z = d in porous medium of porosity ε and medium permeability k_1. The layer is rotating with angular velocity Ω(0,0, Ω). Choose a Cartesian system of coordinate O (x, y, z) rotating with layer, with the origin half – way between the planes, Oz vertically upwards and Ox, Oy be two perpendicular horizontal directions. The layer is heated and saluted from below such that a uniform temperature gradient $\beta\left(=\left|\frac{dT}{dz}\right|\right)$ where T denote the temperature. Let the system is acted upon by linear variable gravity force $\vec{g}$ (0,0,g (z)), where $g(z)=g_0(1+Mz)>0$, M is gravity parameter and g_0 is the value of g at z = 0.

Let $p,\rho,T,\alpha,\mu,\upsilon$ and κ be the pressure, density, temperature and thermal coefficient of expansion, viscosity, kinematic viscosity and thermal diffusivity of fluid respectively.
As the fluid flow through a porous medium the gross effect is represented by Darcy's law[1856]. According to which the usually viscous term is replaced by the resistance term $-\left(\frac{\mu}{k_1}\right)\vec{q}$, in the equation of motion, $\vec{q}$ is filter velocity of fluid; the fluid velocity $\vec{v}$ and filter velocity $\vec{q}$ are connected by relation $\vec{v}=\frac{\vec{q}}{\varepsilon}$.

The equation of motion, continuity and heat conduction for Maxwellian Visco-elastic fluid through porous medium are

$$\frac{\rho}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{d\vec{q}}{dt}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla p+\rho\vec{g}+\frac{2\rho}{\varepsilon}\left(\vec{q}\times\vec{\Omega}\right)\right]-\frac{\mu}{k_1}\vec{q} \qquad (1)$$

$$\nabla.\vec{q}=0 \qquad (2)$$

$$E\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T=\kappa\nabla^2 T \qquad (3)$$

where $\frac{d}{dt}=\frac{\partial}{\partial t}+\frac{1}{\varepsilon}(\vec{q}\nabla)$ stands for convection derivative.

The equation of state is

$$\rho=\rho_0\left[1-\alpha(T-T_0)\right] \qquad (4)$$

where the suffix zero refers to values at reference level z = 0, i.e. ρ_0, T_0 stands for density, temperature at lower boundary z =0.

The steady state solution is

$\vec{q}=(0,0,0), T=T_0-\beta z$, $\rho=\rho_0(1+\alpha\beta z)$,

where $\beta=\frac{T_0-T_1}{d}$ is the magnitude of uniform temperature gradient, which is to be maintained.

Let $\delta\rho, \delta p, \theta$ denote respectively the perturbation in density, pressure, and temperature. Then the linearised perturbations equations of flow through porous medium, following the Boussineq approximations are,

$$\frac{\rho_0}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{d\vec{q}}{dt}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla\delta p+\rho\delta\vec{g}\frac{2\rho_0}{\varepsilon}\left(\vec{q}\times\vec{\Omega}\right)\right]-\frac{\mu}{k_1}\vec{q} \qquad (5)$$

$$E\frac{\partial\theta}{\partial t}=\beta w+\kappa\nabla^2\theta \qquad (6)$$

$$\nabla.\vec{q}=0 \qquad (7)$$

The change in density $\delta\rho$ caused by the perturbation in temperature θ is given by

$$\delta\rho=-\rho_0\alpha\theta. \qquad (8)$$

In the Cartesian form equation (5) – (7) can be written as

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial}{\partial x}\delta p-\frac{2\Omega}{\varepsilon}v\right]=-\frac{\nu}{k_1}u \qquad (9)$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial v}{\partial t}+\frac{1}{\rho_0}\frac{\partial}{\partial y}\delta p+\frac{2\Omega}{\varepsilon}u\right]=-\frac{\nu}{k_1}v \qquad (10)$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial w}{\partial t}+\frac{1}{\rho_0}\frac{\partial}{\partial z}\delta p-\vec{g}\alpha\theta\right]=-\frac{\nu}{k_1}w \qquad (11)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{12}$$

$$E\frac{\partial \gamma}{\partial t}=\beta w+\kappa\nabla^2\theta \tag{13}$$

Operating equation (9) by $\frac{\partial}{\partial x}$ and equation (10) by $\frac{\partial}{\partial y}$; then adding and making use of equation (12), we get

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right)-\frac{1}{\rho_0}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\delta p+\frac{2\Omega}{\varepsilon}\varsigma\right]=-\frac{\nu}{k_1}\frac{\partial w}{\partial z} \tag{14}$$

Now eliminating δp from (11) and (14), we get

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial}{\partial t}\left(\nabla^2 w\right)-\vec{g}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\alpha\theta+\frac{2\Omega}{\varepsilon}\frac{\partial \varsigma}{\partial z}\right]=-\frac{\nu}{k_1}\nabla^2 w \tag{15}$$

Again operating equation (9) by $-\frac{\partial}{\partial y}$ and equation (10) by $\frac{\partial}{\partial x}$, we get

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial \varsigma}{\partial t}-\frac{2\Omega}{\varepsilon}\frac{\partial w}{\partial z}\right]=-\frac{\nu}{k_1}\varsigma \tag{16}$$

where $\varsigma=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$, is z-component of vorticity.
Equations (13) can be written as

$$\left\{E\frac{\partial}{\partial t}-\kappa\nabla^2\right\}\theta=\beta w \tag{17}$$

3 Dispersion relation

Analyzing the disturbances into the normal modes and assuming that the perturbed quantities are of the form

$$\left[w,\theta,\gamma,\varsigma\right]=\left[W(z),\Theta(z),\Gamma(z),Z(z)\right]\exp\left(ik_x x+ik_y y+nt\right) \tag{18}$$

where k_x, k_y are horizontal wave numbers in x and y direction respectively, $k^2=k_x^2+k_y^2$ is the resultant wave number, n is growth rate of disturbances.

Using equation (18), equations (15) – (17) becomes

$$\left(1+\lambda n\right)\left[\frac{1}{\varepsilon}n\left(\frac{d^2}{dz^2}-k^2\right)W+g_0k^2\alpha\theta\left(1+Mz\right)+\frac{2\Omega}{\varepsilon}\frac{dZ}{dz}\right]=-\frac{\nu}{k_1}\left\{\frac{d^2}{dz^2}-k^2\right\}w \tag{19}$$

$$\left(1+\lambda n\right)\left[\frac{1}{\varepsilon}nZ-\frac{2\Omega}{\varepsilon}\frac{dW}{dz}\right]=-\frac{\nu}{k_1}Z \tag{20}$$

$$\left\{En-\kappa\left(\frac{d^2}{dz^2}-k^2\right)\right\}\Theta=\beta W \tag{21}$$

Expressing the coordinate (x, y, z) = (x*d, y*d, z*d), D* = d/dz* in new unit of length 'd' thereafter dropping the superscript for simplicity and also putting

a = kd, $\sigma=\frac{nd^2}{\nu}$, $p_1=\frac{\nu}{\kappa}$ is the Prandtl number, $P_l=\frac{k_1}{d^2}$ is the dimensionless medium permeability and $F=\frac{\lambda\nu}{d^2}$.

Equations (19) – (21) in non-dimensional form can be written as

$$\left[\frac{\sigma}{\varepsilon}+\frac{\left(1+F\sigma\right)^{-1}}{P_l}\right]\left(D^2-a^2\right)W=-\frac{2d^3\Omega}{\varepsilon\nu}DZ-\frac{g_0a^2d^2}{\nu}\left(1+Mz\right)\alpha\Theta \tag{22}$$

$$\left[\frac{\sigma}{\varepsilon}+\frac{\left(1+F\sigma\right)^{-1}}{P_l}\right]Z=\frac{2d\Omega}{\varepsilon\nu}DW \tag{23}$$

$$\left[D^2-a^2-\sigma E_1p_1\right]\Theta=-\frac{\beta d^2}{\kappa}W \tag{24}$$

we consider the case where both the boundaries are free and perfect conductor of heat, while adjoining medium is assumed to be electrically non-conducting. Thus boundary conditions for this case are

$$W=D^2W=DZ=Q=0 \text{ at } z=0 \text{ and } z=1. \tag{25}$$

Eliminating Θ, Z between (22) – (24), we get

$$\left\langle\left(D^2-a^2-\sigma E_1p_1\right)\left(D^2-a^2\right)\left[\frac{\sigma}{\varepsilon}+\frac{\left(1+F\sigma\right)^{-1}}{P_l}\right]^2\right\rangle W$$
$$=-\frac{T_A}{\varepsilon}\left(D^2-a^2-\sigma E_1p_1\right)D^2W+\left[\frac{\sigma}{\varepsilon}+\frac{\left(1+F\sigma\right)^{-1}}{P_l}\right]a^2R\left(1+Mz\right)W \tag{26}$$

where $R = \frac{g_0 \alpha \beta d^4}{\kappa \nu}$ is the thermal Rayleigh number, $T_A = \left(\frac{2\Omega d^2}{\nu} \right)^2$ is Taylor number.

Using the boundary conditions (25) it can be shown that all the even order derivative of W vanish at the boundary and hence the proper solution of equation (26) characterising lowest mode is

$$W = W_0 sinz \tag{27}$$

where W_0 is constant. Substituting the (27) in equation (26) and letting

$a^2 = \pi^2 x, R_1 = \frac{R}{\pi^4}, i\sigma = \frac{\sigma}{\pi^2}$ and $\mathrm{P} = \pi^2 P_l$.

We obtain the following dispersion relation

$$R_1 = \frac{(1+x+i\sigma_1 E_1 p_1)(1+x)\left\{\left(\frac{i\sigma_1}{\varepsilon} + \frac{(1+i\sigma_1\pi^2 F)^{-1}}{P}\right)^2\right\} + \frac{T_A}{\varepsilon}\{(1+x+i\sigma_1 E_1 p_1)\}}{x\left\{\frac{i\sigma_1}{\varepsilon} + \frac{(1+i\sigma_1\pi^2 F)^{-1}}{P}\right\}\left(\frac{M}{4}+\frac{1}{2}\right)}. \tag{28}$$

4 Stationary convection

When the instability sets in as a stationary convection, the marginal state will be characterized by $\sigma = 0$. On putting $\sigma = 0$ ($\sigma_1 = 0$) in equation (28) reduces to

$$R_1 = \frac{(1+x)^2}{x}\left(\frac{4}{2+M}\right)\frac{1}{P} + \frac{(1+x)}{x\varepsilon}\left(\frac{4}{2+M}\right)PT_A \tag{29}$$

Thus in the stationary convection the Visco- elastic parameter F vanish with σ and thus Maxwellian Visco-elastic fluid behaves like an ordinary Newtonian fluid.

To study the effect of variable gravity field, rotation and medium permeability, we examine the nature of $\frac{dR_1}{dM}, \frac{dR_1}{dT_A}$ and $\frac{dR_1}{dP}$ analytically.

Equation (29) yield,

$$\frac{dR_1}{dM} = -\left(\frac{2}{2+M}\right)^2 \frac{(1+x)^2}{xP} - \left(\frac{2}{2+M}\right)^2 \frac{(1+x)}{x\varepsilon} PT_A < 0,$$

thus variable gravity has destabilizing effect on the thermal convection in porous medium.

Also from equation (29), we have

$$\frac{dR_1}{dT_A}=\left(\frac{4}{2+M}\right)\frac{(1+x)}{x\varepsilon}P>0,$$

thus rotation has stabilizing effect on the thermal convection in porous medium.
Also from equation (29), we have

$$\frac{dR_1}{dP}=\frac{(1+x)}{x}\left(\frac{4}{2+M}\right)\left\{\frac{1}{P^2}-\frac{T_A}{\varepsilon}\right\}<0,$$

thus medium permeability have stabilizing effect if $\frac{P^2T_A}{\varepsilon}<1$ and destabilizing effect if $\frac{P^2T_A}{\varepsilon}>1$ on the thermal convection in porous medium

Thus in stationary convection, Maxwellian Visco-elastic fluid behaves like an ordinary Newtonian fluid and variable gravity; suspended particles and medium permeability have destabilizing effect on the system.

5 Oscillatory modes and the 'Principle of exchange of Stabilities'

Here we examine the possibility of oscillatory modes, if any, on the stability problem due to suspended particles, variable gravity field and medium permeability. Multiplying the equation (22) by W* (the complex conjugate of W), integrating over range of z and making use boundary condition (27); we get

$$\left[\frac{\sigma}{\varepsilon}+\frac{(1+F\sigma)^{-1}}{P_l}\right]\int_0^1\left(|DW|^2+a^2|W^2|\right)dz-d^2\left[\frac{\sigma^*}{\varepsilon}+\frac{(1+F\sigma^*)^{-1}}{P_l}\right]\int_0^1(|Z|)^2dz$$
$$-\frac{g_0a^2}{\upsilon}\frac{\kappa}{\beta}\left\{\int_0^1(1+Mz)|D\Theta|^2+a^2|\Theta|^2\,dz+\sigma E_1p_1\int_0^1(1+Mz)|\Theta|^2\,dz\right\}=0 \qquad (30)$$

Now for neutral mode, we must have σ = iσ$_i$ with σ$_i$ is real, and the real and imaginary part of equation (30), we have

$$\frac{1}{P_l}\left(\frac{1}{1+\sigma_i^2F^2}\right)\left\{\int_0^1\left(|DW|^2+a^2|W|^2\right)dz-d^2\int_0^1|Z|^2\,dz\right\}-\frac{g_0a^2\kappa}{\upsilon\beta}\int_0^1(1+Mz)|\Theta|^2dz=0, \qquad (31)$$

and

$$\sigma_i \left[\begin{matrix} \frac{1}{\varepsilon} - \frac{F}{P_l}\left(\frac{1}{1+\sigma_i^2 F^2}\right)\left\{\int_0^1 |DW|^2 + a^2 |W|^2\, dz + d^2 \int_0^1 |Z|^2 dz\right\} \\ + \frac{g_0 a^2 \kappa}{\vartheta\beta} E_1 p_1 \int_0^1 (1+Mz)|\Theta|^2 dz \end{matrix} \right] = 0. \tag{32}$$

From equation (32), it follow that σi=0 or σi # 0, which mean that modes may be non oscillatory or oscillatory.

The term inside the bracket is non zero if $\frac{1}{\varepsilon} > \frac{F}{P_l}$, which implies that $\sigma_I = 0$, thus the mode are non oscillatory and principle of exchange of stabilities is satisfied. Thus $\frac{1}{\varepsilon} > \frac{F}{P_l}$ is the necessary condition for the validity of principle of exchange of stabilities for Maxwellian Visco-elastic fluid in porous.

6 Sufficient conditions for the stability

Multiplying equation (30) by σ*, the complex conjugate of σ and dividing by $|\sigma|^2$, we have

$$\begin{gathered} \left[\frac{1}{\varepsilon} + \frac{\sigma^*}{|\sigma|^2}\frac{(1+F\sigma)^{-1}}{P_l}\right]_1 \int_0^1 |DW|^2 + a^2|W|^2\, dz - d^2 \left[\frac{1}{\varepsilon} + \frac{\sigma^*}{|\sigma|^2}\frac{\left(1+F\sigma^*\right)^{-1}}{P_l}\right]\int_0^1 |DZ|^2 dz \\ = \frac{\sigma^*}{|\sigma|^2}\frac{g_0 a^2 \kappa}{\vartheta\beta}\left\{ \begin{matrix} \int_0^1 (1+Mz)\left(|D\Theta|^2 + a^2|\Theta|^2 \right) dz \\ + E_1 p_1 \int_0^1 (1+Mz)|\Theta|^2 dz \end{matrix} \right\} = 0 \end{gathered} \tag{33}$$

Letting $\sigma = \sigma_r + i\sigma_i$ in equation (35) where σ_r, σ_i are real and equating parts, we get

$$\begin{gathered} \sigma_r \left\langle \begin{matrix} \left[\frac{1}{|\sigma|^2 P_l\left(1+F^2\sigma_r^2\right)}\right]\left\{\int_0^1 \left(|DW|^2 + a^2|W|^2\right)dz - d^2 \int_1^2 |Z|^2\, dz\right\} \\ - \frac{g_0 a^2 \kappa}{\vartheta\beta}\int_0^1 (1+Mz))\left(|D\Theta|^2 + a^2|\Theta|^2\right)dz \end{matrix} \right\rangle \\ = -\left\langle \begin{matrix} \left[\frac{1}{\varepsilon} + \frac{\sigma^2 F}{P_l\left(1+F^2\sigma_r^2\right)}\right]\left\{\int_0^1 \left(|DW|^2 + a^2|W|^2\right)dz - d^2 \int_1^2 |Z|^2\, dz\right\} \\ - \frac{g_0 a^2 \kappa}{\vartheta\beta} E_1 p_1 \int_0^1 (1+Mz)|\Theta|^2\, dz \end{matrix} \right\rangle . \end{gathered} \tag{34}$$

From equation (34) it follow that σ_r, is negative or positive, therefore system may be stable or unstable.

7 Conclusion

Thermal instability in the linear variable gravitational field of a rotating Maxwelian Visco-elastic fluid in porous medium has been studied. In case of stationary convection, Maxwellian Visco-elastic fluid behaves like an ordinary Newtonian fluid. And it is found that rotation has stabilizing while variable gravity field has destabilizing effect on the system and medium permeability have stabilizing/destabilizing effect on the system depending upon some condition. The necessary condition for the validity of principle of exchange of stabilities for Maxwellian Visco-elastic fluid in porous medium is found to be $\frac{1}{\varepsilon} > \frac{F}{P_l}$. The system may be stable or unstable

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