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On an algorithm of the dynamic reconstruction of inputs in systems with time-delay

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ABSTRACT

An algorithm of the dynamic reconstruction of inputs in systems described by differential equations with time-delay is presented. This algorithm is stable with respect to information noises and computation errors; it is based on appropriate modifications of the principle of extremal aiming, which is known in the theory of guaranteed control.

Keywords: Reconstruction, Feedback control.

MSC 2010 codes: 34K29, 93B52.

1 Introduction

We consider the problem of the stable reconstruction of an unknown input in a dynamic system from the results of inaccurate observations of its trajectory. Let us explain the essence of the problem. A dynamic system is described by a vector nonlinear differential equation with time delay. The trajectory of the system depends on a time-varying input, which is interpreted in what follows as a control. Both the input and the trajectory are not known a priori. The phase states of the system are measured during the operation of the system. The measurements are, generally speaking, inaccurate. It is required to design an algorithm for the approximate reconstruction of the input. The algorithm must be dynamic and stable. The dynamic property means that the current values of the approximation of the corresponding coordinates are produced in real time, while the stability property means that the approximations are as accurate as possible if the measurements are accurate enough.

We adopt the following notation:

R^n is the n-dimensional space with the Euclidean norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$;

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$L_2(T;R^n)$ is the Hilbert space of all functions integrable with the square of their norm and mapping the set T to the space R^n (with the norm $|\cdot|_{L_2(T;R^n)}$);

$C(T;R^n)$ is the Banach space of all continuous functions mapping the set T to the space R^n with the sup-norm $|\cdot|_{C(T;R^n)}$;

B' is the transposed matrix B;

N is the set of positive integers;

$W^{1,\infty}(T;R^n)$ is the Banach space of all differentiable functions whose first derivatives belong to $L_\infty(T;R^n)$.

2 Problem statement and solution method

The problem discussed in the present paper can be formulated as follows. There is a dynamic system Σ operating on a time interval $T=[t_0,\vartheta]$, where $\vartheta<+\infty$. We assume that Σ is described by the system of differential equations with time-delay

$$\dot{x}(t)=f(t,x(t),x(t-\tau))+Bu(t)+F(t),\quad t\in T,\quad x(t_0+s)=x_0(s),\quad s\in[-\tau,0], \tag{2.1}$$

where $x(t)\in R^n$, $u(t)\in R^n$, and $F(\cdot)\in C(T;R^n)$ is a given function, $\tau=const>0$ is a delay. The trajectory $x(t)=x(t;t_0,x_0(s),u(\cdot))\in R^n$, $t\in T$, depends on the initial state $x_0(s)$, $s\in[-\tau,0]$, and on the time-varying unknown input $u(\cdot)\in C(T;R^n)$. We assume that the function $x(s)$ is continuously differentiable. Let $\Delta=\{\tau_i\}_{i=0}^m$ be a uniform partition of the interval T with a step δ; we have $\tau_{i+1}=\tau_i+\delta$ and $\tau_m=\vartheta$. The trajectory of the system $x(t)$ is measured at the times τ_i with an error. The results of these inaccurate measurements are vectors $\xi_i^h\in R^n$ satisfying the inequalities

$$\left\|\xi_i^h-x(\tau_i)\right\|\le h,\quad i\in[0:m-1], \tag{2.2}$$

where $h\in(0,1)$ is the value of information error. It is required to design an algorithm that reconstructs in real time the control $u(\cdot)$ that generates $x(\cdot)$ from the results of inaccurate measurements of the trajectory.

Before presenting a strict mathematical formulation of the problem, we will describe the method of its solution. This method is based on a known principle of positional control, namely, on the principle of auxiliary models (Krasovskii and Subbotin 1988; Kryazhimskii and Osipov 1983). Let us formulate this principle in a form convenient for us. Let Σ be described by system (2.1), where the vector function $f:T\times R^n\times R^n\to R^n$ is continuous by the first argument, satisfies the local Lipschitz condition in the second and third arguments, and conforms to the corresponding growth conditions:

$$\|f(t,x,y)\|\le c(1+\|x\|+\|y\|)\quad \forall x,y\in R^n,\ c=const>0.$$

The solution of system (2.1) corresponding to the initial state $x_0(s)$ and control $u(\cdot)\in C(T;R^n)$, as well as solutions of all systems of differential equations considered below, is understood in the sense of Carathéodory. Solutions of system (2.1) are denoted by $x(t;t_0,x_0(s),u(\cdot))$. We first fix a family $\{\Delta_h\}$ of partitions of the interval T into half-open intervals $[\tau_{h,i},\tau_{h,i+1})$:

$$\Delta_h=\left\{\tau_{h,i}\right\}_{i=0}^{m_h},\quad \tau_{h,i+1}=\tau_{h,i}+\delta,\quad \delta=\delta(h),\quad \tau_{h,0}=\tau_0,\quad \tau_{h,m_h}=\vartheta. \tag{2.3}$$

Then, choose a system M (called a model) whose motion $w^h(t)$, $t\in T$, is a solution of a specially chosen ordinary differential equation

$$\dot{w}^h(t)=\Phi(t,w^h(t),\xi_i^h,\xi_{i-k_h}^h,v^h(t)),\quad t\in[\tau_{h,i},\tau_{h,i+1}),\quad i\in[0:m_h-1], \tag{2.4}$$

with the initial condition

$$w^h(t_0)=w_0^h.$$

To simplify the exposition, we assume that $\tau=\delta(h)k_h$, where $k_h\in N$. Here, $v^h(t)$ is a control and $w^h(t)$ is a vector whose dimension is equal to the dimension of the vector x. The notation $w^h(t)=w^h(t;w_0^h,v^h(\cdot))$ is used for the solution of system (2.4) (with the initial condition w_0^h).

After the model has been defined (i.e., equation (2.4) has been written), the solution algorithm is identified with the law of forming feedback controls in the model. The procedure of controlling the model is preceded by the choice of its initial state w_0^h. In accordance with the terminology adopted in the theory of guaranteed control (Krasovskii and Subbotin 1988; Kryazhimskii and Osipov 1983), the laws of forming the controls $v^h(\cdot)$ in the model are called strategies and are identified with pairs

$$S_h=(\Delta_h,U_h),$$

where Δ_h is defined according to (2.3) and U_h is a function that maps each position $q^{(i)}(\cdot)=\left\{\tau_i,\xi_i^h,w^h(\tau_i)\right\}$, $i\in[0:m-1]$, to a vector

$$U_h(q^{(i)}(\cdot))=v_i^h. \tag{2.5}$$

Thus, the triple (Δ_h,M,U_h) for every $h\in(0,1)$ defines some algorithm D_h on the set of measurements $\xi(\cdot)\in\Xi(x(\cdot),h)$, which forms the output

$$D_h\xi^h(\cdot)=\left\{w^h(\cdot),v^h(\cdot)\right\},\quad h\in(0,1), \tag{2.6}$$

according to feedback principle (2.4),(2.5). Here, $\Xi(x(\cdot),h)$ denotes the set of all piecewise constant functions $\xi^h(\cdot)$,

$$\xi^h(t)=\xi_i^h,\quad t\in\delta_{h,i}=[\tau_{h,i},\tau_{h,i+1}),$$

satisfying inequalities (2.2) for $\tau_i=\tau_{h,i}\in\Delta_h$. The control $v^h(\cdot)$ is also piecewise constant:

$$v^h(t)=v_i^h,\quad t\in\delta_{h,i}. \tag{2.7}$$

The algorithm D_h (for fixed h) works by the following scheme. A partition $\Delta=\Delta_h=\{\tau_i\}_{i=0}^m$ ($\tau_i=\tau_{h,i}$, $m=m_h$) of the interval T and an auxiliary system, i.e., a model M, are chosen and fixed before the initial time t_0. The algorithm D_h is decomposed into $m-1$ identical steps. The ith step, $i\geq 0$, is performed on the time interval $[\tau_i,\tau_{i+1})$. The following operations are carried out during this step. At time τ_i, the output $x(\tau_i)$ is measured (with an error); i.e., a vector ξ_i^h with property (2.2) is found. Then, a control in model (2.4) is found by rule (2.5), (2.7) and the memory is modified; i.e., the segment $w^h(t)=w^h(t;\tau_i,w^h(\tau_i),v_i^h)$, $t\in[\tau_i,\tau_{i+1}]$, of the trajectory of the model is formed instead of $w^h(\tau_i)$. The procedure stops at time ϑ.
The problem of the dynamic reconstruction consists in designing a family of algorithms $D_h=(\Delta_h,M,U_h)$ (2.3)–(2.5), (2.7), $h\in(0,1)$, such that

$$v^h(\cdot)\to u(\cdot)\ \text{in}\ C(T;R^n)\ \text{as}\ h\to 0. \tag{2.8}$$

A family of algorithms $D_h=(\Delta_h,M,U_h)$ (2.3)–(2.5), (2.7), $h\in(0,1)$, with property (2.8) is called a reconstructing family. The problem in question consists in designing a reconstructing family of algorithms.

Problems of reconstructing of unknown characteristics of a dynamical system, through measurements of a part of its phase coordinates are embedded into the theory of inverse problems of dynamics. This theory is intensively developed at the present time. One of approaches to solving similar problems based on methods of the theory of positional control (Krasovskii and Subbotin 1988) was suggested in (Kryazhimskii and Osipov 1983) and developed in (Osipov and Kryazhimskii 1995; Maksimov 2002; Osipov *et al*. 2011; Blizorukova *et al*. 2002; Maksimov 2004; Maksimov 2007). In the present paper following the researches in this field, an algorithm of dynamical reconstruction of a control of a time-delay system is designed. This algorithm is dynamical and work in the “real time” mode. It is stable with respect to informational noises and computational errors.

The algorithms suggested in the works cited above realize the reconstruction process in the mean-square metric. In this paper, a solving algorithm for reconstructing unknown input in the uniform metric is presented. We will consider a time-delay system. For other dynamical algorithms reconstructing unknown characteristics (in the L_2-metric) of the time-delay system, see (Blizorukova *et al*. 2002; Maksimov 2004; Maksimov 2007).

3 The solving algorithm

In what follows, we assume that the following condition is valid.

Condition 1. The matrix B is nondegenerate.

As mentioned above, to solve the problem, we must define a family of algorithms D_h, $h \in (0,1)$, consisting of (a) a family of partitions Δ_h of the time interval T of form (2.3); (b) some auxiliary system (model) M of form (2.4); (c) the law of forming the control in the model by the feedback principle U_h (2.5), (2.7). At first, we fixed a family of partitions of the interval T:

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h)$$

(see (2.3)). Note that, in virtue of the fact that the function f is Lipschitz, it is possible to give a number $M > 0$, for which the following inequalities

$$\|\dot{x}(t)\| \le M \quad \text{for a.a.} \quad t \in T, \tag{3.1}$$

$$\|f(t, x(t), x(t-\tau)) - f(\tau_i, \xi_i^h, \xi_{i-k_h})\| \le M(\delta + h + \omega(\delta)) \text{ for } t \in \delta_i = [\tau_i, \tau_{i+1}) \tag{3.2}$$

are true. Here, $\tau_i = \tau_{h,i}$, $\omega(\delta)$ is the modulo of continuity of the function $t \to f(t, x(t), x(t-\tau))$, $t \in T$, i. e.,

$$\omega(\delta) = \sup\{\|f(t, x(t), x(t-\tau)) - f(t-\delta, x(t-\delta), x(t-\delta-\tau))\| : t \in [\delta, \vartheta]\}.$$

As the model M, we take a linear system described by the following ordinary differential equation

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h, \xi_{i-k_h}^h) + Bv^h(t) \quad \text{for a.a.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}), \tag{3.3}$$

$i \in [0 : m-1]$, $m = m_h$, with the initial condition

$$w^h(0) = x_0(0).$$

Let a function $\alpha(h) : (0,1) \to (0,1)$ be fixed. The law of forming the control in the model is defined as follows:

$$U_h(q^{(i)}(\cdot)) = v_i^h = -\frac{1}{\alpha} B'[w^h(\tau_i) - \xi_i^h] \quad \text{for} \quad t \in \delta_i, \ \alpha = \alpha(h). \tag{3.4}$$

Let the control $v^h(t)$ be defined in equation (3.3) by formula (3.4). In this case, the control in the model is found by the feedback principle. Thus, equation (3.3) has the form

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h, \xi_{i-k_h}^h) - \frac{1}{\alpha} BB'[w^h(\tau_i) - \xi_i^h] \quad \text{for a. a.} \quad t \in \delta_i. \tag{3.5}$$

Lemma 1. Let the following conditions

$$\alpha(h) \to 0, \quad \delta(h) \to 0, \quad \delta(h)\alpha^{-1}(h) \to 0, \quad h\alpha^{-1}(h) \to 0 \text{ as } h \to 0 \tag{3.6}$$

be valid. Then, uniformly with respect to $h \in (0,1)$ and $\xi^h(\cdot) \in \Xi(x(\cdot), h)$, the inequality

$$\int_{\tau_i}^{\tau_{i+1}} \|\dot{w}^h(s)\| ds \le C\delta \tag{3.7}$$

is fulfilled. Here, $C = \text{const} > 0$, $\delta = \delta(h)$, $\tau_i = \tau_{h,i}$.

Proof. Using (3.5), we obtain the following equalities

$$\frac{d}{dt}[w^h(t) - x(t)] = f(\tau_i, \xi_i^h, \xi_{i-k_h}^h) - \frac{1}{\alpha} BB'[w^h(\tau_i) - \xi_i^h] - f(t, x(t), x(t-\tau)) - Bu(t) =$$
$$= -\frac{1}{\alpha} BB'[w^h(t) - x(t)] + \Psi_h^{(1)}(s) \quad \text{for a. a.} \quad t \in \delta_i$$

and

$$w^h(0) - x(0) = 0.$$

Here,

$$\Psi_h^{(1)}(s) = \Psi_h(s) + \frac{1}{\alpha} BB'[w^h(s) - w^h(\tau_i)],$$
$$\Psi_h(s) = -\frac{1}{\alpha} BB'[x(s) - \xi_i^h] + [f(\tau_i, \xi_i^h, \xi_{i-k_h}^h) - f(s, x(s), x(s-\tau))] - Bu(s), \quad s \in \delta_i.$$

Note that, due to (3.1), (3.2) and (3.6), the family of functions $\Psi_h(\cdot)$ is bounded:

$$\|\Psi_h(s)\| \le M^{(1)} \quad \text{for almost all} \quad t \in T \quad \text{and all} \quad h \in (0,1) \tag{3.8}$$

uniformly with respect to h. Then, we have

$$w^h(t) - x(t) = \int_0^t e^{-\frac{1}{\alpha}BB'(t-s)} \Psi_h^{(1)}(s)\, ds, \quad t \in T. \tag{3.9}$$

Denote

$$\mu(t) = \max_{0 \le \tau \le t} \|w^h(\tau) - x(\tau)\|, \quad f_h(t) = f(\tau_i, \xi_i^h, \xi_{i-k_h}^h) \quad \text{for} \quad t \in \delta_i.$$

The following estimates are true:

$$\frac{1}{\alpha}\|BB'\|\int_{\tau_i}^{\tau_{i+1}}\|\dot{w}^h(s)\|ds \le \frac{K_0}{\alpha}\int_{\tau_i}^{\tau_{i+1}}\left\|f_h(s)-\frac{1}{\alpha}BB'[w^h(\tau_i)-\xi_i^h]\right\|ds \le \tag{3.10}$$
$$\le K_1\frac{\delta}{\alpha}+K_2\frac{\delta}{\alpha^2}(\mu(\tau_i)+h), \quad \mu(\tau_i)\le\mu(\tau_{i+1}).$$

Moreover, we have

$$\|\Psi_h^{(1)}(t)\| \le \|\Psi_h(t)\|+\frac{1}{\alpha}\|BB'\|\int_{\tau_i}^{\tau_{i+1}}\|\dot{w}^h(s)\|ds \quad \text{при} \quad t\in\delta_i. \tag{3.11}$$

Therefore, from (3.9)–(3.11), we deduce that

$$\mu(t)\le K_3\left(\frac{\delta}{\alpha}+\frac{\delta}{\alpha^2}\mu(\tau_i)+\frac{\delta h}{\alpha^2}\right)\int_0^t\left\|e^{-\frac{1}{\alpha}BB'(t-s)}\right\|ds+ \tag{3.12}$$
$$+\int_0^t\left\|e^{-\frac{1}{\alpha}BB'(t-s)}\right\|\|\Psi_h(s)\|ds, \quad t\in\delta_i.$$

Using (3.8), we obtain

$$\int_0^t\left\|e^{-\frac{1}{\alpha}BB'(t-s)}\right\|\|\Psi_h(s)\|ds \le K_4\int_0^t\left\|e^{-\frac{1}{\alpha}BB'(t-s)}\right\|ds. \tag{3.13}$$

In virtue of condition 1, the matrix BB' is positive definite. In this case, all the eigenvalues of this matrix is real and the least value (denote it by ν) is positive. Then, the following inequality

$$\int_0^t\left\|e^{-\frac{1}{\alpha}BB'(t-s)}\right\|ds \le K_5\int_0^t e^{-\frac{\nu}{\alpha}(t-s)}\,ds= \tag{3.14}$$
$$=K_5\frac{\alpha}{\nu}e^{-\frac{\nu}{\alpha}(t-s)}\Big|_0^t=K_5\frac{\alpha}{\nu}(1-e^{-\frac{\nu}{\alpha}t})\le K_6\alpha$$

is fulfilled. From (3.13) and (3.14), it follows that

$$\int_0^t\left\|e^{-\frac{1}{\alpha}BB'(t-s)}\right\|\|\Psi_h(s)\|ds \le K_7\alpha. \tag{3.15}$$

In turn, assuming $t=\tau_i$ and taking into account (3.15), from (3.12) we derive

$$\left(1-\frac{K_3K_6\delta}{\alpha}\right)\mu(\tau_i)\le K_8\left(\alpha+\delta+\frac{\delta h}{\alpha}\right).$$

Therefore, for sufficiently small h (for example, such that $1-\frac{K_3K_6\delta}{\alpha}\ge\frac{1}{2}$), we have

$$\mu(\tau_i) \le K_9\left(\alpha+\delta+\frac{\delta h}{\alpha}\right) \le K_{10}(\alpha+\delta) \tag{3.16}$$

(see (3.6)). By analogy with (3.10), we obtain

$$\int_{\tau_i}^{\tau_{i+1}} \|\dot{w}^h(s)\|\,ds \le K_{11}\left\{\delta+\frac{\delta}{\alpha}(\mu(\tau_i)+h)\right\}.$$

In addition, using (3.6) again, we have

$$\delta+\frac{\delta}{\alpha}(\mu(\tau_i)+h) \le \delta+K_{10}\frac{\delta}{\alpha}(\alpha+\delta) \le K_{12}\delta.$$

Hence,

$$\int_{\tau_i}^{\tau_{i+1}} \|\dot{w}^h(s)\|\,ds \le K_{13}\delta.$$

Inequality (3.7) is established. The lemma is proved.
Lemma 1 implies the following theorem.

Theorem 2. If $u(\cdot)\in W^{1,\infty}(T;R^n)$ and agreement relations (3.6) for the parameters are fulfilled, then, whatever $\varepsilon>0$ maybe, the convergence $v^h(\cdot)\to u(\cdot)$ in $C([\varepsilon,\vartheta];R^n)$ takes place as $h\to 0$. If, in addition, $u(0)=0$, then $v^h(\cdot)\to u(\cdot)$ in $C(T;R^n)$. The following estimate for the convergence rate is valid:

$$\|v^h(t)-u(t)\| \le$$
$$\le \tilde{c}_1\alpha(h)+\tilde{c}_2(h+\delta(h))\alpha^{-1}(h)+\tilde{c}_3\omega(\delta(h))+\tilde{c}_4\left\|e^{-\frac{1}{\alpha}BB't}Bu(0)\right\|.$$

Proof. It is easily seen that the equality

$$\frac{1}{\alpha}BB'[w^h(t)-x(t)] = \int_0^t\left(\frac{d}{ds}e^{-\frac{1}{\alpha}BB'(t-s)}\right)\Psi_h^{(1)}(s)\,ds = \tag{3.17}$$
$$= -\int_0^t\left(\frac{d}{ds}e^{\frac{1}{\alpha}BB'(t-s)}\right)Bu(s)\,ds+\sum_{j=1}^{3}\int_0^t\left(\frac{d}{ds}e^{-\frac{1}{\alpha}BB'(t-s)}\right)\gamma_\delta^{(j)}(s)\,ds$$

holds. Here,

$$\gamma_\delta^{(1)}(s)=\frac{1}{\alpha}BB'[w^h(s)-w^h(\tau_i)],$$
$$\gamma_\delta^{(2)}(s)=-\frac{1}{\alpha}BB'[x(s)-\xi_i^h],$$
$$\gamma_\delta^{(3)}(s)=f(\tau_i,\xi_i^h,\xi_{i-k_h}^h)-f(s,x(s),x(s-\tau)) \quad \text{for a.a.} \quad s\in\delta_i.$$

From (3.7), we have

$$\left\|\gamma_\delta^{(1)}(s)\right\| \le C_1 \frac{\delta}{\alpha}, \quad s \in T. \tag{3.18}$$

Taking into account (2.2) and (3.1), we conclude that

$$\left\|\gamma_\delta^{(2)}(s)\right\| \le C_2 \frac{\delta + h}{\alpha}, \quad s \in T. \tag{3.19}$$

Moreover (see (3.2)),

$$\left\|\gamma_\delta^{(3)}(s)\right\| \le M(\delta + h + \omega(\delta)), \quad s \in T. \tag{3.20}$$

In this case, from (3.18)–(3.20), taking into account (3.14) and (3.15), we derive

$$\left\|\sum_{j=1}^{3} \int_0^t \left(\frac{d}{ds} e^{-\frac{1}{\alpha}BB'(t-s)}\right)\gamma_\delta^{(j)}(s)\, ds\right\| \le \tag{3.21}$$
$$\le \rho(h,\alpha,\delta) = C_3\left(\delta + h + \omega(\delta) + \frac{\delta + h}{\alpha}\right).$$

Integrating the first term in the right-hand side of equality (3.17) by parts, we obtain

$$-\int_0^t \left(\frac{d}{ds} e^{-\frac{1}{\alpha}BB'(t-s)}\right)Bu(s)\, ds = \tag{3.22}$$
$$= e^{-\frac{1}{\alpha}BB't}Bu(0) - Bu(t) + \int_0^t e^{-\frac{1}{\alpha}BB'(t-s)}B\dot{u}(s)\, ds.$$

Moreover, from (2.2), (3.1), and (3.7), for all $t \in \delta_i$, we derive the estimate

$$\left\|\frac{1}{\alpha}BB'\left\{[w^h(t) - x(t)] - [w^h(\tau_i) - \xi_i^h]\right\}\right\| \le \tag{3.23}$$
$$\le \frac{C_4}{\alpha}\left\{\int_{\tau_i}^{\tau_{i+1}} \|\dot{w}^h(s)\|\, ds + h + \delta\right\} \le C_5 \frac{h + \delta}{\alpha}.$$

From (3.17), (3.21), (3.23), and (3.6), due to the boundedness of $\dot{u}(\cdot)$ ($\dot{u}(\cdot) \in L_\infty(T; R^n)$), it follows that the inequality

$$\left\|-\frac{1}{\alpha}BB'[w^h(\tau_i) - x(\tau_i)] - Bu(t)\right\| \le \tag{3.24}$$
$$\le \rho(h,\delta,\alpha) + C_5 \frac{h + \delta}{\alpha} + C_6\alpha + \left\|e^{-\frac{1}{\alpha}BB't}Bu(0)\right\|$$

is true. The conclusion of the theorem follows from (3.24). The theorem is proved.
Theorem 2 implies the next one.

Theorem 3. Let $u(\cdot)\in W^{1,\infty}(T;R^n)$, agreement relations (3.6) for the parameters be fulfilled, and $u(0)=0$. Then, the family of algorithms $D_h=(\Delta_h, M, U_h)$ of form (2.3), (3.3), (2.7), and (3.4) is reconstructing.

Along with measuring the phase states at discrete time instants (see (2.2)), one can consider the case when measuring the phase states $x(t)$ is performed "continuously". Namely, it is assumed that at every time instant $t\in T$ we measure the phase states of system (2.1); as a result, we obtain vectors $\xi^h(t)\in R^n$ satisfying the equality

$$\left\|\xi^h(t)-x(t)\right\|\le h,\quad t\in T,$$

where the functions $\xi^h(t),\ t\in T,$ are Lebesgue measurable.

In this case, it is possible also to consider the problem of reconstructing the input $u(\cdot)$ and to apply the scheme used above for its solving. Indeed, as a model, we take the n-dimensional system

$$\dot{w}^h(t)=f(t,\xi^h(t),\xi^h(t-\tau))+Bv^h(t)\quad \text{for a.a.}\quad t\in T \tag{3.25}$$

with the initial condition

$$w^h(0)=x_0(0).$$

Thus, the model is described by the linear ordinary differential equation. The control $v=v^h(\cdot)$ in model (3.25) is defined by the rule

$$v^h(t)=-\alpha^{-1}B'(w^h(t)-\xi^h(t)),\quad \alpha=\alpha(h). \tag{3.26}$$

It is easily seen that the function $v^h(t)$ of form (3.26) is calculated as follows:

$$v^h(t)=\operatorname{argmin}\left\{\alpha\|v\|^2+2(\xi^h(t)-w^h(t)),B'v):v\in R^n\right\}.$$

For such choice of the control $v^h(\cdot)$, system (3.25) takes the form

$$\dot{w}^h(t)=f(t,\xi^h(t))-\alpha^{-1}BB'(w^h(t)-\xi^h(t)),\quad t\in T.$$

The solution of this system is denoted by $w^h(\cdot)$.

Theorem 4. Let $\alpha(h)\to 0$ and $h\alpha^{-1}(h)\to 0$ as $h\to 0$. If $u(\cdot)\in W^{1,\infty}(T;R^n)$, then, whatever $\varepsilon>0$ maybe, the convergence $v^h(\cdot)\to u(\cdot)$ in $C([\varepsilon,\vartheta];R^n)$ takes place as $h\to 0$. If, in addition, $u(0)=0$, then $v^h(\cdot)\to u(\cdot)$ in $C(T;R^n)$. The following estimate for the convergence rate is valid:

$$\left\|v^h(t)-u(t)\right\|\le c_1\alpha(h)+c_2h\alpha^{-1}(h)+c_3\left\|e^{-\frac{1}{\alpha}BB't}Bu(0)\right\|$$

Proof. For almost all $t \in T$, we have

$$\frac{d}{dt}[w^h(t) - x(t)] = f(t, \xi^h(t), \xi^h(t-\tau)) - f(t, x(t), x(t-\tau)) - Bu(t) - \frac{1}{\alpha}BB'[w^h(t) - \xi^h(t)],$$

i.e.

$$\frac{d}{dt}[w^h(t) - x(t)] = -\frac{1}{\alpha}BB'[w^h(t) - \xi^h(t)] + \rho_h(t).$$

Here,

$$\rho_h(t) = f(t, \xi^h(t), \xi^h(t-\tau)) - f(t, x(t), x(t-\tau)) - Bu(t).$$

Consequently,

$$\frac{1}{\alpha}BB'[w^h(t) - x(t)] = \int_0^t (\frac{d}{ds}e^{-\frac{1}{\alpha}BB'(t-s)})\rho_h(s)\,ds = \tag{3.27}$$
$$= -\int_0^t (\frac{d}{ds}e^{-\frac{1}{\alpha}BB'(t-s)})Bu(s)\,ds + \sum_{j=1}^{2}\int_0^t (\frac{d}{ds}e^{-\frac{1}{\alpha}BB'(t-s)})\rho_\delta^{(j)}(s)\,ds.$$

Here,

$$\rho_\delta^{(1)}(s) = \frac{1}{\alpha}BB'[x(s) - \xi^h(s)],$$
$$\rho_\delta^{(2)}(s) = f(s, \xi^h(s), \xi^h(\mathrm{s}-\tau)) - f(s, x(s), x(\mathrm{s}-\tau)) \quad \text{for a.a.} \quad s \in T.$$

Using (3.1) and (3.2), we derive

$$\left\|\rho_\delta^{(1)}(s)\right\| \le c_1\frac{h}{\alpha}, \quad \left\|\rho_\delta^{(2)}(s)\right\| \le c_2 h \quad \text{for a.a.} \quad s \in T.$$

In this case, the latter inequalities imply

$$\left\|\sum_{j=1}^{2}\int_0^t (\frac{d}{ds}e^{-\frac{1}{\alpha}BB'(t-s)})\rho_\delta^{(j)}(s)\,ds\right\| \le c_3\frac{h}{\alpha}. \tag{3.28}$$

Integrating the first term in the right-hand side of equality (3.27)) by parts, we obtain (3.22). From (3.27), (3.28), and the boundedness of $\dot{u}(\cdot)$, it follows that

$$\left\|-\frac{1}{\alpha}BB'[w^h(t) - x(t)] - Bu(t)\right\| \le c_4\frac{h}{\alpha} + c_5\alpha + \left\|e^{-\frac{1}{\alpha}BB't}Bu(0)\right\|. \tag{3.29}$$

Estimate (3.29) completes the proof of the theorem.

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