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The similarity solutions of concentration dependent diffusion equation

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A B S T R A C T

This paper deals with the similarity solution of concentration dependent diffusion equation. A fundamental transport process in environmental fluid mechanics is diffusion. A well- known example is the diffusion of perfume in an empty room. The physical phenomenon of the solute transport due to combined effect of diffusion and convection in a medium is represented by the partial differential equation. It is a diffusion equation with constant diffusion coefficient. It is a parabolic type partial differential equation. It is based on the principle of conservation of mass. It is derived using Fick's law. The solution is obtained by using the method of group invariance under an infinitesimal transformation. The solution is represented in the form of Hermite polynomial which is well suited for meaningful interpretation of the response of the physical phenomenon. It is found in good agreement with results of earlier researchers. It is more classical than other results obtained by earlier researchers. It has wide applications in sold physic, petroleum engineering, chemical engineering and bioscience.

Keywords: Diffusion, Heat, Infinitesimal Transformation, Similarity Solution.

MSC 2010 codes: 76M55, 76R50, 54H15

1 Introduction

The diffusion is one of the transport processes that occur in nature. Diffusion is as central to our daily lives. The diffusion plays a key role in many process as diverse as intermixing of gases and liquids, permeation of atoms or molecules through membranes, evaporation of liquids, drying of timbers, doping silicon wafers to make semiconductor devices, transport of thermal neutrons in nuclear power reactors and heat transform. In this paper we review the heat equation. The heat or diffusion equation describes the heat transport due to combined effect of diffusion and convection in a medium. It is a partial differential equation of parabolic

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type, derived on the principle of conservation of mass using Fick's law. There are many researchers, who have discussed about this topic from different aspects, such as, for example, Janavicius and Turskiene (2005) have obtained analytical solution of nonlinear diffusion equation; Nataliya M. Ivanova (2008) have derived the exact solution of diffusion-convection equation. The systematical investigation of invariant solutions of different diffusion equation was started by the case of linear heat equation. (Miller, 1977; Olver, 1986). The infinite series solution of the diffusion equations are obtained by Carslow and Jaeger (1959); Crank (1975) and others. Talaat et al. (2007) have derived the analytical and numerical solution of one dimensional Burgers' equation. Chauhan (2013) has obtained the similarity solution of the diffusion equation using the technique of infinitesimal transformations of groups. In the mathematical theory of diffusion, the diffusion coefficient can be taken as constant in some cases, such as diffusion in dilute solution. In other cases, such as diffusion in high polymers, the diffusion coefficient depends on the concentrations of diffusing substance Fischer et al. (1979). This paper presents similarity solution of one dimensional diffusion equation with constant diffusion coefficient. This solution is obtained by using a technique of infinitesimal transformations of groups. The solution obtained is physically consistent with results of earlier researchers and which is more classical than other results obtained by the earlier researchers.

2 Mathematical formulation

Consider a metal rod of the length L, insulated except at its ends, lies along the x-axis with its left end at coordinate 0 and its right end at coordinate L. Suppose that the mass density ρ (units of mass divided by units of length) and thermal conductivity $K_0((energy \times length)/(time \times temprature))$ and specific heat c $(energy/(mass \times temperature)$ at each point in the rod depend only on the x-coordinate of the point. Let e, φ, and Q be as follows. The thermal energy density $(energy/length)$ at t (units of time after the time origin) at points with first coordinate x is $e(x,t)$. The heat flux $(energy/time)$ to the right at time t through the cross section consisting of points with first coordinate x is $\varphi(x,t)$. (A negative value for $\varphi(x,t)$ indicates heat flow to the left.) The heat energy being generated per unit time inside the rod at time t at points with first coordinate x is $Q(x,t)$. (A negative value for Q indicates a heat sink.) Suppose that $0 \le a \le b \le L$. Conservation of thermal energy tells us that the time-rate-of-change in thermal energy in the section of the rod consisting of points with first coordinate x satisfying $a \le x \le b$ is the net heat energy flowing per unit time across the boundaries of this section plus the net heat energy being generated internally in the section. Thus

$$\frac{d}{dt}\int_a^b e(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)dx$$

Assuming that e and φ have continuous first order partial derivatives, we have,

$$\int_a^b \frac{\partial e(x,t)}{\partial t}dt = -\int_a^b \frac{\partial \phi(a,t)}{\partial x}dx + \int_a^b Q(x,t)dx$$

Thus

$$\int_a^b \left(\frac{\partial e(x,t)}{\partial t} + \frac{\partial \phi(a,t)}{\partial x} + Q(x,t) \right) dx = 0$$

Since this is true for each choice of a and b with $0 \leq a \leq b \leq L$, if Q is continuous and e and φ have continuous first order partials, it follows that,

$$\frac{\partial e(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial t} + Q(x,t) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

By definition, the temperature u is given by,

$$e(x,t) = c(x)\rho(x)u(x,t) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

So,

$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial t} + Q(x,t) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

According to Fourier's law of heat transfer: Rate of heat transfer proportional to negative temperature gradient,

$$\varphi(x,t) = -K_0 \frac{\partial u(x,t)}{\partial x} \text{ for } 0 \leq x \leq L \text{ and } t \geq 0. \quad (1)$$

Therefore, we have the concentration dependent diffusion equation in one dimension,

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q \text{ for } 0 \leq x \leq L \text{ and } t \geq 0.$$

where c, ρ and K_0 are constant and there are no internal sinks or sources, so that Q is zero, we have,

$$\frac{\partial u}{\partial t} + k \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \text{ for } 0 \leq x \leq L \text{ and } t \geq 0. \quad (2)$$

where $k = \frac{c\rho}{K_0}$ is the thermal diffusivity.

Since the slice was chosen arbitrarily, the diffusion equation (2) applies throughout the metal rod.

3 Similarity solution

We consider all possible groups of infinitesimal transformation that will reduce the heat(diffusion) equation (2) to an ordinary differential equation. On applying such technique to a given differential equation, it may turn out that for some or all of the groups other than the linear and spiral groups, the boundary condition cannot be transformed although the partial differential equation can be transformed into an ordinary differential equation. For such cases, we are at least assured that the groups of infinitesimal transformations that remain are the groups possible for the given boundary value problems.

A similarity analysis of the diffusion equation from this point of view is apparently not covered in the literature. The one-dimensional form of the heat (diffusion) equation in rectangular coordinate is chosen because of its simplicity. Extension of analyses to equations expressed in other coordinates can readily be made.

Consider the one dimensional concentration dependent diffusion equation (2) in the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{3}$$

where k is constant, on which an infinitesimal transformation is to be made on the dependent and independent variables and derivatives of the dependent variable with respect to the independent variable. The infinitesimal transformations are

$$G = \begin{cases} \bar{x} = x + \epsilon X \\ \bar{u} = u + \epsilon U \\ \bar{t} = t + \epsilon T \end{cases} \tag{4}$$

where generators X, T and U are functions of x, t and u. Invariance of equation (1) under (2) gives,

$$\frac{\partial \bar{u}}{\partial \bar{t}} = k\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \tag{5}$$

Applying transformations (4) into the (5) we get the group of infinitesimal transformation explicitly is

$$\begin{cases} X = \frac{a_0 x}{2} + a_2 \\ U = a_0 u + a_1 \\ T = a_0 t + a_3 \end{cases} \tag{6}$$

Thus, the characteristic equations are

$$\frac{dx}{\frac{a_0 x}{2}+a_2} = \frac{dt}{a_0 t+a_3} = \frac{du}{a_0 u+a_1} \tag{7}$$

From these, we can have the similarity variable

$$\eta = \frac{\frac{a_0 x}{2}+a_2}{\sqrt{a_0 t+a_3}} \tag{8}$$

also we have $a_0 u + a_1 = (a_0 t + a_3)F(\eta)$ (9)

Let us discuss two different cases:

Case I: If $a_0 = a_1 = a_2 = a_3 = a$, then we have,

$$\eta = \frac{\frac{ax}{2}+a}{\sqrt{at+a}} = \frac{\sqrt{a}(x+2)}{2\sqrt{t+1}} \text{ and } au + a = (at + a)F(\eta) \tag{10}$$

And we get the ordinary differential equation,

$$akF''(\eta)+\frac{\sqrt{a}}{2}\eta F'(\eta)-F(\eta)=0 \tag{11}$$

This is a second order nonlinear ordinary differential equation and the solution of this equation is given by (Murphy, 1960),

$$F(\eta)=e^{-\frac{\eta^2}{4k\sqrt{a}}}\left(c_1H_{-\frac{2}{\sqrt{a}}-a}\left(\frac{\eta}{2\sqrt{k}\sqrt[4]{a}}\right)+c_2\,{}_1F_1\left\{\frac{1}{2}+\frac{1}{\sqrt{a}};\frac{1}{2};\frac{\eta^2}{4k\sqrt{a}}\right\}\right) \tag{12}$$

$$u=(t+1)e^{-\frac{\sqrt{a}(x+2)^2}{8k(t+1)}}\left(c_1H_{-\frac{2}{\sqrt{a}}-a}\left(\frac{\sqrt[4]{a}(x+2)}{4\sqrt{k(t+1)}}\right)+c_2\,{}_1F_1\left\{\frac{1}{2}+\frac{1}{\sqrt{a}};\frac{1}{2};\frac{\sqrt{a}(x+2)^2}{8k(t+1)}\right\}\right)-1 \tag{13}$$

where c_1 and c_2 are constants, $H_n(x)$ is the n^{th} Hermite polynomial in x and ${}_1F_1(a;b;x)$ is Kummer confluent hypergeometric function.

Case II: If $a_0=2$ and $a_1=a_2=a_3=a$, then we have,

$$\eta=\frac{x+a}{\sqrt{2t+a}} \text{ and } 2u+a=(2t+a)F(\eta) \tag{14}$$

And we get,

$$kF''(\eta)+2\eta F'(\eta)-2F(\eta)=0 \tag{15}$$

which is also a second order nonlinear ordinary differential equation and the solution of this equation is given by(Murphy, 1960),

$$F(\eta)=\eta\left[c_1+c_2\left(-\sqrt{\frac{\pi}{k}}erf\left(\frac{\eta}{\sqrt{k}}\right)-\frac{e^{-\frac{x^2}{k}}}{\eta}\right)\right] \tag{16}$$

$$\therefore u=\frac{\eta(2t+a)}{2}\left[c_1+c_2\left(-\sqrt{\frac{\pi}{k}}erf\left(\frac{\eta}{\sqrt{k}}\right)-\frac{e^{-\frac{x^2}{k}}}{\eta}\right)\right]-\frac{a}{2} \tag{17}$$

where c_1, c_2 are constants and $\eta=\frac{x+a}{\sqrt{2t+a}}$.

4 Conclusion

We have discussed the general form of the one dimensional diffusion equation of concentration and obtained its determining equations. According to Lie symmetry operators, we have obtained the characteristic equations. Using the method of group invariance under an infinitesimal transformation we have obtained, via classical approach, the similarity solution of the problem in the form of Hermite polynomial. The method is worth further study in more physical problems.

References

Carslow, H. S. and Jaeger, J. C. (1959): Conduction of heat in solids; Chap. 3, 2nd ed., Clarendon press, Oxford.

Chauhan, K. J.(2013): Use of technique of infinitesimal transformations of groups for finding the similarity solution of heat equation, Journal of Science, Govt. of Gujarat, Gandhinagar, 1 (3).

Crank, J. (1975): The Mathematics of Diffusion, Chap. 4, 2nd ed, Clarendon press, Oxford.

Fischer, H. B., List, E. J., Koh, R. C. Y., Imberger, J. and Brooks, N. A. (1979): Mixing in inland and coastal waters, Academic press, Inc., NY.

Janavicius, A. J. and Turskiene, S. (2005): Analytical solution of nonlinear diffusion equation in two dimensional surface, Actaphysicapolonica A,, 108 (6).

Miller, W. (1977): Symmetry and separation of variable, reading, Addition-Wesley.

Murphy, M. (1960): Ordinary differential equations and their solutions, Van Nostrandreinhold company, NY.

Nataliya, M. Ivanova (2008): Exact solution of diffusion-convection equation; Dynamics of PDE, 5 (2), pp. 139-171.

Olver, P. (1986): Applications of Lie groups of differential equations, Springer-Verlag, New York.

Talaat, S. Et Danat and Ramadan, Mohamed A. (2007): On the analytical and numerical solutions of the one dimensional non-linear Burgers' equation; The Open Applied Mathematics Journal, 1, pp. 1-8.