

# Hydromagnetic flow of a thin nanoliquid film over an unsteady stretching sheet

Research Article

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**Abstract:** A numerical model is developed to study nanoliquid film flow over an unsteady stretching sheet in presence of hydromagnetic have been investigated. Similarity transformations are used to convert unsteady boundary layer equations to a system of non-linear ordinary differential equations. The resulting non-linear ordinary differential equations are solved numerically using Runge-kutta-Fehlberg and Newton-Raphson schemes. A relationship between film thickness  $\beta$  and the unsteadiness parameter  $S$  is found, the effect of unsteadiness parameter  $S$ , and the hydromagnetic parameter  $M$ , on the velocity and temperature distributions are presented. Present analysis shows that the combined effect of hydromagnetic field and viscous dissipation has a significant influence in controlling the dynamics of the considered problem. Comparison with known results for certain particular cases is in excellent agreement.

**MSC:** 76A20 • 76D10 • 76W05 • 80A20 • 82D80**Keywords:** Boundary layer flow • Heat transfer • Hydromagnetic • Nanoliquid • Thin film • Unsteady stretching sheet© 2016 The Author(s). This is an open access article under the CC BY-NC-ND license (<https://creativecommons.org/licenses/by-nc-nd/3.0/>).

## 1. Introduction

The study of boundary layer flow over a stretching sheet is currently attracting the attention of a growing number of researchers because of its immense potential to be used as a technological tool in many engineering applications. The aim in any extrusion process is to achieve a good quality surface of the extrudate. It is very important to control the drag and the heat flux for better product quality. The study of laminar flow of a thin liquid film over stretching sheet is currently attracting the attention of a growing number of researchers because of the immense potential of nanofluids to be used as technological tools in many engineering applications. The main application of such flows is in coating process such as in wire and fiber coatings. Other applications can be found in food processing, transpiration cooling, reactor fluidization and so on. All coating processes demand a smooth glossy finish to meet the requirements for best appearance and optimum performance such as low friction, transparency and strength. All coating processes demand a smooth glossy surface to meet the requirements for best appearance and optimum service properties such as low friction, transparency and strength. The problem of extrusion of thin surface layers needs special attention to gain some knowledge for controlling the coating product efficiently.

The rate of heat within the thin liquid film has a direct bearing on the success of the coating process and the chemical characteristics of the product. The hydrodynamics of thin liquid film flow over a stretching sheet was first studied by Wang [1] who reduced the unsteady Nernst-Stokes equations to a non-linear ordinary differential equations by means of similarity transformation and solved the same using a kind of multiple shooting method. Dandpat et al. [2] analyzed thermocapillarity flow in a liquid film on a horizontal stretching sheet. Thermocapillarity influence to thicken the film and increase the rate of heat transfer between the sheet and film. Wang [3] analyzed the thin film flow over a horizontal stretching sheet. Analytically solved by homotopy analysis method it is helpful to understand the

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flow and heat transfer mechanisms of the liquid film. Dandpat et al. [4] studied two dimensional laminar thin film flow on an unsteady stretching sheet. Momentum equations are solved analytically in flow situations by using singular perturbation method. Santra et al. [5] analyzed the thermocapillary effects on unsteady thin film flow over a heated horizontal stretching sheet. The effect of thermocapillary in the temperature distribution in stretching direction decreases at higher values of Prandtl number and Biot number. Noor et al. [6] studied the effects of thermocapillary and magnetic field in a thin film flow over an unsteady stretching surface. Analytical method is solved by homotopy analysis method to show the flow and heat transfer rate of the thin liquid film.

Aforementioned studies primarily concerned with the laminar flow of Newtonian fluids. However, in the recent past, nanofluids have attracted the attention of the science and engineering community because of their possible applications in industries. Nanotechnology is an emerging science that is extensive use in industry due to the unique chemical and physical properties that nano-sized materials possess. These fluids are colloidal suspensions, typically metals, oxides, carbides, or carbon nanotubes in a base fluid, etc. Common base fluids include water and ethylene glycol. Nanofluids have properties that make them potentially useful in many heat transfer processes including microelectronics, fuel cells, pharmaceutical processes and hybridpowered engines. They exhibit enhanced thermal conductivities and heat transfer coefficients compared to the base fluids. For this reason nanofluids can often be preferred to conventional coolants like oil, water and ethylene glycol mixtures [7]. Khanafer et al. [8] analyzed heat transfer performance of nanofluids inside an enclosure taking into account the solid particle dispersion. He presented a heat transfer correlation of the average Nusselt number for various Grashoff number and volume fraction is presented. Oztop et al [9] examined the natural convection heat transfer in a partially heated rectangular enclosure filled with nanofluids. Nanofluid is a key factor for heat transfer enhancement. The highest values are obtained when using *Cu* nanoparticles. Hamad [10] studied the convective flow and heat transfer of an incompressible viscous nanofluid past a semi-finite vertical stretching sheet in presence of magnetic field. It is seen in the increase of the magnetic field parameter, the momentum boundary layer thickness decreases, while thermal boundary layer thickness increases. The heat transfer rates decreases as the nanoparticle volume fraction increases. Uddin et al. [11] study the steady two dimensional MHD boundary layer flow of an electrical conducting newtonian nanofluid over a solid stationary vertical plate in quiescent fluid taking into account the Newtonian heating boundary condition investigated numerically. Kameswaran et al. [12] investigated the effects of various material and physical parameters such as thermal radiation and nanoparticle volume fraction on heat and mass transfer characteristics in two water-based nanofluids with suction/injection and radiation and solute concentration. The momentum and energy equations are coupled and nonlinear. Ting et al. [13] investigated the forced convection heat transfer and flow characteristics of water-based  $Al_2O_3$  nanofluids inside a horizontal circular tube in the laminar flow regime under the constant wall temperature boundary condition. Heat transfer coefficient of nanofluids increases with increasing Reynolds number or particle volume concentration. Increasing nanoparticle concentration leads to an increase in pressure drop for nanofluids. Dawood et al. [14] studied the three dimensional laminar mixed convective heat transfer at different nanofluids flow in an elliptic annulus with constant heat flux. Mohammadpour et al. [15] studied the magnetic field effects on the forced convection flow of a nanofluid over a stretching surface in the presence of heat generation/absorption. The values of temperature profiles increase with increasing heat generation/absorption and volume fraction of the nanoparticles but they decrease with increasing velocity ratio parameter and temperature index parameter. Saryazdi [16] developed the forced convection flow of a nanofluid in a constant-wall temperature circular tube filled with a porous medium is considered. The flow is steady and Brinkman- Forchheimer extended Darcy equation model is employed. The thermal-equilibrium model is assumed between nanofluid and solid phase. It is also assumed that nanoparticles are distributed non-uniformly inside the pipe, hence the particles volume fraction equation is also coupled with the governing equations. The results show that the Nusselt number is increased with increasing particles volume fraction, the wall shear stresses are also increased. Maity et al. [17] investigated two-dimensional flow of a thin nanoliquid film over an unsteady stretching sheet is studied under the assumption of planar film thickness when the sheet is heated/cooled along the stretching direction. It is observed that there exists a boundary demarcating the region of heat transfer within the film. One side of this boundary heat is transported into the film, while on the other side heat is transported out of the film. Depending on the nanomaterials, this delineated boundary is either squeezed or enlarged. Mahesha et al. [18] studied the laminar flow of a thin film of a nanoliquid over an unsteady stretching sheet is considered. An effective medium theory (EMT) based model is used for the thermal conductivity of the nanoliquid. A parametric study that deals with the effect of the unsteadiness parameter and the nanoparticle volume fraction on the dynamics of the liquid film.

The purpose of present study is to give numerical analysis of hydromagnetic flow of a thin nanoliquid film over an unsteady stretching sheet in presence of viscous dissipation with uniform film thickness. The governing equations are transformed into highly non-linear ordinary differential equations and then solved numerically by using Runge-Kutta-Fehlberg and newton-Raphason schemes based on shooting technique. Numerical computation has been carried out for thermal boundary layer for various values of flow parameters.

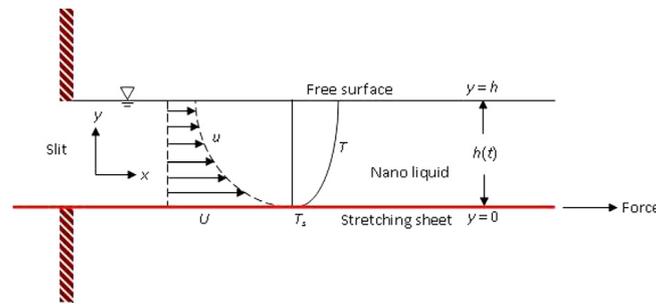


Fig. 1. Schematic representation of a liquid film on an elastic sheet

## 2. Mathematical formulation

Let us consider a thin elastic sheet which emerges from a narrow slit at the origin of a Cartesian co-ordinate system for investigations as shown schematically in Fig. 1. The continuous sheet at  $y = 0$  is parallel with the  $x$ -axis and moves in its own plane with the velocity.

$$U(x, t) = \frac{bx}{1 - \alpha t}, \quad (1)$$

where  $b$  and  $\alpha$  are both positive constants with dimension per time. The surface temperature  $T_s$  of the stretching sheet is assumed to vary with the distance  $x$  from the slit as

$$T_s(x, t) = T_0 - T_{ref} \left[ \frac{bx^2}{2\nu} \right] (1 - \alpha t)^{-\frac{3}{2}}, \quad (2)$$

where  $T_0$  is the temperature at the slit and  $T_{ref}$  can be taken as a constant reference temperature such that  $0 \leq T_{ref} \leq T_0$ . The term  $\frac{bx^2}{\nu(1 - \alpha t)}$  can be recognized as the local Reynolds number based on the surface velocity  $U$ . The expression (1) for the velocity of the sheet  $U(x, t)$  reflects that the elastic sheet which is fixed at the origin is stretched by applying a force in the positive  $x$ -direction and the effective stretching rate  $\frac{b}{(1 - \alpha t)}$  increase with time as  $0 \leq \alpha < 1$ . With the same analogy the expression for the surface temperature  $T_s(x, t)$  given by Eq. (2) represents a situation in which the sheet temperature decreases from  $T_0$  at the slit in proportion to  $x^2$  and such that the amount of temperature reduction along the sheet increases with time. The applied hydromagnetic field is assumed to be of variable kind and is chosen in its special form as

$$B(x, t) = B_0(1 - \alpha t)^{-\frac{1}{2}}. \quad (3)$$

The particular form of the expressions for  $U(x, t)$ ,  $T_s(x, t)$  and  $B(x, t)$  are chosen so as to facilitate the construction of a new similarity transformation which enables in transforming the governing partial differential equations of momentum and heat transport into a set of non-linear ordinary differential equations.

The thermo-physical properties of the nanoliquid (homogeneous mixture of base liquid and nanoparticle) are given in Table 1.

A nanoliquid is an engineered colloidal suspension of nanoparticles in a base liquid and exhibits a significant enhancement in thermal conductivity at modest nanoparticle concentrations. The mechanism leading to this anomalous increase in the thermal performance is still under scrutiny, but recently, several authors proposed two different models of nanoliquids to resolve this issue. The first model consolidates the effects of Brownian motion and thermophoresis (see [7]), in the energy equation. Another model is based on effective medium theory (EMT) like the Maxwell-Garnett theory for the electrical conductivity and dielectric constant of the medium. In this model the macroscopic properties of the nanoliquid such as density, heat capacity, thermal conductivity and so on are expressed in terms of the properties and relative fractions of its components, namely, base liquid and the suspended nanoparticles. In this study we make use of the latter model for the proposed problem along with the following assumptions;

1. The flow is laminar and the nanoliquid is in-compressible.
2. The nanoliquid is non-volatile so that the effect of latent heat due to evaporation is negligible.
3. The buoyancy effect is negligible due to the relatively thin liquid film, but it is not so thin that intermolecular forces come into play.
4. The influence of surface tension on the flow is negligible.

**Table 1.** Thermo-physical properties of liquid and nanoparticle

Physical properties	Base liquid	Nanoparticle Cu
$C_p(Jkg^{-1}K^{-1})$	4179	385
$\rho(kgm^{-3})$	997.1	8933
$k(Wm^{-1}K^{-1})$	0.613	400

The standard boundary layer approximation, based on the scale of analysis, we can write the governing equations

$$\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = 0 \quad (4)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho_{nf}} u \quad (5)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{K_{nf}}{(\rho C_p)_{nf}} \frac{\partial^2 T}{\partial y^2} + \frac{\mu_{nf}}{(\rho C_p)_{nf}} \left( \frac{\partial u}{\partial y} \right)^2 \quad (6)$$

where  $u$  and  $v$  are the velocity components along the  $x$  and  $y$  axes respectively.  $T$  is temperature.

The physical properties characterizing the base liquid and the nanoparticles, namely, density, dynamic viscosity, thermal conductivity, thermal diffusivity and heat capacitance are assumed to be constants while those of the nanoliquid are assumed to be functions of the volume fraction  $\phi$ . The effective density of the nanoliquid is given by

$$\rho_{nf} = (1 - \phi)\rho_f + \phi\rho_s \quad (7)$$

Although the use of the above thermal conductivity model is restricted to nanoparticles of spherical shape it is found to be very appropriate for studying heat transfer enhancement using nanoliquids. The effective viscosity of the nanoliquid as given by Brinkman is

$$\mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}} \quad (8)$$

The effective thermal conductivity of the nanoliquid is approximated by the Maxwell-Garnett model as

$$K_{nf} = K_f \left[ \frac{K_s + 2K_f - 2\phi(K_f - K_s)}{K_s + 2K_f + \phi(K_f - K_s)} \right] \quad (9)$$

The heat capacitance of the nanoliquid is expressed as (see [8]).

$$(\rho C_p)_{nf} = (1 - \phi)(\rho C_p)_{nf} + (\rho C_p)_s \quad (10)$$

The associated boundary conditions for Eqs. (4)-(6) are

$$u = U, v = 0, T = T_w \text{ at } y = 0 \quad (11)$$

$$\frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0 \text{ at } y = h \quad (12)$$

$$v = \frac{\partial h}{\partial t} \text{ at } y = h \quad (13)$$

We introduce the following similarity variables

$$\psi(x, y, t) = \left( \frac{\nu_f b}{1 - \alpha t} \right)^{1/2} x f(\eta) \quad (14)$$

$$T(x, y, t) = T_0 - T_{ref} \left[ \frac{bx^2}{2\nu_f} \right] (1 - \alpha t)^{-3/2} \theta(\eta) \quad (15)$$

$$\eta = \left[ \frac{b}{v_f(1-\alpha t)} \right]^{1/2} y \quad (16)$$

The velocity components  $u$  and  $v$  in terms of the Stream function  $\psi(x, y, t)$  are given by

$$u = \frac{\partial \psi}{\partial y} = \left( \frac{bx}{1-\alpha t} \right) f'(\eta) \quad (17)$$

$$v = -\frac{\partial \psi}{\partial x} = -\left( \frac{v_f b}{1-\alpha t} \right)^{1/2} f(\eta) \quad (18)$$

Assuming  $\eta = \beta$  at free the free surface and using Eq. (16)

$$\beta = \sqrt{\frac{b}{v_f}} (1-\alpha t)^{-1/2} h \quad (19)$$

which gives

$$\frac{dh}{dt} = -\frac{\alpha\beta}{2} \sqrt{\frac{v_f}{b}} (1-\alpha t)^{-1/2} \quad (20)$$

Substituting similarity variable (14)-(16) into Eqs. (4)-(6), the continuity Eq. (3) automatically satisfied and the momentum and the energy equations are reduced to

$$f''' + \phi_1 \left[ f f'' - (f')^2 - S \left( \frac{\eta}{2} f'' + f' \right) + \frac{1}{\phi_2} Mn f' \right] = 0 \quad (21)$$

$$\theta'' + Pr \left( \frac{K_n f}{K_f} \right) \phi_3 \left[ f \theta' - 2 f' \theta - \frac{S}{2} (3\theta + \eta \theta) + \frac{1}{\phi_4} Ec (f'')^2 \right] = 0 \quad (22)$$

subject to the boundary conditions

$$f(0) = 0, f'(0) = \theta(0) = 1, \quad (23)$$

$$f''(\beta) = \theta'(\beta) = 0, \quad (24)$$

$$f(\beta) = \frac{S\beta}{2} \quad (25)$$

where unsteady parameter  $S = \alpha/b$  and hydromagnetic parameter  $Mn = \sigma B_0^2 / \rho_f b$

$$\phi_1 = (1-\phi)^{2.5} \left[ (1-\phi) + \phi \left( \frac{\rho_s}{\rho_f} \right) \right] \quad (26)$$

$$\phi_2 = 1 - \phi + \phi \left( \frac{\rho_s}{\rho_f} \right) \quad (27)$$

$$\phi_3 = 1 - \phi + \phi \left( \frac{(\rho C_p)_s}{(\rho C_p)_f} \right) \quad (28)$$

$$\phi_4 = (1-\phi)^{2.5} \left[ 1 - \phi + \phi \left( \frac{(\rho C_p)_s}{(\rho C_p)_f} \right) \right] \quad (29)$$

### 3. Numerical solution

The system of non-linear differential equations Eq. (21) and (22) subjected to the boundary conditions Eqs. (23)-(25) are solved numerically, using Runge-Kutta-Fehlberg and Newton-Raphson schemes based shooting method. (see [20]). In this method, third order non-linear ordinary differential equation Eq. (21) and second order non-linear ordinary differential equation Eq. (22) have been reduced to first order differential equations as follows:

$$\frac{df_0}{d\eta} = f_1, \quad (30)$$

$$\frac{df_1}{d\eta} = f_2, \quad (31)$$

$$\frac{df_2}{d\eta} = \phi_1 \left[ S \left( f_1 + \frac{\eta}{2} f_2 \right) + (f_1)^2 - f_0 f_2 + \frac{1}{\phi_2} M f_1 \right], \quad (32)$$

$$\frac{d\theta_0}{d\eta} = \theta_1, \quad (33)$$

$$\frac{d\theta_1}{d\eta} = \phi_3 Pr \left( \frac{K_f}{K_{nf}} \right) \left[ \frac{S}{2} (3\theta_0 + \eta\theta_1) + 2f_1\theta_0 - \theta_1 f_0 - \frac{1}{\phi_4} Ec f_2^2 \right]. \quad (34)$$

Corresponding boundary conditions take the form,

$$f_1(0) = 1, \quad f_0(0) = 0, \quad \theta_0(0) = 1, \quad (35)$$

$$f_2(\beta) = 0, \quad \theta_1(\beta) = 0, \quad (36)$$

$$f_0(\beta) = \frac{S\beta}{2}. \quad (37)$$

Here  $f_0(\eta) = f(\eta)$  and  $\theta_0(\eta) = \theta(\eta)$ . The above boundary value problem is first converted into an initial value problem by appropriately guessing the missing slopes  $f_2(0)$  and  $\theta_1(0)$ . The resulting IVP is solved by shooting method for a set of parameters appearing in the governing equations and a known value of  $S$ . The value of  $\beta$  is so adjusted that condition Eq. (37) holds. This is done on the trial and error basis. The value for which condition Eq. (37) holds is taken as the appropriate film thickness and the IVP is finally solved using this value of  $\beta$ . The step length of  $h = 0.01$  is employed for the computation purpose. The convergence criterion largely depends on fairly good guesses of the initial conditions in the shooting technique. The iterative process is terminated until the relative difference between the current and the previous iterative values of  $f(\beta)$  matches with the value of  $\frac{S\beta}{2}$  up to a tolerance of  $10^{-6}$ . Once the convergence is achieved we employed shooting technique with the Runge-Kutta-Fehlberg and Newton-Raphson schemes to determine the unknown in order to convert the boundary value problem to initial value problem. Once all initial conditions are determined, the resulting differential equations were integrated using initial value solver. For this purpose Runge-Kutta-Fehlberg scheme was used.

### 4. Results and discussions

In this work we analyzed Hydromagnetic flow of a thin nanoliquid film over an unsteady stretching sheet is investigated, in presence of viscous dissipation. Both numerical and analytical solutions are presented. The similarity transformations were used to transform the governing partial differential equations into a system of non-linear ordinary differential equations. The accuracy of the method was established. The numerical solution obtained by shooting method together with Runge-Kutta-Fehlberg and Newton-Raphson schemes.

The three main parameters that affect the flow and heat transfer are the unsteadiness parameter  $S$ , film thickness  $\beta$  and the nanoparticle volume fraction  $\phi$ . The Prandtl number  $Pr$  for the base liquid water is usually around 7. Using the definition of Prandtl number and the thermo-physical properties of water as listed in Table 1 along with  $\mu_f = 1 \times 10^{-3}$  Pa s at  $20^\circ C$  the Prandtl number of water is calculated to be  $Pr = 6.8173$ . This value has been used throughout our computations.

It is note worthy to mention that the solution exists only for small value of unsteadiness parameter  $0 \leq S \leq 2$ . Moreover, when  $S \rightarrow 0$  the solution approaches to the analytical solution obtained by Crane [19] with infinitely thick layer of fluid ( $\beta \rightarrow \infty$ ). The other limiting solution corresponding to  $S \rightarrow 2$  represents a liquid film of infinitesimal thickness ( $\beta \rightarrow 0$ ). The numerical results are obtained for  $0 \leq S \leq 2$  and  $0 \leq \phi \leq 0.5$ . The numerical results were validated by comparing with the results reported by Wang [3] in the case of a clear liquid through Table 2 and Table 3. The effects of the unsteadiness parameter  $S$  and nanoparticle volume fraction  $\phi$  on the flow and heat transfer are shown in Figs. 2-14.

Fig. 2 shows the variation of dimensionless film thickness  $\beta$  against the unsteadiness parameter  $S$  for different values of nanoparticle volume fraction  $\phi$  in case of *Cu-water*. It is clear that the fluid layer attains infinite thickness for the steady case ( $S \rightarrow 0$ ) and vanishing the thickness  $S \rightarrow 2$ . For *Cu-water* nanoliquid, increasing values of  $\phi$  results decrease in the film thickness.

The effect of nanoparticle volume fraction on the axial velocity profiles  $f'(\eta)$ , are shown different values of  $S$  in Figs. 3-5 for *Cu-water*. We recall from Eq. 8 that increasing values of  $\phi$  contribute to the enhancement of nanoliquid viscosity. As the viscosity increases it offers considerable drag to fluid flow thereby slowing down its motion. In the case of a *Cu-water* nanoliquid one witnesses this scenario as shown figures. The unsteadiness parameter has an increasing effect on  $f'(\eta)$  in the *Cu-water* nanoliquids considered in the study.

The effect of the nanoparticle volume fraction on the temperature profiles  $\theta(\eta)$  are predicted for different values of  $S$ , in Figs. 6-8 for *Cu-water* nanoliquid. We note here that the inclusion of nano-sized particles in water like cooling liquids greatly enhances their thermal conductivity thereby resulting in increased heat transfer rates. In case of *Cu-water* nanoliquids, increasing the values of the volume fraction results in thickening of the thermal boundary layer for any given value of the unsteadiness parameter. Because the thermal conductivity of Cu is larger. We observe the significant variation of temperature profiles against volume fraction in case of *Cu-water* nanoliquid. The unsteadiness parameter has a decreasing effect on  $\theta(\eta)$  profiles in case of *Cu-water* nanoliquids.

The effect of hydromagnetic on velocity profile for different values of unsteadiness parameter  $S$  in Figs. 9-11 in case of *Cu-water* nanoliquid. The influence of hydrodynamic produces a Lorentz force that opposes the motion and hence decreases the velocity. The boundary condition at the free surface corresponding to a certain film thickness. The film thickness is also decreases for each value of unsteadiness parameter. As a result boundary layer thickness is decreases with increase in the hydromagnetic.

The effect of hydromagnetic on temperature profile for different values of unsteadiness parameter  $S$  in Figs. 12-14 in case of *Cu-water* nanoliquid. The dimensionless temperature is higher at the surface and it decreases with the hydromagnetic distance inside the thermal boundary layer (as shown in Figs. 12-14). Due to a decrease in the dimensionless film thickness, the dimensionless temperature increases with the increase in the hydromagnetic each value of unsteadiness parameter and results the thermal boundary layer thickness increases.

Comparison of values of the dimensionless film thickness  $\beta$ , wall skin friction coefficient  $-f''(0)$ , obtained in the present study corresponding to the clear liquid case  $\phi = 0$  with those obtained by Wang [3] is given in Table 2. From the table one can easily infer that our results are good agreement with those Wang [3].

The effect of the unsteadiness parameter  $S$  on  $\theta'(0)$  is presented in Table 3. It is found that this quantity decreases dramatically as  $S$  enlarges for the *Cu-water* nanofluids, its value can reduce about 10 times when  $S$  increases from 0.4 to 1.8 for nanofluid. For any a given value of  $S$ , it is found from the table that the value of  $-\theta'(0)$  decreases monotonously with  $\phi$  increasing.

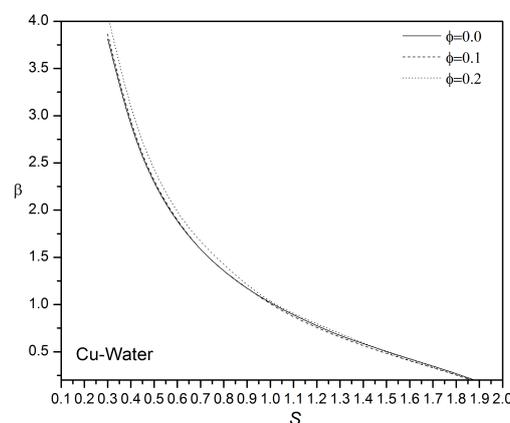


Fig. 2. Variation of diemensionaless film thickness  $\beta$  with  $S$  for different values of  $\phi$

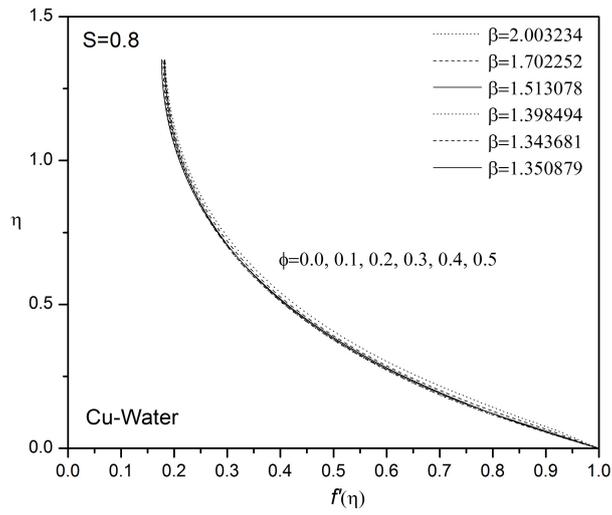


Fig. 3. Effect of  $\phi$  on the axial velocity profile  $f'(\eta)$  for  $S = 0.8$

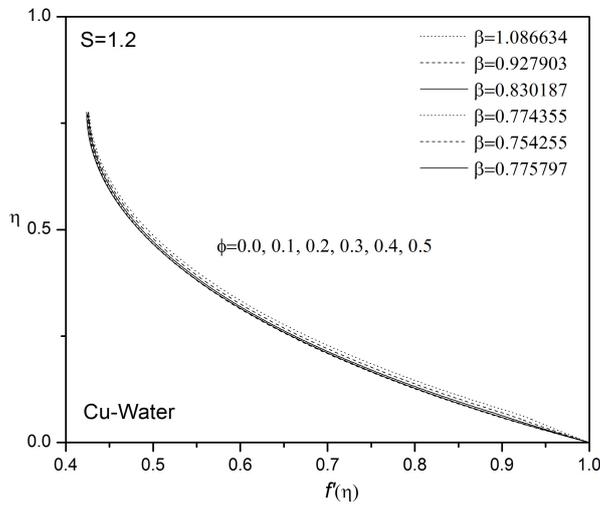


Fig. 4. Effect of  $\phi$  on the axial velocity profile  $f'(\eta)$  for  $S = 1.2$

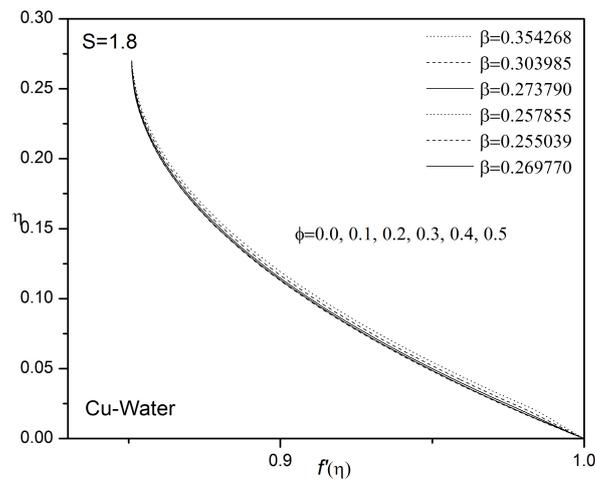


Fig. 5. Effect of  $\phi$  on the axial velocity profile  $f'(\eta)$  for  $S = 1.8$

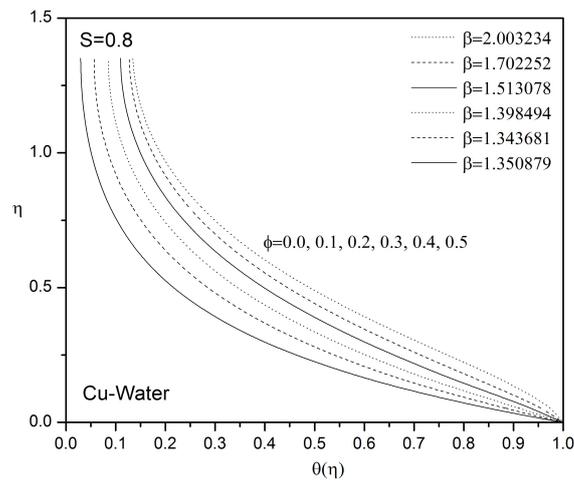


Fig. 6. Effect of  $\phi$  on the temperature profile  $\theta(\eta)$  for  $S = 0.8$

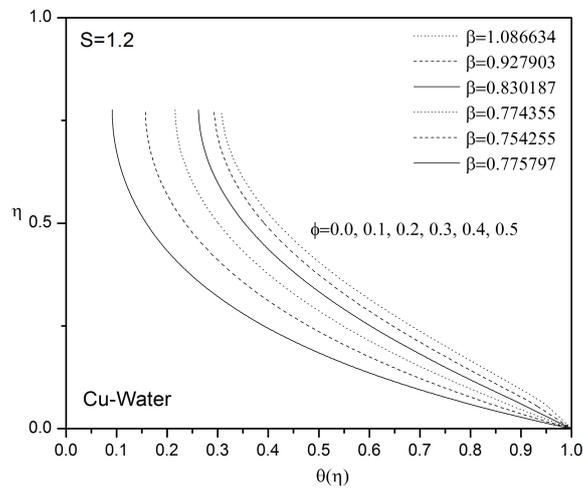


Fig. 7. Effect of  $\phi$  on the temperature profile  $\theta(\eta)$  for  $S = 1.2$

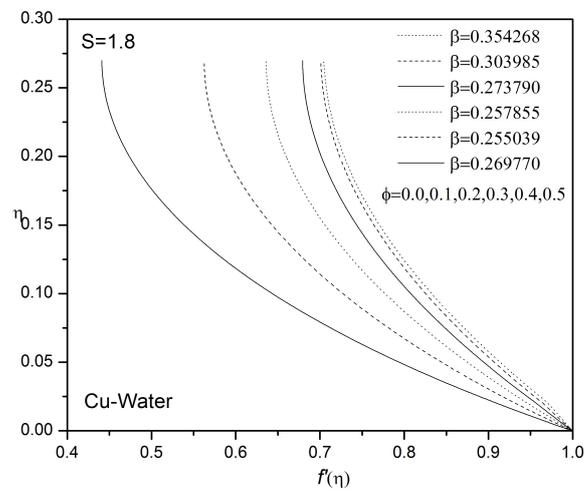


Fig. 8. Effect of  $\phi$  on the temperature profile  $\theta(\eta)$  for  $S = 1.8$

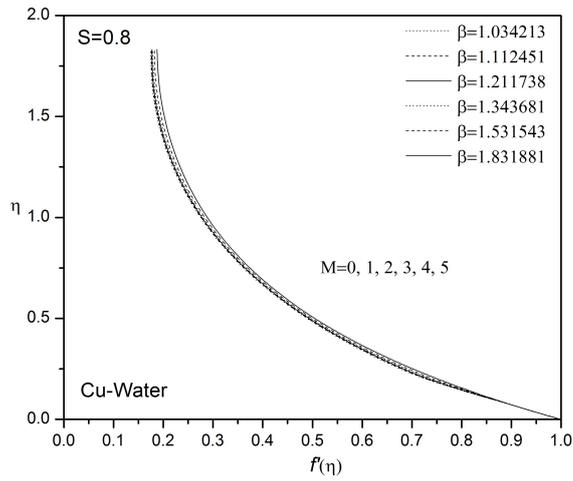


Fig. 9. Effect of magnetic field on the velocity profile  $f'(\eta)$  for  $S = 0.8$

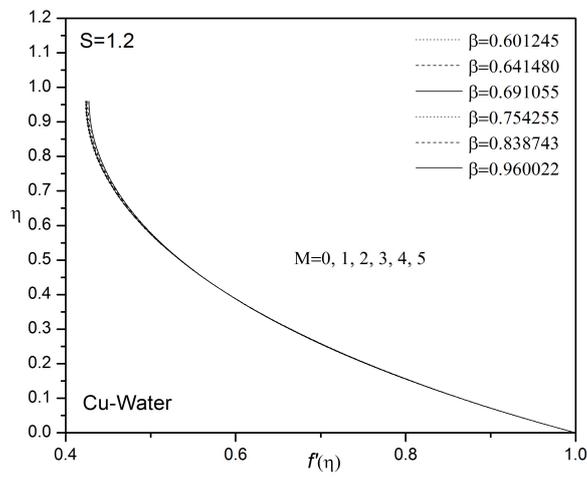


Fig. 10. Effect of magnetic field on the velocity profile  $f'(\eta)$  for  $S = 1.2$

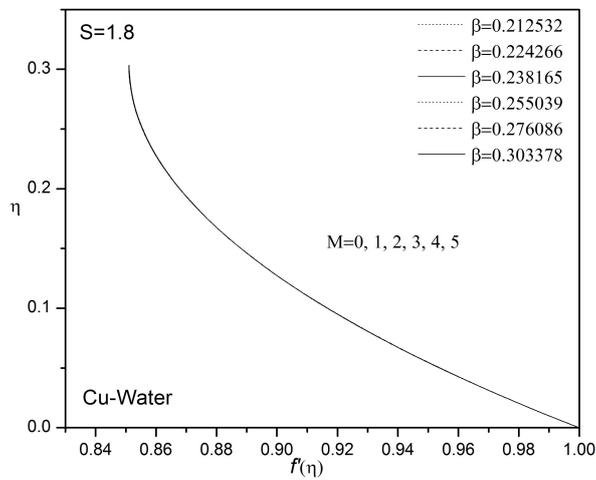


Fig. 11. Effect of magnetic field on the velocity profile  $f'(\eta)$  for  $S = 1.8$

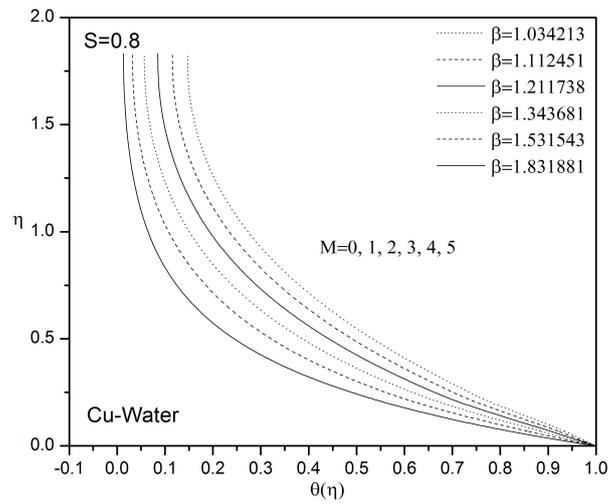


Fig. 12. Effect of magnetic field on the temperature profile  $\theta(\eta)$  for  $S = 0.8$

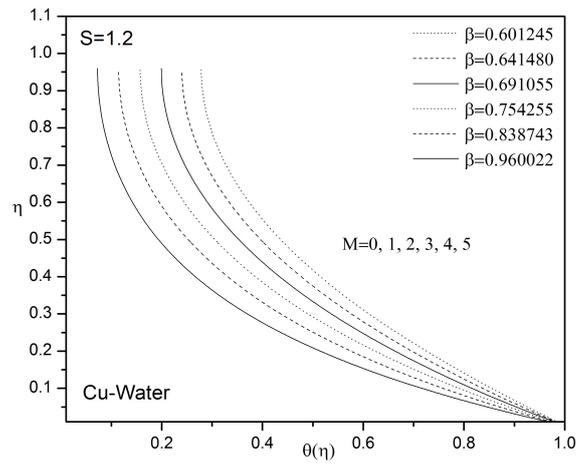


Fig. 13. Effect of magnetic field on the temperature profile  $\theta(\eta)$  for  $S = 1.2$

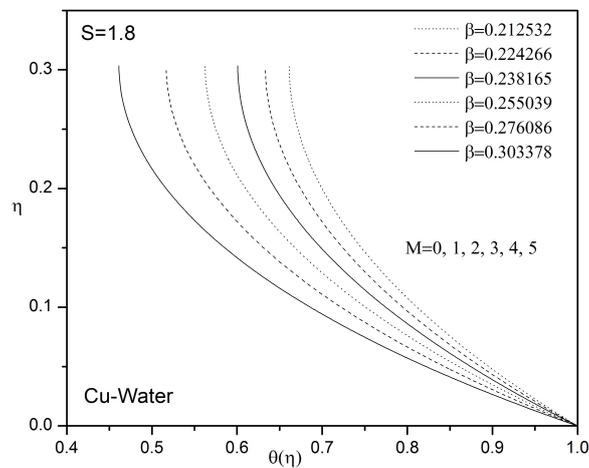


Fig. 14. Effect of magnetic field on the temperature profile  $\theta(\eta)$  for  $S = 1.8$

**Table 2.** Values of diemensional film thickness  $\beta$  and wall skin friction coefficient  $f''(0)$ 

S	Wang [3]		Mahesha [18]		Present results	
	$\beta$	$f''(0)$	$\beta$	$f''(0)$	$\beta$	$f''(0)$
0.4	5.122490	-6.699120	4.981455	-5.6494483	3.036445	-5.31347511
0.3	3.909500	-4.551080	3.887050	-4.5303940	4.072910	-6.99970313
0.5	-	-	-	-	2.402799	-4.27626138
0.6	3.131250	-3.742330	3.131713	-3.7427896	1.968154	-3.55625746
0.7	2.577010	-3.149650	2.576998	-3.1496174	1.648154	-3.01694590
0.8	2.151990	-2.680940	2.151994	-2.6809660	1.398494	-2.58553571
0.9	1.815990	-2.296830	1.815985	-2.2968230	1.198575	-2.22994879
1.0	1.543620	-1.972380	1.543618	-1.9723877	1.033532	-1.92609024
1.1	1.318100	-1.691360	1.318100	-1.6913640	0.894229	-1.65986787
1.2	1.127780	-1.442631	1.127780	-1.4426237	0.774355	-1.42156091
1.3	0.964219	-1.218322	0.964219	1.2183210	0.669321	-1.20464394
1.4	0.821032	-1.012784	0.821032	-1.0127798	0.575614	-1.00421618
1.5	0.693144	-0.821842	0.693144	-0.8218414	0.490359	-0.81674195
1.6	0.376173	-0.642397	0.376173	-0.6423970	0.410968	-0.63958950
1.7	0.465770	-0.472094	0.465770	-0.4720941	0.334711	-0.47073755
1.8	0.356389	-0.309137	0.356389	-0.3091369	0.257855	-0.30860086
1.9	0.237000	-0.152134	0.237000	-0.1521331	0.172543	-0.15201038

**Table 3.** Values of temperature gradient  $-\beta\theta'(0)$  for various values of S and  $\phi$ 

Type of fluid	S	$\phi = 0.0$	$\phi = 0.1$	$\phi = 0.2$	$\phi = 0.3$	$\phi = 0.4$	$\phi = 0.5$
Cu – Water	0.4	7.43386850	5.42181284	4.02359327	2.92236731	1.92960699	0.88294071
	0.6	5.32025619	3.90180884	2.93313991	2.18275489	1.51893417	0.83196790
	0.8	4.20204422	3.09342240	2.35450450	1.79541835	1.31413854	0.83014017
	1.0	3.47841537	2.57714738	1.99275305	1.56377401	1.20834128	0.86721472
	1.2	2.93677954	2.19631513	1.72975420	1.39994434	1.14178464	0.91255523
	1.4	2.46244075	1.85777727	1.48773194	1.23853877	1.05941196	0.92078434
	1.6	1.95947913	1.47784694	1.19349217	1.01419483	0.90046772	0.83286974
	1.8	1.28529219	0.94208856	0.75334917	0.64510400	0.58729902	0.56817502

## 5. Conclusions

The problem of Hydromagnetic flow of a thin nanoliquid film over an unsteady stretching sheet is investigated, in presence of viscous dissipation is analyzed. The efficient numerical method of Runge-Kutta-Fehlberg and Newton-Raphson schemes based on shooting technique is used to solve the governing equations. The thermo-physical properties of the nanoliquid were assumed to be functions of properties of the components and their volume fraction. A similarity solution that depends on the unsteadiness parameter  $S$ , hydromagnetic parameter  $M$  and the nanoparticle volume fraction  $\phi$  was presented. Some of the important findings of the investigation are listed as follows:

1. Free surface velocity decreases with unsteadiness parameter  $S = 0.8, 1.2, 1.8$  and hydromagnetic parameter.
2. The wall temperature gradient  $-\theta'(0)$  is decreasing function of  $\phi$  in  $Cu$  nanoliquid.
3. The wall shear stress  $-f''(0)$  is an increasing function of  $\phi$  in case of  $Cu - water$  nanoliquid.
4. The unsteadiness parameter  $S$  always succeeds in thickening the momentum boundary layer and thinning the thermal boundary layer of nanoliquids.
5. The dimensionless film thickness  $\beta$  decreases with nanoparticle volume fraction  $\phi$  in the case of  $Cu$ -water nanoliquid.

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