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Traveling wave solutions of 7th order Kaup Kuperschmidt and Lax equations of fractional-order

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ABSTRACT

In this paper, we develop a new scheme (U'/U)-expansion method to construct generalized solitary wave solutions of nonlinear 7th order Kaup Kuperschmidt and Lax equations of fractional-order. The proposed scheme constructs the solitary wave solutions of nonlinear equations efficiently. The method appears to be easier and more convenient by means of a symbolic computation system.

Keywords: Euler function, Mittag-Leffer function, Fractional derivative, MAPLE 13, 7th order Kaup Kuperschmidt and Lax equations, (U'/U)-expansion method.

MSC 2010 codes: 35K99, 35P99, 35P05.

1 Introduction

Nonlinear partial differential equations are useful in describing the various phenomena in disciplines. Due to dynamical balance between the nonlinear and dispersive effects, these waves retain a stable wave form. The 7th order Kaup Kuperschmidt equation (Wazwaz, 2009) of fractional order is given as follows:

$$D_t^{\alpha}u + 2016u^3D_x^{\alpha}u + 630(D_x^{\alpha}u)^3 + 2268uD_x^{\alpha}uD_x^{2\alpha}u + 504u^2D_x^{3\alpha}u + 252D_x^{2\alpha}uD_x^{3\alpha}u + 147D_x^{\alpha}uD_x^{4\alpha}u + 42uD_x^{5\alpha}u + D_x^{7\alpha}u = 0, \quad 0 < \alpha \leq 1. \tag{1}$$

and the standard form of the 7th order Lax equation (Xia and Bin, 2004) of fractional-order is given by

$$D_t^{\alpha}u + 140u^3D_x^{\alpha}u + 70(D_x^{\alpha}u)^3 + 280uD_x^{\alpha}uD_x^{2\alpha}u + 70u^2D_x^{3\alpha}u + 70D_x^{2\alpha}uD_x^{3\alpha}u + 42D_x^{\alpha}uD_x^{4\alpha}u + 14uD_x^{5\alpha}u + D_x^{7\alpha}u = 0, \quad 0 < \alpha \leq 1. \tag{2}$$

These equations play a major role in the study of nonlinear dispersive waves (Abdulloev et al, 1976; Bona et al, 1985) because of its description a larger number of important physical phenomena, such as shallow water waves and ion acoustic plasma waves. These equations

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have solitary wave solutions similar to those of the KdVs (Benjamin et al., 1972; Bona et al, 1980). Ganji et al. (2010) construct the solitary wave solution of 7^{th} order Kaup Kuperschmidt equation using Exp-function method. Later on, Salas and Gómez (2010) apply Cole-Hopf transformation to find new traveling wave solution of 7^{th} order Kaup Kuperschmidt equation, Salas et al. (2010) construct new travelling wave solution of 7^{th} order Lax equation and Jafari et al. (2008) applied He's Variational Iteration Method for solving seventh order Lax equation for $\alpha = 1$.

The phenomena of nonlinear science play an important role in applied mathematics and mathematical physics. The appearance of solitary wave in nature is rather frequent, especially in fluids, plasmas, solid state physics, condensed matter physics, optical fibers, chemical kinematics, electrical circuits, bio-genetics, elastic media etc. Recently, it is to be noted that the fractional partial differential equations (FDEs) are applied in many fields of science. The exact solution of nonlinear fractional partial differential equations (Hammouch and Mekkaoui, 2012) has a great importance in nonlinear science. Later on, Wang et al. (2008) presented a reliable technique which is called the (G'/G)-expansion method and obtained exact traveling wave solutions for the nonlinear evolution equations (NLEEs). In this method, second order linear ordinary differential equation with constant coefficients $G''(\eta) + \lambda G'(\eta) + \mu G(\eta) = 0$, is used, as an auxiliary equation. In the subsequent development, this work has been used to obtain exact traveling wave solutions for the nonlinear differential equations, (see Feng et al., 2011; Zhao et al., 2011; Akbar et al., 2012) and the references therein. There are many methods such as, Hirota's bilinear (Wazwaz, 2009, 2012), Exp-function (Mohyud-Din et al., 2009, 2010; Mohyud-Din, 2009), sin-cosine (Wazwaz, 2006), tanh function (Wazwaz, 2005, 2005a), general algebraic (Wang, et al., 2006), extended tanh function (Fan, 2000; El-Wakil and Abdon, 2007), (G'/G)-Expansion (Wang, et al., 2008a; Ozis and Aslan, 2010; Zayed and Al-Joudi, 2010; Naher et al., 2011), F-expansion (Liu and Yang, 2004; Chen et al., 2005), homogeneous balance (Fan, 2000), Backlund transformation (Tam and Hu, 2002), modified Exp-function (Usman et al., 2013) etc. It is important to observe that there exist some fundamental relationships among many complex nonlinear partial differential equations (NPDEs) and some basic and soluble nonlinear ordinary differential equations (NODEs), such as the sine-Gordon equation, sinh-Gordon equation, Riccati equation, Weierstrass elliptic equation etc. Therefore, it is natural to use the solutions of these nonlinear ODEs to construct exact solutions of various intricate nonlinear partial differential equations. Inspired and motivated by ongoing research in this area, we develop a new approach, (U'/U)-expansion method for traveling wave solutions of 7th order Kaup Kuperschmidt and Lax equations of fractional order. In this method, we use $D_\xi^\alpha U(\xi) = AU(\xi) + B$, as auxiliary equation to construct the traveling wave solution of 7th order Kaup Kuperschmidt and Lax equations of fractional-order. The purposed technique expresses the solutions in term of rational exponential functions. Numerical results are very encouraging.

2 Preliminaries

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

2.1 Definition

A real function $f\ (t), t\ >\ 0$, is said to be in the space $c_{\mu}, \mu \in R$ if there exists a real number $p(>\ \mu)$, such that $f(t) = t^{p} f_{1}(t)$, where $f_{1}(t) \in C[0, \infty]$, and it is said to be in the space C_{μ}^{m} iff $f^{m} \in C_{\mu}, m \in N$.

2.2 Definition

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu}, \mu \geq -1$, is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, t > 0,$$
$$J^{0} f(t) = f(t).$$

Properties of the operator J^{α} can be found in Luchko and Groreflo (1998); Miller and Ross (1993); Oldham, and Spanier (1974), we mention only the following: For $f \in C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t).$$

$$J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t).$$

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by M. Caputo (1967) in his work on the theory of viscoelasticity.

2.3 Definition

The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^{\alpha} f(t) = J^{m-\alpha} D^{m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} (t-\tau)^{m-\alpha-1} f^{m}(t) dt,$$

for $m - 1 < \alpha \leq 1m, m \in N, t > 0, f \in C_{-1}^{m}$.

3 Analysis of Adomian's Decomposition Method (ADM)

Consider the nonlinear differential equation,

$$Lu + Ru + Nu = g, \tag{3}$$

where L is, mostly the lower order derivative which is assumed to be invertible, R is other linear differential operator, non linear term and g is a source term. We next apply the inverse operator L^{-1} to both sides of equation (3) and using the given condition to obtain,

$$u = f - L^{-1}(Ru + Nu) \tag{4}$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed. As indicated before, Adomian's method defines the solution u by an infinite series of components given by,

$$u = \sum_{n=x0}^{\infty} u_n, \tag{5}$$

where the components u_0, u_1, u_2 ... are usually recurrently determined. Substituting (5) into both sides of (4) leads to,

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right) + N(\sum_{n=0}^{\infty} u_n)\right). \tag{6}$$

Accordingly, the formal recursive relation is defined by

$$u_0 = f,$$

$$u_{k+1=} - L^{-1}(Ru_k + Nu_k).$$

The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n,$$

where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in Wazwaz and Gorguis (2004); Wazwaz (1995) which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\right) N\left(\sum_{i=0}^{n} (\lambda^i u_i)\right)_{\lambda=0}, \quad n = 0,1,2,3, \ldots$$

Consider the differential equation of fraction order

$$D_\xi^\alpha U = AU + B, \quad 0 < \alpha \le 1, \tag{7}$$

subject to the initial condition

$$U(0) = f.$$

Appling both sides of J^{α} both sides of Eq. (7) using the given initial condition we have

$$U(\xi)=\frac{B}{\Gamma(\alpha+1)}x^{\alpha}+U(0)+J^{\alpha}[AU].$$

Following the discussion presented in the decomposition method section, we can obtain the recurrence relation

$$U_0(\xi)=\frac{B}{\Gamma(\alpha+1)}x^{\alpha}+U(0)=\frac{B}{\Gamma(\alpha+1)}x^{\alpha}+f,$$

$$U_{k+1}(\xi)=J^{\alpha}[AU_k].$$

Case 1: Consider that $f=0, A=1$ and $B=1$ we have the above recurrence relation

$$U_0(\xi)=U(0)=\frac{1}{\Gamma(\alpha+1)}x^{\alpha},$$

$$U_1(\xi)=J^{\alpha}[U_0]=\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha},$$

$$U_2(\xi)=J^{\alpha}[U_1]=\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha},$$

$$U_3(\xi)=J^{\alpha}[U_2]=\frac{1}{\Gamma(4\alpha+1)}x^{4\alpha},$$

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The series solution is given as

$$U(\xi)=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}+\cdots.=-1+E_{\alpha}(\xi^{\alpha}).$$

where E_{α} denote the Mittage-Leffler function (Hammouch and Mekkaoui, 2012), given as

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}.$$

Generally we have

$$U(\xi)=-\frac{B}{A}+E_{\alpha}[(A\xi)^{\alpha}].$$

$$U(\xi)=-\frac{B}{A}+E_{\alpha}[(A\xi)^{\alpha}].$$

4 Analysis of (*U´/U*)-expansion method

In order to simultaneously obtain more periodic wave solutions expressed in rational hyperbolic function and rational trigonometry function to nonlinear equations, we introduce a (*U´/U*)-expansion method. We briefly show what (*U´/U*)-expansion method is and how to use it to obtain various periodic wave solutions to nonlinear equations. Suppose a nonlinear equation for $U(x,t)$ is given by We consider the general nonlinear PDE of the type

$$P\left(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xx}, u_{yy}, u_{xx}, u_{zz}, u_{xt}, u_{yt}, u_{zt}, u_{xy}, u_{xz}, u_{yz}, \dots\right) = 0 \qquad (8)$$

where P is a polynomial in its arguments. The essence of the (*U´/U*)-expansion method can be presented in the following steps:

Step 1: Seek Solitary wave solutions of Eq. (8) by taking

$$u(x,y,z,t) = u(\xi), \qquad \xi = kx + ly + mz + \omega t,$$

and transform Eq. (8) to the ordinary differential equation.

$$Q(u, \omega u', ku', lu', mu', \omega^2 u'', k^2 u'', l^2 u'', \dots) = 0 \qquad (9)$$

where ω is constant and where prime denotes the derivative with respect to ξ.

Step 2: If possible, integrate Eq. (9) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3: According to (*U´/U*)-expansion method, we assume that the wave solution can be expressed in the following form

$$u(\xi) = \sum_{n=0}^{M} a_n \left(\frac{U'}{U}\right)^n \qquad (10)$$

where U is the solution of first order nonlinear equation in the form

$$D_\xi{}^\alpha U = AU + B \qquad (11)$$

where A and B are real constants, M is a positive integer to be determined and the Eq. (11) has solution

$$U(\xi) = -\frac{B}{A} + E_\alpha[(A\xi)^\alpha].$$

Step 4: Determine M. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Eq. (8).

Step 5: Substituting (10) into Eq. (9) with (11) will yields an algebraic equation involving power of U. Equating the coefficients of like power of U to zero gives a system of algebraic equations for a_i, k, l, m and ω. Then, we solve the system with the aid of a computer algebra system (CAS), such as MAPLE 13, to determine these constants.

Step 6: Putting these constant into Eq. (10), coupled with the well known solutions of Eq. (11), we can obtained the more general type and new exact traveling wave solution of the nonlinear partial differential equation (8).

5 Application

In this section, we apply the purposed technique to construct solitary wave solution of nonlinear evolution equations. Numerical results are very encouraging.

5.1 Kaup Kuperschmidt Equation

Consider the 7th order Kaup Kuperschmidt equation (1) of fraction order

$$D_t^\alpha u + 2016u^3 D_x^\alpha u + 630(D_x^\alpha u)^3 + 2268 u D_x^\alpha u D_x^{2\alpha} u + 504 u^2 D_x^{3\alpha} u + 252 D_x^{2\alpha} u D_x^{3\alpha} u + 147 D_x^\alpha u D_x^{4\alpha} u + 42 u D_x^{5\alpha} u + D_x^{7\alpha} u = 0, \qquad 0 < \alpha \le 1, \tag{12}$$

To convert Eq. (12) into ODE we use following transformation

$$u(x,t) = u(\xi), \qquad \xi = kx + \omega t, \tag{13}$$

where k and ω are arbitrary constant. Substituting Eq. (13) into Eq. (12) and using the chain rule we obtained

$$\omega^\alpha D_\xi^\alpha u + 2016 k^\alpha u^3 D_\xi^\alpha u + 630 k^{3\alpha} \left(D_\xi^\alpha u\right)^3 + 2268 k^{3\alpha} u D_\xi^\alpha u D_\xi^{2\alpha} u + 504 k^{3\alpha} u^2 D_\xi^{3\alpha} u + 252 k^{5\alpha} D_\xi^{2\alpha} u D_\xi^{3\alpha} u + 147 k^{5\alpha} D_\xi^\alpha u D_\xi^{4\alpha} u + 42 k^{5\alpha} u D_\xi^{5\alpha} u + k^{7\alpha} D_\xi^{7\alpha} u = 0, \qquad 0 < \alpha \le 1 \tag{14}$$

By applying the homogenous balancing principle for $\alpha = 1$, we have

$$M + 7 = 3M + M + 1,$$

$$M = 2. \tag{15}$$

Using the value of M into Eq. (10), we obtained the trail solution

$$u = a_0 + a_1 \left(\frac{U'}{U}\right) + a_2 \left(\frac{U'}{U}\right)^2. \tag{16}$$

Putting Eq. (16) into Eq. (14) coupled with Eq. (11); the Eq. (16) yields an algebraic equation involving power of U as

$$-\frac{1}{U^9}[(AU+B)B(C_0U^0+C_1U^1+C_2U^2+C_3U^3+C_4U^4+C_5U^5+C_6U^6+C_7U^7)]=0.$$

Compare the like powers of U we have system of equations

U^0: $4032k^{\alpha}a_2^4B^7+44352B^7k^{3\alpha}a_2^3+101808B^7k^{5\alpha}a_2^2+40320k^{7\alpha}a_2B^7=0,$

U^1: $5040k^{7\alpha}a_1B^6+239904B^6k^{3\alpha}Aa_2^3+\cdots+146160k^{7\alpha}a_2AB^6=0,$

U^2: $84672k^{\alpha}a_1B^5a_2^3A+378000B^5k^{3\alpha}Aa_1a_2^2+\cdots+206640k^{7\alpha}a_2A^2B^5=0,$

U^3: $52920B^4k^{3\alpha}a_0a_1a_2+60480k^{\alpha}a_0a_2^3B^4A+\cdots+16800k^{7\alpha}a_1A^2B^4=0,$

U^4: $120960k^{\alpha}a_0a_1Aa_2^2B^3+139104B^3k^{3\alpha}a_0a_1Aa_2+\cdots+8400k^{7\alpha}a_1A^3B^3=0,$

U^5: $7602k^{7\alpha}a_2A^5B^2+145152B^2k^{3\alpha}A^5a_2^3+\cdots+6300k^{5\alpha}a_0a_1A^2B^2=0,$

U^6: $4032k^{\alpha}a_0^3a_2B+6048k^{\alpha}a_0^2a_1^2B+\cdots+1260k^{5\alpha}a_0a_1A^3B=0,$

U^7: $4032k^{\alpha}a_0^3a_2A+6048k^{\alpha}a_0^2a_1^2A+\cdots+3024k^{3\alpha}a_0a_1A^4a_2+\omega^{\alpha}a_1=0,$

Solving the above system for unknown parameters, we have

1st Solution Set:

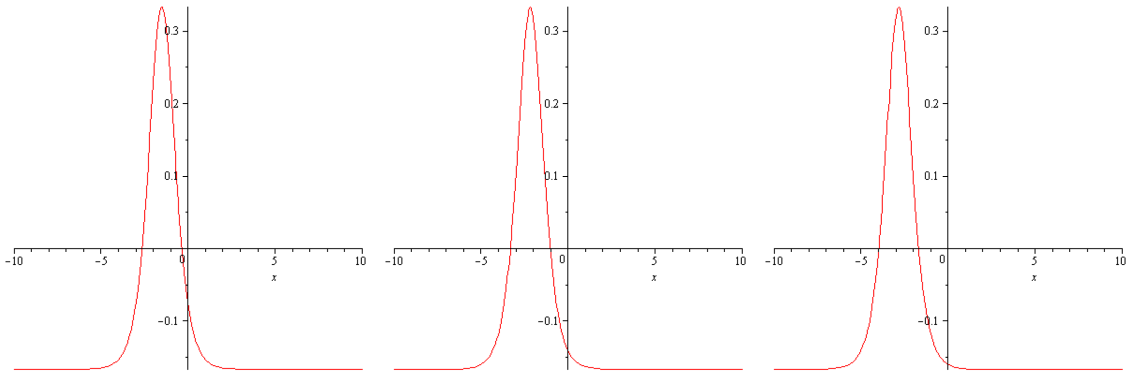
$$k=k,\omega=\mathrm{e}^{\frac{\ln\left(-\frac{1}{48}A^6\right)+7\alpha\ln(k)}{\alpha}},a_0=-\frac{1}{24}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2A^2,a_1=\frac{1}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2A,$$

$$a_2=-\frac{1}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2.$$

Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11)

$$u_1(\xi)=-\frac{1}{8}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2A^2+\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2A\left(\frac{U'}{U}\right)-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right)=\frac{A^{\alpha}E_{\alpha}[(A\xi)^{\alpha}]}{-\frac{B}{A}+E_{\alpha}[(A\xi)^{\alpha}]}.$ and $\xi=kx+\mathrm{e}^{\frac{\ln\left(-\frac{1}{16}A^4\right)+5\alpha\ln(k)}{\alpha}}t.$



$t = 0$ $t = 0.5$ $t = 1$

Figure 1: Solitary wave solution for $t = 0, t = 0.5$ and $t = 1$.

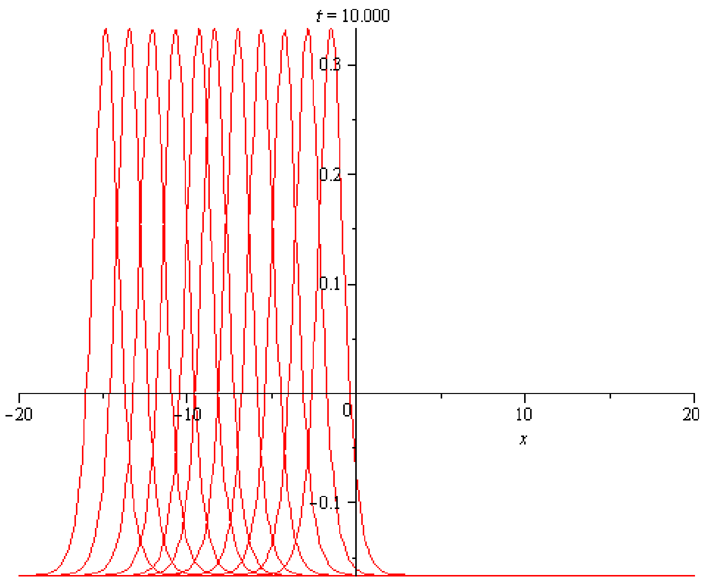


Figure 2: Combined graph for various of t.

5.2 Lax Equation

Consider the 7th order Lax equation (2) of fraction order

$$D_t^{\alpha}u + 140u^3 D_x^{\alpha}u + 70(D_x^{\alpha}u)^3 + 280uD_x^{\alpha}uD_x^{2\alpha}u + 70u^2 D_x^{3\alpha}u + 70D_x^{2\alpha}uD_x^{3\alpha}u + 42D_x^{\alpha}uD_x^{4\alpha}u + 14uD_x^{5\alpha}u + D_x^{7\alpha}u = 0, \ \ 0 < \alpha \leq 1. \quad (17)$$

To convert Eq. (17) into ODE we use following transformation

$$u(x,t) = u(\xi), \qquad \xi = kx + \omega t \quad (18)$$

where k and ω are arbitrary constant. Substituting Eq. (18) into Eq. (17) and using the chain rule we obtained

$$\omega^{\alpha}D_{\xi}^{\alpha}u+140k^{\alpha}u^{3}D_{\xi}^{\alpha}u+70k^{3\alpha}\left(D_{\xi}^{\alpha}u\right)^{3}+280k^{3\alpha}uD_{\xi}^{\alpha}uD_{\xi}^{2\alpha}u+ 70k^{3\alpha}u^{2}D_{\xi}^{3\alpha}u+70k^{5\alpha}D_{\xi}^{2\alpha}uD_{\xi}^{3\alpha}u+42k^{5\alpha}D_{\xi}^{\alpha}uD_{\xi}^{4\alpha}u+14k^{5\alpha}uD_{\xi}^{5\alpha}u+ k^{7\alpha}D_{\xi}^{7\alpha}u=0, \qquad 0<\alpha\leq 1. \tag{19}$$

By applying the homogenous balancing principle for $\alpha=1$, we have

$$M+7=3M+M+1,$$

$$M=2. \tag{20}$$

Using the value of M into Eq. (10), we obtained the trail solution

$$u=a_0+a_1\left(\frac{U'}{U}\right)+a_2\left(\frac{U'}{U}\right)^2 \tag{21}$$

Putting Eq. (21) into Eq. (19) coupled with Eq. (11); the Eq. (19) yields an algebraic equation involving power of U as

$$-\frac{1}{U^9}(AU+B)B[C_0U^0+C_1U^1+C_2U^2+C_3U^3+C_4U^4+C_5U^5+C_6U^6+C_7U^7]=0.$$

Compare the like powers of U we have system of equations

U^0: $\quad 280k^{\alpha}a_2^4B^7+5600B^7k^{3\alpha}a_2^3+30240B^7k^{5\alpha}a_2^2+40320k^{7\alpha}a_2B^7=0,$

U^1: $\quad 980k^{\alpha}a_1B^6a_2^3+1960k^{\alpha}a_2^4AB^6+\cdots+146160k^{7\alpha}a_2AB^6+5040k^{7\alpha}a_1B^6=0,$

U^2: $\quad 840k^{\alpha}a_0a_2^3B^5+1260k^{\alpha}a_1^2B^5a_2^2+\cdots+48300B^5k^{3\alpha}Aa_1a_2^2+86184B^5k^{5\alpha}a_1Aa_2=0,$

U^3: $\quad 700k^{\alpha}a_1^3B^4a_2+9800k^{\alpha}a_2^4A^3B^4+\cdots+4200k^{\alpha}a_0a_2^3AB^4+6300k^{\alpha}a_1^2AB^4a_2^2=0,$

U^4: $\quad 840k^{\alpha}a_0^2a_2^2B^3+9800k^{\alpha}a_2^4A^4B^3+\cdots+8400k^{\alpha}a_0a_1B^3a_2^2A+18480B^3k^{3\alpha}a_0a_1Aa_2=0,$

U^5: $\quad 2520B^2k^{3\alpha}A^2a_1^3+18900B^2k^{3\alpha}A^5a_2^3+\cdots+420k^{\alpha}a_1^4AB^2+5880k^{\alpha}a_2^4A^5B^2=0,$

U^6: $\quad 546Bk^{5\alpha}a_1^2A^4+840Bk^{3\alpha}a_1^3A^3+\cdots+420k^{\alpha}a_1^4A^2B+280k^{\alpha}a_0^3a_2B=0,$

U^7: $\quad 700k^{\alpha}a_1^3A^4a_2+980k^{\alpha}a_2^3A^6a_1+\cdots+k^{7\alpha}a_1A^6+2k^{7\alpha}a_2A^7+\omega^{\alpha}a_1=0,$

Solving the above system for unknown parameters, we have

1st Solution Set:

$$k=\left(\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}},\omega=\mathrm{e}^{\frac{\ln\left(-\frac{20}{27A}a_0^2\sqrt{\Omega}\left(\frac{71}{3}a_0+\frac{14}{3}\mathrm{i}\,a_0\sqrt{5}\right)\right)}{\alpha}},a_0=a_0,a_1=\frac{2}{3}\frac{\Omega}{A},a_2=-\frac{2}{3}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_1(\xi)=a_0+\frac{2}{3}\frac{\Omega}{A}\left(\frac{U'}{U}\right)-\frac{2}{3}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right)=\frac{A^\alpha E_\alpha[(A\xi)^\alpha]}{-\frac{B}{A}+E_\alpha[(A\xi)^\alpha]},\xi=\left(\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}x+\mathrm{e}^{\frac{\ln\left(-\frac{20}{27A}a_0^2\sqrt{\Omega}\left(\frac{71}{3}a_0+\frac{14}{3}\mathrm{i}\,a_0\sqrt{5}\right)\right)}{\alpha}}t.$

and $\Omega=-15a_0+3\mathrm{i}\,a_0\sqrt{5}$.

2nd Solution Set:

$$k=\left(-\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}},\omega=\mathrm{e}^{\frac{\ln\left(\frac{20}{27A}a_0^2\sqrt{\Omega}\left(\frac{71}{3}a_0+\frac{14}{3}\mathrm{i}\,a_0\sqrt{5}\right)\right)}{\alpha}},a_0=a_0,a_1=\frac{2}{3}\frac{\Omega}{A},a_2=-\frac{2}{3}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_2(\xi)=a_0+\frac{2}{3}\frac{\Omega}{A}\left(\frac{U'}{U}\right)-\frac{2}{3}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right)=\frac{A^\alpha E_\alpha[(A\xi)^\alpha]}{-\frac{B}{A}+E_\alpha[(A\xi)^\alpha]},\xi=\left(-\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}x+\mathrm{e}^{\frac{\ln\left(\frac{20}{27A}a_0^2\sqrt{\Omega}\left(\frac{71}{3}a_0+\frac{14}{3}\mathrm{i}\,a_0\sqrt{5}\right)\right)}{\alpha}}t.$

and $\Omega=-15a_0+3\mathrm{i}\,a_0\sqrt{5}$.

3rd Solution Set:

$$k=\left(\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}},\omega=\mathrm{e}^{\frac{\ln\left(-\frac{20}{27A}a_0^2\sqrt{\Omega}\left(\frac{71}{3}a_0-\frac{14}{3}\mathrm{i}\,a_0\sqrt{5}\right)\right)}{\alpha}},a_0=a_0,a_1=\frac{2}{3}\frac{\Omega}{A},a_2=-\frac{2}{3}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_3(\xi)=a_0+\frac{2}{3}\frac{\Omega}{A}\left(\frac{U'}{U}\right)-\frac{2}{3}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right)=\frac{A^\alpha E_\alpha[(A\xi)^\alpha]}{-\frac{B}{A}+E_\alpha[(A\xi)^\alpha]},\xi=\left(\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}x+\mathrm{e}^{\frac{\ln\left(-\frac{20}{27A}a_0^2\sqrt{\Omega}\left(\frac{71}{3}a_0-\frac{14}{3}\mathrm{i}\,a_0\sqrt{5}\right)\right)}{\alpha}}t.$

and $\Omega=-15a_0-3\mathrm{i}\,a_0\sqrt{5}$.

4th Solution Set:

$$k=\left(-\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}},\omega=\mathrm{e}^{\frac{\ln\left(\frac{20}{27A}a_0^2\sqrt{\Omega}\left(\frac{71}{3}a_0-\frac{14}{3}\mathrm{i}\,a_0\sqrt{5}\right)\right)}{\alpha}},a_0=a_0,a_1=\frac{2}{3}\frac{\Omega}{A},a_2=-\frac{2}{3}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_4(\xi)=a_0+\frac{2}{3}\frac{\Omega}{A}\left(\frac{U'}{U}\right)-\frac{2}{3}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right)=\frac{A^\alpha E_\alpha[(A\xi)^\alpha]}{-\frac{B}{A}+E_\alpha[(A\xi)^\alpha]},\xi=\left(-\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}x+\mathrm{e}^{\frac{\ln\left(\frac{20}{27A}a_0^2\sqrt{\Omega}\left(\frac{71}{3}a_0-\frac{14}{3}\mathrm{i}\,a_0\sqrt{5}\right)\right)}{\alpha}}t.$

and $\Omega=-15a_0-3\mathrm{i}\,a_0\sqrt{5}.$

5th Solution Set:

$$k=k,\omega=\mathrm{e}^{\frac{\ln\left(-70\left[\mathrm{e}^{\alpha\ln(k)}\right]^2a_0^2A^2-\left[\mathrm{e}^{\alpha\ln(k)}\right]^6A^6-14\left[\mathrm{e}^{\alpha\ln(k)}\right]^4a_0A^4-140a_0^3\right)+\alpha\ln(k)}{\alpha}},a_0=a_0,$$

$$a_1=2\left[\mathrm{e}^{\alpha\ln(k)}\right]^2A,a_2=-2\left[\mathrm{e}^{\alpha\ln(k)}\right]^2.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_5(\xi)=a_0+2\left[\mathrm{e}^{\alpha\ln(k)}\right]^2A\left(\frac{U'}{U}\right)-2\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\left(\frac{U'}{U}\right)^2,$$

where

$$\left(\frac{U'}{U}\right)=\frac{A^\alpha E_\alpha[(A\xi)^\alpha]}{-\frac{B}{A}+E_\alpha[(A\xi)^\alpha]},\xi=kx+\mathrm{e}^{\frac{\ln\left(-70\left[\mathrm{e}^{\alpha\ln(k)}\right]^2a_0^2A^2-\left[\mathrm{e}^{\alpha\ln(k)}\right]^6A^6-14\left[\mathrm{e}^{\alpha\ln(k)}\right]^4a_0A^4-140a_0^3\right)+\alpha\ln(k)}{\alpha}}t.$$

6th Solution Set:

$$k=\left(\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}},\omega=0,a_0=a_0,a_1=\frac{2}{9}\frac{\Omega}{A},a_2=-\frac{2}{9}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_6(\xi)=a_0+\frac{2}{3}\frac{\Omega}{A}\left(\frac{U'}{U}\right)-\frac{2}{9}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right)=\frac{A^\alpha E_\alpha[(A\xi)^\alpha]}{-\frac{B}{A}+E_\alpha[(A\xi)^\alpha]},\xi=\left(\frac{1}{3}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}x.$

and $\Omega = 3\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} - \frac{42a_0^2}{\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}} - 42a_0.$

8th Solution Set:

$$k = \left(\frac{1}{6}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}, \omega = 0, a_0 = a_0, a_1 = \frac{1}{18}\frac{\Omega}{A}, a_2 = -\frac{1}{18}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_8(\xi) = a_0 + \frac{1}{18}\frac{\Omega}{A}\left(\frac{U'}{U}\right) - \frac{1}{18}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right) = \frac{A^\alpha E_\alpha[(A\xi)^\alpha]}{-\frac{B}{A}+E_\alpha[(A\xi)^\alpha]}, \xi = \left(\frac{1}{6}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}} x, \Omega = \tau_1 + \mathrm{i}\,\tau_2,$

$$\tau_1 = -6\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} + \frac{84a_0^2}{\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}} - 168a_0,$$

and $\tau_2 = 18\mathrm{i}\sqrt{3}\left[\frac{1}{3}\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} + \frac{14a_0^2}{3\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}}\right].$

9th Solution Set:

$$k = \left(-\frac{1}{6}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}, \omega = 0, a_0 = a_0, a_1 = \frac{1}{18}\frac{\Omega}{A}, a_2 = -\frac{1}{18}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_9(\xi) = a_0 + \frac{1}{18}\frac{\Omega}{A}\left(\frac{U'}{U}\right) - \frac{1}{18}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right) = \frac{A^\alpha E_\alpha[(A\xi)^\alpha]}{-\frac{B}{A}+E_\alpha[(A\xi)^\alpha]}, \xi = \left(-\frac{1}{6}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}} x, \Omega = \tau_1 + \mathrm{i}\,\tau_2,$

$$\tau_1 = -6\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} + \frac{84a_0^2}{\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}} - 168a_0,$$

and $\tau_2 = 18\mathrm{i}\sqrt{3}\left[\frac{1}{3}\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} + \frac{14a_0^2}{3\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}}\right].$

10th Solution Set:

$$k = \left(\frac{1}{6}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}, \omega = 0, a_0 = a_0, a_1 = \frac{1}{18}\frac{\Omega}{A}, a_2 = -\frac{1}{18}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_{10}(\xi) = a_0 + \frac{1}{18}\frac{\Omega}{A}\left(\frac{U'}{U}\right) - \frac{1}{18}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right) = \frac{A^{\alpha}E_{\alpha}[(A\xi)^{\alpha}]}{-\frac{B}{A}+E_{\alpha}[(A\xi)^{\alpha}]}, \xi = \left(\frac{1}{6}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}} x, \Omega = \tau_1 + \mathrm{i}\,\tau_2,$

$$\tau_1 = -6\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} + \frac{84a_0^2}{\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}} - 168a_0,$$

and $\tau_2 = -18\mathrm{i}\sqrt{3}\left[\frac{1}{3}\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} + \frac{14a_0^2}{3\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}}\right].$

11th Solution Set:

$$k = \left(-\frac{1}{6}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}}, \omega = 0, a_0 = a_0, a_1 = \frac{1}{18}\frac{\Omega}{A}, a_2 = -\frac{1}{18}\frac{\Omega}{\mathrm{A}^2}.$$

Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11)

$$u_{11}(\xi) = a_0 + \frac{1}{18}\frac{\Omega}{A}\left(\frac{U'}{U}\right) - \frac{1}{18}\frac{\Omega}{A^2}\left(\frac{U'}{U}\right)^2,$$

where $\left(\frac{U'}{U}\right) = \frac{A^{\alpha}E_{\alpha}[(A\xi)^{\alpha}]}{-\frac{B}{A}+E_{\alpha}[(A\xi)^{\alpha}]}, \xi = \left(-\frac{1}{6}\frac{\sqrt{\Omega}}{A}\right)^{\frac{1}{\alpha}} x, \Omega = \tau_1 + \mathrm{i}\,\tau_2,$

$$\tau_1 = -6\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} + \frac{84a_0^2}{\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}} - 168a_0,$$

and $\tau_2 = -18\mathrm{i}\sqrt{3}\left[\frac{1}{3}\left(-224a_0^3 + 42\sqrt{30}a_0^3\right)^{\frac{1}{3}} + \frac{14a_0^2}{3\left(-224a_0^3+42\sqrt{30}a_0^3\right)^{\frac{1}{3}}}\right].$

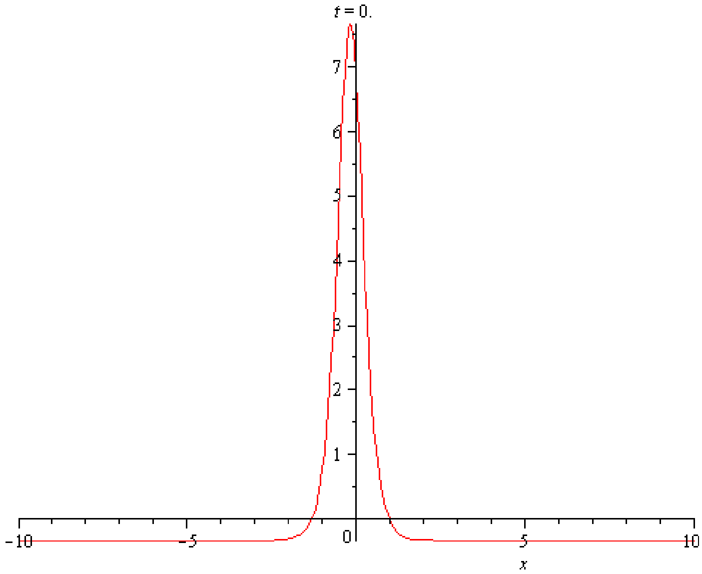


Figure 3: Solitary wave solution for $t = 0$

6 Conclusions

(U'/U)-expansion method is applied to obtain generalized solitary solutions of nonlinear 7th order Kaup Kuperschmidt and Lax equations of fractional order. Figure 1 show the travelling wave solutions for different values of t, Fig. 2, contained the combined graph of 7th order Kaup Kuperschmidt of fractional order for $x \in [-20,20]$ and $t \in [0,1]$ which obey the properties of solitary wave i.e., that maintains its shape while it travels at constant speed.

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