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# The new systematic procedure in group theoretic methods with applications to class of boundary layer flows

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**ABSTRACT**

The present paper is made of the formalism of a new systematic group theoretic procedure for reducing the number of independent variables from a system of partial differential equations with the set of assigned auxiliary conditions. The present procedure has an advantage to group theoretic techniques developed in the past. The present technique is supported by an application to system of nonlinear partial differential equations governing a steady two dimensional boundary layer flow of incompressible second order fluid past a stretching sheet.

***Keywords:*** Symmetry transformations, absolute invariants, group theoretic method.

***MSC 2010 codes:*** 54H15, 37J15.



## 1 Introduction

The appropriate change of variables is perhaps one of the most useful methods available for the treatment of partial differential equations of mathematical physics and engineering boundary value problems. The most general criterion for a transformation is simply to convert given problem into a simpler problem in some sense or the other, that is, either to a form which will yield to more standard solution technique or make it possible to reduce to a form which has been previously solved in connection with a related or similar problem. Conformal transformation and hodograph transformations are well known methods which transform a given problem into a form which yields to a classical technique; such as, free parameter and separation of variables. Generally, various transformation techniques can be classified into three groups: transformation of independent variables, transformation of dependent variables and the mixed transformations of both independent and dependent variables. However, all three groups have common goal: to find a relation or more specifically a basis of comparison between physical and mathematical problems. This broad concept of comparison defines similarity technique.

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The methods to find similarity transformations are classified into: (i) direct methods and (ii) group-theoretic methods; the direct methods such as free parameter, separation of variables and dimensional analysis do not invoke group invariance. Group-theoretic methods, on the other hand, are mathematically more elegant and systematic as its theory is derived from the concept of invariance of continuous group transformations. In some group-theoretic procedures namely: (Birkhoff, 1960) and (Morgan, 1952) method, (Hellums and Churchill, 1964) method, the specific form of the group is assumed in priori. On the other hand, procedure such as finite group method of (Moran and Gaggioli, 1968) and (Moran, Gaggioli and Scholten, 1968), infinite group methods by (Lie, 1881), (Bluman and Cole, 1974), (Bluman and Kumei, 1980), (Bluman and Anco, 2002) and (Na and Hansen, 1971) a general group of transformation is defined and similarity solutions are systematically deduced. In the present paper the procedure proposed by (Morgan, 1952) which was further extended by (Moran and Gaggioli, 1968) and (Moran, Gaggioli and Scholten, 1968), is now implemented in an alternate way for a special group. It is hoped that it will be used in wider class of group transformations and as result the present developed similarity technique has application to a wide range of physical problems. The present paper also overcomes the inspection and or trial in tabular approach of (Moran and Marshek, 1975). The general concept of present technique along with its application is discussed in detail in subsequent sections.

## 2 Preliminaries

We give below a detailed account of some important work from literature laid foundation for the procedure presented here, Different methods for determining similarity transformations, the group theoretic methods one of these methods which arisen in many literatures. In (Morgan, 1952), his theory is a significant contribution to an area of applied mathematics in which there is continuing interest. He considered one-parameter group $S$; $G$ and $E^k$ of the form

$$E^k = \begin{cases} G_1 = \begin{cases} S_1 : \{ \bar{x}_i = f_i(x_1, x_2, ..., x_n; a) \quad i = 1, 2, ..., n \quad n \geq 2 \\ \bar{u}^j = h_j(u^j; a) \quad j = 1, 2, ..., m \quad m \geq 1 \end{cases} \\ \hline \left( \dfrac{\partial^l u^j}{\partial(x_1)^{l_1} ... \partial(x_n)^{l_n}} \right) = h^{(l)}_{(j; l_1, ..., l_n)} \Big( \dfrac{\partial^l u^j}{\partial(x_1)^l}, \dfrac{\partial^l u^j}{\partial(x_1)^{l-1} \partial(x_2)}, ..., \dfrac{\partial^l u^j}{\partial(x_n)^l}, \\ \qquad \dfrac{\partial^{l-1} u^j}{\partial(x_1)^{l-1}}, ..., u^j, x_1, ..., x_m; a \Big) \\ \qquad \sum_{\text{i}} l_i = l \prec k \end{cases} \tag{1}$$

where the functions $f$ and $h$ are continuous in the parameter $a$ ; the identity element is denoted by $a_0$. Given any set of functions $\{u^j\}$, $u^j = u^j(x_1, ..., x_n)$ in class $C^k$ , and consider the set of function $\{\bar{u}_j\}$ defined by

$$\bar{u}^j(\bar{x}_1, ..., \bar{x}_n) = h_j(u^j[f_1(\bar{x}_1, ..., \bar{x}_n; a), ..., f_n(\bar{x}_1, ..., \bar{x}_n; a)]; a) .$$

$E^k$ is continuous group, called the $k$th enlargement of $G$ for $\{u^j\}$, where the functions $h^l_{(j;l_1,...,l_n)}$ are defined so that

$$\overline{\left(\frac{\partial^l u^j}{\partial(x_1)^{l_1}...\partial(x_n)^{l_n}}\right)} = \frac{\partial^l \overline{u}^j}{\partial(\overline{x}_1)^{l_1}...\partial(\overline{x}_n)^{l_n}}$$

Furthermore, according to elementary group theory (Eisenhart, 1961), with $x_i$ , $i=1,2,...,n$ as independent variables and $u^j$, $\mathrm{j}=1,2,...,\mathrm{m}.$ as independent variables, $G$ has $n+m-1$ functionally independent absolute invariant designated by

$$\Omega_1(x_1,x_2,...,x_n),...,\Omega_{n-1}(x_1,x_2,...,x_n);$$
$$g_j(u^1,u^2,...,u^m;x_1,x_2,...,x_n), \qquad \mathrm{j}=1,2,..,\mathrm{m}$$

and $g_j$ can be selected so that the Jacobian $\dfrac{\partial(g_1,...,g_m)}{\partial(u^1,...,u^m)} \neq 0$, and the rank of the Jacobian matri $\dfrac{\partial(\Omega_1,...,\Omega_n)}{\partial(x_1,...,x_n)}$ is equal to $n-1$.

Now consider the differential form,

$$\varphi\left(x_1,...,x_n,u^1,...,u^m,...,\frac{\partial^k u^1}{\partial(x_1)^k},...,\frac{\partial^k u^m}{\partial(x_n)^k}\right) \tag{2}$$

Following Morgan, $\phi$ is said to be conformably invariant under $E^k$ if

$$\varphi(\overline{z}_1,\overline{z}_2...,\overline{z}_p) = F(z_1,z_2...,z_p;a)\varphi(z_1,z_2...,z_p) \tag{3}$$

for some function $F$. Where the arguments $z_1,z_2...,z_p$ are the variables $x_1,x_2...,x_n$ , the functions $u^1,u^2,...,u^m$ and the derivatives thereof up to the $k$th order. Also, a system of partial differential equations

$$\varphi_\gamma\left\{x_1,...,x_n,u^1,...,u^m,...,\frac{\partial^k u^1}{\partial(x_1)^k},...,\frac{\partial^k u^m}{\partial(x_n)^k}\right\}, \text{ where } \gamma=1,2,...N\leq m \tag{4}$$

is said to be invariant under $G$ if each of the differential forms $\phi_1,\phi_2...,\phi_N$ is conformally invariant under $E^k$.

## 2.1 Morgan's Theorem (Morgan, 1952)

If each of the forms $\phi_\gamma$ are conformally invariant under the group $E^k$ , then the invariant solutions of $\phi_\gamma=0$ can be expressed in terms of the solutions of a new system of PDEs with $\eta_1,...,\eta_{n-1}$ as independent variables and $F_1,...,F_m$ as dependent variables

$$\overline{\varphi}_{\gamma}\left\{\eta_1,\eta_2,...,\eta_{n-1},F_1,...,F_m,\frac{\partial^k F_1}{\partial(\eta_1)^k},...,\frac{\partial^k F_m}{\partial(\eta_{n-1})^k}\right\}, \tag{5}$$

for each $\gamma = 1,2,...N \le m$, such that

$$\begin{cases}\eta_i = \Omega_i, & \text{i} = 1,2,...,\text{n}-1\\ F_{\text{j}}(\eta_1,\eta_2,...,\eta_{n-1}) = g_j, & \text{j} = 1,2,...,\text{m}\end{cases}$$

In the original procedure proposed by Morgan the group is deduced for the partial differential equation only, this leads to wastage of efforts if the auxiliary conditions are not satisfied; and there is no systematic approach to determine the absolute invariance, it is yielded by inspection and/or trial. These drawbacks are overcome by (Moran and Gaggioli, 1968) using the deduced group for auxiliary conditions as well as partial differential equation and absolute invariance is determined by a systematic procedure

$$E_r^k : \begin{cases} G_r = \begin{cases} S_r : \left\{\overline{x}_i = f_i = C^{x_i}(a^1,a^2...,a^r)x_i + K^{x_i}(a^1,a^2...,a^r)\right. \\ \overline{u}^j = h_j = C^{u^j}(a^1,a^2...,a^r)u^j + K^{u^j}(a^1,a^2...,a^r)\end{cases} \\ \overline{\left(\frac{\partial^l u^j}{\partial(x_1)^{l_1}...\partial(x_n)^{l_n}}\right)} = h^{(l)}_{(j;l_1,...,l_n)}\left(\frac{\partial^l u^j}{\partial(x_1)^l},\frac{\partial^l u^j}{\partial(x_1)^{l-1}...\partial(x_2)^1},...,\frac{\partial^l u^j}{\partial(x_n)^l},\right. \\ \qquad \left.\frac{\partial^{l-1}u^j}{\partial(x_1)^{l-1}},...,u^j,x_1,...,x_m;a^1,a^2...,a^r\right) \\ \qquad \sum_{\text{i}} l_i = l \prec k \end{cases} \tag{6}$$

with the property $C^{S_i}(a_0^1,a_0^2...,a_0^r) = 1$ and $K^{S_i}(a_0^1,a_0^2...,a_0^r) = 0$, $\text{S}_i = x_1,x_2,...,x_n,u^1,u^2,...,u^m$ where $a_0^1,a_0^2...,a_0^r$ are a values of parameters corresponding to the identical transformation. However, the other steps remain unchanged; as a result, many of the manipulations become easier.

Gaggioli and Moran (1968) for complete sets of absolute invariants invoked a basic theorem from group theory which refers to the symbol $X$ of the group $G_1$ in (1),

$$X = \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial a}\right)_{a_0}\frac{\partial}{\partial x_i} + \sum_{j=1}^{n}\left(\frac{\partial h_j}{\partial a}\right)_{a_0}\frac{\partial}{\partial u^j} \tag{7}$$

*as following*: ***A function*** $F$ ***is an absolute invariant of*** $G$ ***if and only if*** $XF = 0$**.**

In (Moran and Marshek, 1975) presented the tabular approach, he proposed to proceed in a converse manner. That is, the group can be specified and the general form of the partial differential equations which is transformed invariantly under the group can be associated with a specified group.

## 3 Proposed method

The proposed method starts with the infinitesimal group for the group $G$ in (6) which according to the relation between the global group and the infinitesimal group be formed. For that the $k$th expanded group of the infinitesimal group of $G$ in (6) in case one-parameter group becomes as following

$$E^{k}\quad \begin{cases} G_1 = \begin{cases} S_1 : \{ \overline{x}_i = x_i + \varepsilon\ \xi_i + O(\varepsilon^2) \\ \overline{u}^j = u^j + \varepsilon\ \zeta^j + O(\varepsilon^2) \end{cases} \\ \overline{\left(u_i^j\right)} = u_i^j + \varepsilon \zeta^j_{[i]} + O(\varepsilon^2) \\ \vdots \\ \overline{\left(u^j_{i_1 i_2 \ldots i_k}\right)} = u^j_{i_1 i_2 \ldots i_k} + \varepsilon \zeta^j_{[i_1 i_2 \ldots i_k]} + O(\varepsilon^2) \\ \qquad\qquad i_1, i_2, .., i_k = 1,2,...,k;\ \ \mathrm{j}=1,2,...,\mathrm{m} \end{cases} \tag{8}$$

Where $u^j_{i_1 i_2 \ldots i_k} = \dfrac{\partial^k u^j}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}}$ and

$$\xi_i = \left.\frac{\partial \overline{x}_i}{\partial a}\right|_{a=a_0} = \left.\frac{\partial C^{x_i}}{\partial a}\right|_{a=a_0} x_i + \left.\frac{\partial K^{x_i}}{\partial a}\right|_{a=a_0},\ \ \zeta^j = \left.\frac{\partial \overline{u}^j}{\partial a}\right|_{a=a_0} = \left.\frac{\partial C^{u^j}}{\partial a}\right|_{a=a_0} u^j + \left.\frac{\partial K^{u^j}}{\partial a}\right|_{a=a_0}$$

Putting $\alpha_i = \left[\dfrac{\partial C^{x_i}}{\partial a}\right]_{a=a_0},\ \beta_i = \left[\dfrac{\partial k^{x_i}}{\partial a}\right]_{a=a_0},\ \alpha_{n+j} = \left[\dfrac{\partial C^{u^j}}{\partial a}\right]_{a=a_0},\ \beta_{n+j} = \left[\dfrac{\partial k^{u^j}}{\partial a}\right]_{a=a_0}$ to write

$$\xi_i = \alpha_i x_i + \beta_i \ \text{ and } \ \zeta^j = \alpha_{n+j} u^j + \beta_{n+j}\,.$$

The derivations $\zeta^j_{[i]}$ and $\zeta^j_{[i_1 i_2 \ldots i_k]}$ is calculated using some formulas which nicely presented in (Bluman and Kumei, 1980).

Following (Bluman and Cole, 1974) and (Marshek, 1971), the necessary and sufficient conditions for (4) to become invariant in the form under an extended group (8) in case one-parameter group is

$$X\varphi_\gamma = 0, \quad \text{where} \quad \gamma = 1,2,...,N \tag{9}$$

Or in expanded form

$$\sum_{i=1}^{n} (\alpha_i x_i + \beta_i) \frac{\partial \phi_\gamma}{\partial x_i} + \ldots + \sum_{j}^{m} (\alpha_{n+j} u^j + \beta_{n+j}) \frac{\partial \phi_\gamma}{\partial u^j} + \sum_{j=1}^{m} \sum_{i=1}^{n} \zeta^j_{[i]} \frac{\partial \phi_\gamma}{\partial u_i^j} + \ldots. + \sum_{j=1}^{m} \sum_{i=1}^{n} \zeta^j_{[i_1 i_2 \ldots i_k]} \frac{\partial \phi_\gamma}{\partial u^j_{i_1 i_2 \ldots i_k}} = 0$$

Herein, we will restrict ourselves to the one-parameter group $G_1$ in (8). Equation (9) has $p-1$ independent invariants determined upon establishing $p-1$ independent solutions to the following associated system,

$$\frac{dx_1}{\alpha_1 x_1+\beta_1}=\dots=\frac{dx_n}{\alpha_n x_n+\beta_n}=\frac{du^1}{\alpha_{n+1}u^1+\beta_{n+1}}=\dots=\frac{du^m}{\alpha_{n+m}u^m+\beta_{n+m}}=\frac{du_i^j}{\zeta_{[i]}^j}=\dots=\frac{du_{i_1 i_2\dots i_k}^j}{\zeta_{[i_1 i_2\dots i_k]}^j} \quad (10)$$

where $p$ is number of independent variables, dependent variables and the derivatives thereof up to the *k*th order.

Determining $p-1$ independent solutions of the (3.3), $\lambda_1,\dots,\lambda_{p-1}$ (say), is the first step in establishing our formulation. Following (Emanuel, 2001), (Marshek, 1971) and (Cohen, 1911).

It is well known that an arbitrary function $\psi$ obtained by equating an $p-1$ independent invariants of extended group (8) to zero is invariant; that is,

$$\psi(\lambda_1,\lambda_2,\dots,\lambda_{p-1})=0 \quad (11)$$

Let us denoted to the first $n+m-1$ arguments of $\lambda_1,\dots,\lambda_{p-1}$ by $\Omega_1,\dots,\Omega_{n-1},g_1,\dots,g_m$, where $\Omega_\sigma(\sigma=1,2,\dots,n-1)$ satisfying the first $n$ equations of (3.3) and $g_j(j=1,2,\dots,m)$ satisfying the first $n+m$ equations of (10). i.e., the absolute invariants of independent and dependent variables, which form a complete set of the absolute invariants to $G_1$.

$\psi$ which be invariant under the extended one-parameter group of $G_1$, can generally be expressed in terms of new variables $(\eta_1,\dots,\eta_{n-1},F_1,\dots,F_m)$ generated by transform the first $n+m-1$ arguments $\Omega_1,\dots,\Omega_{n-1},g_1,\dots,g_m$, which refer to complete set of an absolute invariants to $G_1$, as following:

$$\begin{cases}\eta_i=\Omega_i, & i=1,2,\dots,n-1\\ F_j(\eta_1,\eta_2,\dots,\eta_{n-1})=g_j, & j=1,2,\dots,m\end{cases} \quad (12)$$

That is, if $\eta_q(q=1,2,\dots,n-1)$ are the absolute invariants of independent variables, then

$$g_j=F_j(\eta_1,\eta_2,\dots,\eta_{n-1}), \qquad j=1,2,\dots,m$$

For more details see (Abd-el-Malek and Badran, 1991); (Abd-el-Malek et al., 1990, 1990a); (Moran and Gaggioli, 1968); (Moran et al., 1968) and (Moran and Marshek, 1975).

This guarantees invariant solution under the group [see (Moran, Gaggioli and Scholten, 1968), page-64]. The result is

$$\psi(\lambda_1,\lambda_2,\dots,\lambda_{p-1})=\overline{\psi}(\eta_1,\eta_2,\dots,\eta_{n-1},F_1,\dots,F_m) \quad (13)$$

where the $\overline{\psi}$ is similarity representation and a new variables $(\eta_1,\eta_2,\dots,\eta_{n-1},F_1,\dots,F_m)$ are similarity variables or similarity transformation of $\psi$ (i.e., the variables of the similarity representation $\overline{\psi}$ ).

### 3.1 Remark

The importance of the absolute invariants of independent variables lies in the fact that they become new variables. Those new variables are called similarity variables if the problem expressible in terms of them, otherwise they are not similarity variables.

The new approach proposed to proceed in a converse manner, i.e., if $\phi_\gamma$ are expressed in terms of the $n+m-1$ new variables which be formed from translate the complete set of the absolute invariants of $G_1$ according to the transformation (12), then $\phi_\gamma$ is invariance under this group.

### 3.2 Lemma

If by inspection $\phi_\gamma$ has the form $\psi$, then $\phi_\gamma$ is expressed in terms of the new variables $(\eta_1,...,\eta_{n-1},F_1,...,F_m)$ generated from the absolute invariants. Conversely, if $\phi_\gamma$ is expressed in terms of $(\eta_1,...,\eta_{n-1},F_1,...,F_m)$, then $\phi_\gamma$ has the form $\psi$ .
The proof is obvious from the above discuss**.**

### 3.3 Theorem

If $\phi_\gamma$ are expressed in terms of the new variables $(viz\ \eta_1,...,\eta_{n-1},F_1,...,F_m)$, then $\phi_\gamma$ is invariant under $G_1$.
According to ((Bluman and Anco, 2002), Theorem (4.3-1); page:330), $G_1$ leaves the system $\phi_\gamma$ invariant if and only if $X^k\phi_\gamma=0$, for each $\gamma=1,2,...,\mathrm{N}$, i.e., if $\phi_\gamma$ satisfy $X^k\phi_\gamma=0$, then the system is invariant under the given group.

***Proof*:**
Suppose that $\phi_\gamma$ expressed in terms of the $n+m-1$ new variables $(viz\ \eta_1,...,\eta_{n-1},F_1,...,F_m)$, which be found from translate the complete set of the absolute invariants of $G_1$, according to the transformation (12). Then lemma 3.2 shows that $\phi_\gamma$ has the form $\psi$ and so

$$\varphi_\gamma(x_1,...,x_n,u^1,...,u^m,\partial\mathrm{u},...,\partial^k\mathrm{u})=\bar{\psi}(\eta_1,...,\eta_{n-1},F_1,...,F_m)=0 \qquad \gamma=1,2,...,N$$

Let $z_1,...,z_p$ are the variables $x_1,...,x_n$, functions $u^1,...,u^m$, and the derivatives thereof up to the $k$th order, $\varsigma_1,...,\varsigma_p$ are $\xi_1,...,\xi_n,\zeta^1,...,\zeta^m,\zeta^j_{[i]},...,\zeta^j_{[\gamma_1\gamma_2\cdots\gamma_k]}$. We will show that $\phi_\gamma$ is invariant under this group, i.e., we will show that $\phi_\gamma$ satisfy $X^k\phi_\gamma=0$**.**

$$X^k\varphi_\gamma = X^k\overline{\psi}$$
$$= X^k\psi$$
$$= \varsigma_1(z_1,...,z_p)\frac{\partial\psi}{\partial z_1} + .... + \varsigma_p(z_1,...,z_p)\frac{\partial\psi}{\partial z_p}$$
$$= \varsigma_1(z_1,...,z_p)(\frac{\partial\psi}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial z_1} + ... + \frac{\partial\psi}{\partial\lambda_{p-1}}\frac{\partial\lambda_{p-1}}{\partial z_1}) + ....$$
$$+ \varsigma_p(z_1,...,z_p)(\frac{\partial\psi}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial z_p} + ... + \frac{\partial\psi}{\partial\lambda_{p-1}}\frac{\partial\lambda_{p-1}}{\partial z_p})$$
$$= \frac{\partial\psi}{\partial\lambda_1}(\varsigma_1\frac{\partial\lambda_1}{\partial z_1} + ... + \varsigma_p\frac{\partial\lambda_1}{\partial z_p}) + .... + \frac{\partial\psi}{\partial\lambda_{p-1}}(\varsigma_1\frac{\partial\lambda_{p-1}}{\partial z_1} + ... + \varsigma_p\frac{\partial\lambda_{p-1}}{\partial z_p})$$
$$= \frac{\partial\psi}{\partial\lambda_1}(0) + .... + \frac{\partial\psi}{\partial\lambda_{p-r}}(0)$$
$$= 0 \qquad \text{[since } \lambda_1,...,\lambda_{\text{p-1}} \text{ are solutions of (9)]}$$

for each $\gamma = 1,2,...,N$. Thus $X^k\phi_\gamma = 0$ hold. Hence $\phi's$ are invariant under the given group.

Now we can get $n+m-1$ independent solutions $\lambda_s$ $(\text{s} = 1,2,...,n+m-1)$ of (10), for the independent and dependent variables from which we can derived the new variables in the general manner. We suppose $x_k$ is the independent variable to be eliminated. There are many sets of invariants and corresponding new variables:

**Set 1**: If $\alpha_k \neq 0$ the invariants of $G_1$, are

$$\Omega_\gamma = \begin{cases} A(x_k + \frac{\beta_k}{\alpha_k})^{-\frac{\alpha_\gamma}{\alpha_k}}(x_\gamma + \frac{\beta_\gamma}{\alpha_\gamma}), & if\ \ \alpha_\gamma \neq 0 \\ A\ln(x_k + \frac{\beta_k}{\alpha_k})^{-\frac{\beta_\gamma}{\alpha_k}} x_\gamma & if\ \ \alpha_\gamma = 0 \end{cases}$$

$$g_j = \begin{cases} B\left(x_k + \frac{\beta_k}{\alpha_k}\right)^{-\frac{\alpha_{n+j}}{\alpha_k}}\left(u^j + \frac{\beta_{n+j}}{\alpha_{n+j}}\right), & \text{if}\ \ \alpha_{n+j} \neq 0 \\ B\ \ln\left(x_k + \frac{\beta_k}{\alpha_k}\right)^{-\frac{\beta_{n+j}}{\alpha_k}} u^j & \text{if}\ \ \alpha_{n+j} = 0 \\ u^j & \text{if}\ \ \alpha_{n+j} = \beta_{n+j} = 0 \end{cases}$$

which generate the following transformations in term of new variables:

$$\eta_\gamma = A\left(x_k + \frac{\beta_k}{\alpha_k}\right)^{-\frac{\alpha_\gamma}{\alpha_k}}\left(x_\gamma + \frac{\beta_\gamma}{\alpha_\gamma}\right), \qquad u^{\text{j}} = R_1 \tag{14}$$

and

$$\eta_\gamma = A\ \ln\left(x_k + \frac{\beta_k}{\alpha_k}\right)^{-\frac{\beta_\gamma}{\alpha_k}} x_\gamma, \qquad u^j = R_1 \tag{15}$$

where

$$R_1 = \begin{cases} \dfrac{1}{B}\left(x_k + \dfrac{\beta_k}{\alpha_k}\right)^{\frac{\alpha_{n+j}}{\alpha_k}} F_j(\eta_1,\eta_2,...,\eta_{n-1}) - \dfrac{\beta_{n+j}}{\alpha_{n+j}}, & \text{if } \alpha_{n+j} \neq 0 \\ \dfrac{1}{B}\ln\left(x_k + \dfrac{\beta_k}{\alpha_k}\right)^{\frac{\beta_{n+j}}{\alpha_k}} F_j(\eta_1,\eta_2,...,\eta_{n-1}) & \text{if } \alpha_{n+j} = 0 \\ F_j(\eta_1,\eta_2,...,\eta_{n-1}) & \text{if } \alpha_{n+j} = \beta_{n+j} = 0 \end{cases}$$

**<u>Set 2</u>**: If $\alpha_k = 0$ the invariants of $G_1$, are

$$\Omega_\gamma = \begin{cases} Ae^{\frac{\alpha_\gamma}{\beta_k}x_k}\left(x_\gamma + \dfrac{\beta_\gamma}{\alpha_\gamma}\right), & if\ \ \alpha_\gamma \neq 0 \\ A\ (\beta_k x_\gamma\text{-}\beta_\gamma x_k) & \text{if } \alpha_\gamma = 0 \end{cases}$$

$$g_j = \begin{cases} Be^{-\frac{\alpha_{n+j}}{\beta_k}x_k}\left(u^j + \dfrac{\beta_{n+j}}{\alpha_{n+j}}\right), & \text{if } \alpha_{n+j} \neq 0 \\ B\ (\beta_{n+j}x_k\text{-}\beta_k u^j) & \text{if } \alpha_{n+j} = 0 \\ u^j & \text{if } \alpha_{n+j} = \beta_{n+j} = 0 \end{cases}$$

which generate the following transformations in term of new variables:

$$\eta_\gamma = Ae^{-\frac{\alpha_\gamma}{\beta_k}x_k}\left(x_\gamma + \frac{\beta_\gamma}{\alpha_\gamma}\right), \qquad u^{\mathrm{j}} = R_2 \tag{16}$$

and

$$\eta_\gamma = A\ (\beta_k x_\gamma\text{-}\beta_\gamma x_k)\ , \qquad u^{\mathrm{j}} = R_2 \tag{17}$$

where

$$R_2 = \begin{cases} \dfrac{1}{B}e^{-\frac{\alpha_{n+j}}{\beta_k}x_k} F_j(\eta_1,\eta_2,...,\eta_{n-1}) - \dfrac{\beta_{n+j}}{\alpha_{n+j}}, & if\ \ \alpha_{n+j} \neq 0 \\ \dfrac{1}{\beta_k}\left\{\beta_{n+j}x_k\text{-}\dfrac{1}{B}F_j(\eta_1,\eta_2,...,\eta_{n-1})\right\} & if\ \ \alpha_{n+j} = 0 \\ F_j(\eta_1,\eta_2,...,\eta_{n-1}) & \text{if } \alpha_{n+j} = \beta_{n+j} = 0 \end{cases}$$

where $\gamma = 1,2,...,n-1$, $(\gamma \neq k)$ and $j = 1,2,...,m$.

# 4 Illustration

The new method has applications to a wide range of physical problems. However, to explain it, we have considered the problem of the flow of an incompressible second-order fluid past a stretching sheet see (Rajagopal, Na and Gupta, 1984). The steady two-dimensional boundary layer equations for this fluid were derived by (Beard and Waters, 1964). In usual notation these equations are

$$u_x + u_y = 0 \tag{18}$$

$$uu_x + v\,u_y = U_\infty \frac{dU_\infty}{dx} + \nu u_{yy} - k[\frac{\partial}{\partial x}(uu_{yy}) + u_y u_{yy} + v\,u_{yyy}] \tag{19}$$

where $u \;\; v$ are the velocity components in the $x, y$ directions, respectively, $\nu$ the kinematic viscosity and $k$ the elasticity parameter. For deriving these equations it was assumed that in addition to the usual boundary layer approximations the contribution due to normal stresses to be of the same order of magnitude as that due to shear stresses. The subjected boundary conditions are

$$u = cx, \;\; v = 0 \;\text{ at }\; y = 0; \;\; u \to 0 \;\text{ as } y \to 0, \; c > 0 \tag{20}$$

## 4.1 The group of transformations

The procedure is initiated with the group $G_1,$ a class of one-parameter group of the form

$$G_1 = \begin{cases} S_1 = \begin{cases} \bar{x} = C^x(a)x + K^x(a) \\ \bar{y} = C^y(a)y + K^y(a) \end{cases} \\ \\ u = C^u(a)u + K^u(a) \\ v = C^v(a)v + K^v(a) \end{cases} \tag{21}$$

Where $C's$ and $K's$ are real-valued and at least differentiable in the real argument $a$ .

## 4.2 Complete set of absolute invariants

By application of a basic theorem in group theory, see (Moran and Gaggioli, 1968) and (Moran et at., 1968), the absolute invariants of independent variables and dependent variables of (21), is determining as solutions of (10) from which the new variables is obtained respectively according to (14)-(17) as following: for $\alpha_1 \neq 1$

$$\eta = A\left(x + \frac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_2}{\alpha_1}}\left(y + \frac{\beta_2}{\alpha_2}\right), \quad \text{with } \alpha_1 \neq 0 \text{ and } \alpha_2 \neq 0 \tag{22}$$

$$u = \begin{cases} \dfrac{1}{B}\left(x + \dfrac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_3}{\alpha_1}} F(\eta) - \dfrac{\beta_3}{\alpha_3}, & \text{if } \alpha_3 \neq 0 \\ B\,F(\eta) & \text{if } \alpha_3 = \beta_3 = 0 \end{cases} \tag{23}$$

and

$$v=\begin{cases}\dfrac{1}{C}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_4}{\alpha_1}}G(\eta)-\dfrac{\beta_4}{\alpha_4}, & \text{if } \alpha_4\neq 0\\ C\,G(\eta) & \text{if } \alpha_4=\beta_4=0\end{cases} \tag{24}$$

where $A$, $B$ and $C$ are arbitrary constants.

### 4.3 The similarity representation

We will start with the auxiliary conditions, i.e., (we will examine whether the auxiliary conditions are appropriate, or disproportionate to (22)-(24), because these auxiliary conditions must transform properly, otherwise no similarity solution is possible). Then we get

$$\left.\begin{array}{l} F(\eta)=1, \quad G(\eta)=0, \ \text{at} \ \eta=0, \text{ when } \alpha_3=\alpha_1, \quad B=c \ \text{ and } \ \beta_2=0,\\ F(\eta)=0, \quad \text{as} \ \eta\to\infty \ \text{ when } \ \beta_1=0\end{array}\right\} \tag{25}$$

Now substituting Eqs. (22)-(24) in the Eqs. (18) and (19) yields, after dividing the two equations by $B\left(x+\frac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_3}{\alpha_1}-1}$ and $B^2\left(x+\frac{\beta_1}{\alpha_1}\right)^{\frac{2\alpha_3}{\alpha_1}-1}$ respectively, and rearranging the terms,

$$\alpha_3F-\alpha_2\eta F'+\frac{AC}{B}\alpha_1\left(x+\frac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_4}{\alpha_1}-\frac{\alpha_2}{\alpha_1}-\frac{\alpha_3}{\alpha_1}+1}G'=0 \tag{26}$$

$$\left.\begin{array}{l}
\dfrac{\alpha_3}{\alpha_1}F^2-\dfrac{\alpha_2}{\alpha_1}\eta FF'+\dfrac{AC}{B}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_4}{\alpha_1}-\frac{\alpha_2}{\alpha_1}-\frac{\alpha_3}{\alpha_1}+1}GF'-\dfrac{A^2}{B}\nu\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}-\frac{2\alpha_2}{\alpha_1}+1}F''+\\
A^2k\left[\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{2\alpha_2}{\alpha_1}}\left\{-\dfrac{\alpha_2}{\alpha_1}\eta(F'F''+FF''')+\left(\dfrac{2\alpha_3}{\alpha_1}-\dfrac{2\alpha_2}{\alpha_1}\right)FF''\right\}\right.\\
\left.+\left(\dfrac{CA^2}{B}\right)\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_4}{\alpha_1}-\frac{3\alpha_2}{\alpha_1}-\frac{\alpha_3}{\alpha_1}+1}(F'G''+GF''')\right]\\
-\dfrac{1}{B}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}}\dfrac{\beta_3}{\alpha_3}\left(\dfrac{\alpha_3}{\alpha_1}F-\dfrac{\alpha_2}{\alpha_1}\eta F'\right)-\dfrac{A}{B}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}-\frac{\alpha_2}{\alpha_1}+1}\dfrac{\beta_4}{\alpha_4}F'\\
\dfrac{A^2}{B}k\dfrac{\beta_3}{\alpha_3}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}-2\frac{\alpha_2}{\alpha_1}}\left[\left(\dfrac{2\alpha_2}{\alpha_1}-\dfrac{\alpha_3}{\alpha_1}\right)F''+\dfrac{\alpha_2}{\alpha_1}\eta F'''\right]\dfrac{A^3}{B}k\dfrac{\beta_4}{\alpha_4}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}-3\frac{\alpha_2}{\alpha_1}+1}F'''=0
\end{array}\right\} \tag{27}$$

For (26) and (27) to be reduced an expression in a single independent invariant, it is necessary that the coefficients should be constants or functions of $\eta$ alone. Thus,

$$\left.\begin{array}{ll}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_4}{\alpha_1}-\frac{\alpha_2}{\alpha_1}-\frac{\alpha_3}{\alpha_1}+1}=c_1; & \left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}-\frac{2\alpha_2}{\alpha_1}+1}=c_2\\ \left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{2\alpha_2}{\alpha_1}}=c_3; & \left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}}\dfrac{\beta_3}{\alpha_3}=c_4\\ \left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}-\frac{\alpha_2}{\alpha_1}+1}\dfrac{\beta_4}{\alpha_4}=c_5; & \dfrac{\beta_3}{\alpha_3}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}-2\frac{\alpha_2}{\alpha_1}}=c_6\\ \dfrac{\beta_4}{\alpha_4}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_3}{\alpha_1}-3\frac{\alpha_2}{\alpha_1}+1}=c_7 & \end{array}\right\} \tag{28}$$

Hence the only possible values of $\alpha's$ and $\beta's$ are

$$\alpha_2=\alpha_4=0,$$
$$\beta_3=\beta_4=0.$$

In the light of the above results, the transformations (22)-(24) take the following form

$$\left.\begin{array}{l}\eta=Ay,\\ u=\dfrac{1}{B}\left(x+\dfrac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_3}{\alpha_1}}F(\eta), \qquad \alpha_3\neq 0\\ v=G(\eta),\end{array}\right\} \tag{29}$$

which are called the similarity transformations, and equations (4.9) and (4.10), readily give the similarity representation,

$$F+G'=0 \tag{30}$$

$$F^2+F\,F'-F''+k\left[2F\,F''+F'G''+GF'''\right]=0 \tag{31}$$

Using (30), (31) can be expressed in the terms of $G$ alone, i.e. we get

$$G'^2+G'\,G''+G'''+k\left[2G'\,G'''-G''^2-GG''''\right]=0$$

Since $F=-G'$, and the following boundary conditions:

$$G'(\eta)=1, \quad G(\eta)=0, \quad \text{at} \quad \eta=0$$
$$G'(\eta)=0, \quad \text{as} \ \eta\to\infty$$

where $A=\left(\dfrac{c}{\nu}\right)^{\frac{1}{2}}$, $B=c$, $AC=c$ and $K_1=k\dfrac{c}{\nu}$.

Thus the transformations (29) are useful in getting similarity solution.

## 5 Conclusion

In the procedure proposed here, we have given new variables which are more general in nature, in comparison with Moran. Also, in this procedure new variables (transformations) are used to reduce the partial differential equation and auxiliary condition to a simpler form which gives rise to a condition, which is referred to as a restriction by (Moran et al., 1968). In the illustration has given, we discuss a problem for which similarity solution does not exist for certain new variables, this was found with less efforts, if other methods are followed then this finding might have reached after a long futile exercise.

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