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One-dimensional ground water recharge through porous media

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ABSTRACT

In this paper a one-dimensional ground water recharge through porous media has been discussed. The unsteady and unsaturated flow of water through soils is due to content changes as a function of time and the entire pore spaces are not completely filled with flowing liquid respectively. Knowledge concerning such flow helps some workers like hydrologist, agriculturalists, many fields of science and engineering. The water infiltration system and the underground disposal of seepage and waste water are encountered by these flows, which are described by nonlinear partial differential equation.

Keywords: Ground water Recharge, Porous Media, S.O.R.

MSC 2010 codes: 76S05, 76S99.

1 Introduction

Engineers in several fields have to learn the mechanism of drainage and to apply to problems of water supply, land reclamation and stabilization of foundations and sub grade, and also to the fields of petroleum production and agriculture.

Drainage in general is any provision for the removal of excess water. The common objective of land projects to prevent or eliminate either water logging or inundation or otherwise productive land. Drainage of projected land refers principally to the disposal of surplus natural water adversely affecting irrigation. Practically every area where irrigation has been carried on for time has been affected by high water table. Therefore provision for adequate drainage is an essential part of planning, construction and operation of an irrigation project.

For agriculture purpose, the continued presence of water in excess of that needed for vegetation is harmful. Prolonged saturation of soil excludes air essentially for healthy plant

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growth and the soil becomes cold, sour and unproductive. Consequently unsaturated or irrigated soils is a necessary evil, so to this type of drainage where originally saturation conditions are existing up to the top.
In the present paper we have derived the mathematical model that conform the hydrological situation of one dimensional vertical ground water recharge by spreading [Verma (1969)]. Such flow is of great importance in water resource science, soil engineering and agricultural sciences. The numerical solution is obtained by Successive over Relaxation (S.O.R.) method [Shastry (2006)].

2 Statement of the problem

In the investigated mathematical model, we consider that the groundwater recharge takes place over a large basin of such geological configuration that the sides are limited by rigid boundaries and the bottom by a thick layer of water table. In this case, the flow is assumed vertically downwards through unsaturated porous media.

It is assumed that the diffusivity co-efficient is equivalent to its average value over the whole range of moisture content, and the permeability of the media is continuous linear function of the moisture content. The theoretical formulation of the problem yields a nonlinear partial differential equation.

2.1 Mathematical formulation of the problem

From Klute (1952), the equation of continuity for an unsaturated medium is given by

$$\frac{\partial}{\partial t}(\rho_s \theta) = -\nabla \cdot M \tag{1}$$

where ρ_s the bulk density of the medium is, θ is its moisture content on a dry weight basis, and M is the mass flux of moisture.

From Darcy's law [Verma (1969), Rijik (1960), Fox (1962)] for the motion of water in a porous medium, we get,

$$\vec{V} = -k\nabla\varphi \tag{2}$$

where $\nabla\varphi$ represents the gradient of the moisture potential, the volume flux of moisture, and k the co-efficient of aqueous conductivity.

Combining equations (1) and (2) we obtain,

$$\frac{\partial}{\partial t}(\rho_s \theta) = \nabla \cdot (\rho k \nabla\varphi) \tag{3}$$

where $M = \rho\vec{V}$, ρ is the flux density.

Since in the present case, we consider that the flow takes place only in the vertical direction, equation (3) reduces to,

$$\rho_s \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(\rho k \frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial z}(\rho k g) \tag{4}$$

where ψ the capillary pressure potential, g is the gravitational constant and $\varphi = \psi - gz$ [5-9] the positive direction of the z-axis is the same as that of the gravity.

Considering θ and ψ to be connected by a single valued function, we may write (4) as,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D \frac{\partial \theta}{\partial z}\right) - \frac{\rho}{\rho_s}\frac{\partial k}{\partial z} \tag{5}$$

Where $D = \frac{\rho}{\rho_s} k \frac{\partial \psi}{\partial \theta}$ and is called diffusivity co-efficient.

Replacing D by average value D_a and assuming $k = k_0\theta$, we have

$$\frac{\partial \theta}{\partial t} = D_a \frac{\partial^2 \theta}{\partial z^2} - \frac{\rho}{\rho_s} k_0 \frac{\partial \theta}{\partial z} \tag{6}$$

Considering the water table to be situated at a depth L. and putting:

$$\frac{z}{L} = \xi, \qquad \frac{tD_a}{L^2} = T, \qquad \beta = \frac{\rho}{\rho_s}\frac{k_0}{D_a}$$

We may write the boundary value problem as:

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial \xi^2} - \beta \frac{\partial \theta}{\partial \xi} \tag{7}$$

Where ξ = Penetration depth (dimensionless), T = Time (dimensionless), β = Flow parameter(cm^2), ρ = Mass density of water (gm), ρ_s = Bulk density of the medium on dry weight basis $(gmcm^{-3})$, k_0= Slope of the permeability vs moisture content plot $(cmsec^{-1})$, D_a= Average value of the diffusivity co-efficient over the whole range of Moisture content $(\mathrm{cm^2 sec^{-1}})$

It may be mentioned for definiteness that a set of appropriate boundary conditions are

$$\theta(0, T) = \theta_0, \ \frac{\partial \theta}{\partial \xi}(1, T) = 0 \tag{8}$$

$$\theta(\xi, 0) = 0 \tag{9}$$

where the moisture content throughout the region is zero initially, at the layer z =0 it is θ_0, and at the water table (z = L) it is assumed to remain 100% throughout the process of investigation. It may be remarked that the effect of capillary action at the stationary groundwater level, being small, is neglected.

The following values of the various parameters have been considered in the analysis:

Let $\beta = 3.5$,$\theta_0 = 0.5$, $h = {}^{1}/_{10}$ and $k = 0.1$

The numerical values are shown by the table. Curves indicating the behavior of the moisture content corresponding to various time period.

3 Mathematical solution of the problem

Using Crank- Nikolson method [see Shastry (2006)], we have,

$$\theta_{i,j+1} = \theta_{i,j} + \frac{k}{2h^2}\left(\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j} + \theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}\right) - \frac{\beta k}{2h}\left(\theta_{i+1,j} - \theta_{i,j} + \theta_{i+1,j+1} - \theta_{i,j+1}\right)$$

Let

$$r = \frac{k}{h^2}$$

and

$$c_i = \theta_{i,j} + \frac{r}{2}\left(\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}\right) - \frac{\beta rh}{2}\left(\theta_{i+1,j} - \theta_{i,j}\right)$$

Using Successive –Over Relaxation Method, we have

$$\theta_{i,j+1} = (1-\omega)\theta_{i,j} + \omega\left[\frac{r}{2\left(1 + r - \frac{\beta rh}{2}\right)}\left(\theta_{i+1,j} - \theta_{i-1,j+1}\right) + \frac{c_i}{\left(1 + r - \frac{\beta rh}{2}\right)} - \frac{\beta rh}{2\left(1 + r - \frac{\beta rh}{2}\right)}\theta_{i+1,j}\right]$$

Choose $k = 0.1$, $h = 0.1$, $\beta = 3.5$, $\omega = 1.4$,

$$\theta_{i,j+1} = -0.4\theta_{i,j} + 1.4\left[0.351135\theta_{i+1,j} + 0.540540\theta_{i-1,j+1} + \frac{c_i}{9.25}\right]$$

where $c_i = -7.250\theta_{i,j} + 3.250\theta_{i+1,j} + 5\theta_{i-1,j}$

Numerical calculation at different values of T and ξ are shown in the Table 1.

4 Result and discussions

The solution is obtained with $\beta = 3.5$, $\theta_0 = 0.5$, h= 0.1, k=0.1. We consider that the sides of begin are limited by rigid boundaries and bottom by a thick layer of water table so that water flows only in positive direction.

From Figure 1, it is clear that $\theta = \theta_0 = 0.5$ at layer x=0 and at water table (x=1), it is assumed to remain 100% throughout the process of investigation. Initially it is 0.5 and then increases to 1. Keeping T constant, for different values of X, moisture content θ is increasing.

From Figure 2, it is clear that initially $\theta = 0$ throughout the region. As time (T) increases, the moisture content θ also increases at each point of the basin and after some time, it become constant.

Table 1: Different values of T and ξ

	T=0.1	T=0.2	T=0.3	T=0.4
ξ	**θ**			
0.0	0.5	0.5	0.5	0.5
0.1	0.62162	0.52809	0.52887	0.51169
0.2	0.71366	0.5721	0.55578	0.53003
0.3	0.78331	0.62262	0.58422	0.55207
0.4	0.83602	0.67387	0.6151	0.57637
0.5	0.8759	0.72252	0.64797	0.60228
0.6	0.90608	0.7668	0.68189	0.6185
0.7	0.92892	0.80596	0.70479	0.7119
0.8	0.94621	0.82869	0.79529	0.72082
0.9	0.94794	0.90753	0.8539	0.85257
1.0	1.0	1.0	1.0	1.0

5 Figures

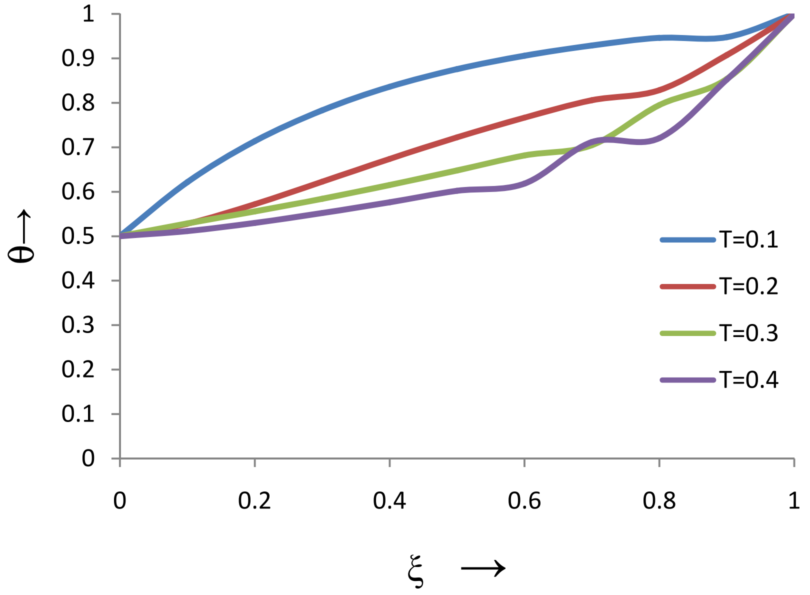

Figure 1: Penetration depth (ξ) → moisture(θ)

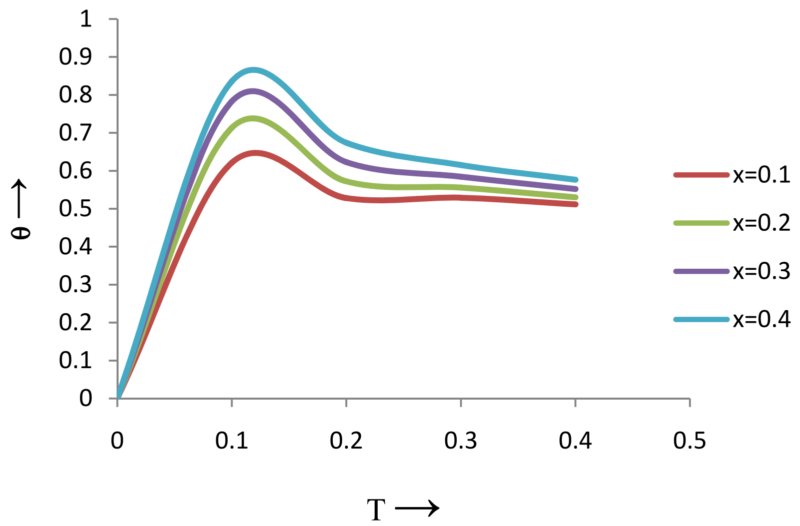

Figure 2: Time (T) → moisture(θ)

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