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A Study on Generalized Fibonacci Polynomials: Sum Formulas

Research Article

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Abstract: In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} z^k W_k$, $\sum_{k=1}^{n} z^k W_{-k}$, $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=1}^{n} z^k W_{-k}^2$, $\sum_{k=0}^{n} z^k W_k^3$, $\sum_{k=1}^{n} z^k W_{-k}^3$, $\sum_{k=0}^{n} kz^k W_k$, $\sum_{k=1}^{n} kz^k W_{-k}$, $\sum_{k=0}^{n} kz^k W_k^2$, $\sum_{k=1}^{n} kz^k W_{-k}^2$, $\sum_{k=0}^{n} kz^k W_k^3$ and $\sum_{k=1}^{n} kz^k W_{-k}^3$ for generalized Fibonacci polynomials are presented.

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1. Introduction: Generalized Fibonacci (Horadam) Polynomials

The generalized Fibonacci polynomials (or Horadam polynomials or x-Horadam numbers or generalized $(r(x), s(x))$-polynomials or $(r(x), s(x))$ Horadam polynomials or 2-step Fibonacci polynomials)

$$\{W_n(W_0, W_1; r(x), s(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n(x) = r(x)W_{n-1}(x) + s(x)W_{n-2}(x), \quad W_0(x) = a(x), W_1(x) = b(x), \ \ n \geq 2 \tag{1}$$

where $W_0(x), W_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x) \neq 0$, $s(x) \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = -\frac{r(x)}{s(x)} W_{-(n-1)}(x) + \frac{1}{s(x)} W_{-(n-2)}(x)$$

for $n = 1, 2, 3, \ldots$ when $s(x) \neq 0$. Therefore, recurrence (1) holds for all integers n. Note that $W_{-n}(x)$ need not to be a polynomial in the ordinary sense. For some references on special cases of Horadam polynomials see [3–5, 15, 18, 19] for papers and [1, 2, 6–8, 16, 17] for books.

NOTE: For the sake of simplicity throughout the rest of the paper, we use

$$W_n, r, s, W_0, W_1$$

instead of

$$W_n(x), r(x), s(x), W_0(x), W_1(x)$$

respectively. For example, we write

$$W_n = rW_{n-1} + W_{n-2}, \quad W_0 = a, W_1 = b, \ \ n \geq 2$$

for the equation (1).

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2. Sum Formulas 2: The case that r and s are arbitrary

In this section, we present the sum formulas $\sum_{k=0}^{n} z^k W_k$, $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_k^3$ and $\sum_{k=1}^{n} z^k W_{-k}$, $\sum_{k=1}^{n} z^k W_{-k}^2$, $\sum_{k=1}^{n} z^k W_{-k}^3$ of generalized Fibonacci polynomials with positive subscripts and negative subscripts.

2.1. Sum Formulas $\sum_{k=0}^{n} z^k W_k$, $\sum_{k=0}^{n} z^k W_{2k}$ and $\sum_{k=0}^{n} z^k W_{2k+1}$ of Generalized Fibonacci (Horadam) Polynomials with Positive Subscripts

In the next Theorem, we need the following Remark.

Remark 2.1.

(a) Solving the equation

$$sz^2 + rz - 1 = 0$$

we find the roots as

$$z_1 = a = \frac{1}{2s}\left(-r + \sqrt{r^2+4s}\right),$$
$$z_2 = b = \frac{1}{2s}\left(-r - \sqrt{r^2+4s}\right).$$

If $r^2+4s = 0$, i.e., $s = -\frac{1}{4}r^2$ then $a = b = -\frac{r}{2s}$. If $r^2+4s \neq 0$, i.e., $s \neq -\frac{1}{4}r^2$ then $a \neq b$ and in this case $z \neq -\dfrac{r}{2s}$, i.e., $r + 2sz \neq 0$ for $z = a$ or $z = b$. We can list some properties:

- There are some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $z = a$ or $z = b$ such that

$$sz^2 + rz - 1 = u(z-a)(z-b) = 0.$$

- There are some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $z = a$, that is $z = -\frac{r}{2s}$, such that

$$sz^2 + rz - 1 = u(z-a)^2 = 0.$$

(b) Solving the equation

$$r^2z - s^2z^2 + 2sz - 1 = 0$$

we find the roots as

$$z_1 = a = \frac{1}{2s^2}\left(r^2 + 2s + r\sqrt{r^2+4s}\right),$$
$$z_2 = b = \frac{1}{2s^2}\left(r^2 + 2s - r\sqrt{r^2+4s}\right).$$

If $r^2+4s = 0$, i.e., $s = -\frac{1}{4}r^2$ then $a = b = \frac{1}{2s^2}\left(r^2+2s\right) = \frac{4}{r^2}$. If $r^2+4s \neq 0$, i.e., $s \neq -\frac{1}{4}r^2$ then $a \neq b$ and in this case $z \neq \frac{1}{2s^2}\left(r^2+2s\right)$, i.e., $r^2 - 2zs^2 + 2s \neq 0$ for $z = a$ or $z = b$. We can list some properties:

- There are some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $z = a$ or $z = b$ such that

$$r^2z - s^2z^2 + 2sz - 1 = u(z-a)(z-b) = 0.$$

- There are some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $z = a$, that is $z = \frac{4}{r^2}$, such that

$$r^2z - s^2z^2 + 2sz - 1 = u(z-a)^2 = 0.$$

The following Theorem presents some summing formulas of generalized Fibonacci polynomials with positive subscripts.

Theorem 2.1.
Let z be a non-zero complex (or reel) number. For $n \geq 0$ we have the following formulas:

(a)

(i) *If* $sz^2 + rz - 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Phi_1}{sz^2 + rz - 1}$$

where
$\Phi_1 = z^{n+2} W_{n+2} + z^{n+1}(1 - rz) W_{n+1} - zW_1 + (rz - 1) W_0.$

(ii) *If* $sz^2 + rz - 1 = u(z-a)(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $z = a$ *or* $z = b$*, that is, if*

$$z = a = \frac{1}{2s}\left(-r + \sqrt{r^2 + 4s}\right),$$
$$z = b = \frac{1}{2s}\left(-r - \sqrt{r^2 + 4s}\right),$$

provided that $r^2 + 4s \neq 0$*, then*

$$\sum_{k=0}^{n} z^k W_k = \frac{(n+2) z^{n+1} W_{n+2} + (n - nrz - 2rz + 1) z^n W_{n+1} - W_1 + r W_0}{r + 2sz}.$$

(iii) *If* $sz^2 + rz - 1 = u(z-a)^2 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $z = a$*, that is, if*

$$z = -\frac{r}{2s}$$

provided that $r^2 + 4s = 0$*, then*

$$\sum_{k=0}^{n} z^k W_k = \frac{(n+1)(n+2) z^n W_{n+2} - (n+1)(r(n+2) z^n - nz^{n-1}) W_{n+1}}{2s}.$$

(b)

(i) *If* $r^2 z - s^2 z^2 + 2sz - 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Phi_2}{r^2 z - s^2 z^2 + 2sz - 1}$$

where
$\Phi_2 = -z^{n+1}(sz - 1) W_{2n+2} + rsz^{n+2} W_{2n+1} - rzW_1 + (r^2 z + sz - 1) W_0.$

(ii) *If* $r^2 z - s^2 z^2 + 2sz - 1 = u(z-a)(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $z = a$ *or* $z = b$*, i.e.,* $z = a$ *or* $z = b$*, that is, if*

$$z = a = \frac{1}{2s^2}\left(r^2 + 2s + r\sqrt{r^2 + 4s}\right),$$
$$z = b = \frac{1}{2s^2}\left(r^2 + 2s - r\sqrt{r^2 + 4s}\right),$$

provided that $r^2 + 4s \neq 0$*, then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{(n - 2sz - nsz + 1) z^n W_{2n+2} + rs(n+2) z^{n+1} W_{2n+1} - rW_1 + (r^2 + s) W_0}{r^2 - 2zs^2 + 2s}.$$

(iii) *If* $r^2 z - s^2 z^2 + 2sz - 1 = u(z-a)^2 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $z = a$*, that is, if*

$$z = \frac{4}{r^2}$$

provided that $r^2 + 4s = 0$*, then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{(n+1)\left(s(n+2) z^n - nz^{n-1}\right) W_{2n+2} - rs(n+1)(n+2) z^n W_{2n+1}}{2s^2}.$$

(c)

(i) *If* $r^2 z - s^2 z^2 + 2sz - 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Phi_3}{r^2 z - s^2 z^2 + 2sz - 1}$$

where

$\Phi_3 = r z^{n+1} W_{2n+2} - s z^{n+1}(sz-1) W_{2n+1} + (sz-1) W_1 - rszW_0.$

(ii) *If* $r^2 z - s^2 z^2 + 2sz - 1 = u(z-a)(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$, *that is, if*

$$z = a = \frac{1}{2s^2}\left(r^2 + 2s + r\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2s^2}\left(r^2 + 2s - r\sqrt{r^2+4s}\right),$$

provided that $r^2 + 4s \neq 0$, *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{r(n+1)z^n W_{2n+2} + s(n - nsz - 2sz + 1)z^n W_{2n+1} + sW_1 - rsW_0}{r^2 - 2zs^2 + 2s}.$$

(iii) *If* $r^2 z - s^2 z^2 + 2sz - 1 = u(z-a)^2 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *that is, if*

$$z = \frac{4}{r^2}$$

provided that $r^2 + 4s = 0$, *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{-nr(n+1)z^{n-1} W_{2n+2} + s(n+1)\left(s(n+2)z^n - nz^{n-1}\right) W_{2n+1}}{2s^2}.$$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[9], Theorem 2.1.].

(a)

(i) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$sW_{n-2} = W_n - rW_{n-1}$$

we obtain

$$\begin{aligned} sz^1 W_1 &= z^1 W_3 - rz^1 W_2 \\ sz^2 W_2 &= z^2 W_4 - rz^2 W_3 \\ &\vdots \\ sz^{n-1} W_{n-1} &= z^{n-1} W_{n+1} - rz^{n-1} W_n \\ sz^n W_n &= z^n W_{n+2} - rz^n W_{n+1}. \end{aligned}$$

If we add the equations side by side, we get

$$\sum_{k=0}^{n} z^k W_k = \frac{z^{n+2} W_{n+2} + z^{n+1}(1 - rz) W_{n+1} - zW_1 + (rz-1) W_0}{sz^2 + rz - 1}.$$

(ii) Suppose that $r^2 + 4s \neq 0$. Then, from Remark 2.1, $a \neq b$. We use (a) (i). For $z = a$ and $z = b$, the right hand side of the sum formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule. We get (a) (ii), by using (a) (i), as

$$\begin{aligned} \sum_{k=0}^{n} a^k W_k &= \left.\frac{\frac{d}{dz}\left(z^{n+2} W_{n+2} + z^{n+1}(1 - rz) W_{n+1} - zW_1 + (rz-1) W_0\right)}{\frac{d}{dz}\left(sz^2 + rz - 1\right)}\right|_{z=a} \\ &= \left.\frac{(n+2)z^{n+1} W_{n+2} + (n - nrz - 2rz + 1)z^n W_{n+1} - W_1 + rW_0}{r + 2sz}\right|_{z=a} \\ &= \frac{(n+2)a^{n+1} W_{n+2} + (n - nra - 2ra + 1)a^n W_{n+1} - W_1 + rW_0}{r + 2sa} \end{aligned}$$

and

$$\sum_{k=0}^{n} b^k W_k = \left.\frac{\frac{d}{dz}\left(z^{n+2}W_{n+2}+z^{n+1}(1-rz)W_{n+1}-zW_1+(rz-1)W_0\right)}{\frac{d}{dz}\left(sz^2+rz-1\right)}\right|_{z=b}$$
$$= \left.\frac{(n+2)z^{n+1}W_{n+2}+(n-nrz-2rz+1)z^nW_{n+1}-W_1+rW_0}{r+2sz}\right|_{z=b}$$
$$= \frac{(n+2)b^{n+1}W_{n+2}+(n-nrb-2rb+1)b^nW_{n+1}-W_1+rW_0}{r+2sb}.$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.1, $a=b$. We use (a) (i). For $z=a$, the right hand side of the sum formula given in (a) (i) is an indeterminate form. We can use L'Hospital rule (twice). Then we get, by using (a) (i),

$$\sum_{k=0}^{n} a^k W_k = \left.\frac{\frac{d^2}{dz^2}\left(z^{n+2}W_{n+2}+z^{n+1}(1-rz)W_{n+1}-zW_1+(rz-1)W_0\right)}{\frac{d^2}{dz^2}\left(sz^2+rz-1\right)}\right|_{z=a}$$
$$= \left.\frac{(n+1)(n+2)z^nW_{n+2}-(n+1)(r(n+2)z^n-nz^{n-1})W_{n+1}}{2s}\right|_{z=a}$$
$$= \frac{(n+1)(n+2)a^nW_{n+2}-(n+1)(r(n+2)a^n-na^{n-1})W_{n+1}}{2s}.$$

(b) (i) and (c) (i) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we obtain

$$\begin{aligned} rz^1W_3 &= z^1W_4 - sz^1W_2 \\ rz^2W_5 &= z^2W_6 - sz^2W_4 \\ rz^3W_7 &= z^3W_8 - sz^3W_6 \\ &\vdots \\ rz^{n-1}W_{2n-1} &= z^{n-1}W_{2n} - sz^{n-1}W_{2n-2} \\ rz^nW_{2n+1} &= z^nW_{2n+2} - sz^nW_{2n}. \end{aligned}$$

Now, if we add the above equations side by side, we get

$$r(-W_1+\sum_{k=0}^{n} z^kW_{2k+1}) = (z^nW_{2n+2}-W_2-z^{-1}W_0+\sum_{k=0}^{n} z^{k-1}W_{2k}) - s(-W_0+\sum_{k=0}^{n} z^kW_{2k}). \quad (2)$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we write the following obvious equations;

$$\begin{aligned} rz^1W_2 &= z^1W_3 - sz^1W_1 \\ rz^2W_4 &= z^2W_5 - sz^2W_3 \\ rz^3W_6 &= z^3W_7 - sz^3W_5 \\ &\vdots \\ rz^{n-1}W_{2n-2} &= z^{n-1}W_{2n-1} - sz^{n-1}W_{2n-3} \\ rz^nW_{2n} &= z^nW_{2n+1} - sz^nW_{2n-1} \end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$r(-W_0+\sum_{k=0}^{n} z^kW_{2k}) = (-W_1+\sum_{k=0}^{n} z^kW_{2k+1}) - s(-z^{n+1}W_{2n+1}+\sum_{k=0}^{n} z^{k+1}W_{2k+1}). \quad (3)$$

Then, solving the system (2)-(3), the required results of (b) (i) and (c) (i) follow.

(b)

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.1, $a\neq b$. We use (b) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule. We get (b) (ii), by using (b) (i), as

$$\begin{aligned}\sum_{k=0}^{n}a^kW_{2k} &= \left.\frac{\frac{d}{dz}\left(-z^{n+1}(sz-1)W_{2n+2}+rsz^{n+2}W_{2n+1}-rzW_1+(r^2z+sz-1)W_0\right)}{\frac{d}{dz}\left(r^2z-s^2z^2+2sz-1\right)}\right|_{z=a}\\ &= \left.\frac{(n-2sz-nsz+1)z^nW_{2n+2}+rs(n+2)z^{n+1}W_{2n+1}-rW_1+(r^2+s)W_0}{r^2-2zs^2+2s}\right|_{z=a}\\ &= \frac{(n-2sa-nsa+1)a^nW_{2n+2}+rs(n+2)a^{n+1}W_{2n+1}-rW_1+(r^2+s)W_0}{r^2-2as^2+2s}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n}b^kW_{2k} &= \left.\frac{\frac{d}{dz}\left(-z^{n+1}(sz-1)W_{2n+2}+rsz^{n+2}W_{2n+1}-rzW_1+(r^2z+sz-1)W_0\right)}{\frac{d}{dz}\left(r^2z-s^2z^2+2sz-1\right)}\right|_{z=b}\\ &= \left.\frac{(n-2sz-nsz+1)z^nW_{2n+2}+rs(n+2)z^{n+1}W_{2n+1}-rW_1+(r^2+s)W_0}{r^2-2zs^2+2s}\right|_{z=b}\\ &= \frac{(n-2sb-nsb+1)b^nW_{2n+2}+rs(n+2)b^{n+1}W_{2n+1}-rW_1+(r^2+s)W_0}{r^2-2bs^2+2s}.\end{aligned}$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.1, $a=b$. We use (b) (i). For $z=a$, the right hand side of the sum formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get, by using (b) (i),

$$\begin{aligned}\sum_{k=0}^{n}a^kW_{2k} &= \left.\frac{\frac{d^2}{dz^2}\left(-z^{n+1}(sz-1)W_{2n+2}+rsz^{n+2}W_{2n+1}-rzW_1+(r^2z+sz-1)W_0\right)}{\frac{d^2}{dz^2}\left(r^2z-s^2z^2+2sz-1\right)}\right|_{z=a}\\ &= \left.\frac{-(n+1)\left(s(n+2)z^n-nz^{n-1}\right)W_{2n+2}+rs(n+1)(n+2)z^nW_{2n+1}}{-2s^2}\right|_{z=a}\\ &= \left.\frac{(n+1)\left(s(n+2)z^n-nz^{n-1}\right)W_{2n+2}-rs(n+1)(n+2)z^nW_{2n+1}}{2s^2}\right|_{z=a}\\ &= \frac{(n+1)\left(s(n+2)a^n-na^{n-1}\right)W_{2n+2}-rs(n+1)(n+2)a^nW_{2n+1}}{2s^2}.\end{aligned}$$

(c)

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.1, $a\neq b$. We use (c) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule. We get (c) (ii), by using (c) (i), as

$$\begin{aligned}\sum_{k=0}^{n}a^kW_{2k+1} &= \left.\frac{\frac{d}{dz}\left(rz^{n+1}W_{2n+2}-sz^{n+1}(sz-1)W_{2n+1}+(sz-1)W_1-rszW_0\right)}{\frac{d}{dz}\left(r^2z-s^2z^2+2sz-1\right)}\right|_{z=a}\\ &= \left.\frac{r(n+1)z^nW_{2n+2}+s(n-nsz-2sz+1)z^nW_{2n+1}+sW_1-rsW_0}{r^2-2zs^2+2s}\right|_{z=a}\\ &= \frac{r(n+1)a^nW_{2n+2}+s(n-nsa-2sa+1)a^nW_{2n+1}+sW_1-rsW_0}{r^2-2as^2+2s}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n}b^kW_{2k+1} &= \left.\frac{\frac{d}{dz}\left(rz^{n+1}W_{2n+2}-sz^{n+1}(sz-1)W_{2n+1}+(sz-1)W_1-rszW_0\right)}{\frac{d}{dz}\left(r^2z-s^2z^2+2sz-1\right)}\right|_{z=b}\\ &= \left.\frac{r(n+1)z^nW_{2n+2}+s(n-nsz-2sz+1)z^nW_{2n+1}+sW_1-rsW_0}{r^2-2zs^2+2s}\right|_{z=b}\\ &= \frac{r(n+1)b^nW_{2n+2}+s(n-nsb-2sb+1)b^nW_{2n+1}+sW_1-rsW_0}{r^2-2bs^2+2s}.\end{aligned}$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.1, $a=b$. We use (c) (i). For $z=a$, the right hand side of the sum formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get, by using (c) (i),

$$\begin{aligned}
\sum_{k=0}^{n} a^k W_{2k+1} &= \left.\frac{\frac{d^2}{dz^2}\left(rz^{n+1}W_{2n+2}-sz^{n+1}(sz-1)W_{2n+1}+(sz-1)W_1-rszW_0\right)}{\frac{d^2}{dz^2}\left(r^2z-s^2z^2+2sz-1\right)}\right|_{z=a}\\
&= \left.\frac{nr(n+1)z^{n-1}W_{2n+2}-s(n+1)\left(s(n+2)z^n-nz^{n-1}\right)W_{2n+1}}{-2s^2}\right|_{z=a}\\
&= \left.\frac{-nr(n+1)z^{n-1}W_{2n+2}+s(n+1)\left(s(n+2)z^n-nz^{n-1}\right)W_{2n+1}}{2s^2}\right|_{z=a}\\
&= \frac{-nr(n+1)a^{n-1}W_{2n+2}+s(n+1)\left(s(n+2)a^n-na^{n-1}\right)W_{2n+1}}{2s^2}. \square
\end{aligned}$$

2.2. Sum Formulas $\sum_{k=1}^{n} z^k W_{-k}$, $\sum_{k=1}^{n} z^k W_{-2k}$ and $\sum_{k=1}^{n} z^k W_{-2k+1}$ of Generalized Fibonacci (Horadam) Polynomials with Negative Subscripts

In the next Theorem, we need the following Remark.

Remark 2.2.

(a) Solving the equation

$$s+rz-z^2=0$$

we find the roots as

$$\begin{aligned}
z_1 = a &= \frac{1}{2}\left(r+\sqrt{r^2+4s}\right),\\
z_2 = b &= \frac{1}{2}\left(r-\sqrt{r^2+4s}\right).
\end{aligned}$$

If $r^2+4s=0$, i.e., $s=-\frac{1}{4}r^2$ then $a=b=\frac{r}{2}$. If $r^2+4s\neq 0$, i.e., $s\neq -\frac{1}{4}r^2$ then $a\neq b$ and in this case $z\neq\frac{r}{2}$, i.e., $r-2z\neq 0$ for $z=a$ or $z=b$. We can list some properties:

- There are some $u,a,b\in\mathbb{C}$ with $u\neq 0$ and $a\neq b$, i.e., $z=a$ or $z=b$ such that

$$s+rz-z^2=u(z-a)(z-b)=0.$$

- There are some $u,a\in\mathbb{C}$ with $u\neq 0$, i.e., $z=a$, that is $z=\frac{r}{2}$, such that

$$s+rz-z^2=u(z-a)^2=0.$$

(b) Solving the equation

$$r^2z+2sz-s^2-z^2=0$$

we find the roots as

$$\begin{aligned}
z_1 = a &= \frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),\\
z_2 = b &= \frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right).
\end{aligned}$$

If $r^2+4s=0$, i.e., $s=-\frac{1}{4}r^2$ then $a=b=\frac{1}{2}\left(r^2+2s\right)=\frac{1}{4}r^2$. If $r^2+4s\neq 0$, i.e., $s\neq -\frac{1}{4}r^2$ then $a\neq b$ and in this case $z\neq\frac{1}{2}(r^2+2s)$, i.e., $r^2-2zs^2+2s\neq 0$ for $z=a$ or $z=b$. We can list some properties:

- There are some $u,a,b\in\mathbb{C}$ with $u\neq 0$ and $a\neq b$, i.e., $z=a$ or $z=b$ such that

$$r^2z+2sz-s^2-z^2=u(z-a)(z-b)=0.$$

- There are some $u,a\in\mathbb{C}$ with $u\neq 0$, i.e., $z=a$, that is $z=\frac{1}{4}r^2$, such that

$$r^2z+2sz-s^2-z^2=u(z-a)^2=0.$$

The following Theorem presents some linear summing formulas of generalized Fibonacci polynomials with negative subscripts.

Theorem 2.2.
Let z be a non-zero complex (or real) number. For $n\geq 1$ we have the following formulas:

(a) (i) *If $s+rz-z^2\neq 0$, then*

$$\sum_{k=1}^{n} z^k W_{-k}=\frac{\Phi_4}{s+rz-z^2}$$

where
$\Phi_4=-z^{n+1}(s+rz)W_{-n-1}-sz^{n+2}W_{-n-2}+zW_1+z(z-r)W_0.$

(ii) *If $s+rz-z^2=u(z-a)(z-b)=0$ for some $u,a,b\in\mathbb{C}$ with $u\neq 0$ and $a\neq b$, i.e., $z=a$ or $z=b$, that is, if*

$$z=a=\frac{1}{2}\left(r+\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}\left(r-\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, then

$$\sum_{k=1}^{n} z^k W_{-k}=\frac{-(n(s+rz)+s+2rz)z^nW_{-n-1}-s(n+2)z^{n+1}W_{-n-2}+W_1+(2z-r)W_0}{r-2z}.$$

(iii) *If $s+rz-z^2=u(z-a)^2=0$ for some $u,a\in\mathbb{C}$ with $u\neq 0$, i.e., $z=a$, that is, if*

$$z=\frac{r}{2}$$

provided that $r^2+4s=0$, then

$$\sum_{k=1}^{n} z^k W_{-k}=\frac{(n+1)(r(n+2)z^n+nsz^{n-1})W_{-n-1}+s(n+1)(n+2)z^nW_{-n-2}-2W_0}{2}.$$

(b) (i) *If $r^2z+2sz-s^2-z^2\neq 0$ then*

$$\sum_{k=1}^{n} z^k W_{-2k}=\frac{\Phi_5}{r^2z+2sz-s^2-z^2}$$

where
$\Phi_5=z^{n+1}(s-z)W_{-2n}-rsz^{n+1}W_{-2n-1}+rzW_1+z(z-s-r^2)W_0.$

(ii) *If $r^2z+2sz-s^2-z^2=u(z-a)(z-b)=0$ for some $u,a,b\in\mathbb{C}$ with $u\neq 0$ and $a\neq b$, i.e., $z=a$ or $z=b$, that is, if*

$$z=a=\frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, then

$$\sum_{k=1}^{n} z^k W_{-2k}=\frac{(n(s-z)+s-2z)z^nW_{-2n}-rs(n+1)z^nW_{-2n-1}+rW_1+(-r^2+2z-s)W_0}{r^2+2s-2z}.$$

(iii) *If $r^2z+2sz-s^2-z^2=u(z-a)^2=0$ for some $u,a\in\mathbb{C}$ with $u\neq 0$, i.e., $z=a$, that is, if*

$$z=\frac{1}{4}r^2$$

provided that $r^2+4s=0$, then

$$\sum_{k=1}^{n} z^k W_{-2k}=\frac{(n+1)((n+2)z^n-nsz^{n-1})W_{-2n}+nrs(n+1)z^{n-1}W_{-2n-1}-2W_0}{2}.$$

(c) **(i)** *If* $r^2z+2sz-s^2-z^2\neq 0$ *then*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\Phi_6}{r^2z+2sz-s^2-z^2}$$

where

$\Phi_6 = -rz^{n+2}W_{-2n}+sz^{n+1}(s-z)W_{-2n-1}+z(z-s)W_1+rszW_0.$

(ii) *If* $r^2z+2sz-s^2-z^2 = u(z-a)(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$*, that is, if*

$$z=a=\frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$*, then*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{-r(n+2)z^{n+1}W_{-2n}+s(n(s-z)+s-2z)z^nW_{-2n-1}+(2z-s)W_1+rsW_0}{r^2+2s-2z}.$$

(iii) *If* $r^2z+2sz-s^2-z^2=u(z-a)^2=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, that is, if*

$$z=\frac{1}{4}r^2$$

provided that $r^2+4s=0$*, then*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{r(n+1)(n+2)z^nW_{-2n}+s(n+1)\left((n+2)z^n-nsz^{n-1}\right)W_{-2n-1}-2W_1}{2}.$$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[9], Theorem 3.1.].

(a)

(i) Using the recurrence relation

$$\begin{aligned} W_{-n+2} &= r\times W_{-n+1}+s\times W_{-n} \\ &\Rightarrow W_{-n} = -\frac{r}{s}\times W_{-n+1}+\frac{1}{s}W_{-n+2} \\ &\Rightarrow W_{-n} = -\frac{r}{s}\times W_{-(n-1)}+\frac{1}{s}W_{-(n-2)} \end{aligned}$$

i.e.

$$sW_{-n} = W_{-n+2}-rW_{-n+1}$$

or

$$W_{-n} = \frac{1}{s}W_{-n+2}-\frac{r}{s}W_{-n+1}$$

we obtain

$$\begin{aligned} sz^nW_{-n} &= z^nW_{-n+2}-rz^nW_{-n+1} \\ sz^{n-1}W_{-n+1} &= z^{n-1}W_{-n+3}-rz^{n-1}W_{-n+2} \\ sz^{n-2}W_{-n+2} &= z^{n-2}W_{-n+4}-rz^{n-2}W_{-n+3} \\ &\vdots \\ sz^5W_{-5} &= z^5W_{-3}-r\times z^5W_{-4} \\ sz^4W_{-4} &= z^4W_{-2}-r\times z^4W_{-3} \\ sz^3W_{-3} &= z^3W_{-1}-r\times z^3W_{-2} \\ sz^2W_{-2} &= z^2W_0-r\times z^2W_{-1} \\ sz^1W_{-1} &= z^1W_1-r\times z^1W_0 \\ sz^{-1}W_0 &= z^{-1}W_2-r\times z^{-1}W_1 \end{aligned}$$

If we add the equations side by side, we get

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{-z^{n+1}(s+rz)W_{-n-1}-sz^{n+2}W_{-n-2}+zW_1+z(z-r)W_0}{s+rz-z^2}.$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.2, $a\neq b$. We use (a) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule. We get (a) (ii), by using (a) (i), as

$$\sum_{k=1}^{n} a^k W_{-k} = \left.\frac{\frac{d}{dz}\left(-z^{n+1}(s+rz)W_{-n-1}-sz^{n+2}W_{-n-2}+zW_1+z(z-r)W_0\right)}{\frac{d}{dz}\left(s+rz-z^2\right)}\right|_{z=a}$$
$$= \left.\frac{-(n(s+rz)+s+2rz)z^n W_{-n-1}-s(n+2)z^{n+1}W_{-n-2}+W_1+(2z-r)W_0}{r-2z}\right|_{z=a}$$
$$= \frac{-(n(s+ra)+s+2ra)a^n W_{-n-1}-s(n+2)a^{n+1}W_{-n-2}+W_1+(2a-r)W_0}{r-2a}$$

and

$$\sum_{k=1}^{n} b^k W_{-k} = \left.\frac{\frac{d}{dz}\left(-z^{n+1}(s+rz)W_{-n-1}-sz^{n+2}W_{-n-2}+zW_1+z(z-r)W_0\right)}{\frac{d}{dz}\left(s+rz-z^2\right)}\right|_{z=b}$$
$$= \left.\frac{-(n(s+rz)+s+2rz)z^n W_{-n-1}-s(n+2)z^{n+1}W_{-n-2}+W_1+(2z-r)W_0}{r-2z}\right|_{z=b}$$
$$= \frac{-(n(s+rb)+s+2rb)b^n W_{-n-1}-s(n+2)b^{n+1}W_{-n-2}+W_1+(2b-r)W_0}{r-2b}.$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.2, $a=b$. We use (a) (i). For $z=a$, the right hand side of the sum formula given in (a) (i) is an indeterminate form. We can use L'Hospital rule (twice). Then we get, by using (a) (i),

$$\sum_{k=1}^{n} a^k W_{-k} = \left.\frac{\frac{d^2}{dz^2}\left(-z^{n+1}(s+rz)W_{-n-1}-sz^{n+2}W_{-n-2}+zW_1+z(z-r)W_0\right)}{\frac{d^2}{dz^2}\left(s+rz-z^2\right)}\right|_{z=a}$$
$$= \left.\frac{-(n+1)(r(n+2)z^n+nsz^{n-1})W_{-n-1}-s(n+1)(n+2)z^nW_{-n-2}+2W_0}{-2}\right|_{z=a}$$
$$= \left.\frac{(n+1)(r(n+2)z^n+nsz^{n-1})W_{-n-1}+s(n+1)(n+2)z^nW_{-n-2}-2W_0}{2}\right|_{z=a}$$
$$= \left.\frac{(n+1)(r(n+2)a^n+nsa^{n-1})W_{-n-1}+s(n+1)(n+2)a^nW_{-n-2}-2W_0}{2}\right|_{z=a}.$$

(b) (i) and (c) (i) Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$rW_{-n+1}=W_{-n+2}-sW_{-n}$$

we obtain

$$\begin{aligned}
rz^nW_{-2n+1} &= z^nW_{-2n+2}-sz^nW_{-2n}\\
rz^{n-1}W_{-2n+3} &= z^{n-1}W_{-2n+4}-sz^{n-1}W_{-2n+2}\\
rz^{n-2}W_{-2n+5} &= z^{n-2}W_{-2n+6}-sz^{n-2}W_{-2n+4}\\
rz^{n-3}W_{-2n+7} &= z^{n-3}W_{-2n+8}-sz^{n-3}W_{-2n+6}\\
&\vdots\\
rz^3W_{-5} &= z^3W_{-4}-sz^3W_{-6}\\
rz^2W_{-3} &= z^2W_{-2}-sz^2W_{-4}\\
rzW_{-1} &= zW_0-szW_{-2}
\end{aligned}$$

If we add the equations side by side, we get

$$r\sum_{k=1}^{n} z^kW_{-2k+1} = (-z^{n+1}W_{-2n}+zW_0+\sum_{k=1}^{n} z^{k+1}W_{-2k})-s(\sum_{k=1}^{n} z^kW_{-2k}). \tag{4}$$

Similarly, using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n}$$

i.e.

$$\begin{aligned} rW_{-n+1} &= W_{-n+2} - sW_{-n} \Rightarrow rW_{-2n+1} = W_{-2n+2} - sW_{-2n} \\ &\Rightarrow rW_{-2n+1-1} = W_{-2n+2-1} - sW_{-2n-1} \\ &\Rightarrow rW_{-2n} = W_{-2n+1} - sW_{-2n-1} \end{aligned}$$

we obtain

$$\begin{aligned} rz^nW_{-2n} &= z^nW_{-2n+1} - sz^nW_{-2n-1} \\ rz^{n-1}W_{-2n+2} &= z^{n-1}W_{-2n+3} - sz^{n-1}W_{-2n+1} \\ rz^{n-2}W_{-2n+4} &= z^{n-2}W_{-2n+5} - sz^{n-2}W_{-2n+3} \\ rz^{n-3}W_{-2n+6} &= z^{n-3}W_{-2n+7} - sz^{n-3}W_{-2n+5} \\ &\vdots \\ rz^4W_{-8} &= z^4W_{-7} - sz^4W_{-9} \\ rz^3W_{-6} &= z^3W_{-5} - sz^3W_{-7} \\ rz^2W_{-4} &= z^2W_{-3} - sz^2W_{-5} \\ rzW_{-2} &= zW_{-1} - szW_{-3} \end{aligned}$$

If we add the equations side by side, we get

$$r\sum_{k=1}^{n} z^kW_{-2k} = (\sum_{k=1}^{n} z^kW_{-2k+1}) - s(z^nW_{-2n-1} - W_{-1} + \sum_{k=1}^{n} z^{k-1}W_{-2k+1}). \tag{5}$$

Since

$$\sum_{k=1}^{n} z^{k-1}W_{-2k+1} = z^{-1}\sum_{k=1}^{n} z^kW_{-2k+1}$$

and

$$W_{-1} = (-\frac{r}{s}\times W_0 + \frac{1}{s}W_1)$$

it follows that

$$r\sum_{k=1}^{n} z^kW_{-2k} = (\sum_{k=1}^{n} z^kW_{-2k+1}) - s(z^nW_{-2n-1} - (-\frac{r}{s}\times W_0 + \frac{1}{s}W_1) + z^{-1}\sum_{k=1}^{n} z^kW_{-2k+1}). \tag{6}$$

Then, solving system (5)-(6) the required result of (b) (i) and (c) (i) follow.

(b)

(ii) Suppose that $r^2+4s \neq 0$. Then, from Remark 2.2, $a \neq b$. We use (b) (i). For $z = a$ and $z = b$, the right hand side of the sum formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule. We get (b) (ii), by using (b) (i), as

$$\begin{aligned} \sum_{k=1}^{n} a^kW_{-2k} &= \left.\frac{\frac{d}{dz}\left(z^{n+1}(s-z)W_{-2n} - rsz^{n+1}W_{-2n-1} + rzW_1 + z(z-s-r^2)W_0\right)}{\frac{d}{dz}\left(r^2z + 2sz - s^2 - z^2\right)}\right|_{z=a} \\ &= \left.\frac{(n(s-z)+s-2z)z^nW_{-2n} - rs(n+1)z^nW_{-2n-1} + rW_1 + (-r^2+2z-s)W_0}{r^2+2s-2z}\right|_{z=a} \\ &= \frac{(n(s-a)+s-2a)a^nW_{-2n} - rs(n+1)a^nW_{-2n-1} + rW_1 + (-r^2+2a-s)W_0}{r^2+2s-2a} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{n} b^kW_{-2k} &= \left.\frac{\frac{d}{dz}\left(z^{n+1}(s-z)W_{-2n} - rsz^{n+1}W_{-2n-1} + rzW_1 + z(z-s-r^2)W_0\right)}{\frac{d}{dz}\left(r^2z + 2sz - s^2 - z^2\right)}\right|_{z=b} \\ &= \left.\frac{(n(s-z)+s-2z)z^nW_{-2n} - rs(n+1)z^nW_{-2n-1} + rW_1 + (-r^2+2z-s)W_0}{r^2+2s-2z}\right|_{z=b} \\ &= \frac{(n(s-b)+s-2b)b^nW_{-2n} - rs(n+1)b^nW_{-2n-1} + rW_1 + (-r^2+2b-s)W_0}{r^2+2s-2b}. \end{aligned}$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.2, $a=b$. We use (b) (i). For $z=a$, the right hand side of the sum formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get, by using (b) (i),

$$\begin{aligned}\sum_{k=1}^{n} a^k W_{-2k} &= \left.\frac{\frac{d^2}{dz^2}\left(z^{n+1}(s-z)W_{-2n}-rsz^{n+1}W_{-2n-1}+rzW_1+z(z-s-r^2)W_0\right)}{\frac{d^2}{dz^2}\left(r^2z+2sz-s^2-z^2\right)}\right|_{z=a} \\ &= \left.\frac{-(n+1)((n+2)z^n-nsz^{n-1})W_{-2n}-nrs(n+1)z^{n-1}W_{-2n-1}+2W_0}{-2}\right|_{z=a} \\ &= \left.\frac{(n+1)((n+2)z^n-nsz^{n-1})W_{-2n}+nrs(n+1)z^{n-1}W_{-2n-1}-2W_0}{2}\right|_{z=a} \\ &= \frac{(n+1)((n+2)a^n-nsa^{n-1})W_{-2n}+nrs(n+1)a^{n-1}W_{-2n-1}-2W_0}{2}.\end{aligned}$$

(c)

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.2, $a\neq b$. We use (c) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule. We get (c) (ii), by using (c) (i), as

$$\begin{aligned}\sum_{k=1}^{n} a^k W_{-2k+1} &= \left.\frac{\frac{d}{dz}\left(-rz^{n+2}W_{-2n}+sz^{n+1}(s-z)W_{-2n-1}+z(z-s)W_1+rszW_0\right)}{\frac{d}{dz}\left(r^2z+2sz-s^2-z^2\right)}\right|_{z=a} \\ &= \left.\frac{-r(n+2)z^{n+1}W_{-2n}+s(n(s-z)+s-2z)z^nW_{-2n-1}+(2z-s)W_1+rsW_0}{r^2+2s-2z}\right|_{z=a} \\ &= \frac{-r(n+2)a^{n+1}W_{-2n}+s(n(s-a)+s-2a)a^nW_{-2n-1}+(2a-s)W_1+rsW_0}{r^2+2s-2a}.\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n} b^k W_{-2k+1} &= \left.\frac{\frac{d}{dz}\left(-rz^{n+2}W_{-2n}+sz^{n+1}(s-z)W_{-2n-1}+z(z-s)W_1+rszW_0\right)}{\frac{d}{dz}\left(r^2z+2sz-s^2-z^2\right)}\right|_{z=b} \\ &= \left.\frac{-r(n+2)z^{n+1}W_{-2n}+s(n(s-z)+s-2z)z^nW_{-2n-1}+(2z-s)W_1+rsW_0}{r^2+2s-2z}\right|_{z=b} \\ &= \frac{-r(n+2)b^{n+1}W_{-2n}+s(n(s-b)+s-2b)b^nW_{-2n-1}+(2b-s)W_1+rsW_0}{r^2+2s-2b}.\end{aligned}$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.2, $a=b$. We use (c) (i). For $z=a$, the right hand side of the sum formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get, by using (c) (i),

$$\begin{aligned}\sum_{k=1}^{n} a^k W_{-2k+1} &= \left.\frac{\frac{d^2}{dz^2}\left(-rz^{n+2}W_{-2n}+sz^{n+1}(s-z)W_{-2n-1}+z(z-s)W_1+rszW_0\right)}{\frac{d^2}{dz^2}\left(r^2z+2sz-s^2-z^2\right)}\right|_{z=a} \\ &= \left.\frac{-r(n+1)(n+2)z^nW_{-2n}-s(n+1)\left((n+2)z^n-nsz^{n-1}\right)W_{-2n-1}+2W_1}{-2}\right|_{z=a} \\ &= \left.\frac{r(n+1)(n+2)z^nW_{-2n}+s(n+1)\left((n+2)z^n-nsz^{n-1}\right)W_{-2n-1}-2W_1}{2}\right|_{z=a} \\ &= \frac{r(n+1)(n+2)a^nW_{-2n}+s(n+1)\left((n+2)a^n-nsa^{n-1}\right)W_{-2n-1}-2W_1}{2}. \square\end{aligned}$$

2.3. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$ and $\sum_{k=0}^{n} z^k W_{k+1}W_k$ of Generalized Fibonacci (Horadam) Polynomials with Positive Subscripts

In the next Theorem, we need the following Remark.

Remark 2.3.
Solving the equation

$$(sz+1)(-s^2z^2+r^2z+2sz-1)=0$$

we find the roots as

$$z_1 = a = \frac{1}{2s^2}\left(r^2 + 2s + r\sqrt{r^2 + 4s}\right),$$
$$z_2 = b = \frac{1}{2s^2}\left(r^2 + 2s - r\sqrt{r^2 + 4s}\right),$$
$$z_3 = c = -\frac{1}{s}.$$

If $r^2 + 4s = 0$, i.e., $s = -\frac{1}{4}r^2$ then $a = b = \frac{1}{2s^2}\left(r^2 + 2s\right) = -\frac{1}{s} = \frac{4}{r^2}$ i.e., $a = b = c = -\frac{1}{s}$.

If $r^2 + 4s \neq 0$, i.e., $s \neq -\frac{1}{4}r^2$ then $a \neq b$ and in this case we must have $a \neq b \neq c$ because $a \neq c = -\frac{1}{s}$ otherwise solving the equation

$$a = c = -\frac{1}{s}$$

i.e.,

$$\frac{1}{2s^2}\left(r^2 + 2s + r\sqrt{r^2 + 4s}\right) = -\frac{1}{s}$$

we find that $s = -\frac{1}{4}r^2$ which contradicts with the assumption $r^2 + 4s \neq 0$. We can list some properties:

- There are some $u, a, b, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $z = a$ or $z = b$ or $z = c$ such that
$$(sz + 1)(-s^2z^2 + r^2z + 2sz - 1) = u(z - a)(z - b)(z - c) = 0.$$
- There are no some $u, a, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq c$, i.e., $z = a$ or $z = c$ such that
$$(sz + 1)(-s^2z^2 + r^2z + 2sz - 1) = u(z - a)(z - c)^2 = 0.$$
- There are some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $z = a$, that is $z = -\frac{1}{s}$, such that
$$(sz + 1)(-s^2z^2 + r^2z + 2sz - 1) = u(z - a)^3 = 0.$$

The following theorem presents some sum formulas of generalized Fibonacci polynomials with positive subscripts.

Theorem 2.3.
Let z be a non-zero complex (or real) number. For $n \geq 0$ we have the following formulas:

(a)

(i) *If $(sz + 1)(-s^2z^2 + r^2z + 2sz - 1) \neq 0$ then*

$$\sum_{k=0}^{n} z^k W_k^2 = \frac{\Phi_7}{(sz + 1)(-s^2z^2 + r^2z + 2sz - 1)}$$

where
$\Phi_7 = -(sz - 1)z^{n+2}W_{n+2}^2 - (r^2sz^2 + r^2z + sz - 1)z^{n+1}W_{n+1}^2 + 2rsz^{n+3}W_{n+2}W_{n+1} + z(sz - 1)W_1^2 + (r^2sz^2 + r^2z + sz - 1)W_0^2 - 2rsz^2W_1W_0.$

(ii) *If $(sz + 1)(-s^2z^2 + r^2z + 2sz - 1) = u(z - a)(z - b)(z - c) = 0$ for some $u, a, b, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $z = a$ or $z = b$ or $z = c$, that is, if*

$$z = a = \frac{1}{2s^2}\left(r^2 + 2s + r\sqrt{r^2 + 4s}\right),$$
$$z = b = \frac{1}{2s^2}\left(r^2 + 2s - r\sqrt{r^2 + 4s}\right),$$
$$z = c = -\frac{1}{s},$$

provided that $r^2 + 4s \neq 0$, then

$$\sum_{k=0}^{n} z^k W_k^2 = \frac{\Phi_8}{-3s^3z^2 + 2r^2sz + 2s^2z + r^2 + s}$$

where
$\Phi_8 = (n - nsz - 3sz + 2)z^{n+1}W_{n+2}^2 - (n(r^2sz^2 + r^2z + sz - 1) + 2r^2z + 3r^2sz^2 + 2sz - 1)z^nW_{n+1}^2 + 2rs(n + 3)z^{n+2}W_{n+2}W_{n+1} + (2sz - 1)W_1^2 + (2r^2sz + r^2 + s)W_0^2 - 4rszW_1W_0.$

(iii) *If* $(sz+1)(-s^2z^2+r^2z+2sz-1)=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *that is, if*

$$z=-\frac{1}{s}$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=0}^{n} z^k W_k^2=\frac{\Phi_9}{-6s^3}$$

where
$\Phi_9=-(n+2)(n+1)(s(n+3)z^n-nz^{n-1})W_{n+2}^2-(n+1)(n^2(r^2sz^2+r^2z+sz-1)+n(5r^2sz^2+2r^2z+2sz+1)+6sr^2z^2)z^{n-2}W_{n+1}^2+2rs(n+1)(n+2)(n+3)z^nW_{n+2}W_{n+1}.$

(b)

(i) *If* $(sz+1)(-s^2z^2+r^2z+2sz-1)\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k=\frac{\Phi_{10}}{(sz+1)(-s^2z^2+r^2z+2sz-1)}$$

where
$\Phi_{10}=rz^{n+2}W_{n+2}^2+rs^2z^{n+3}W_{n+1}^2-(s^2z^2+r^2z-1)z^{n+1}W_{n+2}W_{n+1}-rzW_1^2-rs^2z^2W_0^2+(s^2z^2+r^2z-1)W_1W_0.$

(ii) *If* $(sz+1)(-s^2z^2+r^2z+2sz-1)=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *that is, if*

$$z=a=\frac{1}{2s^2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2s^2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$
$$z=c=-\frac{1}{s},$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k=\frac{\Phi_{11}}{-3s^3z^2+2r^2sz+2s^2z+r^2+s}$$

where
$\Phi_{11}=r(n+2)z^{n+1}W_{n+2}^2+rs^2(n+3)z^{n+2}W_{n+1}^2-(n(s^2z^2+r^2z-1)+2r^2z+3s^2z^2-1)z^nW_{n+2}W_{n+1}-rW_1^2-2rs^2zW_0^2+(r^2+2zs^2)W_1W_0.$

(iii) *If* $(sz+1)(-s^2z^2+r^2z+2sz-1)=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *that is, if*

$$z=-\frac{1}{s}$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k=\frac{\Phi_{12}}{-6s^3}$$

where
$\Phi_{12}=nr(n+1)(n+2)z^{n-1}W_{n+2}^2+rs^2(n+1)(n+2)(n+3)z^nW_{n+1}^2-(n+1)(n^2(s^2z^2+r^2z-1)+n(5s^2z^2+2r^2z+1)+6s^2z^2)z^{n-2}W_{n+2}W_{n+1}.$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[11], Theorem 2.1.].

(a) (i) and (b) (i) Using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n$$

i.e.

$$sW_n=W_{n+2}-rW_{n+1}$$

we obtain

$$\begin{aligned}
s^2 z^n W_n^2 &= z^n W_{n+2}^2 + r^2 z^n W_{n+1}^2 - 2rz^n W_{n+2} W_{n+1} \\
s^2 z^{n-1} W_{n-1}^2 &= z^{n-1} W_{n+1}^2 + r^2 z^{n-1} W_n^2 - 2rz^{n-1} W_{n+1} W_n \\
s^2 z^{n-2} W_{n-2}^2 &= z^{n-2} W_n^2 + r^2 z^{n-2} W_{n-1}^2 - 2rz^{n-2} W_n W_{n-1} \\
s^2 z^{n-3} W_{n-3}^2 &= z^{n-3} W_{n-1}^2 + r^2 z^{n-3} W_{n-2}^2 - 2rz^{n-3} W_{n-1} W_{n-2} \\
&\vdots \\
s^2 z^2 W_2^2 &= z^2 W_4^2 + r^2 z^2 W_3^2 - 2rz^2 W_4 W_3 \\
s^2 z^1 W_1^2 &= z^1 W_3^2 + r^2 z^1 W_2^2 - 2rz^1 W_3 W_2 \\
s^2 z^0 W_0^2 &= z^0 W_2^2 + r^2 z^0 W_1^2 - 2rz^0 W_2 W_{12}.
\end{aligned}$$

If we add the above equations side by side, we get

$$s^2 \sum_{k=0}^{n} z^k W_k^2 = \sum_{k=2}^{n+2} z^{k-2} W_k^2 + r^2 \sum_{k=1}^{n+1} z^{k-1} W_k^2 - 2r \sum_{k=1}^{n+1} z^{k-1} W_{k+1} W_k. \tag{7}$$

Note that

$$\begin{aligned}
\sum_{k=2}^{n+2} z^{k-2} W_k^2 &= -z^{-2} W_0^2 - z^{-1} W_1^2 + z^{n-1} W_{n+1}^2 + z^n W_{n+2}^2 + z^{-2} \sum_{k=0}^{n} z^k W_k^2 \\
\sum_{k=1}^{n+1} z^{k-1} W_k^2 &= -z^{-1} W_0^2 + z^n W_{n+1}^2 + z^{-1} \sum_{k=0}^{n} z^k W_k^2 \\
\sum_{k=1}^{n+1} z^{k-1} W_{k+1} W_k &= -z^{-1} W_1 W_0 + z^n W_{n+2} W_{n+1} + z^{-1} \sum_{k=0}^{n} z^k W_{k+1} W_k.
\end{aligned}$$

If we put them into the (7) we get

$$\begin{aligned}
s^2 \sum_{k=0}^{n} z^k W_k^2 &= (-z^{-2} W_0^2 - z^{-1} W_1^2 + z^{n-1} W_{n+1}^2 + z^n W_{n+2}^2 + z^{-2} \sum_{k=0}^{n} z^k W_k^2) \\
&\quad + r^2 (-z^{-1} W_0^2 + z^n W_{n+1}^2 + z^{-1} \sum_{k=0}^{n} z^k W_k^2) \\
&\quad - 2r(-z^{-1} W_1 W_0 + z^n W_{n+2} W_{n+1} + z^{-1} \sum_{k=0}^{n} z^k W_{k+1} W_k).
\end{aligned} \tag{8}$$

Next we obtain $\sum_{k=0}^{n} W_{k+1} W_k$. Multiplying the both side of the recurrence relation

$$sW_n = W_{n+2} - rW_{n+1}$$

by W_{n+1} we get

$$sW_{n+1} W_n = W_{n+2} W_{n+1} - rW_{n+1}^2.$$

Then using last recurrence relation, we obtain

$$\begin{aligned}
sz^n W_{n+1} W_n &= z^n W_{n+2} W_{n+1} - rz^n W_{n+1}^2 \\
sz^{n-1} W_n W_{n-1} &= z^{n-1} W_{n+1} W_n - rz^{n-1} W_n^2 \\
sz^{n-2} W_{n-1} W_{n-2} &= z^{n-2} W_n W_{n-1} - rz^{n-2} W_{n-1}^2 \\
&\vdots \\
sz^2 W_3 W_2 &= z^2 W_4 W_3 - rz^2 W_3^2 \\
sz^1 W_2 W_1 &= z^1 W_3 W_2 - rz^1 W_2^2 \\
sz^0 W_1 W_0 &= z^0 W_2 W_1 - rz^0 W_1^2
\end{aligned}$$

If we add the above equations side by side, we get

$$s \sum_{k=0}^{n} z^k W_{k+1} W_k = \sum_{k=1}^{n+1} z^{k-1} W_{k+1} W_k - r \sum_{k=1}^{n+1} z^{k-1} W_k^2. \tag{9}$$

Note that

$$\sum_{k=1}^{n+1} z^{k-1} W_{k+1} W_k = -z^{-1} W_1 W_0 + z^n W_{n+2} W_{n+1} + z^{-1} \sum_{k=0}^{n} z^k W_{k+1} W_k,$$

$$\sum_{k=1}^{n+1} z^{k-1} W_k^2 = -z^{-1} W_0^2 + z^n W_{n+1}^2 + z^{-1} \sum_{k=0}^{n} z^k W_k^2.$$

If we put them into the (9) then we obtain

$$s\sum_{k=0}^{n} z^k W_{k+1} W_k = (-z^{-1} W_1 W_0 + z^n W_{n+2} W_{n+1} + z^{-1} \sum_{k=0}^{n} z^k W_{k+1} W_k) \\ -r(-z^{-1} W_0^2 + z^n W_{n+1}^2 + z^{-1} \sum_{k=0}^{n} z^k W_k^2). \tag{10}$$

Then, solving the system (8)-(10), the required results of (a) (i) and (b) (i).

(a)

(ii) Suppose that $r^2 + 4s \neq 0$. Then, from Remark 2.3, $a \neq b \neq c$. We use (a) (i). For $z = a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (a) (ii) by using

$$\sum_{k=0}^{n} a^k W_k^2 = \left. \frac{\frac{d}{dz}(\Phi_7)}{\frac{d}{dz}\left((sz+1)(-s^2z^2 + r^2z + 2sz - 1)\right)} \right|_{z=a} = \left. \frac{\Phi_8}{-3s^3z^2 + 2r^2sz + 2s^2z + r^2 + s} \right|_{z=a}.$$

The proof for the case $z = b$ and $z = c$ are the same.

(iii) Suppose that $r^2 + 4s = 0$. Then, from Remark 2.3, $a = b = c$. We use (a) (i). For $z = a = -\frac{1}{s}$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (a) (iii) by using

$$\sum_{k=0}^{n} a^k W_k^2 = \left. \frac{\frac{d^3}{dz^3}(\Phi_7)}{\frac{d^3}{dz^3}\left((sz+1)(-s^2z^2 + r^2z + 2sz - 1)\right)} \right|_{z=a} = \left. \frac{\Phi_9}{-6s^3} \right|_{z=a}.$$

(b)

(ii) Suppose that $r^2 + 4s \neq 0$. Then, from Remark 2.3, $a \neq b \neq c$. We use (b) (i). For $z = a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) (ii) by using

$$\sum_{k=0}^{n} a^k W_k^2 = \left. \frac{\frac{d}{dz}(\Phi_{10})}{\frac{d}{dz}\left((sz+1)(-s^2z^2 + r^2z + 2sz - 1)\right)} \right|_{z=a} = \left. \frac{\Phi_{11}}{-3s^3z^2 + 2r^2sz + 2s^2z + r^2 + s} \right|_{z=a}.$$

The proof for the case $z = b$ and $z = c$ are the same.

(iii) Suppose that $r^2 + 4s = 0$. Then, from Remark 2.3, $a = b = c$. We use (b) (i). For $z = a = -\frac{1}{s}$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (b) (iii) by using

$$\sum_{k=0}^{n} a^k W_k^2 = \left. \frac{\frac{d^3}{dz^3}(\Phi_{10})}{\frac{d^3}{dz^3}\left((sz+1)(-s^2z^2 + r^2z + 2sz - 1)\right)} \right|_{z=a} = \left. \frac{\Phi_{12}}{-6s^3} \right|_{z=a}. \quad \square$$

2.4. Sum Formulas $\sum_{k=1}^{n} z^k W_{-k}^2$ and $\sum_{k=1}^{n} z^k W_{-k+1} W_{-k}$ of Generalized Fibonacci (Horadam) Polynomials with Negative Subscripts

In the next Theorem, we need the following Remark.

Remark 2.4.
Solving the equation

$$(z+s)(-z^2+r^2z+2sz-s^2)=0$$

we find the roots as

$$z_1 = a = \frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z_2 = b = \frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$
$$z_3 = c = -s.$$

If $r^2+4s=0$, i.e., $s=-\frac{1}{4}r^2$ then $a=b=\frac{1}{2}\left(r^2+2s\right)=-s$, i.e., $a=b=c=-s$.

If $r^2+4s\neq 0$, i.e., $s=-\frac{1}{4}r^2$ then $a\neq b$ and in this case we must have $a\neq b\neq c$ because $a\neq c=-s$ otherwise solving the equation

$$a=c=-s$$

i.e.,

$$\frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right)=-s$$

we find that $s=-\frac{1}{4}r^2$ which contradicts with the assumption $r^2+4s\neq 0$. We can list some properties:

- There are some $u,a,b,c\in\mathbb{C}$ with $u\neq 0$ and $a\neq b\neq c$, i.e., $z=a$ or $z=b$ or $z=c$ such that
$$(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)(z-b)(z-c)=0.$$
- There are no some $u,a,c\in\mathbb{C}$ with $u\neq 0$ and $a\neq c$, i.e., $z=a$ or $z=c$ such that
$$(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)(z-c)^2=0.$$
- There are some $u,a\in\mathbb{C}$ with $u\neq 0$, i.e., $z=a$, that is $z=-s$ such that
$$(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)^3=0.$$

The following theorem presents some sum formulas of generalized Fibonacci polynomials with negative subscripts.

Theorem 2.4.
Let z be a non-zero complex (or real) number. For $n\geq 1$ we have the following formulas:

(a) **(i)** *If $(z+s)(-z^2+r^2z+2sz-s^2)\neq 0$ then*

$$\sum_{k=1}^{n} z^k W_{-k}^2 = \frac{\Phi_{13}}{(z+s)(-z^2+r^2z+2sz-s^2)}$$

where
$\Phi_{13}=(s-z)z^{n+1}W_{-n+1}^2+(-z^2+r^2z+sz+r^2s)z^{n+1}W_{-n}^2-2rsz^{n+1}W_{-n+1}W_{-n}+z(z-s)W_1^2+z(z^2-r^2z-sz-r^2s)W_0^2+2rszW_1W_0.$

(ii) *If $(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)(z-b)(z-c)=0$ for some $u,a,b,c\in\mathbb{C}$ with $u\neq 0$ and $a\neq b\neq c$, i.e., $z=a$ or $z=b$ or $z=c$, that is, if*

$$z=a=\frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$
$$z=c=-s.$$

provided that $r^2+4s\neq 0$, then

$$\sum_{k=1}^{n} z^k W_{-k}^2 = \frac{\Phi_{14}}{-3z^2+2r^2z+2sz+r^2s+s^2}$$

where
$\Phi_{14}=(n(s-z)-2z+s)z^nW_{-n+1}^2+(n(-z^2+r^2z+sz+r^2s)-3z^2+2r^2z+2sz+r^2s)z^nW_{-n}^2-2(n+1)rsz^nW_{-n+1}W_{-n}+(2z-s)W_1^2+(3z^2-2r^2z-2sz-sr^2)W_0^2+2rsW_1W_0.$

(iii) *If* $(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, that is, if*

$$z=-s,$$

provided that $r^2+4s=0$*, then*

$$\sum_{k=1}^{n} z^k W_{-k}^2=\frac{\Phi_{15}}{-6}$$

where
$\Phi_{15}=-n(n+1)((n+2)z-s(n-1))z^{n-2}W_{-n+1}^2+(n+1)(n^2(-z^2+r^2z+sz+r^2s)-n(5z^2-2r^2z-2sz+r^2s)-6z^2)z^{n-2}W_{-n}^2-2(n-1)n(n+1)rsz^{n-2}W_{-n+1}W_{-n}+6W_0^2.$

(b) **(i)** *If* $(z+s)(-z^2+r^2z+2sz-s^2)\neq 0$ *then*

$$\sum_{k=1}^{n} z^k W_{-k+1}W_{-k}=\frac{\Phi_{16}}{(z+s)(-z^2+r^2z+2sz-s^2)}$$

where
$\Phi_{16}=-rz^{n+2}W_{-n+1}^2-rs^2z^{n+1}W_{-n}^2+(-z^2+r^2z+s^2)z^{n+1}W_{-n+1}W_{-n}+rz^2W_1^2+rs^2zW_0^2-z(-z^2+r^2z+s^2)W_1W_0.$

(ii) *If* $(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$*, that is, if*

$$z=a=\frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$
$$z=c=-s.$$

provided that $r^2+4s\neq 0$*, then*

$$\sum_{k=1}^{n} z^k W_{-k+1}W_{-k}=\frac{\Phi_{17}}{-3z^2+2r^2z+2sz+r^2s+s^2}$$

where
$\Phi_{17}=-(n+2)rz^{n+1}W_{-n+1}^2-(n+1)rs^2z^nW_{-n}^2+(n(-z^2+r^2z+s^2)-3z^2+2r^2z+s^2)z^nW_{-n+1}W_{-n}+2rzW_1^2+rs^2W_0^2-(-3z^2+2r^2z+s^2)W_1W_0.$

(iii) *If* $(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, that is, if*

$$z=-s,$$

provided that $r^2+4s=0$*, then*

$$\sum_{k=1}^{n} z^k W_{-k+1}W_{-k}=\frac{\Phi_{18}}{-6}$$

where
$\Phi_{18}=-rn(n+1)(n+2)z^{n-1}W_{-n+1}^2-rs^2(n-1)n(n+1)z^{n-2}W_{-n}^2-(n+1)(n^2(z^2-r^2z-s^2)+n(5z^2-2r^2z+s^2)+6z^2)z^{n-2}W_{-n+1}W_{-n}+6W_1W_0.$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[11], Theorem 3.1.].

(a) (i) and (b) (i) Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}\Rightarrow W_{-n}=-\frac{r}{s}W_{-n+1}+\frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

we obtain

$$\begin{aligned}
s^2 z^n W_{-n}^2 &= z^n W_{-n+2}^2 + r^2 z^n W_{-n+1}^2 - 2rz^n W_{-n+2} W_{-n+1} \\
s^2 z^{n-1} W_{-n+1}^2 &= z^{n-1} W_{-n+3}^2 + r^2 z^{n-1} W_{-n+2}^2 - 2rz^{n-1} W_{-n+3} W_{-n+2} \\
s^2 z^{n-2} W_{-n+2}^2 &= z^{n-2} W_{-n+4}^2 + r^2 z^{n-2} W_{-n+3}^2 - 2rz^{n-2} W_{-n+4} W_{-n+3} \\
&\vdots \\
s^2 z^3 W_{-3}^2 &= z^3 W_{-1}^2 + r^2 z^3 W_{-2}^2 - 2rz^3 W_{-1} W_{-2} \\
s^2 z^2 W_{-2}^2 &= z^2 W_0^2 + r^2 z^2 W_{-1}^2 - 2rz^2 W_0 W_{-1} \\
s^2 z^1 W_{-1}^2 &= z^1 W_1^2 + r^2 z^1 W_0^2 - 2rz^1 W_1 W_0
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s^2 \sum_{k=1}^{n} z^k W_{-k}^2 &= (z^1 W_1^2 + z^2 W_0^2 + \sum_{k=1}^{n-2} z^{k+2} W_{-k}^2) + r^2 (z^1 W_0^2 \\
&\quad + \sum_{k=1}^{n-1} z^{k+1} W_{-k}^2) - 2r(z^1 W_1 W_0 + \sum_{k=1}^{n-1} z^{k+1} W_{-k+1} W_{-k}).
\end{aligned} \tag{11}$$

Note that

$$\begin{aligned}
\sum_{k=1}^{n-2} z^{k+2} W_{-k}^2 &= -z^{n+1} W_{-n+1}^2 - z^{n+2} W_{-n}^2 + z^2 \sum_{k=1}^{n} z^k W_{-k}^2 \\
\sum_{k=1}^{n-1} z^{k+1} W_{-k}^2 &= -z^{n+1} W_{-n}^2 + z \sum_{k=1}^{n} z^k W_{-k}^2 \\
\sum_{k=1}^{n-1} z^{k+1} W_{-k+1} W_{-k} &= -z^{n+1} W_{-n+1} W_{-n} + z \sum_{k=1}^{n} z^k W_{-k+1} W_{-k}.
\end{aligned}$$

If we put them into the (11) then we obtain

$$\begin{aligned}
s^2 \sum_{k=1}^{n} z^k W_{-k}^2 &= (z^1 W_1^2 + z^2 W_0^2 - z^{n+1} W_{-n+1}^2 - z^{n+2} W_{-n}^2 + z^2 \sum_{k=1}^{n} z^k W_{-k}^2) \\
&\quad + r^2 (z^1 W_0^2 - z^{n+1} W_{-n}^2 + z \sum_{k=1}^{n} z^k W_{-k}^2) - 2r(z^1 W_1 W_0 - z^{n+1} W_{-n+1} W_{-n} \\
&\quad + z \sum_{k=1}^{n} z^k W_{-k+1} W_{-k})
\end{aligned} \tag{12}$$

Next we calculate $\sum_{k=1}^{n} W_{-k+1} W_{-k}$. Multiplying the both side of the recurrence relation

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

by W_{-n+1} we get

$$sW_{-n+1} W_{-n} = W_{-n+2} W_{-n+1} - rW_{-n+1}^2.$$

Then using last recurrence relation, we obtain

$$\begin{aligned}
sz^n W_{-n+1} W_{-n} &= z^n W_{-n+2} W_{-n+1} - rz^n W_{-n+1}^2 \\
sz^{n-1} W_{-n+2} W_{-n+1} &= z^{n-1} W_{-n+3} W_{-n+2} - rz^{n-1} W_{-n+2}^2 \\
sz^{n-2} W_{-n+3} W_{-n+2} &= z^{n-2} W_{-n+4} W_{-n+3} - rz^{n-2} W_{-n+3}^2 \\
&\vdots \\
sz^3 W_{-2} W_{-3} &= z^3 W_{-1} W_{-2} - rz^3 W_{-2}^2 \\
sz^2 W_{-1} W_{-2} &= z^2 W_0 W_{-1} - rz^2 W_{-1}^2 \\
szW_0 W_{-1} &= zW_1 W_0 - rzW_0^2
\end{aligned}$$

If we add the above equations side by side, we get

$$s \sum_{k=1}^{n} z^k W_{-k+1} W_{-k} = (zW_1 W_0 + \sum_{k=1}^{n-1} z^{k+1} W_{-k+1} W_{-k}) - r(zW_0^2 + \sum_{k=1}^{n-1} z^{k+1} W_{-k}^2). \tag{13}$$

Note that

$$\sum_{k=1}^{n-1} z^{k+1}W_{-k+1}W_{-k} = -z^{n+1}W_{-n+1}W_{-n} + z\sum_{k=1}^{n} z^{k}W_{-k+1}W_{-k},$$

$$\sum_{k=1}^{n-1} z^{k+1}W_{-k}^2 = -z^{n+1}W_{-n}^2 + z\sum_{k=1}^{n} z^{k}W_{-k}^2.$$

If we put them into the (13) then we obtain

$$s\sum_{k=1}^{n} z^{k}W_{-k+1}W_{-k} = (zW_1W_0 - z^{n+1}W_{-n+1}W_{-n} + z\sum_{k=1}^{n} z^{k}W_{-k+1}W_{-k}) - r(zW_0^2 - z^{n+1}W_{-n}^2 + z\sum_{k=1}^{n} z^{k}W_{-k}^2). \quad (14)$$

Then, solving the system (12)-(14), the required results of (a) (i) and (b) (i) follow.

(a)

(ii) Suppose that $r^2+4s \neq 0$. Then, from Remark 2.4, $a \neq b \neq c$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (a) (ii) by using

$$\sum_{k=1}^{n} a^{k}W_{-k}^2 = \left.\frac{\frac{d}{dz}(\Phi_{13})}{\frac{d}{dz}\left((z+s)(-z^2+r^2z+2sz-s^2)\right)}\right|_{z=a}$$
$$= \left.\frac{\Phi_{14}}{-3z^2+2r^2z+2sz+r^2s+s^2}\right|_{z=a}.$$

The proof for the case $z=b$ and $z=c$ are the same.

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.4, $a=b=c$. We use (a) (i). For $z=a=-\frac{1}{s}$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (a) (iii) by using

$$\sum_{k=1}^{n} a^{k}W_{-k}^2 = \left.\frac{\frac{d^3}{dz^3}(\Phi_{13})}{\frac{d^3}{dz^3}\left((z+s)(-z^2+r^2z+2sz-s^2)\right)}\right|_{z=a}$$
$$= \left.\frac{\Phi_{15}}{-6}\right|_{z=a}.$$

(b)

(ii) Suppose that $r^2+4s \neq 0$. Then, from Remark 2.4, $a \neq b \neq c$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) (ii) by using

$$\sum_{k=1}^{n} a^{k}W_{-k+1}W_{-k} = \left.\frac{\frac{d}{dz}(\Phi_{16})}{\frac{d}{dz}\left((z+s)(-z^2+r^2z+2sz-s^2)\right)}\right|_{z=a}$$
$$= \left.\frac{\Phi_{17}}{-3z^2+2r^2z+2sz+r^2s+s^2}\right|_{z=a}.$$

The proof for the case $z=b$ and $z=c$ are the same.

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.4, $a=b=c$. We use (b) (i). For $z=a=-\frac{1}{s}$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (b) (iii) by using

$$\sum_{k=1}^{n} a^{k}W_{-k+1}W_{-k} = \left.\frac{\frac{d^3}{dz^3}(\Phi_{16})}{\frac{d^3}{dz^3}\left((z+s)(-z^2+r^2z+2sz-s^2)\right)}\right|_{z=a}$$
$$= \left.\frac{\Phi_{18}}{-6}\right|_{z=a}. \quad \square$$

2.5. Sum Formulas $\sum_{k=0}^{n} z^k W_k^3$, $\sum_{k=0}^{n} z^k W_k^2 W_{k+1}$ and $\sum_{k=0}^{n} z^k W_{k+1}^2 W_k$ of Generalized Fibonacci (Horadam) Polynomials with Positive Subscripts

In the next Theorem, we need the following Remark.

Remark 2.5.
Solving the equation

$$(-s^3z^2 + rsz + 1)(s^3z^2 + r^3z + 3rsz - 1) = 0$$

we find the roots as

$$\begin{aligned}
z_1 = a &= \frac{1}{2s^2}\left(r + \sqrt{r^2+4s}\right),\\
z_2 = b &= \frac{1}{2s^2}\left(r - \sqrt{r^2+4s}\right),\\
z_3 = c &= \frac{1}{2s^3}\left(-r^3 - 3rs + \left(s + r^2\right)\sqrt{r^2+4s}\right),\\
z_4 = d &= \frac{1}{2s^3}\left(-r^3 - 3rs - \left(s + r^2\right)\sqrt{r^2+4s}\right).
\end{aligned}$$

Solving $a = b$ i.e.,

$$\frac{1}{2s^2}\left(r + \sqrt{r^2+4s}\right) = \frac{1}{2s^2}\left(r - \sqrt{r^2+4s}\right)$$

we see that $s = -\frac{1}{4}r^2$.Solving $c = d$, we find that $s = -\frac{1}{4}r^2, s = -r^2$. Solving each case of $a = c$, $a = d$, $b = c$, $b = d$, we get $s = -\frac{1}{4}r^2$.

If $r^2 + s = 0$, i.e., $s = -r^2$ then $a \neq b \neq c$ and $c = d$ and in this case

$$\begin{aligned}
z = a &= \frac{1}{2r^4}\left(r + \sqrt{-3r^2}\right),\\
z = b &= \frac{1}{2r^4}\left(r - \sqrt{-3r^2}\right),\\
z = c = d &= -\frac{1}{r^3}.
\end{aligned}$$

If $r^2 + 4s = 0$, i.e., $s = -\frac{1}{4}r^2$ then $a = b = \dfrac{r}{2s^2} = \frac{8}{r^3}$ and $c = d = \dfrac{-r^3 - 3rs}{2s^3} = \frac{8}{r^3}$ and so in this case we must have $a = b = c = d = \dfrac{r}{2s^2} = \frac{8}{r^3}$.

If $r^2 + 4s \neq 0$ then $a \neq b$ and $c \neq d$ and in this case we must have $a \neq b \neq c \neq d$ because $a \neq c$ otherwise solving the equation

$$a = c$$

i.e.,

$$\frac{1}{2s^2}\left(r + \sqrt{r^2+4s}\right) = \frac{1}{2s^3}\left(-r^3 - 3rs + \left(s + r^2\right)\sqrt{r^2+4s}\right)$$

we find that $s = -\frac{1}{4}r^2$ which contradicts with the assumption $r^2 + 4s \neq 0$

We can list some properties:

- There are some $u, a, b, c, d \in \mathbb{C}$ and $u \neq 0$ and $a \neq b \neq c \neq d$ such that

  $$(-s^3z^2 + rsz + 1)(s^3z^2 + r^3z + 3rsz - 1) = u(z-a)(z-b)(z-c)(z-d) = 0,$$

  i.e., $z = a$ or $z = b$ or $z = c$ or $z = d$.

- There are some $u, a, b, c \in \mathbb{C}$ and $u \neq 0$ and $a \neq b \neq c$ such that

  $$(-s^3z^2 + rsz + 1)(s^3z^2 + r^3z + 3rsz - 1) = u(z-a)(z-b)(z-c)^2 = 0,$$

  i.e., $z = a$ or $z = b$ or $z = c$.

- There are no some $u,a\in\mathbb{C}$ with $u,a,b\in\mathbb{C}$ and $u\neq 0$ and $a\neq b$ such that
$$(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)^3(z-b)=0,$$
i.e., $z=a$ or $z=b$.
- There are some $u,a\in\mathbb{C},u\neq 0$ such that
$$(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)^4=0,$$
i.e., $z=a$, that is $z=\frac{8}{r^3}$.

The following theorem presents some sum formulas of generalized Fibonacci polynomials with positive subscripts.

Theorem 2.5.
Let z be a non-zero complex (or real) number. For $n\geq 0$ we have the following formulas:

(a) (i) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\neq 0$ *then*
$$\sum_{k=0}^{n} z^kW_k^3=\frac{\Phi_{19}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}$$
where
$\Phi_{19}=-(s^3z^2+2rsz-1)z^{n+2}W_{n+2}^3-(-r^3s^3z^3+s^3z^2+3r^2s^2z^2+r^4sz^2+r^3z+2rsz-1)z^{n+1}W_{n+1}^3+3rs(s^2z+r)z^{n+3}W_{n+2}^2W_{n+1}-3rs^2(rsz-1)z^{n+3}W_{n+1}^2W_{n+2}+z(s^3z^2+2rsz-1)W_1^3+(-r^3s^3z^3+s^3z^2+3r^2s^2z^2+r^4sz^2+r^3z+2rsz-1)W_0^3-3rsz^2(r+s^2z)W_1^2W_0+3rs^2z^2(rsz-1)W_0^2W_1.$

(ii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)(z-b)(z-c)(z-d)=0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$*, that is, if*
$$\begin{aligned}z=a&=\frac{1}{2s^2}\left(r+\sqrt{r^2+4s}\right),\\ z=b&=\frac{1}{2s^2}\left(r-\sqrt{r^2+4s}\right),\\ z=c&=\frac{1}{2s^3}\left(-r^3-3rs+\left(s+r^2\right)\sqrt{r^2+4s}\right),\\ z=d&=\frac{1}{2s^3}\left(-r^3-3rs-\left(s+r^2\right)\sqrt{r^2+4s}\right),\end{aligned}$$
provided that $r^2+4s\neq 0$*, then*
$$\sum_{k=0}^{n} z^kW_k^3=\frac{\Phi_{20}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}$$
where
$\Phi_{20}=(-(n+4)s^3z^2-2(n+3)rsz+n+2)z^{n+1}W_{n+2}^3-(-(n+4)r^3s^3z^3+s(n+3)(r^4+3r^2s+s^2)z^2+r(n+2)(r^2+2s)z-n-1)z^nW_{n+1}^3+3rs(n(s^2z+r)+4s^2z+3r)z^{n+2}W_{n+2}^2W_{n+1}+3rs^2(n(1-rsz)-4rsz+3)z^{n+2}W_{n+1}^2W_{n+2}+(3s^3z^2+4rsz-1)W_1^3+(-3r^3s^3z^2+2sz(r^4+3r^2s+s^2)+r^3+2rs)W_0^3-3rs(3zs^2+2r)zW_1^2W_0+3rs^2(3rsz-2)zW_0^2W_1.$

(iii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)(z-b)(z-c)^2=0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$*, then if* $z=a$ *or* $z=b$*, that is, if*
$$\begin{aligned}z=a&=\frac{1}{2r^4}\left(r+\sqrt{-3r^2}\right),\\ z=b&=\frac{1}{2r^4}\left(r-\sqrt{-3r^2}\right),\end{aligned}$$
provided that $r^2+s=0$*, then*
$$\sum_{k=0}^{n} z^kW_k^3=\frac{\Phi_{21}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}$$
where
$\Phi_{21}=(-(n+4)s^3z^2-2(n+3)rsz+n+2)z^{n+1}W_{n+2}^3-(-(n+4)r^3s^3z^3+s(n+3)(r^4+3r^2s+s^2)z^2+r(n+2)(r^2+2s)z-n-1)z^nW_{n+1}^3+3rs(n(s^2z+r)+4s^2z+3r)z^{n+2}W_{n+2}^2W_{n+1}+3rs^2(n(1-rsz)-4rsz+3)z^{n+2}W_{n+1}^2W_{n+2}+$

$(3s^3z^2+4rsz-1)W_1^3+(-3r^3s^3z^2+2sz(r^4+3r^2s+s^2)+r^3+2rs)W_0^3-3rs(3zs^2+2r)zW_1^2W_0+3rs^2(3rsz-2)zW_0^2W_1$.

and if $z=c$ *, that is, if*

$$z=c=-\frac{1}{r^3},$$

provided that $r^2+s=0$*, then*

$$\sum_{k=0}^{n} z^k W_k^3 = \frac{\Phi_{22}}{2s(-6s^5z^2-3rs^2(r^2+2s)z+r^4+3r^2s+2s^2)}$$

where

$\Phi_{22}=(-(n+3)(n+4)s^3z^2-2(n+2)(n+3)rsz+n^2+3n+2)z^nW_{n+2}^3-(-(n+3)(n+4)r^3s^3z^3+(n+2)(n+3)s(r^4+3r^2s+s^2)z^2+(n+1)(n+2)r(r^2+2s)z-n^2-n)z^{n-1}W_{n+1}^3+3rs(n+3)(n(s^2z+r)+4s^2z+2r)z^{n+1}W_{n+2}^2W_{n+1}+3rs^2(n+3)(n(1-rsz)-4rsz+2)z^{n+1}W_{n+1}^2W_{n+2}+2s(3s^2z+2r)W_1^3+(-6r^3s^3z+2r^4s+6r^2s^2+2s^3)W_0^3-6rs(3zs^2+r)W_1^2W_0+6rs^2(3rsz-1)W_0^2W_1$.

(iv) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)^4=0$ *for some* $u,a\in\mathbb{C},u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{r}{2s^2}=\frac{8}{r^3}$$

provided that $r^2+4s=0$*, then*

$$\sum_{k=0}^{n} z^k W_k^3 = \frac{\Phi_{23}}{-24s^6}$$

where

$\Phi_{23} = -(n^2+3n+2)\left((n+3)(n+4)s^3z^2+2(n+3)nrsz-n^2+n\right)z^{n-2}W_{n+2}^3-(n+1)(-(n+2)(n+3)(n+4)r^3s^3z^3+n(n+2)(n+3)s(r^4+3r^2s+s^2)z^2+(n-1)n(n+2)rz(r^2+2s)-n^3+3n^2-2n)z^{n-3}W_{n+1}^3+3rs(n(s^2z+r)+4s^2z)(n^3+6n^2+11n+6)z^{n-1}W_{n+2}^2W_{n+1}-3rs^2(n(rsz-1)+4rsz)(n^3+6n^2+11n+6)z^{n-1}W_{n+1}^2W_{n+2}$.

(b) (i) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k^2 W_{k+1} = \frac{\Phi_{24}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}$$

where

$\Phi_{24}=-r(rsz-1)z^{n+2}W_{n+2}^3-rs^3(rsz-1)z^{n+3}W_{n+1}^3+s(-s^3z^2+2r^3z+1)z^{n+2}W_{n+2}^2W_{n+1}-(-2rs^4z^3+s^3z^2+r^4sz^2+r^3z+2rsz-1)z^{n+1}W_{n+1}^2W_{n+2}+rz(rsz-1)W_1^3+rs^3z^2(rsz-1)W_0^3-sz(-s^3z^2+2r^3z+1)W_1^2W_0+(-2rs^4z^3+s^3z^2+r^4sz^2+r^3z+2rsz-1)W_0^2W_1$.

(ii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)(z-b)(z-c)(z-d)=0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$*, that is, if*

$$z=a=\frac{1}{2s^2}\left(r+\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2s^2}\left(r-\sqrt{r^2+4s}\right),$$
$$z=c=\frac{1}{2s^3}\left(-r^3-3rs+\left(s+r^2\right)\sqrt{r^2+4s}\right),$$
$$z=d=\frac{1}{2s^3}\left(-r^3-3rs-\left(s+r^2\right)\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$*, then*

$$\sum_{k=0}^{n} z^k W_k^2 W_{k+1} = \frac{\Phi_{25}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}$$

where

$\Phi_{25}=r(-(n+3)rsz+n+2)z^{n+1}W_{n+2}^3-rs^3((n+4)rsz-n-3)z^{n+2}W_{n+1}^3+s(-(n+4)s^3z^2+2(n+3)r^3z+n+2)z^{n+1}W_{n+2}^2W_{n+1}-(-2(n+4)rs^4z^3+(n+3)s(r^4+s^2)z^2+r(n+2)(r^2+2s)z-n-1)z^nW_{n+1}^2W_{n+2}+r(2rsz-1)W_1^3+rs^3z(3rsz-2)W_0^3-s(-3s^3z^2+4r^3z+1)W_1^2W_0+(-6rs^4z^2+2r^4sz+2s^3z+r^3+2rs)W_0^2W_1$.

(iii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)(z-b)(z-c)^2=0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z=a=\frac{1}{2r^4}\left(r+\sqrt{-3r^2}\right),$$
$$z=b=\frac{1}{2r^4}\left(r-\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=0}^{n}z^kW_k^2W_{k+1}=\frac{\Phi_{26}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}$$

where

$\Phi_{26}=r(-(n+3)rsz+n+2)z^{n+1}W_{n+2}^3-rs^3((n+4)rsz-n-3)z^{n+2}W_{n+1}^3+s(-(n+4)s^3z^2+2(n+3)r^3z+n+2)z^{n+1}W_{n+2}^2W_{n+1}-(-2(n+4)rs^4z^3+(n+3)s(r^4+s^2)z^2+r(n+2)(r^2+2s)z-n-1)z^nW_{n+1}^2W_{n+2}+r(2rsz-1)W_1^3+rs^3z(3rsz-2)W_0^3-s(-3s^3z^2+4r^3z+1)W_1^2W_0+(-6rs^4z^2+2r^4sz+2s^3z+r^3+2rs)W_0^2W_1$

and if $z=c$, *that is, if*

$$z=c=-\frac{1}{r^3},$$

provided that $r^2+s=0$, *then*

$$\sum_{k=0}^{n}z^kW_k^2W_{k+1}=\frac{\Phi_{27}}{2s(-6s^5z^2-3rs^2(r^2+2s)z+r^4+3r^2s+2s^2)}$$

where

$\Phi_{27}=(n+2)r(-(n+3)rsz+n+1)z^nW_{n+2}^3+(n+3)rs^3(-(n+4)rsz+n+2)z^{n+1}W_{n+1}^3+s(-(n+3)(n+4)s^3z^2+2(n+2)(n+3)r^3z+n^2+3n+2)z^nW_{n+2}^2W_{n+1}-(-2(n+3)(n+4)rs^4z^3+(n+2)(n+3)s(r^4+s^2)z^2+(n+1)(n+2)r(r^2+2s)z-n^2-n)z^{n-1}W_{n+1}^2W_{n+2}+2r^2sW_1^3+2rs^3(3rsz-1)W_0^3-2s(2r^3-3s^3z)W_1^2W_0+(2r^4s-12zrs^4+2s^3)W_0^2W_1$.

(iv) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)^4=0$ *for some* $u,a\in\mathbb{C},u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{r}{2s^2}=\frac{8}{r^3}$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=0}^{n}z^kW_k^2W_{k+1}=\frac{\Phi_{28}}{-24s^6}$$

where

$\Phi_{28}=-n(n^2+3n+2)r((n+3)rsz-n+1)z^{n-2}W_{n+2}^3-(n^3+6n^2+11n+6)rs^3((n+4)rsz-n)z^{n-1}W_{n+1}^3-s(n^2+3n+2)((n+3)(n+4)s^3z^2-2n(n+3)r^3z-n^2+n)z^{n-2}W_{n+2}^2W_{n+1}-(n+1)(-2(n+2)(n+3)(n+4)rs^4z^3+n(n+2)(n+3)s(r^4+s^2)z^2+(n-1)n(n+2)r(r^2+2s)z-n^3+3n^2-2n)z^{n-3}W_{n+1}^2W_{n+2}$.

(c) **(i)** *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\neq 0$ *then*

$$\sum_{k=0}^{n}z^kW_{k+1}^2W_k=\frac{\Phi_{29}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}$$

where

$\Phi_{29}=r(s^2z+r)z^{n+2}W_{n+2}^3+rs^3(s^2z+r)z^{n+3}W_{n+1}^3-(s^3z^2+3r^2s^2z^2+r^3z-1)z^{n+1}W_{n+2}^2W_{n+1}+s^2(-s^3z^2+2r^3z+1)z^{n+2}W_{n+1}^2W_{n+2}-rz(s^2z+r)W_1^3-rs^3z^2(s^2z+r)W_0^3+(s^3z^2+3r^2s^2z^2+r^3z-1)W_1^2W_0-s^2z(-s^3z^2+2r^3z+1)W_0^2W_1$.

(ii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)(z-b)(z-c)(z-d)=0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *that is, if*

$$z=a=\frac{1}{2s^2}\left(r+\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2s^2}\left(r-\sqrt{r^2+4s}\right),$$
$$z=c=\frac{1}{2s^3}\left(-r^3-3rs+\left(s+r^2\right)\sqrt{r^2+4s}\right),$$
$$z=d=\frac{1}{2s^3}\left(-r^3-3rs-\left(s+r^2\right)\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=0}^{n} z^k W_{k+1}^2 W_k = \frac{\Phi_{30}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}$$

where

$\Phi_{30} = r((n+3)s^2z+nr+2r)z^{n+1}W_{n+2}^3+rs^3((n+4)s^2z+nr+3r)z^{n+2}W_{n+1}^3-((n+3)s^2z^2(3r^2+s)+(n+2)r^3z-n-1)z^nW_{n+2}^2W_{n+1}+s^2(-(n+4)s^3z^2+2(n+3)r^3z+n+2)z^{n+1}W_{n+1}^2W_{n+2}-r(2s^2z+r)W_1^3-rs^3(3s^2z+2r)zW_0^3+(2s^2(3r^2+s)z+r^3)W_1^2W_0+s^2(3s^3z^2-4r^3z-1)W_0^2W_1.$

(iii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1) = u(z-a)(z-b)(z-c)^2 = 0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z=a=\frac{1}{2r^4}\left(r+\sqrt{-3r^2}\right),$$
$$z=b=\frac{1}{2r^4}\left(r-\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=0}^{n} z^k W_{k+1}^2 W_k = \frac{\Phi_{31}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}$$

where

$\Phi_{31} = r((n+3)s^2z+nr+2r)z^{n+1}W_{n+2}^3+rs^3((n+4)s^2z+nr+3r)z^{n+2}W_{n+1}^3-((n+3)s^2z^2(3r^2+s)+(n+2)r^3z-n-1)z^nW_{n+2}^2W_{n+1}+s^2(-(n+4)s^3z^2+2(n+3)r^3z+n+2)z^{n+1}W_{n+1}^2W_{n+2}-r(2s^2z+r)W_1^3-rs^3(3s^2z+2r)zW_0^3+(2s^2(3r^2+s)z+r^3)W_1^2W_0+s^2(3s^3z^2-4r^3z-1)W_0^2W_1.$

and if $z=c$, *that is, if*

$$z=c=-\frac{1}{r^3},$$

provided that $r^2+s=0$, *then*

$$\sum_{k=0}^{n} z^k W_{k+1}^2 W_k = \frac{\Phi_{32}}{2s(-6s^5z^2-3rs^2(r^2+2s)z+r^4+3r^2s+2s^2)}$$

where

$\Phi_{32} = (n+2)r((n+3)s^2z+nr+r)z^nW_{n+2}^3+(n+3)rs^3((n+4)s^2z+nr+2r)z^{n+1}W_{n+1}^3-((n+2)(n+3)s^2(3r^2+s)z^2+(n+1)(n+2)r^3z-n^2-n)z^{n-1}W_{n+2}^2W_{n+1}+s^2(-(n+3)(n+4)s^3z^2+2(n+2)(n+3)r^3z+n^2+3n+2)z^nW_{n+1}^2W_{n+2}-2rs^2W_1^3-2rs^3(3zs^2+r)W_0^3+(6r^2s^2+2s^3)W_1^2W_0-2s^2(2r^3-3s^3z)W_0^2W_1.$

(iv) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1) = u(z-a)^4 = 0$ *for some* $u,a\in\mathbb{C}, u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{r}{2s^2}=\frac{8}{r^3}$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=0}^{n} z^k W_{k+1}^2 W_k = \frac{\Phi_{33}}{-24s^6}$$

where

$\Phi_{33} = n(n^2+3n+2)r((n+3)s^2z-r+nr)z^{n-2}W_{n+2}^3+(n^3+6n^2+11n+6)rs^3((n+4)s^2z+nr)z^{n-1}W_{n+1}^3-n(n+1)((n+2)(n+3)s^2(3r^2+s)z^2+(n-1)(n+2)r^3z-n^2+3n-2)z^{n-3}W_{n+2}^2W_{n+1}-s^2(n^2+3n+2)((n+3)(n+4)s^3z^2-2(n+3)nr^3z-n^2+n)z^{n-2}W_{n+1}^2W_{n+2}.$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[14], Theorem 2.1.].

(a) (i), (b) (i) and (c) (i) Using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n$$

i.e.

$$sW_n=W_{n+2}-rW_{n+1}$$

we obtain

$$s^3W_n^3 = W_{n+2}^3 - 3rW_{n+2}^2W_{n+1} + 3r^2W_{n+1}^2W_{n+2} - r^3W_{n+1}^3$$

and so

$$\begin{aligned}
s^3z^nW_n^3 &= z^nW_{n+2}^3 - 3rz^nW_{n+2}^2W_{n+1} + 3r^2z^nW_{n+1}^2W_{n+2} - r^3z^nW_{n+1}^3\\
s^3z^{n-1}W_{n-1}^3 &= z^{n-1}W_{n+1}^3 - 3rz^{n-1}W_{n+1}^2W_n + 3r^2z^{n-1}W_n^2W_{n+1} - r^3z^{n-1}W_n^3\\
s^3z^{n-2}W_{n-2}^3 &= z^{n-2}W_n^3 - 3rz^{n-2}W_n^2W_{n-1} + 3r^2z^{n-2}W_{n-1}^2W_n - r^3z^{n-2}W_{n-1}^3\\
&\vdots\\
s^3z^2W_2^3 &= z^2W_4^3 - 3rz^2W_4^2W_3 + 3r^2z^2W_3^2W_4 - r^3z^2W_3^3\\
s^3z^1W_1^3 &= z^1W_3^3 - 3rz^1W_3^2W_2 + 3r^2z^1W_2^2W_3 - r^3z^1W_2^3\\
s^3z^0W_0^3 &= z^0W_2^3 - 3rz^0W_2^2W_1 + 3r^2z^0W_1^2W_2 - r^3z^0W_1^3
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s^3\sum_{k=0}^{n} z^kW_k^3 &= (z^nW_{n+2}^3 + z^{n-1}W_{n+1}^3 - z^{-1}W_1^3 - z^{-2}W_0^3 + z^{-2}\sum_{k=0}^{n} z^kW_k^3)\\
&\quad -3r(z^nW_{n+2}^2W_{n+1} - z^{-1}W_1^2W_0 + z^{-1}\sum_{k=0}^{n} z^kW_{k+1}^2W_k)\\
&\quad +3r^2(z^nW_{n+1}^2W_{n+2} - z^{-1}W_0^2W_1 + z^{-1}\sum_{k=0}^{n} z^kW_k^2W_{k+1})\\
&\quad -r^3(z^nW_{n+1}^3 - z^{-1}W_0^3 + z^{-1}\sum_{k=0}^{n} z^kW_k^3).
\end{aligned} \tag{15}$$

Next we calculate $\sum_{k=0}^{n} z^kW_{k+1}^2W_k$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$sW_{n+1}^2W_n = W_{n+1}^2W_{n+2} - rW_{n+1}^3$$

and so

$$\begin{aligned}
sz^nW_{n+1}^2W_n &= z^nW_{n+1}^2W_{n+2} - rz^nW_{n+1}^3\\
sz^{n-1}W_n^2W_{n-1} &= z^{n-1}W_n^2W_{n+1} - rz^{n-1}W_n^3\\
sz^{n-2}W_{n-1}^2W_{n-2} &= z^{n-2}W_{n-1}^2W_n - rz^{n-2}W_{n-1}^3\\
&\vdots\\
sz^2W_3^2W_2 &= z^2W_3^2W_4 - rz^2W_3^3\\
sz^1W_2^2W_1 &= z^1W_2^2W_3 - rz^1W_2^3\\
sz^0W_1^2W_0 &= z^0W_1^2W_2 - rz^0W_1^3
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s\sum_{k=0}^{n} z^kW_{k+1}^2W_k &= (z^nW_{n+1}^2W_{n+2} - z^{-1}W_0^2W_1 + z^{-1}\sum_{k=0}^{n} z^kW_k^2W_{k+1})\\
&\quad -r(z^nW_{n+1}^3 - z^{-1}W_0^3 + z^{-1}\sum_{k=0}^{n} z^kW_k^3).
\end{aligned} \tag{16}$$

Next we calculate $\sum_{k=0}^{n} z^kW_k^2W_{k+1}$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1} \Rightarrow s^2W_n^2 = W_{n+2}^2 + r^2W_{n+1}^2 - 2rW_{n+2}W_{n+1}$$

we obtain

$$s^2W_n^2W_{n+1} = W_{n+2}^2W_{n+1} + r^2W_{n+1}^3 - 2rW_{n+1}^2W_{n+2}$$

and so

$$\begin{aligned}
s^2z^nW_n^2W_{n+1} &= z^nW_{n+2}^2W_{n+1} + r^2z^nW_{n+1}^3 - 2rz^nW_{n+1}^2W_{n+2} \\
s^2z^{n-1}W_{n-1}^2W_n &= z^{n-1}W_{n+1}^2W_n + r^2z^{n-1}W_n^3 - 2rz^{n-1}W_n^2W_{n+1} \\
s^2z^{n-2}W_{n-2}^2W_{n-1} &= z^{n-2}W_n^2W_{n-1} + r^2z^{n-2}W_{n-1}^3 - 2rz^{n-2}W_{n-1}^2W_n \\
&\vdots \\
s^2z^2W_2^2W_3 &= z^2W_4^2W_3 + r^2z^2W_3^3 - 2rz^2W_3^2W_4 \\
s^2z^1W_1^2W_2 &= z^1W_3^2W_2 + r^2z^1W_2^3 - 2rz^1W_2^2W_3 \\
s^2z^0W_0^2W_1 &= z^0W_2^2W_1 + r^2z^0W_1^3 - 2rz^0W_1^2W_2
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s^2\sum_{k=0}^{n} z^kW_k^2W_{k+1} &= (z^nW_{n+2}^2W_{n+1} - z^{-1}W_1^2W_0 + z^{-1}\sum_{k=0}^{n} z^kW_{k+1}^2W_k) \\
&\quad + r^2(z^nW_{n+1}^3 - z^{-1}W_0^3 + z^{-1}\sum_{k=0}^{n} z^kW_k^3) \\
&\quad - 2r(z^nW_{n+1}^2W_{n+2} - z^{-1}W_0^2W_1 + z^{-1}\sum_{k=0}^{n} z^kW_k^2W_{k+1}).
\end{aligned} \tag{17}$$

Solving the system (15)-(16)-(17), the required results of (a) (i), (b) (i) and (c) (i) follow.

(a)

(ii) Suppose that $r^2+4s \neq 0$. Then, from Remark 2.5, $a \neq b \neq c \neq d$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (a) (ii) by using

$$\begin{aligned}
\sum_{k=0}^{n} a^kW_k^3 &= \left.\frac{\frac{d}{dz}(\Phi_{19})}{\frac{d}{dz}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a} \\
&= \left.\frac{\Phi_{20}}{-4s^6z^3 - 3rs^3(r^2+2s)z^2 + 2s(r^2+2s)(r^2+s)z + r^3 + 2rs}\right|_{z=a}.
\end{aligned}$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.5, $a \neq b$ and $c=d$. For $z=a$ and $z=b$, as in (a) (ii), we obtain

$$\begin{aligned}
\sum_{k=0}^{n} a^kW_k^3 &= \left.\frac{\frac{d}{dz}(\Phi_{19})}{\frac{d}{dz}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a} \\
&= \left.\frac{\Phi_{21}}{-4s^6z^3 - 3rs^3(r^2+2s)z^2 + 2s(r^2+2s)(r^2+s)z + r^3 + 2rs}\right|_{z=a}
\end{aligned}$$

and

$$\sum_{k=0}^{n} b^kW_k^3 = \left.\frac{\Phi_{21}}{-4s^6z^3 - 3rs^3(r^2+2s)z^2 + 2s(r^2+2s)(r^2+s)z + r^3 + 2rs}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) (iii) by using

$$\begin{aligned}
\sum_{k=0}^{n} c^kW_k^3 &= \left.\frac{\frac{d^2}{dz^2}(\Phi_{19})}{\frac{d^2}{dz^2}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=c} \\
&= \left.\frac{\Phi_{22}}{2s(-6s^5z^2 - 3rs^2(r^2+2s)z + r^4 + 3r^2s + 2s^2)}\right|_{z=c}.
\end{aligned}$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.5, $a=b=c=d$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (a) (iii) by using

$$\sum_{k=0}^{n} a^k W_k^3 = \left.\frac{\frac{d^4}{dz^4}(\Phi_{19})}{\frac{d^4}{dz^4}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a} = \left.\frac{\Phi_{23}}{-24s^6}\right|_{z=a}.$$

(b)

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.5, $a\neq b\neq c\neq d$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) (ii) by using

$$\sum_{k=0}^{n} a^k W_k^2 W_{k+1} = \left.\frac{\frac{d}{dz}(\Phi_{24})}{\frac{d}{dz}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a} = \left.\frac{\Phi_{25}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}\right|_{z=a}.$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.5, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (b) (ii), we obtain

$$\sum_{k=0}^{n} a^k W_k^2 W_{k+1} = \left.\frac{\frac{d}{dz}(\Phi_{24})}{\frac{d}{dz}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a} = \left.\frac{\Phi_{26}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}\right|_{z=a}$$

and

$$\sum_{k=0}^{n} b^k W_k^2 W_{k+1} = \left.\frac{\Phi_{26}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) (iii) by using

$$\sum_{k=0}^{n} c^k W_k^2 W_{k+1} = \left.\frac{\frac{d^2}{dz^2}(\Phi_{24})}{\frac{d^2}{dz^2}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=c} = \left.\frac{\Phi_{217}}{2s(-6s^5z^2-3rs^2(r^2+2s)z+r^4+3r^2s+2s^2)}\right|_{z=c}.$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.5, $a=b=c=d$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (b) (iii) by using

$$\sum_{k=0}^{n} a^k W_k^2 W_{k+1} = \left.\frac{\frac{d^4}{dz^4}(\Phi_{24})}{\frac{d^4}{dz^4}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a} = \left.\frac{\Phi_{28}}{-24s^6}\right|_{z=a}.$$

(c)

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.5, $a\neq b\neq c\neq d$. We use (c) (i). For $z=a$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (c) (ii) by using

$$\sum_{k=0}^{n} a^k W_{k+1}^2 W_k = \left.\frac{\frac{d}{dz}(\Phi_{29})}{\frac{d}{dz}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a} = \left.\frac{\Phi_{30}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}\right|_{z=a}.$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.5, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (c) (ii), we obtain

$$\sum_{k=0}^{n} a^k W_{k+1}^2 W_k = \left.\frac{\frac{d}{dz}(\Phi_{29})}{\frac{d}{dz}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a}$$
$$= \left.\frac{\Phi_{31}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}\right|_{z=a}$$

and

$$\sum_{k=0}^{n} b^k W_{k+1}^2 W_k = \left.\frac{\Phi_{31}}{-4s^6z^3-3rs^3(r^2+2s)z^2+2s(r^2+2s)(r^2+s)z+r^3+2rs}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) (iii) by using

$$\sum_{k=0}^{n} c^k W_{k+1}^2 W_k = \left.\frac{\frac{d^2}{dz^2}(\Phi_{29})}{\frac{d^2}{dz^2}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=c}$$
$$= \left.\frac{\Phi_{32}}{2s(-6s^5z^2-3rs^2(r^2+2s)z+r^4+3r^2s+2s^2)}\right|_{z=c}.$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.5, $a=b=c=d$. We use (c) (i). For $z=a$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) (iii) by using

$$\sum_{k=0}^{n} a^k W_{k+1}^2 W_k = \left.\frac{\frac{d^4}{dz^4}(\Phi_{29})}{\frac{d^4}{dz^4}\left((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\right)}\right|_{z=a}$$
$$= \left.\frac{\Phi_{33}}{-24s^6}\right|_{z=a}. \quad \square$$

2.6. Sum Formulas $\sum_{k=1}^{n} z^k W_{-k}^3$, $\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k}$ and $\sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1}$ of Generalized Fibonacci (Horadam) Polynomials with Negative Subscripts

In the next Theorem, we need the following Remark.

Remark 2.6.
Solving the equation

$$(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=0$$

we find the roots as

$$z_1 = a = \frac{1}{2}s\left(-r+\sqrt{r^2+4s}\right),$$
$$z_2 = b = \frac{1}{2}s\left(-r-\sqrt{r^2+4s}\right),$$
$$z_3 = c = \frac{1}{2}\left(r^3+3rs+(r^2+s)\sqrt{r^2+4s}\right),$$
$$z_4 = d = \frac{1}{2}\left(r^3+3rs-(r^2+s)\sqrt{r^2+4s}\right).$$

Solving $a=b$ i.e.,

$$\frac{1}{2}s\left(-r+\sqrt{r^2+4s}\right) = \frac{1}{2}s\left(-r-\sqrt{r^2+4s}\right),$$

we see that $s=-\frac{1}{4}r^2$. Solving $c=d$, i.e.,

$$\frac{1}{2}\left(r^3+3rs+(r^2+s)\sqrt{r^2+4s}\right) = \frac{1}{2}\left(r^3+3rs-(r^2+s)\sqrt{r^2+4s}\right),$$

we find that $s=-\frac{1}{4}r^2, s=-r^2$. Solving each case of $a=c$, $a=d$, $b=c$, $b=d$, we get $s=-\frac{1}{4}r^2$.

If $r^2+s=0$, i.e., $s=-r^2$ then $a\neq b\neq c$ and $c=d$ and in this case

$$z=a=\frac{1}{2}r^2\left(r-\sqrt{-3r^2}\right),$$
$$z=b=\frac{1}{2}r^2\left(r+\sqrt{-3r^2}\right),$$
$$z=c=d=-r^3.$$

If If $r^2+4s=0$, i.e., $s=-\frac{1}{4}r^2$ then $a=b=c=d=\frac{1}{8}r^3$.

If $r^2+4s\neq 0$ then $a\neq b$ and $c\neq d$ and in this case we must have $a\neq b\neq c\neq d$ because $a\neq c$ otherwise solving the equation

$$a=c$$

i.e.,

$$\frac{1}{2}s\left(-r+\sqrt{r^2+4s}\right)=\frac{1}{2}\left(r^3+3rs+(r^2+s)\sqrt{r^2+4s}\right)$$

we find that $s=-\frac{1}{4}r^2$ which contradicts with the assumption $r^2+4s\neq 0$.

We can list some properties:

- There are some $u,a,b,c,d\in\mathbb{C}$ and $u\neq 0$ and $a\neq b\neq c\neq d$ such that
$$(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)(z-d)=0,$$
i.e., $z=a$ or $z=b$ or $z=c$ or $z=d$.
- There are some $u,a,b,c\in\mathbb{C}$ and $u\neq 0$ and $a\neq b\neq c$ such that
$$(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)^2=0,$$
i.e., $z=a$ or $z=b$ or $z=c$.
- There are no some $u,a\in\mathbb{C}$ with $u,a,b\in\mathbb{C}$ and $u\neq 0$ and $a\neq b$ such that
$$(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)^3(z-b)=0,$$
i.e., $z=a$ or $z=b$.
- There are some $u,a\in\mathbb{C},u\neq 0$ such that
$$(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)^4=0,$$
i.e., $z=a$, that is $z=\frac{1}{8}r^3$.

The following theorem presents some sum formulas of generalized Fibonacci polynomials with negative subscripts.

Theorem 2.6.
Let z be a non-zero complex (or real) number. For $n\geq 1$ we have the following formulas:

(a) (i) *If $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\neq 0$ then*

$$\sum_{k=1}^{n}z^kW_{-k}^3=\frac{\Phi_{34}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}$$

where
$\Phi_{34}=(-z^2+2rsz+s^3)z^{n+1}W_{-n+1}^3+(-z^3+r^3z^2+2rsz^2+s^3z+r^4sz+3r^2s^2z-r^3s^3)z^{n+1}W_{-n}^3-3rs(rz+s^2)z^{n+1}W_{-n+1}^2W_{-n}+3rs^2(-z+rs)z^{n+1}W_{-n}^2W_{-n+1}-z(-z^2+s^3+2rsz)W_1^3+z(z^3-2rsz^2-r^3z^2-r^4sz-3r^2s^2z-s^3z+r^3s^3)W_0^3+3rsz(rz+s^2)W_1^2W_0-3rs^2z(-z+rs)W_0^2W_1.$

(ii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)(z-d)=0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *that is, if*

$$z=a=\frac{1}{2}s\left(-r+\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}s\left(-r-\sqrt{r^2+4s}\right),$$
$$z=c=\frac{1}{2}\left(r^3+3rs+(r^2+s)\sqrt{r^2+4s}\right),$$
$$z=d=\frac{1}{2}\left(r^3+3rs-(r^2+s)\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=1}^{n}z^kW_{-k}^3=\frac{\Phi_{35}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}$$

where

$\Phi_{35}=(-(n+3)z^2+2(n+2)rsz+ns^3+s^3)z^nW_{-n+1}^3+(-(n+4)z^3+r(n+3)(2s+r^2)z^2+s(n+2)(3r^2s+r^4+s^2)z-r^3s^3-nr^3s^3)z^nW_{-n}^3-3rs(n(rz+s^2)+2rz+s^2)z^nW_{-n+1}^2W_{-n}-3rs^2((n+2)z-rs-nrs)z^nW_{-n}^2W_{-n+1}-(-3z^2+4rsz+s^3)W_1^3+(4z^3-3r(r^2+2s)z^2-2s(r^4+3r^2s+s^2)z+r^3s^3)W_0^3+3rs(s^2+2rz)W_1^2W_0+3rs^2(2z-rs)W_0^2W_1.$

(iii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)^2=0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z=a=\frac{1}{2}r^2\left(r-\sqrt{-3r^2}\right),$$
$$z=b=\frac{1}{2}r^2\left(r+\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n}z^kW_{-k}^3=\frac{\Phi_{36}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}$$

where

$\Phi_{36}=(-(n+3)z^2+2(n+2)rsz+ns^3+s^3)z^nW_{-n+1}^3+(-(n+4)z^3+r(n+3)(2s+r^2)z^2+s(n+2)(3r^2s+r^4+s^2)z-r^3s^3-nr^3s^3)z^nW_{-n}^3-3rs(n(rz+s^2)+2rz+s^2)z^nW_{-n+1}^2W_{-n}-3rs^2((n+2)z-rs-nrs)z^nW_{-n}^2W_{-n+1}-(-3z^2+4rsz+s^3)W_1^3+(4z^3-3r(r^2+2s)z^2-2s(r^4+3r^2s+s^2)z+r^3s^3)W_0^3+3rs(s^2+2rz)W_1^2W_0+3rs^2(2z-rs)W_0^2W_1$
and if $z=c$, *that is, if*

$$z=c=-r^3,$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n}z^kW_{-k}^3=\frac{\Phi_{37}}{2(-6z^2+6rsz+3r^3z+r^4s+3r^2s^2+2s^3)}$$

where

$\Phi_{37}=(-(n+2)(n+3)z^2+2(n+1)(n+2)rsz+n^2s^3+ns^3)z^{n-1}W_{-n+1}^3+(-(n+3)(n+4)z^3+r(n+2)(n+3)(2s+r^2)z^2+(n+1)(n+2)s(3r^2s+r^4+s^2)z-n^2r^3s^3-nr^3s^3)z^{n-1}W_{-n}^3-3(n+1)rs(r(n+2)z+ns^2)z^{n-1}W_{-n+1}^2W_{-n}-3(n+1)rs^2((n+2)z-nrs)z^{n-1}W_{-n}^2W_{-n+1}+(6z-4rs)W_1^3+2(6z^2-3rz(r^2+2s)-3r^2s^2-r^4s-s^3)W_0^3+6r^2sW_1^2W_0+6rs^2W_0^2W_1.$

(iv) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)^4=0$ *for some* $u,a\in\mathbb{C}$, $u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{1}{8}r^3$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=1}^{n}z^kW_{-k}^3=\frac{\Phi_{38}}{-24}$$

where

$\Phi_{38}=-n(n+1)((n+2)(n+3)z^2-2(n-1)(n+2)rsz-(n-2)(n-1)s^3)z^{n-3}W_{-n+1}^3+(n+1)(-(n+2)(n+3)(n+4)z^3+n(n+2)(n+3)r(2s+r^2)z^2+(n-1)n(n+2)s(3r^2s+r^4+s^2)z-(n-2)(n-1)nr^3s^3)z^{n-3}W_{-n}^3-3n(n^2-1)rs(r(n+2)z+ns^2-2s^2)z^{n-3}W_{-n+1}^2W_{-n}-3n(n^2-1)rs^2((n+2)z+2rs-nrs)z^{n-3}W_{-n}^2W_{-n+1}+24W_0^3.$

(b) **(i)** *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\neq 0$ *then*

$$\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k} = \frac{\Phi_{39}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}$$

where

$\Phi_{39} = -r(rz+s^2)z^{n+2}W_{-n+1}^3 - rs^3(rz+s^2)z^{n+1}W_{-n}^3 + (-z^2+r^3z+3r^2s^2+s^3)z^{n+2}W_{-n+1}^2 W_{-n} + s^2(-z^2-2r^3z+s^3)z^{n+1}W_{-n}^2 W_{-n+1} + rz^2(rz+s^2)W_1^3 + rs^3z(rz+s^2)W_0^3 - z^2(-z^2+r^3z+3r^2s^2+s^3)W_1^2 W_0 + s^2z(z^2+2r^3z-s^3)W_0^2 W_1.$

(ii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3) = u(z-a)(z-b)(z-c)(z-d) = 0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *that is, if*

$$\begin{aligned} z &= a = \frac{1}{2}s\left(-r+\sqrt{r^2+4s}\right),\\ z &= b = \frac{1}{2}s\left(-r-\sqrt{r^2+4s}\right),\\ z &= c = \frac{1}{2}\left(r^3+3rs+(r^2+s)\sqrt{r^2+4s}\right),\\ z &= d = \frac{1}{2}\left(r^3+3rs-(r^2+s)\sqrt{r^2+4s}\right), \end{aligned}$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k} = \frac{\Phi_{40}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}$$

where

$\Phi_{40} = -r((n+3)rz+ns^2+2s^2)z^{n+1}W_{-n+1}^3 - rs^3((n+2)rz+ns^2+s^2)z^nW_{-n}^3 + (-(n+4)z^2+(n+3)r^3z+(n+2)s^2(s+3r^2))z^{n+1}W_{-n+1}^2W_{-n} - s^2((n+3)z^2+2(n+2)r^3z-ns^3-s^3)z^nW_{-n}^2W_{-n+1} + r(2s^2+3rz)zW_1^3 + rs^3(s^2+2rz)W_0^3 - (3r^3z+6r^2s^2+2s^3-4z^2)zW_1^2W_0 + s^2(4r^3z-s^3+3z^2)W_0^2W_1.$

(iii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3) = u(z-a)(z-b)(z-c)^2 = 0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$\begin{aligned} z &= a = \frac{1}{2}r^2\left(r-\sqrt{-3r^2}\right),\\ z &= b = \frac{1}{2}r^2\left(r+\sqrt{-3r^2}\right), \end{aligned}$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k} = \frac{\Phi_{41}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}$$

where

$\Phi_{41} = -r((n+3)rz+ns^2+2s^2)z^{n+1}W_{-n+1}^3 - rs^3((n+2)rz+ns^2+s^2)z^nW_{-n}^3 + (-(n+4)z^2+(n+3)r^3z+(n+2)s^2(s+3r^2))z^{n+1}W_{-n+1}^2W_{-n} - s^2((n+3)z^2+2(n+2)r^3z-ns^3-s^3)z^nW_{-n}^2W_{-n+1} + r(2s^2+3rz)zW_1^3 + rs^3(s^2+2rz)W_0^3 - (3r^3z+6r^2s^2+2s^3-4z^2)zW_1^2W_0 + s^2(4r^3z-s^3+3z^2)W_0^2W_1$

and if $z=c$, *that is, if*

$$z = c = -r^3,$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k} = \frac{\Phi_{42}}{2(-6z^2+6rsz+3r^3z+r^4s+3r^2s^2+2s^3)}$$

where

$\Phi_{42} = -(n+2)r((n+3)rz+ns^2+s^2)z^nW_{-n+1}^3 - (n+1)rs^3((n+2)rz+ns^2)z^{n-1}W_{-n}^3 + (-(n+3)(n+4)z^2+(n+2)(n+3)r^3z+(n+1)(n+2)s^2(s+3r^2))z^nW_{-n+1}^2W_{-n} - s^2((n+2)(n+3)z^2+2(n+1)(n+2)r^3z-n(n+1)s^3)z^{n-1}W_{-n}^2W_{-n+1} + 2r(s^2+3rz)W_1^3 + 2r^2s^3W_0^3 - 2(-6z^2+3r^3z+3r^2s^2+s^3)W_1^2W_0 + 2s^2(2r^3+3z)W_0^2W_1.$

(iv) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)^4=0$ *for some* $u,a\in\mathbb{C}, u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{1}{8}r^3$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k}=\frac{\Phi_{43}}{-24}$$

where

$\Phi_{43}=-nr(n^2+3n+2)(r(n+3)z+(n-1)s^2)z^{n-2}W_{-n+1}^3-n(n^2-1)rs^3(r(n+2)z+(n-2)s^2)z^{n-3}W_{-n}^3-(n^2+3n+2)((n+3)(n+4)z^2-n(n+3)r^3z-(n-1)ns^2(s+3r^2))z^{n-2}W_{-n+1}^2W_{-n}-n(n+1)s^2((n+2)(n+3)z^2+2(n-1)(n+2)r^3z-(n-2)(n-1)s^3)z^{n-3}W_{-n}^2W_{-n+1}+24W_1^2W_0.$

(c) **(i)** *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\neq 0$ *then*

$$\sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1}=\frac{\Phi_{44}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}$$

where

$\Phi_{44}=r(-z+rs)z^{n+2}W_{-n+1}^3+rs^3(-z+rs)z^{n+1}W_{-n}^3+s(-z^2-2r^3z+s^3)z^{n+1}W_{-n+1}^2W_{-n}+(-z^3+2rsz^2+r^3z^2+s^3z+r^4sz-2rs^4)z^{n+1}W_{-n}^2W_{-n+1}+rz^2(z-rs)W_1^3+rs^3z(z-rs)W_0^3+sz(z^2+2r^3z-s^3)W_1^2W_0+z(z^3-2rsz^2-r^3z^2-r^4sz-s^3z+2rs^4)W_0^2W_1.$

(ii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)(z-d)=0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *that is, if*

$$z=a=\frac{1}{2}s\left(-r+\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}s\left(-r-\sqrt{r^2+4s}\right),$$
$$z=c=\frac{1}{2}\left(r^3+3rs+(r^2+s)\sqrt{r^2+4s}\right),$$
$$z=d=\frac{1}{2}\left(r^3+3rs-(r^2+s)\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1}=\frac{\Phi_{45}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}$$

where

$\Phi_{45}=-r((n+3)z-(n+2)rs)z^{n+1}W_{-n+1}^3-rs^3((n+2)z-(n+1)rs)z^nW_{-n}^3-s((n+3)z^2+2(n+2)r^3z-(n+1)s^3)z^nW_{-n+1}^2W_{-n}+(-(n+4)z^3+(n+3)r(2s+r^2)z^2+(n+2)s(r^4+s^2)z-2(n+1)rs^4)z^nW_{-n}^2W_{-n+1}+r(3z^2-2rsz)W_1^3+rs^3(2z-rs)W_0^3+s(3z^2+4r^3z-s^3)W_1^2W_0+(4z^3-3r(2s+r^2)z^2-2s(r^4+s^2)z+2rs^4)W_0^2W_1.$

(iii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)^2=0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z=a=\frac{1}{2}r^2\left(r-\sqrt{-3r^2}\right),$$
$$z=b=\frac{1}{2}r^2\left(r+\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1}=\frac{\Phi_{46}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}$$

where

$\Phi_{46}=-r((n+3)z-(n+2)rs)z^{n+1}W_{-n+1}^3-rs^3((n+2)z-(n+1)rs)z^nW_{-n}^3-s((n+3)z^2+2(n+2)r^3z-(n+1)s^3)z^nW_{-n+1}^2W_{-n}+(-(n+4)z^3+(n+3)r(2s+r^2)z^2+(n+2)s(r^4+s^2)z-2(n+1)rs^4)z^nW_{-n}^2W_{-n+1}+r(3z^2-2rsz)W_1^3+rs^3(2z-rs)W_0^3+s(3z^2+4r^3z-s^3)W_1^2W_0+(4z^3-3r(2s+r^2)z^2-2s(r^4+s^2)z+2rs^4)W_0^2W_1$

and if $z=c$, *that is, if*

$$z=c=-r^3,$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1} = \frac{\Phi_{47}}{2(-6z^2+6rsz+3r^3z+r^4s+3r^2s^2+2s^3)}$$

where

$\Phi_{47} = -r(n+2)((n+3)z-(n+1)rs)z^n W_{-n+1}^3 - (n+1)rs^3(2z+nz-nrs)z^{n-1}W_{-n}^3 - s((n+2)(n+3)z^2+2(n+1)(n+2)r^3z-n(n+1)s^3)z^{n-1}W_{-n+1}^2 W_{-n} + (-(n+3)(n+4)z^3+(n+2)(n+3)(2s+r^2)rz^2+(n+2)(n+1)(r^4+s^2)sz-2n(n+1)rs^4)z^{n-1}W_{-n}^2 W_{-n+1} + 2r(3z-rs)W_1^3 + 2rs^3W_0^3 + 2s(2r^3+3z)W_1^2 W_0 + (12z^2-6r(2s+r^2)z-2s(r^4+s^2))W_0^2 W_1$

(iv) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3) = u(z-a)^4 = 0$ *for some* $u, a \in \mathbb{C}, u \neq 0$ *i.e.,* $z=a$ *that is, if*

$$z = a = \frac{1}{8}r^3$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1} = \frac{\Phi_{48}}{-24}$$

where

$\Phi_{48} = -nr(n^2+3n+2)((n+3)z-(n-1)rs)z^{n-2}W_{-n+1}^3 - n(n^2-1)rs^3((n+2)z-(n-2)rs)z^{n-3}W_{-n}^3 - n(n+1)s((n+2)(n+3)z^2+2(n-1)(n+2)r^3z-(n-2)(n-1)s^3)z^{n-3}W_{-n+1}^2 W_{-n} + (n+1)(-(n+2)(n+3)(n+4)z^3 + n(n+2)(n+3)(2s+r^2)rz^2 + (n-1)n(n+2)(r^4+s^2)sz - 2(n-2)(n-1)nrs^4)z^{n-3}W_{-n}^2 W_{-n+1} + 24W_0^2 W_1.$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[14], Theorem 3.1.].

(a) (i), (b) (i) and (c) (i) Using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$s^3W_{-n}^3 = W_{-n+2}^3 - 3rW_{-n+2}^2W_{-n+1} + 3r^2W_{-n+1}^2W_{-n+2} - r^3W_{-n+1}^3$$

and so

$$\begin{aligned}
s^3z^nW_{-n}^3 &= z^nW_{-n+2}^3 - 3rz^nW_{-n+2}^2W_{-n+1} + 3r^2z^nW_{-n+1}^2W_{-n+2} - r^3z^nW_{-n+1}^3 \\
s^3z^{n-1}W_{-n+1}^3 &= z^{n-1}W_{-n+3}^3 - 3rz^{n-1}W_{-n+3}^2W_{-n+2} + 3r^2z^{n-1}W_{-n+2}^2W_{-n+3} - r^3z^{n-1}W_{-n+2}^3 \\
s^3z^{n-2}W_{-n+2}^3 &= z^{n-2}W_{-n+4}^3 - 3rz^{n-2}W_{-n+4}^2W_{-n+3} + 3r^2z^{n-2}W_{-n+3}^2W_{-n+4} - r^3z^{n-2}W_{-n+3}^3 \\
&\vdots \\
s^3z^3W_{-3}^3 &= z^3W_{-1}^3 - 3rz^3W_{-1}^2W_{-2} + 3r^2z^3W_{-2}^2W_{-1} - r^3z^3W_{-2}^3 \\
s^3z^2W_{-2}^3 &= z^2W_0^3 - 3rz^2W_0^2W_{-1} + 3r^2z^2W_{-1}^2W_0 - r^3z^2W_{-1}^3 \\
s^3z^1W_{-1}^3 &= z^1W_1^3 - 3rz^1W_1^2W_0 + 3r^2z^1W_0^2W_1 - r^3z^1W_0^3
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s^3\Big(\sum_{k=1}^{n} z^kW_{-k}^3\Big) &= \Big(-z^{n+1}W_{-n+1}^3 - z^{n+2}W_{-n}^3 + z^1W_1^3 + z^2W_0^3 + z^2\sum_{k=1}^{n} z^kW_{-k}^3\Big) \\
&\quad -3r\Big(-z^{n+1}W_{-n+1}^2W_{-n} + z^1W_1^2W_0 + z^1\sum_{k=1}^{n} z^kW_{-k+1}^2W_{-k}\Big) \\
&\quad +3r^2\Big(-z^{n+1}W_{-n}^2W_{-n+1} + z^1W_0^2W_1 + z^1\sum_{k=1}^{n} z^kW_{-k}^2W_{-k+1}\Big) \\
&\quad -r^3\Big(-z^{n+1}W_{-n}^3 + z^1W_0^3 + z^1\sum_{k=1}^{n} z^kW_{-k}^3\Big).
\end{aligned} \tag{18}$$

Next we calculate $\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$sW_{-n+1}^2 W_{-n} = W_{-n+1}^2 W_{-n+2} - rW_{-n+1}^3$$

and so

$$\begin{aligned}
sz^n W_{-n+1}^2 W_{-n} &= z^n W_{-n+1}^2 W_{-n+2} - rz^n W_{-n+1}^3 \\
sz^{n-1} W_{-n+2}^2 W_{-n+1} &= z^{n-1} W_{-n+2}^2 W_{-n+3} - rz^{n-1} W_{-n+2}^3 \\
sz^{n-2} W_{-n+3}^2 W_{-n+2} &= z^{n-2} W_{-n+3}^2 W_{-n+4} - rz^{n-2} W_{-n+3}^3 \\
&\vdots \\
sz^3 W_{-2}^2 W_{-3} &= z^3 W_{-2}^2 W_{-1} - rz^3 W_{-2}^3 \\
sz^2 W_{-1}^2 W_{-2} &= z^2 W_{-1}^2 W_0 - rz^2 W_{-1}^3 \\
sz^1 W_0^2 W_{-1} &= z^1 W_0^2 W_1 - rz^1 W_0^3
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k} = {} & (-z^{n+1} W_{-n}^2 W_{-n+1} + z^1 W_0^2 W_1 + z^1 \sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1}) \\
& -r(-z^{n+1} W_{-n}^3 + z^1 W_0^3 + z^1 \sum_{k=1}^{n} z^k W_{-k}^3).
\end{aligned} \tag{19}$$

Next we calculate $\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$\begin{aligned}
s^2 W_{-n}^2 &= W_{-n+2}^2 - 2rW_{-n+2}W_{-n+1} + r^2 W_{-n+1}^2 \\
&\Rightarrow s^2 W_{-n}^2 W_{-n+1} = W_{-n+2}^2 W_{-n+1} - 2rW_{-n+1}^2 W_{-n+2} + r^2 W_{-n+1}^3
\end{aligned}$$

and so

$$\begin{aligned}
s^2 z^n W_{-n}^2 W_{-n+1} &= z^n W_{-n+2}^2 W_{-n+1} - 2rz^n W_{-n+1}^2 W_{-n+2} + r^2 z^n W_{-n+1}^3 \\
s^2 z^{n-1} W_{-n+1}^2 W_{-n+2} &= z^{n-1} W_{-n+3}^2 W_{-n+2} - 2rz^{n-1} W_{-n+2}^2 W_{-n+3} + r^2 z^{n-1} W_{-n+2}^3 \\
s^2 z^{n-2} W_{-n+2}^2 W_{-n+3} &= z^{n-2} W_{-n+4}^2 W_{-n+3} - 2rz^{n-2} W_{-n+3}^2 W_{-n+4} + r^2 z^{n-2} W_{-n+3}^3 \\
&\vdots \\
s^2 z^3 W_{-3}^2 W_{-2} &= z^3 W_{-1}^2 W_{-2} - 2rz^3 W_{-2}^2 W_{-1} + r^2 z^3 W_{-2}^3 \\
s^2 z^2 W_{-2}^2 W_{-1} &= z^2 W_0^2 W_{-1} - 2rz^2 W_{-1}^2 W_0 + r^2 z^2 W_{-1}^3 \\
s^2 z^1 W_{-1}^2 W_0 &= z^1 W_1^2 W_0 - 2rz^1 W_0^2 W_1 + r^2 z^1 W_0^3
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s^2 \sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1} = {} & (-z^{n+1} W_{-n+1}^2 W_{-n} + z^1 W_1^2 W_0 + z^1 \sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k}) \\
& -2r(-z^{n+1} W_{-n}^2 W_{-n+1} + z^1 W_0^2 W_1 + z^1 \sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1}) \\
& +r^2(-z^{n+1} W_{-n}^3 + z^1 W_0^3 + z^1 \sum_{k=1}^{n} z^k W_{-k}^3).
\end{aligned} \tag{20}$$

Then, solving the system (18)-(19)-(20), the required results of (a) (i), (b) (i) and (c) (i) follow.

(a)

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.6, $a\neq b\neq c\neq d$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (a) (ii) by using

$$\begin{aligned}\sum_{k=1}^{n}a^kW_{-k}^3 &= \left.\frac{\frac{d}{dz}(\Phi_{34})}{\frac{d}{dz}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{29}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=a}.\end{aligned}$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.6, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (a) (ii), we obtain

$$\begin{aligned}\sum_{k=1}^{n}a^kW_{-k}^3 &= \left.\frac{\frac{d}{dz}(\Phi_{35})}{\frac{d}{dz}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{36}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=a}\end{aligned}$$

and

$$\sum_{k=1}^{n}b^kW_{-k}^3 = \left.\frac{\Phi_{36}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) (iii) by using

$$\begin{aligned}\sum_{k=1}^{n}c^kW_{-k}^3 &= \left.\frac{\frac{d^2}{dz^2}(\Phi_{35})}{\frac{d^2}{dz^2}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=c}\\ &= \left.\frac{\Phi_{37}}{2(-6z^2+6rsz+3r^3z+r^4s+3r^2s^2+2s^3)}\right|_{z=c}.\end{aligned}$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.6, $a=b=c=d$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (a) (iii) by using

$$\begin{aligned}\sum_{k=1}^{n}a^kW_{-k}^3 &= \left.\frac{\frac{d^4}{dz^4}(\Phi_{35})}{\frac{d^4}{dz^4}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{38}}{-24}\right|_{z=a}.\end{aligned}$$

(b)

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.6, $a\neq b\neq c\neq d$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) (ii) by using

$$\begin{aligned}\sum_{k=1}^{n}a^kW_{-k+1}^2W_{-k} &= \left.\frac{\frac{d}{dz}(\Phi_{39})}{\frac{d}{dz}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{40}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=a}.\end{aligned}$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.6, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (b) (ii), we obtain

$$\begin{aligned}\sum_{k=1}^{n}a^kW_{-k+1}^2W_{-k} &= \left.\frac{\frac{d}{dz}(\Phi_{39})}{\frac{d}{dz}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{41}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=a}\end{aligned}$$

and

$$\sum_{k=1}^{n} b^k W_{-k+1}^2 W_{-k} = \left.\frac{\Phi_{41}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) (iii) by using

$$\begin{aligned}\sum_{k=1}^{n} c^k W_{-k+1}^2 W_{-k} &= \left.\frac{\frac{d^2}{dz^2}(\Phi_{39})}{\frac{d^2}{dz^2}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=c}\\ &= \left.\frac{\Phi_{41}}{2(-6z^2+6rsz+3r^3z+r^4s+3r^2s^2+2s^3)}\right|_{z=c}.\end{aligned}$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.6, $a=b=c=d$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (b) (iii) by using

$$\begin{aligned}\sum_{k=1}^{n} c^k W_{-k+1}^2 W_{-k} &= \left.\frac{\frac{d^4}{dz^4}(\Phi_{39})}{\frac{d^4}{dz^4}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{43}}{-24}\right|_{z=a}.\end{aligned}$$

(c)

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.6, $a\neq b\neq c\neq d$. We use (c) (i). For $z=a$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (c) (ii) by using

$$\begin{aligned}\sum_{k=1}^{n} a^k W_{-k}^2 W_{-k+1} &= \left.\frac{\frac{d}{dz}(\Phi_{44})}{\frac{d}{dz}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{45}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=a}.\end{aligned}$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.6, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (c) (ii), we obtain

$$\begin{aligned}\sum_{k=1}^{n} a^k W_{-k}^2 W_{-k+1} &= \left.\frac{\frac{d}{dz}(\Phi_{44})}{\frac{d}{dz}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{46}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=a}\end{aligned}$$

and

$$\sum_{k=1}^{n} b^k W_{-k}^2 W_{-k+1} = \left.\frac{\Phi_{46}}{-4z^3+6rsz^2+3r^3z^2+4s^3z+2r^4sz+6r^2s^2z-2rs^4-r^3s^3}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) (iii) by using

$$\begin{aligned}\sum_{k=1}^{n} c^k W_{-k}^2 W_{-k+1} &= \left.\frac{\frac{d^2}{dz^2}(\Phi_{44})}{\frac{d^2}{dz^2}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=c}\\ &= \left.\frac{\Phi_{47}}{2(-6z^2+6rsz+3r^3z+r^4s+3r^2s^2+2s^3)}\right|_{z=c}.\end{aligned}$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.6, $a=b=c=d$. We use (c) (i). For $z=a$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) (iii) by using

$$\begin{aligned}\sum_{k=1}^{n} a^k W_{-k}^2 W_{-k+1} &= \left.\frac{\frac{d^4}{dz^4}(\Phi_{44})}{\frac{d^4}{dz^4}\left((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\right)}\right|_{z=a}\\ &= \left.\frac{\Phi_{48}}{-24}\right|_{z=a}. \qquad \square\end{aligned}$$

3. Sum Formulas 3: The case that r and s are arbitrary

In this section, we present the sum formulas $\sum_{k=0}^{n} kz^k W_k$, $\sum_{k=0}^{n} kz^k W_k^2$, $\sum_{k=0}^{n} kz^k W_k^3$ and $\sum_{k=1}^{n} kz^k W_{-k}$, $\sum_{k=1}^{n} kz^k W_{-k}^2$, $\sum_{k=1}^{n} kz^k W_{-k}^3$ of generalized Fibonacci polynomials with positive subscripts and negative subscripts.

3.1. Sum Formulas $\sum_{k=0}^{n} kz^k W_k$, $\sum_{k=0}^{n} kz^k W_{2k}$ and $\sum_{k=0}^{n} kz^k W_{2k+1}$ of Generalized Fibonacci (Horadam) Polynomials with Positive Subscripts

The following theorem presents some sum formulas of generalized Fibonacci polynomials with positive subscripts.

Theorem 3.1.
Let z be a non-zero complex (or real) number. For $n \geq 0$ we have the following formulas:

(a)

(i) *If $sz^2 + rz - 1 \neq 0$, then*

$$\sum_{k=0}^{n} kz^k W_k = \frac{z(sz^2+rz-1)\frac{d}{dz}\Phi_1 - z(r+2sz)\Phi_1}{(sz^2+rz-1)^2} = \frac{\Lambda_1}{(sz^2+rz-1)^2}$$

where
$\Phi_1 = z^{n+2}W_{n+2} + z^{n+1}(1-rz)W_{n+1} - zW_1 + (rz-1)W_0$ *given in Theorem 2.1 (a) (i)*
and
$\frac{d}{dz}\Phi_1 = \Phi_1'$ *denotes the derivatives of* Φ_1 *with respect to* z
and
$\Lambda_1 = (n(sz^2+rz-1)+rz-2)z^{n+2}W_{n+2} - (n(rz-1)(sz^2+rz-1)+sz^2+r^2z^2-2rz+1)z^{n+1}W_{n+1} + z\left(sz^2+1\right)W_1 - s(rz-2)z^2W_0.$

(ii) *If $sz^2+rz-1 = u(z-a)(z-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $z = a$ or $z = b$, that is, if*

$$z = a = \frac{1}{2s}\left(-r+\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2s}\left(-r-\sqrt{r^2+4s}\right),$$

provided that $r^2+4s \neq 0$, then

$$\sum_{k=0}^{n} kz^k W_k = \frac{\Lambda_2}{2(-2s+6s^2z^2+r^2+6rsz)}$$

where
$\Lambda_2 = (n^3(sz^2+rz-1)+n^2(7sz^2+6rz-5)+n(12sz^2+11rz-8)+2(3rz-2))z^nW_{n+2} - (n^3(rz-1)(sz^2+rz-1)+n^2(-4sz^2+6r^2z^2-8rz+7rsz^3+2)+n(-sz^2+11r^2z^2-10rz+12rsz^3+1)+2z(-2r+3r^2z+3sz))z^{n-1}W_{n+1} + 6szW_1 - 2s(3rz-2)W_0.$

(iii) *If $sz^2+rz-1 = u(z-a)^2 = 0$ for some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $z = a$, that is, if*

$$z = -\frac{r}{2s}$$

provided that $r^2+4s = 0$, then

$$\sum_{k=0}^{n} kz^k W_k = \frac{\Lambda_3}{24s^2}$$

where
$\Lambda_3 = n(n^2+3n+2)(n^2(sz^2+rz-1)+n(7sz^2+4rz-1)+12sz^2+3rz+2)z^{n-2}W_{n+2} - n(n+1)(n^3(rz-1)(sz^2+rz-1)+n^2(-4sz^2+6r^2z^2-4rz+9rsz^3-2)+n(-sz^2+11r^2z^2+2rz+26rsz^3-1)+24srz^3+6r^2z^2+6sz^2+4rz+2)z^{n-3}W_{n+1}.$

(b)

(i) *If* $r^2z-s^2z^2+2sz-1\neq 0$ *then*

$$\sum_{k=0}^{n} kz^kW_{2k}=\frac{z(r^2z-s^2z^2+2sz-1)\frac{d}{dz}\Phi_2-z(r^2-2zs^2+2s)\Phi_2}{(r^2z-s^2z^2+2sz-1)^2}=\frac{\Lambda_4}{(r^2z-s^2z^2+2sz-1)^2}$$

where

$\Phi_2=-z^{n+1}(sz-1)W_{2n+2}+rsz^{n+2}W_{2n+1}-rzW_1+(r^2z+sz-1)W_0$ *given in Theorem 2.1 (b) (i)*

and

$\frac{d}{dz}\Phi_2=\Phi_2'$ *denotes the derivatives of* Φ_2 *with respect to* z

and

$\Lambda_4=-(n(sz-1)(r^2z-s^2z^2+2sz-1)+s^2z^2-2sz+r^2sz^2+1)z^{n+1}W_{2n+2}+rs(n(r^2z-s^2z^2+2sz-1)+r^2z+2sz-2)z^{n+2}W_{2n+1}-rz(sz-1)(sz+1)W_1+sz(s^2z^2-2sz+r^2sz^2+1)W_0.$

(ii) *If* $r^2z-s^2z^2+2sz-1=u(z-a)(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$*, that is, if*

$$z=a=\frac{1}{2s^2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2s^2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$*, then*

$$\sum_{k=0}^{n} kz^kW_{2k}=\frac{\Lambda_5}{2(6s^4z^2-6r^2s^2z-12s^3z+r^4+4r^2s+6s^2)}$$

where

$\Lambda_5=-(n^3(sz-1)(-s^2z^2+r^2z+2sz-1)+n^2(-7s^3z^3+16s^2z^2+6r^2sz^2-3r^2z-11sz+2)+n(-12s^3z^3+23s^2z^2+11r^2sz^2-2r^2z-12sz+1)+2sz(3r^2z+3sz-2))z^{n-1}W_{2n+2}+rs(n^3(-s^2z^2+r^2z+2sz-1)+n^2(-7s^2z^2+6r^2z+12sz-5)+n(-12s^2z^2+11r^2z+22sz-8)+2(3r^2z+6sz-2))z^nW_{2n+1}-6rs^2zW_1+2s^2(3zr^2+3sz-2)W_0.$

(iii) *If* $r^2z-s^2z^2+2sz-1=u(z-a)^2=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, that is, if*

$$z=\frac{4}{r^2}$$

provided that $r^2+4s=0$*, then*

$$\sum_{k=0}^{n} kz^kW_{2k}=\frac{\Lambda_6}{24s^4}$$

where $\Lambda_6=-n(n+1)(n^3(sz-1)(-s^2z^2+r^2z+2sz-1)+n^2(-9s^3z^3+16s^2z^2+6r^2sz^2-5sz-r^2z-2)+n(-26s^3z^3+23s^2z^2+11r^2sz^2+2r^2z+4sz-1)+2sz(3r^2z-12s^2z^2+3sz+2)+2)z^{n-3}W_{2n+2}+nrs(n^2+3n+2)(n^2(-s^2z^2+r^2z+2sz-1)+n(-7s^2z^2+4r^2z+8sz-1)+3z(-4s^2z+2s+r^2)+2)z^{n-2}W_{2n+1}.$

(c)

(i) *If* $r^2z-s^2z^2+2sz-1\neq 0$ *then*

$$\sum_{k=0}^{n} kz^kW_{2k+1}=\frac{z(r^2z-s^2z^2+2sz-1)\frac{d}{dz}\Phi_3-z(r^2-2zs^2+2s)\Phi_3}{(r^2z-s^2z^2+2sz-1)^2}=\frac{\Lambda_7}{(r^2z-s^2z^2+2sz-1)^2}$$

where

$\Phi_3=rz^{n+1}W_{2n+2}-sz^{n+1}(sz-1)W_{2n+1}+(sz-1)W_1-rszW_0$ *given in Theorem 2.1 (c) (i)*

and

$\frac{d}{dz}\Phi_3=\Phi_3'$ *denotes the derivatives of* Φ_3 *with respect to* z

and

$\Lambda_7=r(n(-s^2z^2+r^2z+2sz-1)+s^2z^2-1)z^{n+1}W_{2n+2}-s(n(sz-1)(-s^2z^2+r^2z+2sz-1)+sz(r^2z+sz-2)+1)z^{n+1}W_{2n+1}+z(s-2s^2z+s^3z^2+r^2)W_1-rsz(sz-1)(sz+1)W_0.$

(ii) *If* $r^2z-s^2z^2+2sz-1=u(z-a)(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$, *that is, if*

$$z=a=\frac{1}{2s^2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2s^2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=0}^{n}kz^kW_{2k+1}=\frac{\Lambda_8}{2(6s^4z^2-6r^2s^2z-12s^3z+r^4+4r^2s+6s^2)}$$

where

$\Lambda_8=r(n^3(-s^2z^2+r^2z+2sz-1)+n^2(-4s^2z^2+3r^2z+6sz-2)+n(-s^2z^2+2r^2z+4sz-1)+6s^2z^2)z^{n-1}W_{2n+2}-s(n^3(sz-1)(-s^2z^2+r^2z+2sz-1)+n^2(-7s^3z^3+16s^2z^2+6r^2sz^2-3r^2z-11sz+2)+n(-12s^3z^3+23s^2z^2+11r^2sz^2-2r^2z-12sz+1)+2sz(3r^2z+3sz-2))z^{n-1}W_{2n+1}+2s^2(3sz-2)W_1-6rs^3zW_0.$

(iii) *If* $r^2z-s^2z^2+2sz-1=u(z-a)^2=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *that is, if*

$$z=\frac{4}{r^2}$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=0}^{n}kz^kW_{2k+1}=\frac{\Lambda_9}{24s^4}$$

where

$\Lambda_9=nr(n^2-1)(n^2(-s^2z^2+r^2z+2sz-1)+n(-5s^2z^2+2r^2z+4sz+1)-6s^2z^2+2)z^{n-3}W_{2n+2}-ns(n+1)(n^3(sz-1)(-s^2z^2+r^2z+2sz-1)+n^2(-9s^3z^3+16s^2z^2+6r^2sz^2-r^2z-5sz-2)+n(-26s^3z^3+23s^2z^2+11r^2sz^2+2r^2z+4sz-1)+2sz(-12s^2z^2+3r^2z+3sz+2)+2)z^{n-3}W_{2n+1}.$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[10]].

(a)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=0}^{n}z^kW_k=\frac{\Phi_1}{sz^2+rz-1}$$

where

$\Phi_1=z^{n+2}W_{n+2}+z^{n+1}(1-rz)W_{n+1}-zW_1+(rz-1)W_0$

which is given in Theorem 2.1 (a) (i), then we get

$$\sum_{k=0}^{n}kz^{k-1}W_k=\frac{d}{dz}\left(\frac{\Phi_1}{sz^2+rz-1}\right)$$

$$\Rightarrow$$

$$\sum_{k=0}^{n}kz^kW_k=z\frac{d}{dz}\left(\frac{\Phi_1}{sz^2+rz-1}\right)=z\frac{(sz^2+rz-1)\frac{d}{dz}\Phi_1-\Phi_1\frac{d}{dz}(sz^2+rz-1)}{(sz^2+rz-1)^2}$$
$$=\frac{z(sz^2+rz-1)\frac{d}{dz}\Phi_1-z(r+2sz)\Phi_1}{(sz^2+rz-1)^2}=\frac{\Lambda_1}{(sz^2+rz-1)^2}.$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.1, $a\neq b$. We use (a) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). We get (a) (ii), by using (a) (i), as

$$\sum_{k=0}^{n}ka^kW_k=\left.\frac{\frac{d^2}{dz^2}(\Lambda_1)}{\frac{d^2}{dz^2}(sz^2+rz-1)^2}\right|_{z=a}=\left.\frac{\Lambda_2}{2(-2s+6s^2z^2+r^2+6rsz)}\right|_{z=a}$$

and

$$\sum_{k=0}^{n}kb^kW_k=\left.\frac{\frac{d^2}{dz^2}(\Lambda_1)}{\frac{d^2}{dz^2}(sz^2+rz-1)^2}\right|_{z=b}=\left.\frac{\Lambda_2}{2(-2s+6s^2z^2+r^2+6rsz)}\right|_{z=b}.$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.1, $a=b$. We use (a) (i). For $z=a$, the right hand side of the sum formula given in (a) (i) is an indeterminate form. We can use L'Hospital rule (four times). Then we get, by using (a) (i),

$$\begin{aligned}\sum_{k=0}^{n} ka^k W_k &= \left.\frac{\frac{d^4}{dz^4}(\Lambda_1)}{\frac{d^4}{dz^4}(sz^2+rz-1)^2}\right|_{z=a} \\ &= \left.\frac{\Lambda_3}{24s^2}\right|_{z=a}.\end{aligned}$$

(b)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Phi_2}{r^2z-s^2z^2+2sz-1}$$

where
$\Phi_2=-z^{n+1}(sz-1)W_{2n+2}+rsz^{n+2}W_{2n+1}-rzW_1+(r^2z+sz-1)W_0$
which is given in Theorem 2.1 (b) (i), then we get

$$\begin{aligned}\sum_{k=0}^{n} kz^{k-1} W_{2k} &= \frac{d}{dz}\left(\frac{\Phi_2}{r^2z-s^2z^2+2sz-1}\right) \\ &\Rightarrow \\ \sum_{k=0}^{n} kz^{k} W_{2k} &= z\frac{d}{dz}\left(\frac{\Phi_2}{r^2z-s^2z^2+2sz-1}\right) \\ &= z\frac{(r^2z-s^2z^2+2sz-1)\frac{d}{dz}\Phi_2-\Phi_2\frac{d}{dz}(r^2z-s^2z^2+2sz-1)}{(r^2z-s^2z^2+2sz-1)^2} \\ &= \frac{z(r^2z-s^2z^2+2sz-1)\frac{d}{dz}\Phi_2-z(r^2-2zs^2+2s)\Phi_2}{(r^2z-s^2z^2+2sz-1)^2}=\frac{\Lambda_4}{(r^2z-s^2z^2+2sz-1)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.1, $a\neq b$. We use (b) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). We get (b) (ii), by using (b) (i), as

$$\begin{aligned}\sum_{k=0}^{n} ka^k W_{2k} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_4)}{\frac{d^2}{dz^2}\left((r^2z-s^2z^2+2sz-1)^2\right)}\right|_{z=a} \\ &= \left.\frac{\Lambda_5}{2(6s^4z^2-6r^2s^2z-12s^3z+r^4+4r^2s+6s^2)}\right|_{z=a}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n} kb^k W_{2k} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_4)}{\frac{d^2}{dz^2}\left((r^2z-s^2z^2+2sz-1)^2\right)}\right|_{z=b} \\ &= \left.\frac{\Lambda_5}{2(6s^4z^2-6r^2s^2z-12s^3z+r^4+4r^2s+6s^2)}\right|_{z=b}.\end{aligned}$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.1, $a=b$. We use (b) (i). For $z=a$, the right hand side of the sum formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get, by using (b) (i),

$$\begin{aligned}\sum_{k=0}^{n} ka^k W_{2k} &= \left.\frac{\frac{d^4}{dz^4}(\Lambda_4)}{\frac{d^4}{dz^4}\left((r^2z-s^2z^2+2sz-1)^2\right)}\right|_{z=a} \\ &= \left.\frac{\Lambda_6}{24s^4}\right|_{z=a}.\end{aligned}$$

(c)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Phi_3}{r^2 z - s^2 z^2 + 2sz - 1}$$

where

$\Phi_3 = rz^{n+1} W_{2n+2} - sz^{n+1}(sz-1)W_{2n+1} + (sz-1)W_1 - rszW_0$

which is given in Theorem 2.1 (c) (i), then we get

$$\sum_{k=0}^{n} kz^{k-1} W_{2k+1} = \frac{d}{dz}\left(\frac{\Phi_3}{r^2 z - s^2 z^2 + 2sz - 1}\right)$$

$$\Rightarrow$$

$$\begin{aligned}
\sum_{k=0}^{n} kz^{k} W_{2k+1} &= z\frac{d}{dz}\left(\frac{\Phi_3}{r^2 z - s^2 z^2 + 2sz - 1}\right) \\
&= z\frac{(r^2 z - s^2 z^2 + 2sz - 1)\frac{d}{dz}\Phi_3 - \Phi_3 \frac{d}{dz}(r^2 z - s^2 z^2 + 2sz - 1)}{(r^2 z - s^2 z^2 + 2sz - 1)^2} \\
&= \frac{z(r^2 z - s^2 z^2 + 2sz - 1)\frac{d}{dz}\Phi_3 - z(r^2 - 2zs^2 + 2s)\Phi_3}{(r^2 z - s^2 z^2 + 2sz - 1)^2} \\
&= \frac{\Lambda_7}{(r^2 z - s^2 z^2 + 2sz - 1)^2}.
\end{aligned}$$

(ii) Suppose that $r^2 + 4s \neq 0$. Then, from Remark 2.1, $a \neq b$. We use (c) (i). For $z = a$ and $z = b$, the right hand side of the sum formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). We get (c) (ii), by using (c) (i), as

$$\begin{aligned}
\sum_{k=0}^{n} a^k W_{2k+1} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_7)}{\frac{d^2}{dz^2}\left((r^2 z - s^2 z^2 + 2sz - 1)^2\right)}\right|_{z=a} \\
&= \left.\frac{\Lambda_8}{2(6s^4 z^2 - 6r^2 s^2 z - 12s^3 z + r^4 + 4r^2 s + 6s^2)}\right|_{z=a}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} b^k W_{2k+1} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_7)}{\frac{d^2}{dz^2}\left((r^2 z - s^2 z^2 + 2sz - 1)^2\right)}\right|_{z=b} \\
&= \left.\frac{\Lambda_8}{2(6s^4 z^2 - 6r^2 s^2 z - 12s^3 z + r^4 + 4r^2 s + 6s^2)}\right|_{z=b}.
\end{aligned}$$

(iii) Suppose that $r^2 + 4s = 0$. Then, from Remark 2.1, $a = b$. We use (c) (i). For $z = a$, the right hand side of the sum formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get, by using (c) (i),

$$\begin{aligned}
\sum_{k=0}^{n} ka^k W_{2k+1} &= \left.\frac{\frac{d^4}{dz^4}(\Lambda_7)}{\frac{d^4}{dz^4}\left((r^2 z - s^2 z^2 + 2sz - 1)^2\right)}\right|_{z=a} \\
&= \left.\frac{\Lambda_9}{24s^4}\right|_{z=a}. \quad \square
\end{aligned}$$

Note that the proof of (i) 's of Theorem 3.1 can be done as in the following Remark.

Remark 3.1.

We present the direct proofs of Theorem 3.1 (a) (i), (b) (i) and (c) (i) without using any derivatives.

Proof of Theorem 3.1 (a) (i):

Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$sW_{n-2} = W_n - rW_{n-1}$$

we obtain

$$\begin{aligned} snz^nW_n &= nz^nW_{n+2} - rnz^nW_{n+1} \\ s(n-1)z^{n-1}W_{n-1} &= (n-1)z^{n-1}W_{n+1} - r(n-1)z^{n-1}W_n \\ s(n-2)z^{n-2}W_{n-2} &= (n-2)z^{n-2}W_n - r(n-2)z^{n-2}W_{n-1} \\ s(n-3)z^{n-3}W_{n-3} &= (n-3)z^{n-3}W_{n-1} - r(n-3)z^{n-3}W_{n-2} \\ &\vdots \\ s5z^5W_5 &= 5z^5W_7 - r5z^5W_6 \\ s4z^4W_4 &= 4z^4W_6 - r4z^4W_5 \\ s3z^3W_3 &= 3z^3W_5 - r3z^3W_4 \\ s2z^2W_2 &= 2z^2W_4 - r2z^2W_3 \\ sz^1W_1 &= z^1W_3 - rz^1W_2 \\ s\times 0\times z^0W_0 &= 0\times z^0W_2 - r\times 0\times z^0W_1 \end{aligned}$$

If we add the equations side by side, we get

$$s\sum_{k=0}^{n}kz^kW_k = \sum_{k=3}^{n+2}(k-2)z^{k-2}W_k - r\sum_{k=2}^{n+1}(k-1)z^{k-1}W_k. \tag{21}$$

Note that

$$\begin{aligned} \sum_{k=3}^{n+2}(k-2)z^{k-2}W_k &= z^{-1}W_1 + 2z^{-2}W_0 + (n-1)z^{n-1}W_{n+1} + nz^nW_{n+2} \\ &\quad + z^{-2}\sum_{k=0}^{n}kz^kW_k - 2z^{-2}\sum_{k=0}^{n}z^kW_k, \\ \sum_{k=2}^{n+1}(k-1)z^{k-1}W_k &= z^{-1}W_0 + nz^nW_{n+1} + z^{-1}\sum_{k=0}^{n}kz^kW_k - z^{-1}\sum_{k=0}^{n}z^kW_k. \end{aligned}$$

If we put them in (21) then it follows that

$$\begin{aligned} s\sum_{k=0}^{n}kz^kW_k &= (z^{-1}W_1 + 2z^{-2}W_0 + (n-1)z^{n-1}W_{n+1} + nz^nW_{n+2} \\ &\quad + z^{-2}\sum_{k=0}^{n}kz^kW_k - 2z^{-2}\sum_{k=0}^{n}z^kW_k) - r(z^{-1}W_0 + nz^nW_{n+1} \\ &\quad + z^{-1}\sum_{k=0}^{n}kz^kW_k - z^{-1}\sum_{k=0}^{n}z^kW_k). \end{aligned}$$

Then, if we use Theorem 2.1 (a) (i), the required results of (a) (i) of Theorem 3.1 follows.

Proof of Theorem 3.1 (b) (i) and (c) (i):

Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2}$$

we obtain

$$\begin{aligned} rnz^nW_{2n+1} &= nz^nW_{2n+2} - snz^nW_{2n} \\ r(n-1)z^{n-1}W_{2n-1} &= (n-1)z^{n-1}W_{2n} - s(n-1)z^{n-1}W_{2n-2} \\ &\vdots \\ r4z^4W_9 &= 4z^4W_{10} - s4z^4W_8 \\ r3z^3W_7 &= 3z^3W_8 - s3z^3W_6 \\ r2z^2W_5 &= 2z^2W_6 - s2z^2W_4 \\ rz^1W_3 &= z^1W_4 - sz^1W_2 \end{aligned}$$

Now, if we add the above equations side by side, we get

$$r\sum_{k=0}^{n}kz^{k}W_{2k+1}=\sum_{k=1}^{n+1}(k-1)z^{k-1}W_{2k}-s\sum_{k=0}^{n}kz^{k}W_{2k}. \tag{22}$$

Note that

$$\sum_{k=1}^{n+1}(k-1)z^{k-1}W_{2k}=z^{-1}W_{0}+nz^{n}W_{2n+2}+z^{-1}\sum_{k=0}^{n}kz^{k}W_{2k}-z^{-1}\sum_{k=0}^{n}z^{k}W_{2k}.$$

If we put this in (22) we obtain

$$r\sum_{k=0}^{n}kz^{k}W_{2k+1}=z^{-1}W_{0}+nz^{n}W_{2n+2}+(z^{-1}-s)\sum_{k=0}^{n}kz^{k}W_{2k}-z^{-1}\sum_{k=0}^{n}z^{k}W_{2k}. \tag{23}$$

Similarly, using the recurrence relation

$$W_{n}=rW_{n-1}+sW_{n-2}$$

i.e.

$$rW_{n-1}=W_{n}-sW_{n-2}\Rightarrow rW_{n}=W_{n+1}-sW_{n-1}\Rightarrow rW_{2n}=W_{2n+1}-sW_{2n-1}$$

we write the following obvious equations

$$\begin{aligned}
rnz^{n}W_{2n}&=nz^{n}W_{2n+1}-snz^{n}W_{2n-1}\\
r(n-1)z^{n-1}W_{2n-2}&=(n-1)z^{n-1}W_{2n-1}-s(n-1)z^{n-1}W_{2n-3}\\
&\vdots\\
r6z^{6}W_{12}&=6z^{6}W_{13}-s6z^{6}W_{11}\\
r5z^{5}W_{10}&=5z^{5}W_{11}-s5z^{5}W_{9}\\
r4z^{4}W_{8}&=4z^{4}W_{9}-s4z^{4}W_{7}\\
r3z^{3}W_{6}&=3z^{3}W_{7}-s3z^{3}W_{5}\\
r2z^{2}W_{4}&=2z^{2}W_{5}-s2z^{2}W_{3}\\
rz^{1}W_{2}&=z^{1}W_{3}-sz^{1}W_{1}
\end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$r\sum_{k=0}^{n}kz^{k}W_{2k}=\sum_{k=0}^{n}kz^{k}W_{2k+1}-s\sum_{k=-1}^{n-1}(k+1)z^{k+1}W_{2k+1}. \tag{24}$$

Note that

$$\sum_{k=-1}^{n-1}(k+1)z^{k+1}W_{2k+1}=-(n+1)z^{n+1}W_{2n+1}+z\sum_{k=0}^{n}kz^{k}W_{2k+1}+z\sum_{k=0}^{n}z^{k}W_{2k+1}.$$

If we put this in (24) we obtain

$$r\sum_{k=0}^{n}kz^{k}W_{2k}=(1-sz)\sum_{k=0}^{n}kz^{k}W_{2k+1}+(s(n+1)z^{n+1}W_{2n+1}-sz\sum_{k=0}^{n}z^{k}W_{2k+1}. \tag{25}$$

Then, using Theorem 2.1 (b) (i) and (c) (i) and solving the system (23)-(25), the required result of (b) (i) and (c) (i) of Theorem 3.1 follow.

3.2. Sum Formulas $\sum_{k=1}^{n}kz^{k}W_{-k}$, $\sum_{k=1}^{n}kz^{k}W_{-2k}$ and $\sum_{k=1}^{n}kz^{k}W_{-2k+1}$ of Generalized Fibonacci (Horadam) Polynomials with Negative Subscripts

The following theorem presents some sum formulas of generalized Fibonacci polynomials with negative subscripts.

Theorem 3.2.
Let z be a non-zero complex (or real) number. For $n\geq 1$ we have the following formulas:

(a)

(i) *If* $s+rz-z^2 \neq 0$, *then*

$$\sum_{k=1}^{n} kz^k W_{-k} = \frac{z(s+rz-z^2)\frac{d}{dz}\Phi_4 - z(r-2z)\Phi_4}{(s+rz-z^2)^2} = \frac{\Lambda_{10}}{(s+rz-z^2)^2}$$

where

$\Phi_4 = -z^{n+1}(s+rz)W_{-n-1} - sz^{n+2}W_{-n-2} + zW_1 + z(z-r)W_0$ *given in Theorem 2.2 (a) (i)*

and

$\frac{d}{dz}\Phi_4 = \Phi_4^{'}$ *denotes the derivatives of* Φ_4 *with respect to* z

and

$\Lambda_{10} = -(n(s+rz)(-z^2+rz+s)+sz^2+r^2z^2+2rsz+s^2)z^{n+1}W_{-n-1} - s(n(-z^2+rz+s)+rz+2s)z^{n+2}W_{-n-2} + z(z^2+s)W_1 - sz(r-2z)W_0.$

(ii) *If* $s+rz-z^2 = u(z-a)(z-b) = 0$ *for some* $u,a,b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z=a$ *or* $z=b$, *that is, if*

$$z = a = \frac{1}{2}\left(r+\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2}\left(r-\sqrt{r^2+4s}\right),$$

provided that $r^2+4s \neq 0$, *then*

$$\sum_{k=1}^{n} kz^k W_{-k} = \frac{\Lambda_{11}}{2(6z^2-6rz+r^2-2s)}$$

where

$\Lambda_{11} = -(n^3(rz+s)(-z^2+rz+s)+n^2(-7rz^3-4sz^2+6r^2z^2+8rsz+2s^2)+n(-12rz^3-sz^2+11r^2z^2+10rsz+s^2)+2z(3r^2z+3sz+2rs))z^{n-1}W_{-n-1} - s(n^3(-z^2+rz+s)+n^2(-7z^2+6rz+5s)+n(-12z^2+11rz+8s)+6rz+4s)z^nW_{-n-2} + 6zW_1 + 4sW_0.$

(iii) *If* $s+rz-z^2 = u(z-a)^2 = 0$ *for some* $u,a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z=a$, *that is, if*

$$z = \frac{r}{2}$$

provided that $r^2+4s = 0$, *then*

$$\sum_{k=1}^{n} kz^k W_{-k} = \frac{\Lambda_{12}}{24}$$

where

$\Lambda_{12} = n(n+1)(n^3(rz+s)(z^2-rz-s)+n^2(9rz^3+4sz^2-6r^2z^2-4rsz+2s^2)+n(26rz^3+sz^2-11r^2z^2+2rsz+s^2)+24rz^3-6r^2z^2-6sz^2+4rsz-2s^2)z^{n-3}W_{-n-1} - ns(n^2+3n+2)(n^2(-z^2+rz+s)+n(-7z^2+4rz+s)-12z^2+3rz-2s)z^{n-2}W_{-n-2}.$

(b)

(i) *If* $r^2z+2sz-s^2-z^2 \neq 0$ *then*

$$\sum_{k=1}^{n} kz^k W_{-2k} = \frac{z(r^2z+2sz-s^2-z^2)\frac{d}{dz}\Phi_5 - z(r^2+2s-2z)\Phi_5}{(r^2z+2sz-s^2-z^2)^2} = \frac{\Lambda_{13}}{(r^2z+2sz-s^2-z^2)^2}$$

where

$\Phi_5 = z^{n+1}(s-z)W_{-2n} - rsz^{n+1}W_{-2n-1} + rzW_1 + z(z-s-r^2)W_0$ *given in Theorem 2.2 (b) (i)*

and

$\frac{d}{dz}\Phi_5 = \Phi_5^{'}$ *denotes the derivatives of* Φ_5 *with respect to* z

and

$\Lambda_{13} = (n(s-z)(-z^2+r^2z+2sz-s^2)+2s^2z-r^2z^2-s^3-sz^2)z^{n+1}W_{-2n} + rs(n(z^2-r^2z-2sz+s^2)+s^2-z^2)z^{n+1}W_{-2n-1} - rz(s-z)(s+z)W_1 + sz(z^2-2sz+r^2s+s^2)W_0.$

(ii) *If* $r^2z+2sz-s^2-z^2 = u(z-a)(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $z = a$ *or* $z = b$*, i.e.,* $z = a$ *or* $z = b$*, that is, if*

$$z = a = \frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$

provided that $r^2+4s \neq 0$*, then*

$$\sum_{k=1}^{n} kz^k W_{-2k} = \frac{\Lambda_{14}}{2(6z^2-6r^2z-12sz+r^4+6s^2+4r^2s)}$$

where

$\Lambda_{14} = -(n^3(s-z)(z^2-r^2z-2sz+s^2)+n^2(-7z^3+16sz^2+6r^2z^2-11s^2z-3r^2sz+2s^3)+n(-12z^3+23sz^2+11r^2z^2-2r^2sz-12s^2z+s^3)+2z(3r^2z+3sz-2s^2))z^{n-1}W_{-2n}-rs(n^3(-z^2+r^2z+2sz-s^2)+n^2(-4z^2+3r^2z+6sz-2s^2)+n(-z^2+2r^2z+4sz-s^2)+6z^2)z^{n-1}W_{-2n-1}+6rzW_1-2s(2s-3z)W_0.$

(iii) *If* $r^2z+2sz-s^2-z^2 = u(z-a)^2 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $z = a$*, that is, if*

$$z = \frac{1}{4}r^2$$

provided that $r^2+4s = 0$*, then*

$$\sum_{k=1}^{n} kz^k W_{-2k} = \frac{\Lambda_{15}}{24}$$

where

$\Lambda_{15} = -n(n+1)(n^3(s-z)(z^2-r^2z-2sz+s^2)-n^2(9z^3-16sz^2-6r^2z^2+5s^2z+r^2sz+2s^3)+n(-26z^3+23sz^2+11r^2z^2+4s^2z+2r^2sz-s^3)-24z^3+6r^2z^2+6sz^2+4s^2z+2s^3)z^{n-3}W_{-2n}-nrs(n^2-1)(n^2(-z^2+r^2z+2sz-s^2)+n(-5z^2+2r^2z+4sz+s^2)+2s^2-6z^2)z^{n-3}W_{-2n-1}.$

(c)

(i) *If* $r^2z+2sz-s^2-z^2 \neq 0$ *then*

$$\sum_{k=1}^{n} kz^k W_{-2k+1} = \frac{z(r^2z+2sz-s^2-z^2)\frac{d}{dz}\Phi_6 - z(r^2+2s-2z)\Phi_6}{(r^2z+2sz-s^2-z^2)^2} = \frac{\Lambda_{16}}{(r^2z+2sz-s^2-z^2)^2}$$

where

$\Phi_6 = -rz^{n+2}W_{-2n}+sz^{n+1}(s-z)W_{-2n-1}+z(z-s)W_1+rszW_0$ *given in Theorem 2.2 (c) (i)*

and

$\frac{d}{dz}\Phi_6 = \Phi_6'$ *denotes the derivatives of* Φ_6 *with respect to* z

and

$\Lambda_{16} = r(n(z^2-r^2z-2sz+s^2)-r^2z-2sz+2s^2)z^{n+2}W_{-2n}+s(n(s-z)(-z^2+r^2z+2sz-s^2)+2s^2z-sz^2-r^2z^2-s^3)z^{n+1}W_{-2n-1}+z(sz^2-2s^2z+r^2z^2+s^3)W_1-rsz(s-z)(s+z)W_0.$

(ii) *If* $r^2z+2sz-s^2-z^2 = u(z-a)(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $z = a$ *or* $z = b$*, i.e.,* $z = a$ *or* $z = b$*, that is, if*

$$z = a = \frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$

provided that $r^2+4s \neq 0$*, then*

$$\sum_{k=1}^{n} kz^k W_{-2k+1} = \frac{\Lambda_{17}}{2(6z^2-6r^2z-12sz+r^4+6s^2+4r^2s)}$$

where

$\Lambda_{17} = -r(n^3(-z^2+r^2z+2sz-s^2)+n^2(-7z^2+6r^2z+12sz-5s^2)+n(-12z^2+11r^2z+22sz-8s^2)+6r^2z+12sz-4s^2)z^nW_{-2n}-s(n^3(s-z)(z^2-r^2z-2sz+s^2)+n^2(-7z^3+16sz^2+6r^2z^2-3r^2sz-11s^2z+2s^3)+n(-12z^3+23sz^2+11r^2z^2-2r^2sz-12s^2z+s^3)+6r^2z^2-4s^2z+6sz^2)z^{n-1}W_{-2n-1}+2(3r^2z+3sz-2s^2)W_1+6rszW_0.$

(iii) *If $r^2z+2sz-s^2-z^2=u(z-a)^2=0$ for some $u,a\in\mathbb{C}$ with $u\neq 0$, i.e., $z=a$, that is, if*

$$z=\frac{1}{4}r^2$$

provided that $r^2+4s=0$, then

$$\sum_{k=1}^{n} kz^k W_{-2k+1}=\frac{\Lambda_{18}}{24}$$

where
$\Lambda_{18}=-nr(n^2+3n+2)(n^2(-z^2+r^2z+2sz-s^2)+n(-7z^2+4r^2z+8sz-s^2)-12z^2+3r^2z+6sz+2s^2)z^{n-2}W_{-2n}-ns(n+1)(n^3(s-z)(z^2-r^2z-2sz+s^2)-n^2(9z^3-16sz^2-6r^2z^2+5s^2z+r^2sz+2s^3)+n(-26z^3+23sz^2+11r^2z^2+2r^2sz+4s^2z-s^3)-24z^3+6sz^2+6r^2z^2+4s^2z+2s^3)z^{n-3}W_{-2n-1}$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[10]].

(a)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=1}^{n} z^k W_{-k}=\frac{\Phi_4}{s+rz-z^2}$$

where
$\Phi_4=-z^{n+1}(s+rz)W_{-n-1}-sz^{n+2}W_{-n-2}+zW_1+z(z-r)W_0$
which is given in Theorem 2.2 (a) (i), then we get

$$\sum_{k=1}^{n} kz^{k-1}W_{-k} = \frac{d}{dz}\left(\frac{\Phi_4}{s+rz-z^2}\right)$$

$$\Rightarrow$$

$$\sum_{k=1}^{n} kz^{k}W_{-k} = z\frac{d}{dz}\left(\frac{\Phi_4}{s+rz-z^2}\right)=z\frac{(s+rz-z^2)\frac{d}{dz}\Phi_4-\Phi_4\frac{d}{dz}(s+rz-z^2)}{(s+rz-z^2)^2}$$
$$= \frac{z(s+rz-z^2)\frac{d}{dz}\Phi_4-z(r-2z)\Phi_4}{(s+rz-z^2)^2}=\frac{\Lambda_{10}}{(s+rz-z^2)^2}.$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.2, $a\neq b$. We use (a) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). We get (a) (ii), by using (a) (i), as

$$\sum_{k=1}^{n} ka^k W_{-k}=\left.\frac{\frac{d^2}{dz^2}(\Lambda_{10})}{\frac{d^2}{dz^2}\left((s+rz-z^2)^2\right)}\right|_{z=a}$$
$$=\left.\frac{\Lambda_{11}}{2(6z^2-6rz+r^2-2s)}\right|_{z=a}$$

and

$$\sum_{k=1}^{n} kb^k W_{-k}=\left.\frac{\frac{d^2}{dz^2}(\Lambda_{10})}{\frac{d^2}{dz^2}\left((s+rz-z^2)^2\right)}\right|_{z=b}$$
$$=\left.\frac{\Lambda_{11}}{2(6z^2-6rz+r^2-2s)}\right|_{z=b}.$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.2, $a=b$. We use (a) (i). For $z=a$, the right hand side of the sum formula given in (a) (i) is an indeterminate form. We can use L'Hospital rule (four times). Then we get, by using (a) (i),

$$\sum_{k=1}^{n} ka^k W_{-k}=\left.\frac{\frac{d^4}{dz^4}(\Lambda_{10})}{\frac{d^4}{dz^4}\left((s+rz-z^2)^2\right)}\right|_{z=a}$$
$$=\left.\frac{\Lambda_{12}}{24}\right|_{z=a}.$$

(b)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=1}^{n} z^k W_{-2k} = \frac{\Phi_5}{r^2 z + 2sz - s^2 - z^2}$$

where
$\Phi_5 = z^{n+1}(s-z)W_{-2n} - rsz^{n+1}W_{-2n-1} + rzW_1 + z(z-s-r^2)W_0$
which is given in Theorem 2.2 (b) (i), then we get

$$\sum_{k=1}^{n} kz^{k-1} W_{-2k} = \frac{d}{dz}\left(\frac{\Phi_5}{r^2 z + 2sz - s^2 - z^2}\right)$$
$$\Rightarrow$$
$$\begin{aligned}\sum_{k=1}^{n} kz^{k} W_{-2k} &= z\frac{d}{dz}\left(\frac{\Phi_5}{r^2 z + 2sz - s^2 - z^2}\right)\\ &= z\frac{(r^2z+2sz-s^2-z^2)\frac{d}{dz}\Phi_5 - \Phi_5\frac{d}{dz}(r^2z+2sz-s^2-z^2)}{(r^2z+2sz-s^2-z^2)^2}\\ &= \frac{z(r^2z+2sz-s^2-z^2)\frac{d}{dz}\Phi_5 - z(r^2+2s-2z)\Phi_5}{(r^2z+2sz-s^2-z^2)^2}\\ &= \frac{\Lambda_{13}}{(r^2z+2sz-s^2-z^2)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s \neq 0$. Then, from Remark 2.2, $a \neq b$. We use (b) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). We get (b) (ii), by using (b) (i), as

$$\begin{aligned}\sum_{k=1}^{n} ka^{k} W_{-2k} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{13})}{\frac{d^2}{dz^2}\left((r^2z+2sz-s^2-z^2)^2\right)}\right|_{z=a}\\ &= \left.\frac{\Lambda_{14}}{2(6z^2-6r^2z-12sz+r^4+6s^2+4r^2s)}\right|_{z=a}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n} kb^{k} W_{-2k} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{13})}{\frac{d^2}{dz^2}\left((r^2z+2sz-s^2-z^2)^2\right)}\right|_{z=b}\\ &= \left.\frac{\Lambda_{14}}{2(6z^2-6r^2z-12sz+r^4+6s^2+4r^2s)}\right|_{z=b}.\end{aligned}$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.2, $a=b$. We use (b) (i). For $z=a$, the right hand side of the sum formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get, by using (b) (i),

$$\begin{aligned}\sum_{k=1}^{n} ka^{k} W_{-2k} &= \left.\frac{\frac{d^4}{dz^4}(\Lambda_{13})}{\frac{d^4}{dz^4}\left((r^2z+2sz-s^2-z^2)^2\right)}\right|_{z=a}\\ &= \left.\frac{\Lambda_{15}}{24}\right|_{z=a}.\end{aligned}$$

(c)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\Phi_6}{r^2 z + 2sz - s^2 - z^2}$$

where
$\Phi_6 = -rz^{n+2}W_{-2n} + sz^{n+1}(s-z)W_{-2n-1} + z(z-s)W_1 + rszW_0$

which is given in Theorem 2.2 (c) (i), then we get

$$
\begin{aligned}
\sum_{k=1}^{n} kz^{k-1}W_{-2k+1} &= \frac{d}{dz}\left(\frac{\Phi_6}{r^2z+2sz-s^2-z^2}\right) \\
&\Rightarrow \\
\sum_{k=1}^{n} kz^{k}W_{-2k+1} &= z\frac{d}{dz}\left(\frac{\Phi_6}{r^2z+2sz-s^2-z^2}\right) \\
&= z\frac{(r^2z+2sz-s^2-z^2)\frac{d}{dz}\Phi_6-\Phi_6\frac{d}{dz}(r^2z+2sz-s^2-z^2)}{(r^2z+2sz-s^2-z^2)^2} \\
&= \frac{z(r^2z+2sz-s^2-z^2)\frac{d}{dz}\Phi_6-z(r^2+2s-2z)\Phi_6}{(r^2z+2sz-s^2-z^2)^2} \\
&= \frac{\Lambda_{16}}{(r^2z+2sz-s^2-z^2)^2}.
\end{aligned}
$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.2, $a\neq b$. We use (c) (i). For $z=a$ and $z=b$, the right hand side of the sum formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). We get (c) (ii), by using (c) (i), as

$$
\begin{aligned}
\sum_{k=1}^{n} ka^{k}W_{-2k+1} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{16})}{\frac{d^2}{dz^2}\left((r^2z+2sz-s^2-z^2)^2\right)}\right|_{z=a} \\
&= \left.\frac{\Lambda_{17}}{2(6z^2-6r^2z-12sz+r^4+6s^2+4r^2s)}\right|_{z=a}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k=1}^{n} kb^{k}W_{-2k+1} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{16})}{\frac{d^2}{dz^2}\left((r^2z+2sz-s^2-z^2)^2\right)}\right|_{z=b} \\
&= \left.\frac{\Lambda_{17}}{2(6z^2-6r^2z-12sz+r^4+6s^2+4r^2s)}\right|_{z=b}
\end{aligned}
$$

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.2, $a=b$. We use (c) (i). For $z=a$, the right hand side of the sum formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get, by using (c) (i),

$$
\begin{aligned}
\sum_{k=1}^{n} ka^{k}W_{-2k+1} &= \left.\frac{\frac{d^4}{dz^4}(\Lambda_{16})}{\frac{d^4}{dz^4}\left((r^2z+2sz-s^2-z^2)^2\right)}\right|_{z=a} \\
&= \left.\frac{\Lambda_{18}}{24}\right|_{z=a}. \square
\end{aligned}
$$

Note that the proof of (i) 's of Theorem 3.2 can be done as in the following Remark.

Remark 3.2.

We present the direct proofs of Theorem Theorem 3.2 (a) (i), (b) (i) and (c) (i) without using any derivatives.

Proof of Theorem Theorem 3.2 (a) (i):
Using the recurrence relation

$$
\begin{aligned}
W_{-n+2} &= r\times W_{-n+1}+s\times W_{-n} \\
&\Rightarrow W_{-n}=-\frac{r}{s}\times W_{-n+1}+\frac{1}{s}W_{-n+2} \\
&\Rightarrow W_{-n}=-\frac{r}{s}\times W_{-(n-1)}+\frac{1}{s}W_{-(n-2)}
\end{aligned}
$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

or

$$W_{-n}=\frac{1}{s}W_{-n+2}-\frac{r}{s}W_{-n+1}$$

we obtain

$$\begin{aligned}
snz^{n}W_{-n} &= nz^{n}W_{-n+2}-rnz^{n}W_{-n+1}\\
s(n-1)z^{n-1}W_{-n+1} &= (n-1)z^{n-1}W_{-n+3}-r(n-1)z^{n-1}W_{-n+2}\\
s(n-2)z^{n-2}W_{-n+2} &= (n-2)z^{n-2}W_{-n+4}-r(n-2)z^{n-2}W_{-n+3}\\
&\vdots\\
s\times 5\times z^{5}W_{-5} &= 5\times z^{5}W_{-3}-r\times 5z^{5}W_{-4}\\
s\times 4\times z^{4}W_{-4} &= 4\times z^{4}W_{-2}-r\times 4z^{4}W_{-3}\\
s\times 3\times z^{3}W_{-3} &= 3\times z^{3}W_{-1}-r\times 3z^{3}W_{-2}\\
s\times 2\times z^{2}W_{-2} &= 2\times z^{2}W_{0}-r\times 2z^{2}W_{-1}\\
s\times 1\times z^{1}W_{-1} &= 1\times z^{1}W_{1}-r\times 1\times z^{1}W_{0}
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
& s((n+1)z^{n+1}W_{-n-1}+(n+2)z^{n+2}W_{-n-2}+\sum_{k=1}^{n}kz^{k}W_{-k})\\
&= (z^{1}W_{1}+2z^{2}W_{0}+\sum_{k=1}^{n}(k+2)z^{k+2}W_{-k})\\
&\quad -r((n+2)z^{n+2}W_{-n-1}+z^{1}W_{0}+\sum_{k=1}^{n}(k+1)z^{k+1}W_{-k}).
\end{aligned}$$

Note that since

$$\begin{aligned}
\sum_{k=1}^{n}(k+2)z^{k+2}W_{-k} &= z^{2}\sum_{k=1}^{n}kz^{k}W_{-k}+2z^{2}\sum_{k=1}^{n}z^{k}W_{-k},\\
\sum_{k=1}^{n}(k+1)z^{k+1}W_{-k} &= z\sum_{k=1}^{n}kz^{k}W_{-k}+z\sum_{k=1}^{n}z^{k}W_{-k},
\end{aligned}$$

we have

$$\begin{aligned}
& s((n+1)z^{n+1}W_{-n-1}+(n+2)z^{n+2}W_{-n-2}+\sum_{k=1}^{n}kz^{k}W_{-k})\\
&= z^{1}W_{1}+(2z^{2}-rz^{1})W_{0}-r(n+2)z^{n+2}W_{-n-1}\\
&\quad +(z^{2}-rz)\sum_{k=1}^{n}kz^{k}W_{-k}+(2z^{2}-rz)\sum_{k=1}^{n}z^{k}W_{-k}.
\end{aligned}$$

Then, using Theorem 2.2 (a) (i), the required results of (a) (i) of Theorem 3.2 follows.

Proof of TheoremTheorem 3.2 (b) (i) and (c) (i):

Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$rW_{-n+1}=W_{-n+2}-sW_{-n}$$

we obtain

$$\begin{aligned}
rnz^{n}W_{-2n+1} &= nz^{n}W_{-2n+2}-snz^{n}W_{-2n}\\
r(n-1)z^{n-1}W_{-2n+3} &= (n-1)z^{n-1}W_{-2n+4}-s(n-1)z^{n-1}W_{-2n+2}\\
r(n-2)z^{n-2}W_{-2n+5} &= (n-2)z^{n-2}W_{-2n+6}-s(n-2)z^{n-2}W_{-2n+4}\\
r(n-3)z^{n-3}W_{-2n+7} &= (n-3)z^{n-3}W_{-2n+8}-s(n-3)z^{n-3}W_{-2n+6}\\
&\vdots\\
r\times 3z^{3}W_{-5} &= 3z^{3}W_{-4}-s\times 3\times z^{3}W_{-6}\\
r\times 2z^{2}W_{-3} &= 2z^{2}W_{-2}-s\times 2\times z^{2}W_{-4}\\
r\times 1\times zW_{-1} &= 1\times zW_{0}-s\times 1\times zW_{-2}
\end{aligned}$$

If we add the equations side by side, we get

$$r\sum_{k=1}^{n}kz^{k}W_{-2k+1}=(-(n+1)z^{n+1}W_{-2n}+zW_{0}+\sum_{k=1}^{n}(k+1)z^{k+1}W_{-2k})-s\sum_{k=1}^{n}kz^{k}W_{-2k}.$$

Since

$$\sum_{k=1}^{n}(k+1)z^{k+1}W_{-2k}=z\sum_{k=1}^{n}kz^{k}W_{-2k}+z\sum_{k=1}^{n}z^{k}W_{-2k}$$

it follows that

$$r\sum_{k=1}^{n}kz^{k}W_{-2k+1}=-(n+1)z^{n+1}W_{-2n}+zW_{0}+(z-s)\sum_{k=1}^{n}kz^{k}W_{-2k}+z\sum_{k=1}^{n}z^{k}W_{-2k}. \tag{26}$$

Similarly, using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$\begin{aligned} rW_{-n+1} &= W_{-n+2}-sW_{-n}\Rightarrow rW_{-2n+1}=W_{-2n+2}-sW_{-2n}\\ &\Rightarrow rW_{-2n+1-1}=W_{-2n+2-1}-sW_{-2n-1}\\ &\Rightarrow rW_{-2n}=W_{-2n+1}-sW_{-2n-1}\end{aligned}$$

we obtain

$$\begin{aligned} rnz^{n}W_{-2n} &= n\times z^{n}W_{-2n+1}-snz^{n}W_{-2n-1}\\ r(n-1)z^{n-1}W_{-2n+2} &= (n-1)z^{n-1}W_{-2n+3}-s(n-1)z^{n-1}W_{-2n+1}\\ r(n-2)z^{n-2}W_{-2n+4} &= (n-2)z^{n-2}W_{-2n+5}-s(n-2)z^{n-2}W_{-2n+3}\\ r(n-3)z^{n-3}W_{-2n+6} &= (n-3)z^{n-3}W_{-2n+7}-s(n-3)z^{n-3}W_{-2n+5}\\ &\vdots\\ r\times 4\times z^{4}W_{-8} &= 4z^{4}W_{-7}-s\times 4\times z^{4}W_{-9}\\ r\times 3\times z^{3}W_{-6} &= 3z^{3}W_{-5}-s\times 3\times z^{3}W_{-7}\\ r\times 2\times z^{2}W_{-4} &= 2z^{2}W_{-3}-s\times 2\times z^{2}W_{-5}\\ r\times 1\times zW_{-2} &= 1\times zW_{-1}-s\times 1\times zW_{-3}\end{aligned}$$

If we add the equations side by side, we get

$$r\sum_{k=1}^{n}kz^{k}W_{-2k}=(\sum_{k=1}^{n}kz^{k}W_{-2k+1})-s(nz^{n}W_{-2n-1}+\sum_{k=1}^{n}(k-1)z^{k-1}W_{-2k+1}).$$

Since

$$\sum_{k=1}^{n}(k-1)z^{k-1}W_{-2k+1}=z^{-1}\sum_{k=1}^{n}kz^{k}W_{-2k+1}-z^{-1}\sum_{k=1}^{n}z^{k}W_{-2k+1}$$

it follows that

$$r\sum_{k=1}^{n}kz^{k}W_{-2k}=-snz^{n}W_{-2n-1}+(1-sz^{-1})\sum_{k=1}^{n}kz^{k}W_{-2k+1}+sz^{-1}\sum_{k=1}^{n}z^{k}W_{-2k+1}. \tag{27}$$

Then, using Theorem 2.2 (b) (i) and (c) (i) and solving system (26)-(27) the required result of (b) (i) and (c) (i) of Theorem 3.2 follow.

3.3. Sum Formulas $\sum_{k=0}^{n} kz^k W_k^2$ and $\sum_{k=0}^{n} kz^k W_{k+1}W_k$ of Generalized Fibonacci (Horadam) Polynomials with Positive Subscripts

The following theorem presents some sum formulas of generalized Fibonacci polynomials with positive subscripts. In the sequel, for example, we use the notation

$$\frac{d^2}{dz^2}(\Lambda_{19}) := \left.\frac{d^2}{dz^2}(\Lambda_{19})\right|_{z=a}$$

for $z = a$ as in Theorem 3.3 (a) (ii) and so on.

Theorem 3.3.
Let z be a non-zero complex (or real) number. For $n \geq 0$ we have the following formulas:

(a)

(i) *If $(sz+1)(-s^2z^2+r^2z+2sz-1) \neq 0$ then*

$$\sum_{k=0}^{n} kz^k W_k^2 = \frac{z(sz+1)(-s^2z^2+r^2z+2sz-1)\frac{d}{dz}\Phi_7 - z\Phi_7\frac{d}{dz}((sz+1)(-s^2z^2+r^2z+2sz-1))}{(sz+1)^2(-s^2z^2+r^2z+2sz-1)^2}$$
$$= \frac{\Lambda_{19}}{(sz+1)^2(-s^2z^2+r^2z+2sz-1)^2}$$

where
$\Phi_7 = -(sz-1)z^{n+2}W_{n+2}^2 - (r^2sz^2+r^2z+sz-1)z^{n+1}W_{n+1}^2 + 2rsz^{n+3}W_{n+2}W_{n+1} + z(sz-1)W_1^2 + (r^2sz^2+r^2z+sz-1)W_0^2 - 2rsz^2W_1W_0$ *given in Theorem 2.3 (a) (i)*
and
$\frac{d}{dz}\Phi_7 = \Phi_7^{'}$ *denotes the derivatives of Φ_7 with respect to z*
and
$\Lambda_{19} = (ns^4z^4 - r^2s^2z^3 - nr^2s^2z^3 - 2ns^3z^3 - 2s^2z^2 - 2r^2sz^2 + nr^2z + 2nsz + 4sz + r^2z - n - 2)z^{n+2}W_{n+2}^2 + (nr^2s^4z^5 - s^4z^4 - 2r^2s^3z^4 - nr^4s^2z^4 - r^4s^2z^4 + ns^4z^4 + 2s^3z^3 - 3nr^2s^2z^3 - 2nr^4sz^3 - 2ns^3z^3 - 2r^4sz^3 - 2r^2s^2z^3 - nr^4z^2 - r^4z^2 - 2s^2z^2 + 2nsz + 2nr^2z + 2r^2z + 2sz - n - 1)z^{n+1}W_{n+1}^2 + 2rs(-ns^3z^3 + s^2z^2 + ns^2z^2 + r^2sz^2 + nr^2sz^2 + 2sz + nr^2z + nsz + 2r^2z - n - 3)z^{n+3}W_{n+2}W_{n+1} + z(s^4z^4 - 2s^3z^3 + 2s^2z^2 + 2r^2sz^2 - 2sz + 1)W_1^2 + s^2z^2(r^2s^2z^3 + 2r^2sz^2 + 2s^2z^2 - r^2z - 4sz + 2)W_0^2 - 2rsz^2(s^3z^3 + r^2z + sz - 2)W_1W_0.$

(ii) *If $(sz+1)(-s^2z^2+r^2z+2sz-1) = u(z-a)(z-b)(z-c) = 0$ for some $u, a, b, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $z = a$ or $z = b$ or $z = c$, that is, if*

$$z = a = \frac{1}{2s^2}\left(r^2 + 2s + r\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2s^2}\left(r^2 + 2s - r\sqrt{r^2+4s}\right),$$
$$z = c = -\frac{1}{s},$$

provided that $r^2 + 4s \neq 0$, then

$$\sum_{k=0}^{n} kz^k W_k^2 = \frac{\Lambda_{20}}{2(15s^6z^4 - 20s^4(r^2+s)z^3 + 6s^2(r^4-s^2)z^2 + 6s(r^4+2r^2s+2s^2)z + r^4 - s^2)}$$

where
$\Lambda_{20} = \frac{d^2}{dz^2}(\Lambda_{19}).$

(iii) *If $(sz+1)(-s^2z^2+r^2z+2sz-1) = u(z-a)^3 = 0$ for some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $z = a$, that is, if*

$$z = -\frac{1}{s}$$

provided that $r^2 + 4s = 0$, then

$$\sum_{k=0}^{n} kz^k W_k^2 = \frac{\Lambda_{21}}{720s^6}$$

where
$\Lambda_{21} = \frac{d^6}{dz^6}(\Lambda_{19}).$

(b)

(i) *If* $(sz+1)(-s^2z^2+r^2z+2sz-1)\neq 0$ *then*

$$\sum_{k=0}^{n} kz^kW_{k+1}W_k = \frac{z(sz+1)(-s^2z^2+r^2z+2sz-1)\frac{d}{dz}\Phi_{10}-z\Phi_{10}\frac{d}{dz}((sz+1)(-s^2z^2+r^2z+2sz-1))}{(sz+1)^2(-s^2z^2+r^2z+2sz-1)^2}$$
$$= \frac{\Lambda_{22}}{(sz+1)^2(r^2z-s^2z^2+2sz-1)^2}$$

where
$\Phi_{10} = rz^{n+2}W_{n+2}^2+rs^2z^{n+3}W_{n+1}^2-(s^2z^2+r^2z-1)z^{n+1}W_{n+2}W_{n+1}-rzW_1^2-rs^2z^2W_0^2+(s^2z^2+r^2z-1)W_1W_0$
given in Theorem 2.3 (b) (i)
and
$\frac{d}{dz}\Phi_{10}=\Phi_{10}^{'}$ *denotes the derivatives of* Φ_{10} *with respect to* z
and
$\Lambda_{22} = r(-ns^3z^3+s^3z^3+ns^2z^2+nr^2sz^2+sz+nr^2z+nsz+r^2z-n-2)z^{n+2}W_{n+2}^2+rs^2(-ns^3z^3+s^2z^2+ns^2z^2+r^2sz^2+nr^2sz^2+2sz+nr^2z+nsz+2r^2z-n-3)z^{n+3}W_{n+1}^2-(-ns^5z^5+s^4z^4+ns^4z^4+2r^2s^3z^4+2r^2s^2z^3+2ns^3z^3+2nr^2s^2z^3+nr^4sz^3+nr^4z^2-2ns^2z^2+2r^2sz^2+r^4z^2-2s^2z^2-nsz-2nr^2z-2r^2z+n+1)z^{n+1}W_{n+1}W_{n+2}+rz(-2s^3z^3+s^2z^2+r^2sz^2+1)W_1^2-rs^2z^2(s^3z^3+r^2z+sz-2)W_0^2+sz(s^4z^4+2r^2s^2z^3-r^4z^2-2s^2z^2+2r^2z+1)W_0W_1.$

(ii) *If* $(sz+1)(-s^2z^2+r^2z+2sz-1)=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *that is, if*

$$z=a=\frac{1}{2s^2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2s^2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$
$$z=c=-\frac{1}{s},$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=0}^{n} kz^kW_{k+1}W_k = \frac{\Lambda_{23}}{2(15s^6z^4-20s^4(r^2+s)z^3+6s^2(r^4-s^2)z^2+6s(r^4+2r^2s+2s^2)z+r^4-s^2)}$$

where
$\Lambda_{23}=\frac{d^2}{dz^2}(\Lambda_{22}).$

(iii) *If* $(sz+1)(-s^2z^2+r^2z+2sz-1)=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *that is, if*

$$z=-\frac{1}{s}$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=0}^{n} kz^kW_{k+1}W_k = \frac{\Lambda_{24}}{720s^6}$$

where
$\Lambda_{24}=\frac{d^6}{dz^6}(\Lambda_{22}).$

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[12], Theorem 3.1].

(a)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=0}^{n} z^kW_k^2 = \frac{\Phi_7}{(sz+1)(-s^2z^2+r^2z+2sz-1)}$$

where
$\Phi_7 = -(sz-1)z^{n+2}W_{n+2}^2-(r^2sz^2+r^2z+sz-1)z^{n+1}W_{n+1}^2+2rsz^{n+3}W_{n+2}W_{n+1}+z(sz-1)W_1^2+(r^2sz^2+r^2z+sz-1)W_0^2-2rsz^2W_1W_0$

which is given in Theorem 2.3 (a) (i), then we get

$$\sum_{k=0}^{n} kz^{k-1}W_k^2 = \frac{d}{dz}\left(\frac{\Phi_7}{(sz+1)(-s^2z^2+r^2z+2sz-1)}\right)$$

$$\Rightarrow$$

$$\begin{aligned}\sum_{k=0}^{n} kz^{k}W_k^2 &= z\frac{d}{dz}\left(\frac{\Phi_7}{(sz+1)(-s^2z^2+r^2z+2sz-1)}\right)\\ &= \frac{z(sz+1)(-s^2z^2+r^2z+2sz-1)\frac{d}{dz}\Phi_7 - z\Phi_7\frac{d}{dz}((sz+1)(-s^2z^2+r^2z+2sz-1))}{(sz+1)^2(-s^2z^2+r^2z+2sz-1)^2}\\ &= \frac{\Lambda_{19}}{(sz+1)^2(-s^2z^2+r^2z+2sz-1)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s \neq 0$. Then, from Remark 2.3, $a \neq b \neq c$. We use (a) (i). For $z = a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) (ii) by using

$$\begin{aligned}\sum_{k=0}^{n} a^kW_k^2 &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{19})}{\frac{d^2}{dz^2}\left((sz+1)^2(-s^2z^2+r^2z+2sz-1)^2\right)}\right|_{z=a}\\ &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{19})}{2(15s^6z^4-20s^4(r^2+s)z^3+6s^2(r^4-s^2)z^2+6s(r^4+2r^2s+2s^2)z+r^4-s^2)}\right|_{z=a}.\end{aligned}$$

The proof for the case $z = b$ and $z = c$ are the same.

(iii) Suppose that $r^2+4s = 0$. Then, from Remark 2.3, $a = b = c$. We use (a) (i). For $z = a = -\frac{1}{s}$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (six times). Then we get (a) (iii) by using

$$\begin{aligned}\sum_{k=0}^{n} a^kW_k^2 &= \left.\frac{\frac{d^6}{dz^6}(\Lambda_{19})}{\frac{d^6}{dz^6}\left((sz+1)^2(-s^2z^2+r^2z+2sz-1)^2\right)}\right|_{z=a}\\ &= \left.\frac{\frac{d^6}{dz^6}(\Lambda_{19})}{720s^6}\right|_{z=a}.\end{aligned}$$

(b)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=0}^{n} z^kW_{k+1}W_k = \frac{\Phi_{10}}{(sz+1)(-s^2z^2+r^2z+2sz-1)}$$

where

$\Phi_{10} = rz^{n+2}W_{n+2}^2 + rs^2z^{n+3}W_{n+1}^2 - (s^2z^2+r^2z-1)z^{n+1}W_{n+2}W_{n+1} - rzW_1^2 - rs^2z^2W_0^2 + (s^2z^2+r^2z-1)W_1W_0$

which is given in Theorem 2.3 (b) (i), then we get

$$\sum_{k=0}^{n} kz^{k-1}W_{k+1}W_k = \frac{d}{dz}\left(\frac{\Phi_{10}}{(sz+1)(-s^2z^2+r^2z+2sz-1)}\right)$$

$$\Rightarrow$$

$$\begin{aligned}\sum_{k=0}^{n} kz^{k}W_{k+1}W_k &= z\frac{d}{dz}\left(\frac{\Phi_{10}}{(sz+1)(-s^2z^2+r^2z+2sz-1)}\right)\\ &= \frac{z(sz+1)(-s^2z^2+r^2z+2sz-1)\frac{d}{dz}\Phi_{10} - z\Phi_{10}\frac{d}{dz}((sz+1)(-s^2z^2+r^2z+2sz-1))}{(sz+1)^2(-s^2z^2+r^2z+2sz-1)^2}\\ &= \frac{\Lambda_{22}}{(sz+1)^2(-s^2z^2+r^2z+2sz-1)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s \neq 0$. Then, from Remark 2.3, $a \neq b \neq c$. We use (b) (i). For $z = a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) (ii) by using

$$\begin{aligned}\sum_{k=0}^{n} a^kW_{k+1}W_k &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{22})}{\frac{d^2}{dz^2}\left((sz+1)^2(-s^2z^2+r^2z+2sz-1)^2\right)}\right|_{z=a}\\ &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{22})}{2(15s^6z^4-20s^4(r^2+s)z^3+6s^2(r^4-s^2)z^2+6s(r^4+2r^2s+2s^2)z+r^4-s^2)}\right|_{z=a}.\end{aligned}$$

The proof for the case $z = b$ and $z = c$ are the same.

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.3, $a=b=c$. We use (b) (i). For $z=a=-\frac{1}{s}$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (six times). Then we get (b) (iii) by using

$$\sum_{k=0}^{n} a^k W_{k+1}W_k = \left.\frac{\frac{d^6}{dz^6}(\Lambda_{22})}{\frac{d^6}{dz^6}\left((sz+1)^2(-s^2z^2+r^2z+2sz-1)^2\right)}\right|_{z=a} = \left.\frac{\frac{d^6}{dz^6}(\Lambda_{22})}{720s^6}\right|_{z=a}. \square$$

Note that the proof of (i) 's of Theorem 3.3 can be done as in the following Remark.

Remark 3.3.
We present the direct proofs of Theorem 3.3 (a) (i) and (b) (i) without using any derivatives.
Proof of Theorem 3.3 (a) (i) and (b) (i):
Using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$\begin{aligned} sW_n &= W_{n+2} - rW_{n+1}, \\ s^2W_n^2 &= (W_{n+2}-rW_{n+1})^2 = W_{n+2}^2 + r^2W_{n+1}^2 - 2rW_{n+2}W_{n+1} \end{aligned}$$

we obtain

$$\begin{aligned} s^2nz^nW_n^2 &= nz^nW_{n+2}^2 + nr^2z^nW_{n+1}^2 - 2r\times nz^nW_{n+2}W_{n+1} \\ s^2(n-1)z^{n-1}W_{n-1}^2 &= (n-1)z^{n-1}W_{n+1}^2 + (n-1)r^2z^{n-1}W_n^2 - 2r\times(n-1)z^{n-1}W_{n+1}W_n \\ s^2(n-2)z^{n-2}W_{n-2}^2 &= (n-2)z^{n-2}W_n^2 + (n-2)r^2z^{n-2}W_{n-1}^2 - 2r\times(n-2)z^{n-2}W_nW_{n-1} \\ &\vdots \\ s^23z^3W_3^2 &= 3z^3W_5^2 + 3r^2z^3W_4^2 - 2r\times 3z^3W_5W_4 \\ s^22z^2W_2^2 &= 2z^2W_4^2 + 2r^2z^2W_3^2 - 2r\times 2z^2W_4W_3 \\ s^2z^1W_1^2 &= z^1W_3^2 + r^2z^1W_2^2 - 2r\times z^1W_3W_2 \\ s^2\times 0\times z^0W_0^2 &= 0\times z^0W_2^2 + 0\times r^2z^0W_1^2 - 2r\times 0\times z^0W_2W_1 \end{aligned}$$

If we add the above equations side by side, we get

$$s^2\sum_{k=0}^{n} kz^kW_k^2 = \sum_{k=2}^{n+2}(k-2)z^{k-2}W_k^2 + r^2\sum_{k=1}^{n+1}(k-1)z^{k-1}W_k^2 - 2r\sum_{k=1}^{n+1}(k-1)z^{k-1}W_{k+1}W_k. \tag{28}$$

Note that

$$\begin{aligned} \sum_{k=2}^{n+2}(k-2)z^{k-2}W_k^2 &= 2z^{-2}W_0^2 + z^{-1}W_1^2 + (n-1)z^{n-1}W_{n+1}^2 + nz^nW_{n+2}^2 \\ &\quad + z^{-2}\left(\sum_{k=0}^{n} kz^kW_k^2 - 2\sum_{k=0}^{n} z^kW_k^2\right), \\ \sum_{k=1}^{n+1}(k-1)z^{k-1}W_k^2 &= z^{-1}W_0^2 + nz^nW_{n+1}^2 + z^{-1}\left(\sum_{k=0}^{n} kz^kW_k^2 - \sum_{k=0}^{n} z^kW_k^2\right), \\ \sum_{k=1}^{n+1}(k-1)z^{k-1}W_{k+1}W_k &= z^{-1}W_1W_0 + nz^nW_{n+2}W_{n+1} + z^{-1}\left(\sum_{k=0}^{n} kz^kW_{k+1}W_k - \sum_{k=0}^{n} z^kW_{k+1}W_k\right). \end{aligned}$$

If we put them into the (28) we get

$$\begin{aligned} s^2\sum_{k=0}^{n} kz^kW_k^2 &= nz^nW_{n+2}^2 + ((n-1)z^{n-1} + r^2nz^n)W_{n+1}^2 - 2rnz^nW_{n+2}W_{n+1} + z^{-1}W_1^2 \\ &\quad + (2z^{-2} + r^2z^{-1})W_0^2 - 2rz^{-1}W_1W_0 + (z^{-2} + r^2z^{-1})\sum_{k=0}^{n} kz^kW_k^2 \\ &\quad + (-2z^{-2} - r^2z^{-1})\sum_{k=0}^{n} z^kW_k^2 - 2rz^{-1}\sum_{k=0}^{n} kz^kW_{k+1}W_k + 2rz^{-1}\sum_{k=0}^{n} z^kW_{k+1}W_k \end{aligned} \tag{29}$$

Next we calculate $\sum_{k=1}^{n} kz^k W_{k+1} W_k$. Using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n \Rightarrow sW_n = W_{n+2} - rW_{n+1}$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

and multiplying the both side of the last relations by W_{n+1} we obtain

$$sW_{n+1}W_n = W_{n+2}W_{n+1} - rW_{n+1}^2$$

and so

$$\begin{aligned}
snz^n W_{n+1}W_n &= nz^n W_{n+2}W_{n+1} - r \times nz^n W_{n+1}^2\\
s(n-1)z^{n-1}W_nW_{n-1} &= (n-1)z^{n-1}W_{n+1}W_n - r \times (n-1)z^{n-1}W_n^2\\
s(n-2)Wz_{n-1}^{n-2}W_{n-2} &= (n-2)z^{n-2}W_nW_{n-1} - r(n-2)z^{n-2}W_{n-1}^2\\
&\vdots\\
s \times 3z^3 W_4W_3 &= 3z^3W_5W_4 - r \times 3z^3W_4^2\\
s \times 2z^2 W_3W_2 &= 2z^2W_4W_3 - r \times 2z^2W_3^2\\
sz^1W_2W_1 &= z^1W_3W_2 - rz^1W_2^2\\
s \times 0 \times z^0W_1W_0 &= 0 \times z^0W_2W_1 - r \times 0 \times z^0W_1^2
\end{aligned}$$

If we add the above equations side by side, we get

$$s\sum_{k=0}^{n} kz^kW_{k+1}W_k = \sum_{k=1}^{n+1}(k-1)z^{k-1}W_{k+1}W_k - r\sum_{k=1}^{n+1}(k-1)z^{k-1}W_k^2 \tag{30}$$

Note that

$$\begin{aligned}
\sum_{k=1}^{n+1}(k-1)z^{k-1}W_{k+1}W_k &= z^{-1}W_1W_0 + nz^nW_{n+2}W_{n+1} + z^{-1}\sum_{k=0}^{n}kz^kW_{k+1}W_k - z^{-1}\sum_{k=0}^{n}z^kW_{k+1}W_k\\
\sum_{k=1}^{n+1}(k-1)z^{k-1}W_k^2 &= z^{-1}W_0^2 + nz^nW_{n+1}^2 + z^{-1}\sum_{k=0}^{n}kz^kW_k^2 - z^{-1}\sum_{k=0}^{n}z^kW_k^2.
\end{aligned}$$

If we put them in (30) we obtain

$$\begin{aligned}
s\sum_{k=0}^{n}kz^kW_{k+1}W_k &= -rnz^nW_{n+1}^2 + nz^nW_{n+2}W_{n+1} - rz^{-1}W_0^2 + z^{-1}W_1W_0\\
&\quad + z^{-1}\sum_{k=0}^{n}kz^kW_{k+1}W_k - z^{-1}\sum_{k=0}^{n}z^kW_{k+1}W_k - rz^{-1}\sum_{k=0}^{n}kz^kW_k^2 + rz^{-1}\sum_{k=0}^{n}z^kW_k^2.
\end{aligned} \tag{31}$$

Then, using

$$W_2 = (rW_1 + sW_0)$$

and Theorem 2.3 (a) (i) and (b) (i) and solving the system (29)-(31), the required results of (a) (i) and (b) (i) of Theorem 3.3 follow.

3.4. Sum Formulas $\sum_{k=1}^{n} kz^kW_{-k}^2$ and $\sum_{k=1}^{n} kz^kW_{-k+1}W_{-k}$ of Generalized Fibonacci (Horadam) Polynomials with Negative Subscripts

The following theorem presents some sum formulas of generalized Fibonacci polynomials with negative subscripts.

Theorem 3.4.
Let z be a non-zero complex (or real) number. For $n \geq 1$ we have the following formulas:

(a)

(i) *If* $(z+s)(-z^2+r^2z+2sz-s^2)\neq 0$ *then*

$$\sum_{k=1}^{n} kz^k W_{-k}^2 = \frac{z(z+s)(-z^2+r^2z+2sz-s^2)\frac{d}{dz}\Phi_{13}-z\Phi_{13}\frac{d}{dz}((z+s)(-z^2+r^2z+2sz-s^2))}{(z+s)^2(-z^2+r^2z+2sz-s^2)^2} = \frac{\Lambda_{25}}{(z+s)^2(-z^2+r^2z+2sz-s^2)^2}$$

where

$\Phi_{13}=(s-z)z^{n+1}W_{-n+1}^2+(-z^2+r^2z+sz+r^2s)z^{n+1}W_{-n}^2-2rsz^{n+1}W_{-n+1}W_{-n}+z(z-s)W_1^2+z(z^2-r^2z-sz-r^2s)W_0^2+2rszW_1W_0$ *given in Theorem 2.4 (a) (i)*

and

$\frac{d}{dz}\Phi_{13}=\Phi_{13}'$ *denotes the derivatives of* Φ_{13} *with respect to* z

and

$\Lambda_{25}=(-z^4+nz^4+2sz^3-2nsz^3-nr^2z^3-2r^2sz^2-2s^2z^2+2ns^3z+nr^2s^2z+2s^3z-s^4-ns^4)z^{n+1}W_{-n+1}^2+(nz^5-2nsz^4-2nr^2z^4-2s^2z^3+nr^4z^3+4s^3z^2+r^2s^2z^2+2ns^3z^2+3nr^2s^2z^2+2nr^4sz^2+nr^4s^2z-2r^2s^3z-ns^4z-2s^4z-r^2s^4-nr^2s^4)z^{n+1}W_{-n}^2-2rs(2z^3-nz^3-sz^2-r^2z^2+nr^2z^2+nsz^2+ns^2z+nr^2sz-ns^3-s^3)z^{n+1}W_{-n+1}W_{-n}+z(z^4-2sz^3-2s^3z+2s^2z^2+2r^2sz^2+s^4)W_1^2+s^2z(-4sz^2+2s^2z+r^2s^2-r^2z^2+2z^3+2r^2sz)W_0^2-2rsz(sz^2+r^2z^2+s^3-2z^3)W_1W_0.$

(ii) *If* $(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *that is, if*

$$z=a=\frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$
$$z=c=-s.$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=1}^{n} kz^k W_{-k}^2 = \frac{\Lambda_{26}}{2(15z^4-20(r^2+s)z^3+6(r^4-s^2)z^2+6s(r^4+2r^2s+2s^2)z+r^4s^2-s^4)}$$

where

$\Lambda_{26}=\frac{d^2}{dz^2}(\Lambda_{25}).$

(iii) *If* $(z+s)(-z^2+r^2z+2sz-s^2)=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *that is, if*

$$z=-s,$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=1}^{n} kz^k W_{-k}^2 = \frac{\Lambda_{27}}{720}$$

where

$\Lambda_{27}=\frac{d^6}{dz^6}(\Lambda_{25}).$

(b)

(i) *If* $(z+s)(-z^2+r^2z+2sz-s^2)\neq 0$ *then*

$$\sum_{k=1}^{n} kz^k W_{-k+1}W_{-k} = \frac{z(z+s)(-z^2+r^2z+2sz-s^2)\frac{d}{dz}\Phi_{16}-z\Phi_{16}\frac{d}{dz}((z+s)(-z^2+r^2z+2sz-s^2))}{(z+s)^2(-z^2+r^2z+2sz-s^2)^2} = \frac{\Lambda_{28}}{(z+s)^2(-z^2+r^2z+2sz-s^2)^2}$$

where

$\Phi_{16}=-rz^{n+2}W_{-n+1}^2-rs^2z^{n+1}W_{-n}^2+(-z^2+r^2z+s^2)z^{n+1}W_{-n+1}W_{-n}+rz^2W_1^2+rs^2zW_0^2-z(-z^2+r^2z+s^2)W_1W_0$ *given in Theorem 2.4 (b) (i)*

and

$\frac{d}{dz}\Phi_{16}=\Phi_{16}'$ *denotes the derivatives of* Φ_{16} *with respect to* z

and

$\Lambda_{28}=r(nz^3-z^3-nsz^2-nr^2z^2-nr^2sz-ns^2z-r^2sz-s^2z+2s^3+ns^3)z^{n+2}W_{-n+1}^2+rs^2(nz^3-2z^3-nsz^2+sz^2+r^2z^2-nr^2z^2-ns^2z-nr^2sz+s^3+ns^3)z^{n+1}W_{-n}^2+(nz^5-sz^4-nsz^4-2nr^2z^4+nr^4z^3-2ns^2z^3-2r^2sz^3+r^4sz^2+2ns^3z^2+2s^3z^2+2nr^2s^2z^2+nr^4sz^2-2r^2s^3z+ns^4z-s^5-ns^5)z^{n+1}W_{-n+1}W_{-n}+rz^2(z^3+s^2z+r^2sz-2s^3)W_1^2-rs^2z(r^2z^2+s^3+sz^2-2z^3)W_0^2+sz(2r^2z^3-r^4z^2-2s^2z^2+s^4+z^4+2r^2s^2z)W_1W_0.$

(ii) *If* $(z+s)(-z^2+r^2z+2sz-s^2) = u(z-a)(z-b)(z-c) = 0$ *for some* $u,a,b,c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$*, that is, if*

$$z = a = \frac{1}{2}\left(r^2+2s+r\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2}\left(r^2+2s-r\sqrt{r^2+4s}\right),$$
$$z = c = -s.$$

provided that $r^2+4s \neq 0$*, then*

$$\sum_{k=1}^{n} kz^k W_{-k+1}W_{-k} = \frac{\Lambda_{29}}{2(15z^4-20(r^2+s)z^3+6(r^4-s^2)z^2+6s(r^4+2r^2s+2s^2)z+r^4s^2-s^4)}$$

where
$\Lambda_{29} = \frac{d^2}{dz^2}(\Lambda_{28})$.

(iii) *If* $(z+s)(-z^2+r^2z+2sz-s^2) = u(z-a)^3 = 0$ *for some* $u,a \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $z=a$*, that is, if*

$$z = -s,$$

provided that $r^2+4s = 0$*, then*

$$\sum_{k=1}^{n} kz^k W_{-k+1}W_{-k} = \frac{\Lambda_{30}}{720}$$

where
$\Lambda_{30} = \frac{d^6}{dz^6}(\Lambda_{28})$.

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[12], Theorem 4.1].

(a)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=1}^{n} z^k W_{-k}^2 = \frac{\Phi_{13}}{(z+s)(-z^2+r^2z+2sz-s^2)}$$

where
$\Phi_{13} = (s-z)z^{n+1}W_{-n+1}^2 + (-z^2+r^2z+sz+r^2s)z^{n+1}W_{-n}^2 - 2rsz^{n+1}W_{-n+1}W_{-n} + z(z-s)W_1^2 + z(z^2-r^2z-sz-r^2s)W_0^2 + 2rszW_1W_0$
which is given in Theorem 2.4 (a) (i), then we get

$$\sum_{k=1}^{n} kz^{k-1}W_{-k}^2 = \frac{d}{dz}\left(\frac{\Phi_{13}}{(z+s)(-z^2+r^2z+2sz-s^2)}\right)$$
$$\Rightarrow$$
$$\begin{aligned}\sum_{k=1}^{n} kz^k W_{-k}^2 &= z\frac{d}{dz}\left(\frac{\Phi_{13}}{(z+s)(-z^2+r^2z+2sz-s^2)}\right)\\ &= \frac{z(z+s)(-z^2+r^2z+2sz-s^2)\frac{d}{dz}\Phi_{13} - z\Phi_{13}\frac{d}{dz}((z+s)(-z^2+r^2z+2sz-s^2))}{(z+s)^2(-z^2+r^2z+2sz-s^2)^2}\\ &= \frac{\Lambda_{25}}{(z+s)^2(-z^2+r^2z+2sz-s^2)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s \neq 0$. Then, from Remark 2.4, $a \neq b \neq c$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) (ii) by using

$$\begin{aligned}\sum_{k=1}^{n} a^k W_{-k}^2 &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{25})}{\frac{d^2}{dz^2}\left((z+s)^2(-z^2+r^2z+2sz-s^2)^2\right)}\right|_{z=a}\\ &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{25})}{2(15z^4-20(r^2+s)z^3+6(r^4-s^2)z^2+6s(r^4+2r^2s+2s^2)z+r^4s^2-s^4)}\right|_{z=a}.\end{aligned}$$

The proof for the case $z=b$ and $z=c$ are the same.

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.4, $a=b=c$. We use (a) (i). For $z=a=-\frac{1}{s}$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (six times). Then we get (a) (iii) by using

$$\sum_{k=1}^{n} a^k W_{-k}^2 = \left.\frac{\frac{d^6}{dz^6}(\Lambda_{25})}{\frac{d^6}{dz^6}\left((z+s)^2(-z^2+r^2z+2sz-s^2)^2\right)}\right|_{z=a} = \left.\frac{\frac{d^6}{dz^6}(\Lambda_{25})}{720}\right|_{z=a}.$$

(b)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=1}^{n} z^k W_{-k+1}W_{-k} = \frac{\Phi_{16}}{(z+s)(-z^2+r^2z+2sz-s^2)}$$

where

$\Phi_{16} = -rz^{n+2}W_{-n+1}^2 - rs^2z^{n+1}W_{-n}^2 + (-z^2+r^2z+s^2)z^{n+1}W_{-n+1}W_{-n} + rz^2W_1^2 + rs^2zW_0^2 - z(-z^2+r^2z+s^2)W_1W_0$

which is given in Theorem 2.4 (b) (i), then we get

$$\sum_{k=1}^{n} kz^{k-1} W_{-k+1}W_{-k} = \frac{d}{dz}\left(\frac{\Phi_{16}}{(z+s)(-z^2+r^2z+2sz-s^2)}\right)$$
$$\Rightarrow$$
$$\begin{aligned}\sum_{k=1}^{n} kz^{k} W_{-k+1}W_{-k} &= z\frac{d}{dz}\left(\frac{\Phi_{16}}{(z+s)(-z^2+r^2z+2sz-s^2)}\right)\\ &= \frac{z(z+s)(-z^2+r^2z+2sz-s^2)\frac{d}{dz}\Phi_{16} - z\Phi_{16}\frac{d}{dz}((z+s)(-z^2+r^2z+2sz-s^2))}{(z+s)^2(-z^2+r^2z+2sz-s^2)^2}\\ &= \frac{\Lambda_{28}}{(z+s)^2(-z^2+r^2z+2sz-s^2)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.4, $a\neq b\neq c$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) (ii) by using

$$\begin{aligned}\sum_{k=1}^{n} a^k W_{-k+1}W_{-k} &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{28})}{\frac{d^2}{dz^2}\left((z+s)^2(-z^2+r^2z+2sz-s^2)^2\right)}\right|_{z=a}\\ &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{28})}{2(15z^4-20(r^2+s)z^3+6(r^4-s^2)z^2+6s(r^4+2r^2s+2s^2)z+r^4s^2-s^4)}\right|_{z=a}.\end{aligned}$$

The proof for the case $z=b$ and $z=c$ are the same.

(iii) Suppose that $r^2+4s=0$. Then, from Remark 2.4, $a=b=c$. We use (b) (i). For $z=a=-\frac{1}{s}$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (six times). Then we get (b) (iii) by using

$$\sum_{k=1}^{n} a^k W_{-k+1}W_{-k} = \left.\frac{\frac{d^6}{dz^6}(\Lambda_{28})}{\frac{d^6}{dz^6}\left((z+s)^2(-z^2+r^2z+2sz-s^2)^2\right)}\right|_{z=a} = \left.\frac{\frac{d^6}{dz^6}(\Lambda_{28})}{720}\right|_{z=a}. \square$$

Note that the proof of (i) 's of Theorem 3.4 can be done as in the following Remark.

Remark 3.4.

We present the direct proofs of Theorem 3.4 (a) (i) and (b) (i) without using any derivatives.

Proof of Theorem 3.4 (a) (i) and (b) (i):

Using the recurrence relation

$$W_{-n+2} = r \times W_{-n+1} + s \times W_{-n}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

and using

$$s^2W_{-n}^2 = (W_{-n+2} - rW_{-n+1})^2 = W_{-n+2}^2 + r^2W_{-n+1}^2 - 2rW_{-n+2}W_{-n+1}$$

we obtain

$$\begin{aligned}
s^2nz^nW_{-n}^2 &= nz^nW_{-n+2}^2 + r^2nz^nW_{-n+1}^2 - 2r \times nz^nW_{-n+2}W_{-n+1}\\
s^2(n-1)z^{n-1}W_{-n+1}^2 &= (n-1)z^{n-1}W_{-n+3}^2 + r^2(n-1)z^{n-1}W_{-n+2}^2 - 2r(n-1)z^{n-1}W_{-n+3}W_{-n+2}\\
s^2(n-2)z^{n-2}W_{-n+2}^2 &= (n-2)z^{n-2}W_{-n+4}^2 + r^2(n-2)z^{n-2}W_{-n+3}^2 - 2r(n-2)z^{n-2}W_{-n+4}W_{-n+3}\\
&\vdots\\
s^2 \times 4z^4W_{-4}^2 &= 4z^4W_{-2}^2 + z^4r^24W_{-3}^2 - 2r \times 4z^4W_{-2}W_{-3}\\
s^2 \times 3z^3W_{-3}^2 &= 3z^3W_{-1}^2 + r^23z^3W_{-2}^2 - 2r \times 3z^3W_{-1}W_{-2}\\
s^2 \times 2z^2W_{-2}^2 &= 2z^2W_0^2 + z^2r^22W_{-1}^2 - 2r \times 2z^2W_0W_{-1}\\
s^2z^1W_{-1}^2 &= z^1W_1^2 + r^2z^1W_0^2 - 2rz^1W_1W_0
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s^2\sum_{k=1}^{n} kz^kW_{-k}^2 &= z^1W_1^2 + z\left(2z + r^2\right)W_0^2 - 2rz^1W_1W_0 - (n+1)z^{n+1}W_{-n+1}^2 \\
&\quad -z^{n+1}\left(2z + nr^2 + nz + r^2\right)W_{-n}^2 + 2r(n+1)z^{n+1}W_{-n+1}W_{-n}\\
&\quad +(z^2 + r^2z)\sum_{k=1}^{n} kz^kW_{-k}^2 + (2z^2 + r^2z)\sum_{k=1}^{n} z^kW_{-k}^2 - 2rz\sum_{k=1}^{n} kz^kW_{-k+1}W_{-k}\\
&\quad -2rz\sum_{k=1}^{n} z^kW_{-k+1}W_{-k}.
\end{aligned} \tag{32}$$

Next we calculate $\sum_{k=1}^{n} kW_{-k+1}W_{-k}$. Using the recurrence relation

$$W_{-n+2} = r \times W_{-n+1} + s \times W_{-n}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

and and multiplying the both side of the last relations by W_{-n+1} we get

$$sW_{-n+1}W_{-n} = W_{-n+2}W_{-n+1} - rW_{-n+1}^2$$

and so we obtain

$$\begin{aligned}
s \times nz^nW_{-n+1}W_{-n} &= nz^nW_{-n+2}W_{-n+1} - r \times nz^nW_{-n+1}^2\\
s \times (n-1)z^{n-1}W_{-n+2}W_{-n+1} &= (n-1)z^{n-1}W_{-n+3}W_{-n+2} - r \times (n-1)z^{n-1}W_{-n+2}^2\\
s \times (n-2)z^{n-2}W_{-n+3}W_{-n+2} &= (n-2)z^{n-2}W_{-n+4}W_{-n+3} - r \times (n-2)z^{n-2}W_{-n+3}^2\\
&\vdots\\
s \times 4z^4W_{-3}W_{-4} &= 4z^4W_{-2}W_{-3} - r \times 4z^4W_{-3}^2\\
s \times 3z^3W_{-2}W_{-3} &= 3z^3W_{-1}W_{-2} - r \times 3z^3W_{-2}^2\\
s \times 2z^2W_{-1}W_{-2} &= 2z^2W_0W_{-1} - r \times 2z^2W_{-1}^2\\
szW_0W_{-1} &= zW_1W_0 - r \times zW_0^2
\end{aligned}$$

If we add the above equations side by side, we get

$$s\sum_{k=1}^{n} kz^kW_{-k+1}W_{-k} = -rzW_0^2 + zW_1W_0 + r(n+1)z^{n+1}W_{-n}^2 - (n+1)z^{n+1}W_{-n+1}W_{-n} \tag{33}$$

$$+z\sum_{k=1}^{n} kz^kW_{-k+1}W_{-k} + z\sum_{k=1}^{n} z^kW_{-k+1}W_{-k} - rz\sum_{k=1}^{n} kz^kW_{-k}^2 - rz\sum_{k=1}^{n} z^kW_{-k}^2. \tag{34}$$

Then, using Theorem 2.4 (a) and (b) (i) and solving the system (32)-(33), the required results of (a) (i) and (b) (i) of Theorem 3.4 follow.

3.5. Sum Formulas $\sum_{k=0}^{n} kz^kW_k^3$, $\sum_{k=0}^{n} kz^kW_k^2W_{k+1}$ and $\sum_{k=0}^{n} kz^kW_{k+1}^2W_k$ of Generalized Fibonacci (Horadam) Polynomials with Positive Subscripts

The following theorem presents some sum formulas of generalized Fibonacci polynomials with positive subscripts.

Theorem 3.5.
Let z be a non-zero complex (or real) number. For $n \geq 0$ we have the following formulas:

(a)

(i) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1) \neq 0$ *then*

$$\sum_{k=0}^{n} kz^kW_k^3 = \frac{z(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\frac{d}{dz}\Phi_{19} - z\Phi_{19}\frac{d}{dz}((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1))}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$
$$= \frac{\Lambda_{31}}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Phi_{19} = -(s^3z^2+2rsz-1)z^{n+2}W_{n+2}^3 - (-r^3s^3z^3+s^3z^2+3r^2s^2z^2+r^4sz^2+r^3z+2rsz-1)z^{n+1}W_{n+1}^3 + 3rs(s^2z+r)z^{n+3}W_{n+2}^2W_{n+1} - 3rs^2(rsz-1)z^{n+3}W_{n+1}^2W_{n+2} + z(s^3z^2+2rsz-1)W_1^3 + (-r^3s^3z^3+s^3z^2+3r^2s^2z^2+r^4sz^2+r^3z+2rsz-1)W_0^3 - 3rsz^2(r+s^2z)W_1^2W_0 + 3rs^2z^2(rsz-1)W_0^2W_1$ *given in Theorem 2.5 (a) (i)*
and
$\frac{d}{dz}\Phi_{19} = \Phi_{19}'$ *denotes the derivatives of* Φ_{19} *with respect to* z
and
$\Lambda_{31} = -(n(-s^3z^2+rsz+1)(s^3z^2+2rsz-1)(r^3z+s^3z^2+3rsz-1) - r^3z - 4s^3z^2 + 2s^6z^4 + 8r^2s^2z^2 + 8r^3s^3z^3 + 2r^5s^2z^3 + 6r^2s^5z^4 + 2r^4s^4z^4 - r^3s^6z^5 + 4r^4sz^2 + 8rs^4z^3 - 8rsz + 2)z^{n+2}W_{n+2}^3 - (n(-s^3z^2+rsz+1)(r^3z+s^3z^2+3rsz-1)(r^3z+s^3z^2+3r^2s^2z^2-r^3s^3z^3+r^4sz^2+2rsz-1) - 2r^3z - s^3z^2 + r^6z^2 - s^6z^4 + s^9z^6 - 2r^2s^2z^2 + 16r^3s^3z^3 + 10r^5s^2z^3 + 13r^2s^5z^4 + 10r^4s^4z^4 + 4r^6s^3z^4 - 2r^3s^6z^5 + r^8s^2z^4 - 6r^5s^5z^5 + 3r^2s^8z^6 - 2r^7s^4z^5 + 3r^4s^7z^6 + r^6s^6z^6 + 1 + 2r^4sz^2 + 2r^7sz^3 + 4rs^7z^5 - 4rsz)z^{n+1}W_{n+1}^3 + 3rs(n(-s^3z^2+rsz+1)(r+s^2z)(r^3z+s^3z^2+3rsz-1) - 3r + 2r^4z - 4s^2z + 4s^5z^3 + 6r^3s^2z^2 + 6r^2s^4z^3 + 2r^4s^3z^3 - r^3s^5z^4 + 8rs^3z^2 + r^5sz^2 - rs^6z^4 + 4r^2sz)z^{n+3}W_{n+2}^2W_{n+1} - 3rs^2(n(-s^3z^2+rsz+1)(rsz-1)(r^3z+s^3z^2+3rsz-1) + (2r^3z-s^3z^2-r^2s^2z^2+5rsz-3)(s^3z^2+r^2s^2z^2+rsz-1))z^{n+3}W_{n+1}^2W_{n+2} + z(s^3z^2+1)(-2s^3z^2+s^6z^4+7r^2s^2z^2+3r^4sz^2+4rs^4z^3-4rsz+1)W_1^3 + s^3z^2(-r^3z-4s^3z^2+2s^6z^4+8r^2s^2z^2+8r^3s^3z^3+2r^5s^2z^3+6r^2s^5z^4+2r^4s^4z^4-r^3s^6z^5+4r^4sz^2+8rs^4z^3-8rsz+2)W_0^3 - 3rsz^2(-2r+r^4z-3s^2z+2s^5z^3+s^8z^5+2r^3s^2z^2+5r^2s^4z^3+2r^4s^3z^3+2r^2sz+4rs^3z^2+2rs^6z^4)W_1^2W_0 + 3rs^2z^2(-r^3z-2s^6z^4+4r^2s^2z^2+2r^3s^3z^3+r^5s^2z^3+2r^4sz^2+rs^7z^5-5rsz+2)W_0^2W_1.$

(ii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1) = u(z-a)(z-b)(z-c)(z-d) = 0$ *for some* $u,a,b,c,d \in \mathbb{C}$ *and* $u \neq 0$ *and* $a \neq b \neq c \neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *that is, if*

$$z = a = \frac{1}{2s^2}\left(r+\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2s^2}\left(r-\sqrt{r^2+4s}\right),$$
$$z = c = \frac{1}{2s^3}\left(-r^3-3rs+\left(s+r^2\right)\sqrt{r^2+4s}\right),$$
$$z = d = \frac{1}{2s^3}\left(-r^3-3rs-\left(s+r^2\right)\sqrt{r^2+4s}\right),$$

provided that $r^2+4s \neq 0$, *then*

$$\sum_{k=0}^{n} kz^kW_k^3 = \frac{\Lambda_{32}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2},$$

where
$\Lambda_{32} = \frac{d^2}{dz^2}(\Lambda_{31})$.

(iii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1) = u(z-a)(z-b)(z-c)^2 = 0$ *for some* $u,a,b,c \in \mathbb{C}$ *and* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z = a = \frac{1}{2r^4}\left(r+\sqrt{-3r^2}\right),$$
$$z = b = \frac{1}{2r^4}\left(r-\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=0}^{n} kz^kW_k^3 = \frac{\Lambda_{33}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Lambda_{33} = \frac{d^2}{dz^2}(\Lambda_{31})$
and if $z=c$, *that is, if*

$$z = c = -\frac{1}{r^3},$$

provided that $r^2+s=0$, *then*

$$\sum_{k=0}^{n} kz^kW_k^3 = \frac{\Lambda_{34}}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Lambda_{34} = \frac{d^4}{dz^4}(\Lambda_{31})$.

(iv) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1) = u(z-a)^4 = 0$ *for some* $u,a\in\mathbb{C}, u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z = a = \frac{r}{2s^2} = \frac{8}{r^3}$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=0}^{n} kz^kW_k^3 = \frac{\Lambda_{35}}{40320s^{12}}$$

where
$\Lambda_{35} = \frac{d^8}{dz^8}(\Lambda_{31})$.

(b)

(i) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\neq 0$ *then*

$$\sum_{k=0}^{n} kz^kW_k^2W_{k+1} = \frac{z(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\frac{d}{dz}\Phi_{24} - z\Phi_{24}\frac{d}{dz}((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1))}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2} = \frac{\Lambda_{36}}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Phi_{24} = -r(rsz-1)z^{n+2}W_{n+2}^3 - rs^3(rsz-1)z^{n+3}W_{n+1}^3 + s(-s^3z^2+2r^3z+1)z^{n+2}W_{n+2}^2W_{n+1} - (-2rs^4z^3+s^3z^2+r^4sz^2+r^3z+2rsz-1)z^{n+1}W_{n+1}^2W_{n+2} + rz(rsz-1)W_1^3 + rs^3z^2(rsz-1)W_0^3 - sz(-s^3z^2+2r^3z+1)W_1^2W_0 + (-2rs^4z^3+s^3z^2+r^4sz^2+r^3z+2rsz-1)W_0^2W_1$ *given in Theorem 2.5 (b) (i)*
and
$\frac{d}{dz}\Phi_{24} = \Phi_{24}'$ *denotes the derivatives of* Φ_{24} *with respect to* z
and
$\Lambda_{36} = -r(n(-s^3z^2+rsz+1)(rsz-1)(r^3z+s^3z^2+3rsz-1) - r^3z - 2s^6z^4 + 4r^2s^2z^2 + 2r^3s^3z^3 + r^5s^2z^3 + 2r^4sz^2 + rs^7z^5 - 5rsz + 2)z^{n+2}W_{n+2}^3 - rs^3(n(rsz-1)(-s^3z^2+rsz+1)(r^3z+s^3z^2+3rsz-1) + (s^3z^2+r^2s^2z^2+rsz-1)(2r^3z-s^3z^2-r^2s^2z^2+5rsz-3))z^{n+3}W_{n+1}^3 + s(n(-s^3z^2+rsz+1)(2r^3z-s^3z^2+1)(r^3z+s^3z^2+3rsz-1) - 5r^3z + 4s^3z^2 + 4r^6z^2 - 2s^6z^4 + 2r^3s^3z^3 + 6r^5s^2z^3 - 6r^2s^5z^4 - 2r^4s^4z^4 + 3r^3s^6z^5 + 8r^4sz^2 - 4rs^4z^3 + 2r^7sz^3 + 2rs^7z^5 + 2rsz - 2)z^{n+2}W_{n+2}^2W_{n+1} - (n(-s^3z^2+rsz+1)(r^3z+s^3z^2+3rsz-1)(r^3z+s^3z^2+r^4sz^2-2rs^4z^3+2rsz-1) - 2r^3z - s^3z^2 + r^6z^2 - s^6z^4 + s^9z^6 + 7r^2s^2z^2 + 4r^5s^2z^3 - 5r^2s^5z^4 + r^4s^4z^4 + 4r^6s^3z^4 - 10r^3s^6z^5 + r^8s^2z^4 - 4r^5s^5z^5 + 4r^2s^8z^6 + 3r^4s^7z^6 + 2r^4sz^2 + 8rs^4z^3 + 2r^7sz^3 - 4rs^7z^5 - 4rsz + 1)z^{n+1}W_{n+1}^2W_{n+2} + rz(2s^3z^2 - 3s^6z^4 + 5r^2s^2z^2 - 2r^3s^3z^3 + 2r^2s^5z^4 + r^4s^4z^4 + 2r^4sz^2 - 4rs^4z^3 + 2rs^7z^5 - 2rsz + 1)W_1^3 + rs^3z^2(-r^3z - 2s^6z^4 + 4r^2s^2z^2 + 2r^3s^3z^3 + r^5s^2z^3 + 2r^4sz^2 + rs^7z^5 - 5rsz + 2)W_0^3 + sz(s^3z^2+1)(4r^3z - 2s^3z^2 - 2r^6z^2 + s^6z^4 + 3r^2s^2z^2 - 4r^3s^3z^3 - 3r^4sz^2 + 1)W_1^2W_0 + s^2z^2(rsz-1)(-2s + 3r^5z + 4s^4z^2 - 2s^7z^4 - 6r^2 + 2r^3s^4z^3 + 4r^3sz + r^6sz^2)W_0^2W_1$.

(ii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)(z-b)(z-c)(z-d)=0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$*, that is, if*

$$\begin{aligned} z=a&=\frac{1}{2s^2}\left(r+\sqrt{r^2+4s}\right),\\ z=b&=\frac{1}{2s^2}\left(r-\sqrt{r^2+4s}\right),\\ z=c&=\frac{1}{2s^3}\left(-r^3-3rs+\left(s+r^2\right)\sqrt{r^2+4s}\right),\\ z=d&=\frac{1}{2s^3}\left(-r^3-3rs-\left(s+r^2\right)\sqrt{r^2+4s}\right),\end{aligned}$$

provided that $r^2+4s\neq 0$*, then*

$$\sum_{k=0}^{n}kz^kW_k^2W_{k+1}=\frac{\Lambda_{37}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Lambda_{37}=\frac{d^2}{dz^2}(\Lambda_{36})$.

(iii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)(z-b)(z-c)^2=0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$*, then if* $z=a$ *or* $z=b$*, that is, if*

$$\begin{aligned} z=a&=\frac{1}{2r^4}\left(r+\sqrt{-3r^2}\right),\\ z=b&=\frac{1}{2r^4}\left(r-\sqrt{-3r^2}\right),\end{aligned}$$

provided that $r^2+s=0$*, then*

$$\sum_{k=0}^{n}kz^kW_k^2W_{k+1}=\frac{\Lambda_{38}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Lambda_{38}=\frac{d^2}{dz^2}(\Lambda_{36})$
and if $z=c$ *, that is, if*

$$z=c=-\frac{1}{r^3},$$

provided that $r^2+s=0$*, then*

$$\sum_{k=0}^{n}kz^kW_k^2W_{k+1}=\frac{\Lambda_{39}}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Lambda_{39}=\frac{d^4}{dz^4}(\Lambda_{36})$.

(iv) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)^4=0$ *for some* $u,a\in\mathbb{C},u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{r}{2s^2}=\frac{8}{r^3}$$

provided that $r^2+4s=0$*, then*

$$\sum_{k=0}^{n}kz^kW_k^2W_{k+1}=\frac{\Lambda_{40}}{40320s^{12}}$$

where
$\Lambda_{40}=\frac{d^8}{dz^8}(\Lambda_{36})$.

(c)

(i) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\neq 0$ *then*

$$\sum_{k=0}^{n} kz^kW_{k+1}^2W_k = \frac{z(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\frac{d}{dz}\Phi_{29} - z\Phi_{29}\frac{d}{dz}((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1))}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$
$$= \frac{\Lambda_{41}}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Phi_{29} = r(s^2z+r)z^{n+2}W_{n+2}^3 + rs^3(s^2z+r)z^{n+3}W_{n+1}^3 - (s^3z^2+3r^2s^2z^2+r^3z-1)z^{n+1}W_{n+2}^2W_{n+1} + s^2(-s^3z^2+2r^3z+1)z^{n+2}W_{n+1}^2W_{n+2} - rz(s^2z+r)W_1^3 - rs^3z^2(s^2z+r)W_0^3 + (s^3z^2+3r^2s^2z^2+r^3z-1)W_1^2W_0 - s^2z(-s^3z^2+2r^3z+1)W_0^2W_1$ *given in Theorem 2.5 (c) (i)*
and
$\frac{d}{dz}\Phi_{29} = \Phi_{29}'$ *denotes the derivatives of* Φ_{29} *with respect to z*
and
$\Lambda_{41} = r(n(-s^3z^2+rsz+1)(r+s^2z)(r^3z+s^3z^2+3rsz-1) - 2r + r^4z - 3s^2z + 2s^5z^3 + s^8z^5 + 2r^3s^2z^2 + 5r^2s^4z^3 + 2r^4s^3z^3 + 2r^2sz + 4rs^3z^2 + 2rs^6z^4)z^{n+2}W_{n+2}^3 + rs^3(n(r+s^2z)(-s^3z^2+rsz+1)(r^3z+s^3z^2+3rsz-1) - 3r + 2r^4z - 4s^2z + 4s^5z^3 + 6r^3s^2z^2 + 6r^2s^4z^3 + 2r^4s^3z^3 - r^3s^5z^4 + 4r^2sz + 8rs^3z^2 + r^5sz^2 - rs^6z^4)z^{n+3}W_{n+1}^3 - (n(-s^3z^2+rsz+1)(r^3z+s^3z^2+3rsz-1)(r^3z+s^3z^2+3r^2s^2z^2-1) - 2r^3z - s^3z^2 + r^6z^2 - s^6z^4 + s^9z^6 - 6r^2s^2z^2 + 12r^3s^3z^3 + 6r^5s^2z^3 + 9r^2s^5z^4 + 12r^4s^4z^4 + 4r^6s^3z^4 + 2r^3s^6z^5 + 3r^2s^8z^6 + 3r^4sz^2 + 1)z^{n+1}W_{n+2}^2W_{n+1} + s^2(n(r^3z+s^3z^2+3rsz-1)(-s^3z^2+rsz+1)(2r^3z-s^3z^2+1) - 5r^3z + 4s^3z^2 + 4r^6z^2 - 2s^6z^4 + 2r^3s^3z^3 + 6r^5s^2z^3 - 6r^2s^5z^4 - 2r^4s^4z^4 + 3r^3s^6z^5 + 8r^4sz^2 - 4rs^4z^3 + 2r^7sz^3 + 2rs^7z^5 + 2rsz - 2)z^{n+2}W_{n+1}^2W_{n+2} + rz(r + 2s^2z - 2s^8z^5 + 2r^3s^2z^2 - 4r^2s^4z^3 - 2r^4s^3z^3 - r^3s^5z^4 + r^5sz^2 - 5rs^6z^4)W_1^3 + rs^3z^2(2r - r^4z + 3s^2z - 2s^5z^3 - s^8z^5 - 2r^3s^2z^2 - 5r^2s^4z^3 - 2r^4s^3z^3 - 2r^2sz - 4rs^3z^2 - 2rs^6z^4)W_0^3 + sz(r+s^2z)(2r^3z - 4s^3z^2 - r^6z^2 + 2s^6z^4 + 4r^3s^3z^3 + 3r^5s^2z^3 + 6r^2s^5z^4 + 2)W_1^2W_0 + s^2z(s^3z^2+1)(4r^3z - 2s^3z^2 - 2r^6z^2 + s^6z^4 + 3r^2s^2z^2 - 4r^3s^3z^3 - 3r^4sz^2 + 1)W_0^2W_1.$

(ii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1) = u(z-a)(z-b)(z-c)(z-d) = 0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *that is, if*

$$z = a = \frac{1}{2s^2}\left(r+\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2s^2}\left(r-\sqrt{r^2+4s}\right),$$
$$z = c = \frac{1}{2s^3}\left(-r^3-3rs+\left(s+r^2\right)\sqrt{r^2+4s}\right),$$
$$z = d = \frac{1}{2s^3}\left(-r^3-3rs-\left(s+r^2\right)\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=0}^{n} kz^kW_{k+1}^2W_k = \frac{\Lambda_{42}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Lambda_{42} = \frac{d^2}{dz^2}(\Lambda_{41})$.

(iii) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1) = u(z-a)(z-b)(z-c)^2 = 0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z = a = \frac{1}{2r^4}\left(r+\sqrt{-3r^2}\right),$$
$$z = b = \frac{1}{2r^4}\left(r-\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=0}^{n} kz^kW_{k+1}^2W_k = \frac{\Lambda_{43}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Lambda_{43} = \frac{d^2}{dz^2}(\Lambda_{41})$
and if $z=c$, *that is, if*

$$z = c = -\frac{1}{r^3},$$

provided that $r^2+s=0$, *then*

$$\sum_{k=0}^{n} kz^k W_{k+1}^2 W_k = \frac{\Lambda_{44}}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

where
$\Lambda_{44}=\frac{d^4}{dz^4}(\Lambda_{41})$.

(iv) *If* $(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)=u(z-a)^4=0$ *for some* $u,a\in\mathbb{C}, u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{r}{2s^2}=\frac{8}{r^3}$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=0}^{n} kz^k W_{k+1}^2 W_k = \frac{\Lambda_{45}}{40320s^{12}}$$

where
$\Lambda_{45}=\frac{d^8}{dz^8}(\Lambda_{41})$.

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [[13], Theorem 2.1].

(a)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=0}^{n} z^k W_k^3 = \frac{\Phi_{19}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}$$

where
$\Phi_{19}=-(s^3z^2+2rsz-1)z^{n+2}W_{n+2}^3-(-r^3s^3z^3+s^3z^2+3r^2s^2z^2+r^4sz^2+r^3z+2rsz-1)z^{n+1}W_{n+1}^3+3rs(s^2z+r)z^{n+3}W_{n+2}^2W_{n+1}-3rs^2(rsz-1)z^{n+3}W_{n+1}^2W_{n+2}+z(s^3z^2+2rsz-1)W_1^3+(-r^3s^3z^3+s^3z^2+3r^2s^2z^2+r^4sz^2+r^3z+2rsz-1)W_0^3-3rsz^2(r+s^2z)W_1^2W_0+3rs^2z^2(rsz-1)W_0^2W_1$
which is given in Theorem 2.5 (a) (i), then we get

$$\sum_{k=0}^{n} kz^{k-1} W_k^3 = \frac{d}{dz}\left(\frac{\Phi_{19}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}\right)$$
$$\Rightarrow$$
$$\begin{aligned}\sum_{k=0}^{n} kz^{k} W_k^3 &= z\frac{d}{dz}\left(\frac{\Phi_{19}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}\right)\\ &= \frac{z(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\frac{d}{dz}\Phi_{19} - z\Phi_{19}\frac{d}{dz}((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1))}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\\ &= \frac{\Lambda_{31}}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.5, $a\neq b\neq c\neq d$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) (ii) by using

$$\begin{aligned}\sum_{k=0}^{n} ka^k W_k^3 &= \left.\frac{\frac{d^2}{dz^2}(\Lambda_{31})}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a}\\ &= \left.\frac{\Lambda_{32}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a}.\end{aligned}$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.5, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (a) (ii), we obtain

$$\sum_{k=0}^{n} ka^kW_k^3 = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{31})}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a} = \left.\frac{\Lambda_{33}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a}$$

and

$$\sum_{k=0}^{n} kb^kW_k^3 = \left.\frac{\Lambda_{33}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (a) (iii) by using

$$\sum_{k=0}^{n} kc^kW_k^3 = \left.\frac{\frac{d^4}{dz^4}(\Lambda_{31})}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=c} = \left.\frac{\Lambda_{34}}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=c}.$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.5, $a=b=c=d$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (eight times). Then we get (a) (iii) by using

$$\sum_{k=0}^{n} ka^kW_k^3 = \left.\frac{\frac{d^8}{dz^8}(\Lambda_{31})}{\frac{d^8}{dz^8}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a} = \left.\frac{\Lambda_{35}}{40320s^{12}}\right|_{z=a}.$$

(b)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=0}^{n} z^kW_k^2W_{k+1} = \frac{\Phi_{24}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}$$

where

$\Phi_{24} = -r(rsz-1)z^{n+2}W_{n+2}^3 - rs^3(rsz-1)z^{n+3}W_{n+1}^3 + s(-s^3z^2+2r^3z+1)z^{n+2}W_{n+2}^2W_{n+1} - (-2rs^4z^3+s^3z^2+r^4sz^2+r^3z+2rsz-1)z^{n+1}W_{n+1}^2W_{n+2} + rz(rsz-1)W_1^3 + rs^3z^2(rsz-1)W_0^3 - sz(-s^3z^2+2r^3z+1)W_1^2W_0 + (-2rs^4z^3+s^3z^2+r^4sz^2+r^3z+2rsz-1)W_0^2W_1$

which is given in Theorem 2.5 (b) (i), then we get

$$\sum_{k=0}^{n} kz^{k-1}W_k^2W_{k+1} = \frac{d}{dz}\left(\frac{\Phi_{24}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}\right)$$

$$\Rightarrow$$

$$\begin{aligned}\sum_{k=0}^{n} kz^kW_k^2W_{k+1} &= z\frac{d}{dz}\left(\frac{\Phi_{24}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}\right)\\ &= \frac{z(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\frac{d}{dz}\Phi_{24} - z\Phi_{24}\frac{d}{dz}((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1))}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\\ &= \frac{\Lambda_{36}}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.5, $a\neq b\neq c\neq d$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) (ii) by using

$$\sum_{k=0}^{n} ka^kW_k^2W_{k+1} = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{36})}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a} = \left.\frac{\Lambda_{37}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a}.$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.5, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (b) (ii), we obtain

$$\sum_{k=0}^{n} ka^kW_k^2W_{k+1} = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{36})}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a} = \left.\frac{\Lambda_{38}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a}$$

and

$$\sum_{k=0}^{n} kb^kW_k^2W_{k+1} = \left.\frac{\Lambda 38}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (b) (iii) by using

$$\sum_{k=0}^{n} kc^kW_k^2W_{k+1} = \left.\frac{\frac{d^4}{dz^4}(\Lambda_{36})}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=c} = \left.\frac{\Lambda_{39}}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=c}.$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.5, $a=b=c=d$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (eight times). Then we get (b) (iii) by using

$$\sum_{k=0}^{n} ka^kW_k^2W_{k+1} = \left.\frac{\frac{d^8}{dz^8}(\Lambda_{36})}{\frac{d^8}{dz^8}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a} = \left.\frac{\Lambda_{40}}{40320s^{12}}\right|_{z=a}.$$

(c)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=0}^{n} z^kW_{k+1}^2W_k = \frac{\Phi_{29}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}$$

where

$\Phi_{29} = r(s^2z+r)z^{n+2}W_{n+2}^3+rs^3(s^2z+r)z^{n+3}W_{n+1}^3-(s^3z^2+3r^2s^2z^2+r^3z-1)z^{n+1}W_{n+2}^2W_{n+1}+s^2(-s^3z^2+2r^3z+1)z^{n+2}W_{n+1}^2W_{n+2}-rz(s^2z+r)W_1^3-rs^3z^2(s^2z+r)W_0^3+(s^3z^2+3r^2s^2z^2+r^3z-1)W_1^2W_0-s^2z(-s^3z^2+2r^3z+1)W_0^2W_1$

which is given in Theorem 2.5 (c) (i), then we get

$$\sum_{k=0}^{n} kz^{k-1}W_{k+1}^2W_k = \frac{d}{dz}\left(\frac{\Phi_{29}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}\right)$$

$$\Rightarrow$$

$$\sum_{k=0}^{n} kz^kW_{k+1}^2W_k = z\frac{d}{dz}\left(\frac{\Phi_{29}}{(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)}\right)$$

$$= \frac{z(-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1)\frac{d}{dz}\Phi_{29} - z\Phi_{29}\frac{d}{dz}((-s^3z^2+rsz+1)(s^3z^2+r^3z+3rsz-1))}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}$$

$$= \frac{\Lambda_{41}}{(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}.$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.5, $a\neq b\neq c\neq d$. We use (c) (i). For $z=a$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) (ii) by using

$$\sum_{k=0}^{n} ka^kW_{k+1}^2W_k = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{41})}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a} = \left.\frac{\Lambda_{42}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a}.$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.5, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (c) (ii), we obtain

$$\sum_{k=0}^{n} ka^kW_{k+1}^2W_k = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{41})}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a} = \left.\frac{\Lambda_{43}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a}$$

and

$$\sum_{k=0}^{n} kb^kW_{k+1}^2W_k = \left.\frac{\Lambda_{43}}{\frac{d^2}{dz^2}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) (iii) by using

$$\sum_{k=0}^{n} kc^kW_{k+1}^2W_k = \left.\frac{\frac{d^4}{dz^4}(\Lambda_{41})}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=c} = \left.\frac{\Lambda_{44}}{\frac{d^4}{dz^4}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=c}.$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.5, $a=b=c=d$. We use (c) (i). For $z=a$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (eight times). Then we get (c) (iii) by using

$$\sum_{k=0}^{n} ka^kW_{k+1}^2W_k = \left.\frac{\frac{d^8}{dz^8}(\Lambda_{41})}{\frac{d^8}{dz^8}(-s^3z^2+rsz+1)^2(s^3z^2+r^3z+3rsz-1)^2}\right|_{z=a} = \left.\frac{\Lambda_{45}}{40320s^{12}}\right|_{z=a}. \square$$

Note that the proof of (i) 's of Theorem 3.5 can be done as in the following Remark.

Remark 3.5.

We present the direct proofs of Theorem 3.5 (a) (i), (b) (i) and (c) (i) without using any derivatives.

Proof of Theorem 3.5 (a) (i), (b) (i) and (c) (i):

Using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n$$

i.e.

$$sW_n=W_{n+2}-rW_{n+1}$$

we obtain

$$s^3W_n^3=W_{n+2}^3-3rW_{n+2}^2W_{n+1}+3r^2W_{n+1}^2W_{n+2}-r^3W_{n+1}^3$$

and so

$$\begin{aligned}
s^3\times n\times z^nW_n^3 &= n\times z^nW_{n+2}^3-3r\times n\times z^nW_{n+2}^2W_{n+1}\\
&\quad +3r^2\times n\times z^nW_{n+1}^2W_{n+2}-r^3\times n\times z^nW_{n+1}^3\\
s^3(n-1)z^{n-1}W_{n-1}^3 &= (n-1)z^{n-1}W_{n+1}^3-3r(n-1)z^{n-1}W_{n+1}^2W_n\\
&\quad +3r^2(n-1)z^{n-1}W_n^2W_{n+1}-r^3(n-1)z^{n-1}W_n^3\\
s^3(n-2)z^{n-2}W_{n-2}^3 &= (n-2)z^{n-2}W_n^3-3r(n-2)z^{n-2}W_n^2W_{n-1}\\
&\quad +3r^2(n-2)z^{n-2}W_{n-1}^2W_n-r^3(n-2)z^{n-2}W_{n-1}^3\\
&\vdots\\
s^3\times 2\times z^2W_2^3 &= 2\times z^2W_4^3-3r\times 2\times z^2W_4^2W_3+3r^2\times 2\times z^2W_3^2W_4-r^3\times 2\times z^2W_3^3\\
s^3\times 1\times z^1W_1^3 &= 1\times z^1W_3^3-3r\times 1\times z^1W_3^2W_2+3r^2\times 1\times z^1W_2^2W_3-r^3\times 1\times z^1W_2^3\\
s^3\times 0\times z^0W_0^3 &= 0\times z^0W_2^3-3r\times 0\times z^0W_2^2W_1+3r^2\times 0\times z^0W_1^2W_2-r^3\times 0\times z^0W_1^3
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s^3\sum_{k=0}^{n}kz^kW_k^3 &= (nz^nW_{n+2}^3+(n-1)z^{n-1}W_{n+1}^3-(-1)z^{-1}W_1^3-(-2)z^{-2}W_0^3\\
&\quad +z^{-2}\sum_{k=0}^{n}kz^kW_k^3-2z^{-2}\sum_{k=0}^{n}z^kW_k^3)-3r(nz^nW_{n+2}^2W_{n+1}-(-1)z^{-1}W_1^2W_0\\
&\quad +z^{-1}\sum_{k=0}^{n}kz^kW_{k+1}^2W_k-z^{-1}\sum_{k=0}^{n}z^kW_{k+1}^2W_k)\\
&\quad +3r^2(nz^nW_{n+1}^2W_{n+2}-(-1)z^{-1}W_0^2W_1\\
&\quad +z^{-1}\sum_{k=0}^{n}kz^kW_k^2W_{k+1}-z^{-1}\sum_{k=0}^{n}z^kW_k^2W_{k+1})-r^3(nz^nW_{n+1}^3-(-1)z^{-1}W_0^3\\
&\quad +z^{-1}\sum_{k=0}^{n}kz^kW_k^3-z^{-1}\sum_{k=0}^{n}z^kW_k^3).
\end{aligned} \tag{35}$$

Next we calculate $\sum_{k=0}^{n}kz^kW_{k+1}^2W_k$. Again, using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n$$

i.e.

$$sW_n=W_{n+2}-rW_{n+1}$$

we obtain

$$sW_{n+1}^2W_n=W_{n+1}^2W_{n+2}-rW_{n+1}^3$$

and so

$$\begin{aligned}
s\times n\times z^nW_{n+1}^2W_n &= nz^nW_{n+1}^2W_{n+2}-r\times n\times z^nW_{n+1}^3\\
s(n-1)z^{n-1}W_n^2W_{n-1} &= (n-1)z^{n-1}W_n^2W_{n+1}-r(n-1)z^{n-1}W_n^3\\
s(n-2)z^{n-2}W_{n-1}^2W_{n-2} &= (n-2)z^{n-2}W_{n-1}^2W_n-r(n-2)z^{n-2}W_{n-1}^3\\
&\vdots\\
s\times 2\times z^2W_3^2W_2 &= 2\times z^2W_3^2W_4-r\times 2\times z^2W_3^3\\
s\times 1\times z^1W_2^2W_1 &= 1\times z^1W_2^2W_3-r\times 1\times z^1W_2^3\\
s\times 0\times z^0W_1^2W_0 &= 0\times z^0W_1^2W_2-r\times 0\times z^0W_1^3
\end{aligned}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s\sum_{k=0}^{n}kz^kW_{k+1}^2W_k &= (nz^nW_{n+1}^2W_{n+2}-(-1)z^{-1}W_0^2W_1+z^{-1}\sum_{k=0}^{n}kz^kW_k^2W_{k+1}\\
&\quad -z^{-1}\sum_{k=0}^{n}z^kW_k^2W_{k+1})\\
&\quad -r(nz^nW_{n+1}^3-(-1)z^{-1}W_0^3+z^{-1}\sum_{k=0}^{n}kz^kW_k^3-z^{-1}\sum_{k=0}^{n}z^kW_k^3).
\end{aligned} \tag{36}$$

Next we calculate $\sum_{k=0}^{n} kz^k W_k^2 W_{k+1}$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1} \Rightarrow s^2 W_n^2 = W_{n+2}^2 + r^2 W_{n+1}^2 - 2rW_{n+2}W_{n+1}$$

we obtain

$$s^2 W_n^2 W_{n+1} = W_{n+2}^2 W_{n+1} + r^2 W_{n+1}^3 - 2rW_{n+1}^2 W_{n+2}$$

and so

$$\begin{array}{c}
s^2 \times n \times z^n W_n^2 W_{n+1} = n \times z^n W_{n+2}^2 W_{n+1} + r^2 \times n \times z^n W_{n+1}^3 - 2r \times n \times z^n W_{n+1}^2 W_{n+2} \\
s^2 (n-1) z^{n-1} W_{n-1}^2 W_n = (n-1) z^{n-1} W_{n+1}^2 W_n + r^2 (n-1) z^{n-1} W_n^3 - 2r(n-1) z^{n-1} W_n^2 W_{n+1} \\
s^2 (n-2) z^{n-2} W_{n-2}^2 W_{n-1} = (n-2) z^{n-2} W_n^2 W_{n-1} + r^2 (n-2) z^{n-2} W_{n-1}^3 - 2r(n-2) z^{n-2} W_{n-1}^2 W_n \\
\vdots \\
s^2 \times 3 \times z^3 W_3^2 W_4 = 3 \times z^3 W_5^2 W_4 + r^2 \times 3 \times z^3 W_4^3 - 2r \times 3 \times z^3 W_4^2 W_5 \\
s^2 \times 2 \times z^2 W_2^2 W_3 = 2 \times z^2 W_4^2 W_3 + r^2 \times 2 \times z^2 W_3^3 - 2r \times 2 \times z^2 W_3^2 W_4 \\
s^2 \times 1 \times z^1 W_1^2 W_2 = 1 \times z^1 W_3^2 W_2 + r^2 \times 1 \times z^1 W_2^3 - 2r \times 1 \times z^1 W_2^2 W_3 \\
s^2 \times 0 \times z^0 W_0^2 W_1 = 0 \times z^0 W_2^2 W_1 + r^2 \times 0 \times z^0 W_1^3 - 2r \times 0 \times z^0 W_1^2 W_2
\end{array}$$

If we add the above equations side by side, we get

$$\begin{aligned}
s^2 \sum_{k=0}^{n} kz^k W_k^2 W_{k+1} = {} & (nz^n W_{n+2}^2 W_{n+1} - (-1)z^{-1} W_1^2 W_0 + z^{-1} \sum_{k=0}^{n} kz^k W_{k+1}^2 W_k - z^{-1} \sum_{k=0}^{n} z^k W_{k+1}^2 W_k) \\
& + r^2 (nz^n W_{n+1}^3 - (-1)z^{-1} W_0^3 + z^{-1} \sum_{k=0}^{n} kz^k W_k^3 - z^{-1} \sum_{k=0}^{n} z^k W_k^3) \\
& - 2r(nz^n W_{n+1}^2 W_{n+2} - (-1)z^{-1} W_0^2 W_1 + z^{-1} \sum_{k=0}^{n} kz^k W_k^2 W_{k+1} - z^{-1} \sum_{k=0}^{n} z^k W_k^2 W_{k+1}).
\end{aligned} \tag{37}$$

Using Theorem 2.5 and solving the system (35)-(36)-(37), the required results of (a) (ii), (b) (ii) and (c) (ii) of Theorem 3.5 follow.

3.6. Sum Formulas $\sum_{k=1}^{n} kz^k W_{-k}^3$, $\sum_{k=1}^{n} kz^k W_{-k+1}^2 W_{-k}$ and $\sum_{k=1}^{n} kz^k W_{-k}^2 W_{-k+1}$ of Generalized Fibonacci (Horadam) Polynomials with Negative Subscripts

The following theorem presents some sum formulas of generalized Fibonacci polynomials with negative subscripts.

Theorem 3.6.
Let z be a non-zero complex (or real) number. For $n \geq 1$ we have the following formulas:

(a)

(i) *If $(z^2 + rsz - s^3)(-z^2 + r^3 z + 3rsz + s^3) \neq 0$ then*

$$\begin{aligned}
\sum_{k=1}^{n} kz^k W_{-k}^3 &= \frac{\begin{array}{c} z(z^2 + rsz - s^3)(-z^2 + 3rsz + r^3 z + s^3) \frac{d}{dz} \Phi_{34} \\ -z\Phi_{34} \frac{d}{dz}((z^2 + rsz - s^3)(-z^2 + 3rsz + r^3 z + s^3)) \end{array}}{(z^2 + rsz - s^3)^2 (-z^2 + 3rsz + r^3 z + s^3)^2} \\
&= \frac{\Lambda_{46}}{(z^2 + rsz - s^3)^2 (-z^2 + r^3 z + 3rsz + s^3)^2}
\end{aligned}$$

where
$\Phi_{34} = (-z^2 + 2rsz + s^3) z^{n+1} W_{-n+1}^3 + (-z^3 + r^3 z^2 + 2rsz^2 + s^3 z + r^4 sz + 3r^2 s^2 z - r^3 s^3) z^{n+1} W_{-n}^3 - 3rs(rz + s^2) z^{n+1} W_{-n+1}^2 W_{-n} + 3rs^2(-z + rs) z^{n+1} W_{-n}^2 W_{-n+1} - z(-z^2 + s^3 + 2rsz) W_1^3 + z(z^3 - 2rsz^2 - r^3 z^2 - r^4 sz - 3r^2 s^2 z - s^3 z + r^3 s^3) W_0^3 + 3rsz(rz + s^2) W_1^2 W_0 - 3rs^2 z(-z + rs) W_0^2 W_1$ *given in Theorem 2.6 (a) (i)*
and
$\frac{d}{dz}\Phi_{34} = \Phi_{34}'$ *denotes the derivatives of Φ_{34} with respect to z*

and

$\Lambda_{46} = (n(-s^3+z^2+rsz)(s^3-z^2+2rsz)(r^3z+s^3-z^2+3rsz)-(s^3+z^2)(-2s^3z^2+s^6+z^4+7r^2s^2z^2-4rsz^3+4rs^4z+3r^4sz^2))z^{n+1}W^3_{-n+1}+(n(r^3z+s^3-z^2+3rsz)(-s^3+z^2+rsz)(s^3z-r^3s^3+r^3z^2-z^3+2rsz^2+r^4sz+3r^2s^2z)-s^3(2s^6z-r^3s^6-r^3z^4-4s^3z^3+2z^5+8r^2s^2z^3+8r^3s^3z^2+2r^5s^2z^2-8rsz^4+8rs^4z^2+6r^2s^5z+4r^4sz^3+2r^4s^4z))z^{n+1}W^3_{-n}-3rs(n(rz+s^2)(r^3z+s^3-z^2+3rsz)(-s^3+z^2+rsz)+2rz^5-r^4z^4+3s^2z^4-2s^5z^2-s^8-5r^2s^4z^2-2r^3s^2z^3-2r^4s^3z^2-2rs^6z-2r^2sz^4-4rs^3z^3)z^{n+1}W^2_{-n+1}W_{-n}+3rs^2(n(-z+rs)(r^3z+s^3-z^2+3rsz)(-s^3+z^2+rsz)+2s^6z+r^3z^4-2z^5-4r^2s^2z^3-2r^3s^3z^2-r^5s^2z^2-2r^4sz^3-rs^7+5rsz^4)z^{n+1}W^2_{-n}W_{-n+1}+z(s^3+z^2)(-2s^3z^2+s^6+z^4+7r^2s^2z^2-4rsz^3+4rs^4z+3r^4sz^2)W^3_1+s^3z(2s^6z-r^3s^6-r^3z^4-4s^3z^3+2z^5+8r^2s^2z^3+8r^3s^3z^2+2r^5s^2z^2-8rsz^4+8rs^4z^2+6r^2s^5z+4r^4sz^3+2r^4s^4z)W^3_0-3rsz(-2rz^5+r^4z^4-3s^2z^4+2s^5z^2+s^8+5r^2s^4z^2+2r^3s^2z^3+2r^4s^3z^2+2rs^6z+4rs^3z^3+2r^2sz^4)W^2_1W_0+3rs^2z(rs^7-2s^6z-r^3z^4+2z^5+4r^2s^2z^3+2r^3s^3z^2+r^5s^2z^2-5rsz^4+2r^4sz^3)W^2_0W_1.$

(ii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)(z-d)=0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *that is, if*

$$z=a=\frac{1}{2}s\left(-r+\sqrt{r^2+4s}\right),$$
$$z=b=\frac{1}{2}s\left(-r-\sqrt{r^2+4s}\right),$$
$$z=c=\frac{1}{2}\left(r^3+3rs+(r^2+s)\sqrt{r^2+4s}\right),$$
$$z=d=\frac{1}{2}\left(r^3+3rs-(r^2+s)\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=1}^{n}kz^kW^3_{-k}=\frac{\Lambda_{47}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}$$

where

$\Lambda_{47}=\frac{d^2}{dz^2}(\Lambda_{46})$.

(iii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)^2=0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z=a=\frac{1}{2}r^2\left(r-\sqrt{-3r^2}\right),$$
$$z=b=\frac{1}{2}r^2\left(r+\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n}kz^kW^3_{-k}=\frac{\Lambda_{48}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}$$

where

$\Lambda_{48}=\frac{d^2}{dz^2}(\Lambda_{46})$

and if $z=c$, *that is, if*

$$z=c=-r^3,$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n}kz^kW^3_{-k}=\frac{\Lambda_{49}}{\frac{d^4}{dz^4}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}$$

where

$\Lambda_{49}=\frac{d^4}{dz^4}(\Lambda_{46})$

(iv) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)^4=0$ *for some* $u,a\in\mathbb{C}, u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{1}{8}r^3$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=1}^{n}kz^kW^3_{-k}=\frac{\Lambda_{50}}{40320}$$

where

$\Lambda_{50}=\frac{d^8}{dz^8}(\Lambda_{46})$.

(b)

(i) *If* $(z^2+rsz-s^3)(-z^2+r^3z+3rsz+s^3)\neq 0$ *then*

$$\sum_{k=1}^{n} kz^k W_{-k+1}^2 W_{-k} = \frac{z(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\frac{d}{dz}\Phi_{39} - z\Phi_{39}\frac{d}{dz}((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3))}{(z^2+rsz-s^3)^2(-z^2+3rsz+r^3z+s^3)^2} = \frac{\Lambda_{51}}{(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}$$

where

$\Phi_{39} = -r(rz+s^2)z^{n+2}W_{-n+1}^3 - rs^3(rz+s^2)z^{n+1}W_{-n}^3 + (-z^2+r^3z+3r^2s^2+s^3)z^{n+2}W_{-n+1}^2W_{-n} + s^2(-z^2-2r^3z+s^3)z^{n+1}W_{-n}^2W_{-n+1} + rz^2(rz+s^2)W_1^3 + rs^3z(rz+s^2)W_0^3 - z^2(-z^2+r^3z+3r^2s^2+s^3)W_1^2W_0 + s^2z(z^2+2r^3z-s^3)W_0^2W_1$ *given in Theorem 2.6 (b) (i)*

and

$\frac{d}{dz}\Phi_{39} = \Phi_{39}^{'}$ *denotes the derivatives of* Φ_{39} *with respect to* z

and

$\Lambda_{51} = r(n(rz+s^2)(s^3-z^2-rsz)(r^3z+s^3-z^2+3rsz) - 2s^2z^4 + 2s^8 + 4r^2s^4z^2 - 2r^3s^2z^3 + 2r^4s^3z^2 + r^3s^5z - r^5sz^3 + 5rs^6z - rz^5)z^{n+2}W_{-n+1}^3 + rs^3(n(rz+s^2)(s^3-z^2-rsz)(r^3z+s^3-z^2+3rsz) - 2rz^5 + r^4z^4 - 3s^2z^4 + 2s^5z^2 + s^8 + 5r^2s^4z^2 + 2r^3s^2z^3 + 2r^4s^3z^2 + 4rs^3z^3 + 2r^2sz^4 + 2rs^6z)z^{n+1}W_{-n}^3 + (n(r^3z+s^3-z^2+3rsz)(-s^3+z^2+rsz)(r^3z+3r^2s^2+s^3-z^2) + s(rz+s^2)(-6r^2s^5 - 2r^3z^3 + 4s^3z^2 + r^6z^2 - 2s^6 - 2z^4 - 4r^3s^3z - 3r^5s^2z))z^{n+2}W_{-n+1}^2W_{-n} - s^2(n(-s^3+z^2+rsz)(2r^3z-s^3+z^2)(r^3z+s^3-z^2+3rsz) + (s^3+z^2)(4r^3z^3 - 2s^3z^2 - 2r^6z^2 + s^6 + z^4 + 3r^2s^2z^2 - 4r^3s^3z - 3r^4sz^2))z^{n+1}W_{-n}^2W_{-n+1} + rz^2(rz^5 + 2s^2z^4 - 2s^8 - 4r^2s^4z^2 + 2r^3s^2z^3 - 2r^4s^3z^2 - 5rs^6z - r^3s^5z + r^5sz^3)W_1^3 - rs^3z(-2rz^5 + r^4z^4 - 3s^2z^4 + 2s^5z^2 + s^8 + 5r^2s^4z^2 + 2r^3s^2z^3 + 2r^4s^3z^2 + 2rs^6z + 4rs^3z^3 + 2r^2sz^4)W_0^3 + sz^2(rz+s^2)(6r^2s^5 + 2r^3z^3 - 4s^3z^2 - r^6z^2 + 2s^6 + 2z^4 + 4r^3s^3z + 3r^5s^2z)W_1^2W_0 + s^2z(s^3+z^2)(4r^3z^3 - 2s^3z^2 - 2r^6z^2 + s^6 + z^4 + 3r^2s^2z^2 - 4r^3s^3z - 3r^4sz^2)W_0^2W_1.$

(ii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3) = u(z-a)(z-b)(z-c)(z-d) = 0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *that is, if*

$$z = a = \frac{1}{2}s\left(-r+\sqrt{r^2+4s}\right),$$
$$z = b = \frac{1}{2}s\left(-r-\sqrt{r^2+4s}\right),$$
$$z = c = \frac{1}{2}\left(r^3+3rs+(r^2+s)\sqrt{r^2+4s}\right),$$
$$z = d = \frac{1}{2}\left(r^3+3rs-(r^2+s)\sqrt{r^2+4s}\right),$$

provided that $r^2+4s\neq 0$, *then*

$$\sum_{k=1}^{n} kz^k W_{-k+1}^2 W_{-k} = \frac{\Lambda_{52}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}$$

where

$\Lambda_{52} = \frac{d^2}{dz^2}(\Lambda_{51})$.

(iii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3) = u(z-a)(z-b)(z-c)^2 = 0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z = a = \frac{1}{2}r^2\left(r-\sqrt{-3r^2}\right),$$
$$z = b = \frac{1}{2}r^2\left(r+\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n} kz^k W_{-k+1}^2 W_{-k} = \frac{\Lambda_{53}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}$$

where

$\Lambda_{53} = \frac{d^2}{dz^2}(\Lambda_{51})$

and if $z = c$ *, that is, if*

$$z = c = -r^3,$$

provided that $r^2 + s = 0$*, then*

$$\sum_{k=1}^{n} kz^k W_{-k+1}^2 W_{-k} = \frac{\Lambda_{54}}{\frac{d^4}{dz^4}(z^2 + rsz - s^3)^2(-z^2 + r^3z + 3rsz + s^3)^2}$$

where

$\Lambda_{54} = \frac{d^4}{dz^4}(\Lambda_{51})$.

(iv) *If* $(z^2 + rsz - s^3)(-z^2 + 3rsz + r^3z + s^3) = u(z-a)^4 = 0$ *for some* $u, a \in \mathbb{C}, u \neq 0$ *i.e.,* $z = a$ *that is, if*

$$z = a = \frac{1}{8}r^3$$

provided that $r^2 + 4s = 0$*, then*

$$\sum_{k=1}^{n} kz^k W_{-k+1}^2 W_{-k} = \frac{\Lambda_{55}}{40320}$$

where

$\Lambda_{55} = \frac{d^8}{dz^8}(\Lambda_{51})$.

(c)

(i) *If* $(z^2 + rsz - s^3)(-z^2 + r^3z + 3rsz + s^3) \neq 0$ *then*

$$\sum_{k=1}^{n} kz^k W_{-k}^2 W_{-k+1} = \frac{\begin{array}{c} z(z^2 + rsz - s^3)(-z^2 + 3rsz + r^3z + s^3)\frac{d}{dz}\Phi_{44} \\ -z\Phi_{44}\frac{d}{dz}((z^2 + rsz - s^3)(-z^2 + 3rsz + r^3z + s^3)) \end{array}}{(z^2 + rsz - s^3)^2(-z^2 + 3rsz + r^3z + s^3)^2}$$

$$= \frac{\Lambda_{56}}{(z^2 + rsz - s^3)^2(-z^2 + r^3z + 3rsz + s^3)^2}$$

where

$\Phi_{44} = r(-z + rs)z^{n+2}W_{-n+1}^3 + rs^3(-z + rs)z^{n+1}W_{-n}^3 + s(-z^2 - 2r^3z + s^3)z^{n+1}W_{-n+1}^2 W_{-n} + (-z^3 + 2rsz^2 + r^3z^2 + s^3z + r^4sz - 2rs^4)z^{n+1}W_{-n}^2 W_{-n+1} + rz^2(z - rs)W_1^3 + rs^3z(z - rs)W_0^3 + sz(z^2 + 2r^3z - s^3)W_1^2 W_0 + z(z^3 - 2rsz^2 - r^3z^2 - r^4sz - s^3z + 2rs^4)W_0^2 W_1$ *given in Theorem 2.6 (c) (i)*

and

$\frac{d}{dz}\Phi_{44} = \Phi_{44}^{'}$ *denotes the derivatives of* Φ_{44} *with respect to* z

and

$\Lambda_{56} = r(n(-z + rs)(-s^3 + z^2 + rsz)(r^3z + s^3 - z^2 + 3rsz) - 2rs^7 + 3s^6z - 2s^3z^3 - z^5 - 5r^2s^2z^3 + 2r^3s^3z^2 + 2rsz^4 - 2r^2s^5z - 2r^4sz^3 - r^4s^4z + 4rs^4z^2)z^{n+2}W_{-n+1}^3 + rs^3(n(-z + rs)(-s^3 + z^2 + rsz)(r^3z + s^3 - z^2 + 3rsz) - rs^7 + 2s^6z + r^3z^4 - 2z^5 - 4r^2s^2z^3 - 2r^3s^3z^2 - r^5s^2z^2 - 2r^4sz^3 + 5rsz^4)z^{n+1}W_{-n}^3 - s(n(-s^3 + z^2 + rsz)(2r^3z - s^3 + z^2)(r^3z + s^3 - z^2 + 3rsz) + (s^3 + z^2)(4r^3z^3 - 2s^3z^2 - 2r^6z^2 + s^6 + z^4 + 3r^2s^2z^2 - 4r^3s^3z - 3r^4sz^2))z^{n+1}W_{-n+1}^2 W_{-n} + (n(r^3z + s^3 - z^2 + 3rsz)(-s^3 + z^2 + rsz)(-2rs^4 + s^3z + r^3z^2 - z^3 + 2rsz^2 + r^4sz) + s^2(z - rs)(-2sz^4 - 6r^2z^4 + 3r^5z^3 + 4s^4z^2 - 2s^7 + 4r^3sz^3 + 2r^3s^4z + r^6sz^2))z^{n+1}W_{-n}^2 W_{-n+1} + rz^2(2rs^7 - 3s^6z + 2s^3z^3 + z^5 + 5r^2s^2z^3 - 2r^3s^3z^2 - 2rsz^4 - 4rs^4z^2 + 2r^2s^5z + 2r^4sz^3 + r^4s^4z)W_1^3 + rs^3z(rs^7 - 2s^6z - r^3z^4 + 2z^5 + 4r^2s^2z^3 + 2r^3s^3z^2 + r^5s^2z^2 - 5rsz^4 + 2r^4sz^3)W_0^3 + sz(s^3 + z^2)(4r^3z^3 - 2s^3z^2 - 2r^6z^2 + s^6 + z^4 + 3r^2s^2z^2 - 4r^3s^3z - 3r^4sz^2)W_1^2 W_0 + s^2z(-z + rs)(-2sz^4 - 6r^2z^4 + 3r^5z^3 + 4s^4z^2 - 2s^7 + 4r^3sz^3 + 2r^3s^4z + r^6sz^2)W_0^2 W_1$.

(ii) *If* $(z^2 + rsz - s^3)(-z^2 + 3rsz + r^3z + s^3) = u(z-a)(z-b)(z-c)(z-d) = 0$ *for some* $u, a, b, c, d \in \mathbb{C}$ *and* $u \neq 0$ *and* $a \neq b \neq c \neq d$*, i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *or* $z = d$*, that is, if*

$$z = a = \frac{1}{2}s\left(-r + \sqrt{r^2 + 4s}\right),$$

$$z = b = \frac{1}{2}s\left(-r - \sqrt{r^2 + 4s}\right),$$

$$z = c = \frac{1}{2}\left(r^3 + 3rs + (r^2 + s)\sqrt{r^2 + 4s}\right),$$

$$z = d = \frac{1}{2}\left(r^3 + 3rs - (r^2 + s)\sqrt{r^2 + 4s}\right),$$

provided that $r^2 + 4s \neq 0$*, then*

$$\sum_{k=1}^{n} kz^k W_{-k}^2 W_{-k+1} = \frac{\Lambda_{57}}{\frac{d^2}{dz^2}(z^2 + rsz - s^3)^2(-z^2 + r^3z + 3rsz + s^3)^2}$$

where

$\Lambda_{57} = \frac{d^2}{dz^2}(\Lambda_{56})$.

(iii) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)(z-b)(z-c)^2=0$ *for some* $u,a,b,c\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$, *then if* $z=a$ *or* $z=b$, *that is, if*

$$z=a=\frac{1}{2}r^2\left(r-\sqrt{-3r^2}\right),$$
$$z=b=\frac{1}{2}r^2\left(r+\sqrt{-3r^2}\right),$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n}kz^kW_{-k}^2W_{-k+1}=\frac{\Lambda_{58}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}$$

where
$\Lambda_{58}=\frac{d^2}{dz^2}(\Lambda_{56})$
and if $z=c$, *that is, if*

$$z=c=-r^3,$$

provided that $r^2+s=0$, *then*

$$\sum_{k=1}^{n}kz^kW_{-k}^2W_{-k+1}=\frac{\Lambda_{59}}{\frac{d^4}{dz^4}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}$$

where
$\Lambda_{59}=\frac{d^4}{dz^4}(\Lambda_{56})$.

(iv) *If* $(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)=u(z-a)^4=0$ *for some* $u,a\in\mathbb{C},u\neq 0$ *i.e.,* $z=a$ *that is, if*

$$z=a=\frac{1}{8}r^3$$

provided that $r^2+4s=0$, *then*

$$\sum_{k=1}^{n}kz^kW_{-k}^2W_{-k+1}=\frac{\Lambda_{60}}{40320}$$

where
$\Lambda_{60}=\frac{d^8}{dz^8}(\Lambda_{56})$.

Proof. The cases (i)'s for the generalized Fibonacci (Horadam) numbers is given in Soykan [3.6, Theorem 3.1].

(a)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=1}^{n}z^kW_{-k}^3=\frac{\Phi_{34}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}$$

where
$\Phi_{34}=(-z^2+2rsz+s^3)z^{n+1}W_{-n+1}^3+(-z^3+r^3z^2+2rsz^2+s^3z+r^4sz+3r^2s^2z-r^3s^3)z^{n+1}W_{-n}^3-3rs(rz+s^2)z^{n+1}W_{-n+1}^2W_{-n}+3rs^2(-z+rs)z^{n+1}W_{-n}^2W_{-n+1}-z(-z^2+s^3+2rsz)W_1^3+z(z^3-2rsz^2-r^3z^2-r^4sz-3r^2s^2z-s^3z+r^3s^3)W_0^3+3rsz(rz+s^2)W_1^2W_0-3rs^2z(-z+rs)W_0^2W_1$
which is given in Theorem 2.6 (a) (i), then we get

$$\sum_{k=1}^{n}kz^{k-1}W_{-k}^3=\frac{d}{dz}\left(\frac{\Phi_{34}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}\right)$$
$$\Rightarrow$$
$$\begin{aligned}\sum_{k=1}^{n}kz^kW_{-k}^3 &= z\frac{d}{dz}\left(\frac{\Phi_{34}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}\right)\\ &= \frac{z(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\frac{d}{dz}\Phi_{34}-z\Phi_{34}\frac{d}{dz}((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3))}{(z^2+rsz-s^3)^2(-z^2+3rsz+r^3z+s^3)^2}\\ &= \frac{\Lambda_{46}}{(z^2+rsz-s^3)^2(-z^2+3rsz+r^3z+s^3)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.6, $a\neq b\neq c\neq d$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) (ii) by using

$$\sum_{k=1}^{n} ka^k W_{-k}^3 = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{46})}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{47}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a}.$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.6, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (a) (ii), we obtain

$$\sum_{k=1}^{n} ka^k W_{-k}^3 = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{46})}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{48}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a}$$

and

$$\sum_{k=1}^{n} kb^k W_{-k}^3 = \left.\frac{\Lambda_{48}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=b}.$$

For $z=c$, the right side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (a) (iii) by using

$$\sum_{k=1}^{n} kc^k W_{-k}^3 = \left.\frac{\frac{d^4}{dz^4}(\Lambda_{46})}{\frac{d^4}{dz^4}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=c} = \left.\frac{\Lambda_{49}}{\frac{d^4}{dz^4}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=c}.$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.6, $a=b=c=d$. We use (a) (i). For $z=a$, the right hand side of the formula given in (a) (i) is an indeterminate form. Now, we can use L'Hospital rule (eight times). Then we get (a) (iii) by using

$$\sum_{k=1}^{n} ka^k W_{-k}^3 = \left.\frac{\frac{d^8}{dz^8}(\Lambda_{46})}{\frac{d^8}{dz^8}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{50}}{40320}\right|_{z=a}.$$

(b)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k} = \frac{\Phi_{39}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}$$

where

$\Phi_{39} = -r(rz+s^2)z^{n+2}W_{-n+1}^3 - rs^3(rz+s^2)z^{n+1}W_{-n}^3 + (-z^2+r^3z+3r^2s^2+s^3)z^{n+2}W_{-n+1}^2W_{-n} + s^2(-z^2-2r^3z+s^3)z^{n+1}W_{-n}^2W_{-n+1} + rz^2(rz+s^2)W_1^3 + rs^3z(rz+s^2)W_0^3 - z^2(-z^2+r^3z+3r^2s^2+s^3)W_1^2W_0 + s^2z(z^2+2r^3z-s^3)W_0^2W_1$

which is given in Theorem 2.6 (b) (i), then we get

$$\sum_{k=1}^{n} kz^{k-1} W_{-k+1}^2 W_{-k} = \frac{d}{dz}\left(\frac{\Phi_{39}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}\right)$$

$$\Rightarrow$$

$$\sum_{k=1}^{n} kz^{k} W_{-k+1}^2 W_{-k} = z\frac{d}{dz}\left(\frac{\Phi_{39}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}\right)$$
$$= \frac{z(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\frac{d}{dz}\Phi_{39} - z\Phi_{39}\frac{d}{dz}((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3))}{(z^2+rsz-s^3)^2(-z^2+3rsz+r^3z+s^3)^2}$$
$$= \frac{\Lambda_{51}}{(z^2+rsz-s^3)^2(-z^2+3rsz+r^3z+s^3)^2}.$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.6, $a\neq b\neq c\neq d$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) (ii) by using

$$\sum_{k=1}^{n} ka^kW_{-k+1}^2W_{-k} = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{51})}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{52}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a}.$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.6, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (b) (ii), we obtain

$$\sum_{k=1}^{n} ka^kW_{-k+1}^2W_{-k} = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{51})}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{53}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a}$$

and

$$\sum_{k=1}^{n} kb^kW_{-k+1}^2W_{-k} = \left.\frac{\Lambda_{53}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (b) (iii) by using

$$\sum_{k=1}^{n} kc^kW_{-k+1}^2W_{-k} = \left.\frac{\frac{d^4}{dz^4}(\Lambda_{51})}{\frac{d^4}{dz^4}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=c} = \left.\frac{\Lambda_{54}}{\frac{d^4}{dz^4}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=c}.$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.6, $a=b=c=d$. We use (b) (i). For $z=a$, the right hand side of the formula given in (b) (i) is an indeterminate form. Now, we can use L'Hospital rule (eight times). Then we get (b) (iii) by using

$$\sum_{k=1}^{n} kc^kW_{-k+1}^2W_{-k} = \left.\frac{\frac{d^8}{dz^8}(\Lambda_{51})}{\frac{d^8}{dz^8}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{55}}{40320}\right|_{z=a}.$$

(c)

(i) If we take the derivative (with respect to z) of both sides of the sum formula

$$\sum_{k=1}^{n} z^kW_{-k}^2W_{-k+1} = \frac{\Phi_{44}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}$$

where

$\Phi_{44} = r(-z+rs)z^{n+2}W_{-n+1}^3 + rs^3(-z+rs)z^{n+1}W_{-n}^3 + s(-z^2-2r^3z+s^3)z^{n+1}W_{-n+1}^2W_{-n} + (-z^3+2rsz^2+r^3z^2+s^3z+r^4sz-2rs^4)z^{n+1}W_{-n}^2W_{-n+1} + rz^2(z-rs)W_1^3 + rs^3z(z-rs)W_0^3 + sz(z^2+2r^3z-s^3)W_1^2W_0 + z(z^3-2rsz^2-r^3z^2-r^4sz-s^3z+2rs^4)W_0^2W_1$

which is given in Theorem 2.6 (c) (i), then we get

$$\sum_{k=1}^{n} kz^{k-1}W_{-k}^2W_{-k+1} = \frac{d}{dz}\left(\frac{\Phi_{44}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}\right)$$

$$\Rightarrow$$

$$\begin{aligned}\sum_{k=1}^{n} kz^kW_{-k}^2W_{-k+1} &= z\frac{d}{dz}\left(\frac{\Phi_{44}}{(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)}\right)\\ &= \frac{z(z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3)\frac{d}{dz}\Phi_{44} - z\Phi_{44}\frac{d}{dz}((z^2+rsz-s^3)(-z^2+3rsz+r^3z+s^3))}{(z^2+rsz-s^3)^2(-z^2+3rsz+r^3z+s^3)^2}\\ &= \frac{\Lambda_{56}}{(z^2+rsz-s^3)^2(-z^2+3rsz+r^3z+s^3)^2}.\end{aligned}$$

(ii) Suppose that $r^2+4s\neq 0$. Then, from Remark 2.6, $a\neq b\neq c\neq d$. We use (c) (i). For $z=a$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) (ii) by using

$$\sum_{k=1}^{n} ka^kW_{-k}^2W_{-k+1} = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{56})}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{57}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a}.$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(iii) Suppose that $r^2+s=0$. Then, from Remark 2.6, $a\neq b$ and $c=d$. For $z=a$ and $z=b$, as in (c) (ii), we obtain

$$\sum_{k=1}^{n} ka^kW_{-k}^2W_{-k+1} = \left.\frac{\frac{d^2}{dz^2}(\Lambda_{56})}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{58}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a}$$

and

$$\sum_{k=1}^{n} kb^kW_{-k}^2W_{-k+1} = \left.\frac{\Lambda_{58}}{\frac{d^2}{dz^2}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=b}.$$

For $z=c$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) (iii) by using

$$\sum_{k=1}^{n} kc^kW_{-k}^2W_{-k+1} = \left.\frac{\frac{d^4}{dz^4}(\Lambda_{56})}{\frac{d^4}{dz^4}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=c} = \left.\frac{\Lambda_{59}}{\frac{d^4}{dz^4}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=c}.$$

(iv) Suppose that $r^2+4s=0$. Then, from Remark 2.6, $a=b=c=d$. We use (c) (i). For $z=a$, the right hand side of the formula given in (c) (i) is an indeterminate form. Now, we can use L'Hospital rule (eight times). Then we get (c) (iii) by using

$$\sum_{k=1}^{n} ka^kW_{-k}^2W_{-k+1} = \left.\frac{\frac{d^8}{dz^8}(\Lambda_{56})}{\frac{d^8}{dz^8}(z^2+rsz-s^3)^2(-z^2+r^3z+3rsz+s^3)^2}\right|_{z=a} = \left.\frac{\Lambda_{60}}{40320}\right|_{z=a}. \square$$

Note that the proof of (i) 's of Theorem 3.6 can be done as in the following Remark.

Remark 3.6.

We present the direct proofs of Theorem 3.6 (a) (i), (b) (i) and (c) (i) without using any derivatives.

Proof of Theorem 3.6 (a) (i), (b) (i) and (c) (i):

Using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}\Rightarrow W_{-n}=-\frac{r}{s}W_{-n+1}+\frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

we obtain

$$s^3W_{-n}^3=W_{-n+2}^3-3rW_{-n+2}^2W_{-n+1}+3r^2W_{-n+1}^2W_{-n+2}-r^3W_{-n+1}^3$$

and so

$$
\begin{aligned}
s^3 \times n \times z^n W_{-n}^3 &= nz^n W_{-n+2}^3 - 3r \times n \times z^n W_{-n+2}^2 W_{-n+1} \\
&\quad +3r^2 \times n \times z^n W_{-n+1}^2 W_{-n+2} - r^3 \times n \times z^n W_{-n+1}^3 \\
s^3(n-1)z^{n-1} W_{-n+1}^3 &= (n-1)z^{n-1} W_{-n+3}^3 - 3r(n-1)z^{n-1} W_{-n+3}^2 W_{-n+2} \\
&\quad +3r^2(n-1)z^{n-1} W_{-n+2}^2 W_{-n+3} - r^3(n-1)z^{n-1} W_{-n+2}^3 \\
s^3(n-2)z^{n-2} W_{-n+2}^3 &= (n-2)z^{n-2} W_{-n+4}^3 - 3r(n-2)z^{n-2} W_{-n+4}^2 W_{-n+3} \\
&\quad +3r^2(n-2)z^{n-2} W_{-n+3}^2 W_{-n+4} - r^3(n-2)z^{n-2} W_{-n+3}^3 \\
&\vdots \\
s^3 \times 3 \times z^3 W_{-3}^3 &= 3 \times z^3 W_{-1}^3 - 3r \times 3 \times z^3 W_{-1}^2 W_{-2} \\
&\quad +3r^2 \times 3 \times z^3 W_{-2}^2 W_{-1} - r^3 \times 3 \times z^3 W_{-2}^3 \\
s^3 \times 2 \times z^2 W_{-2}^3 &= 2 \times z^2 W_0^3 - 3r \times 2 \times z^2 W_0^2 W_{-1} + 3r^2 \times 2 \times z^2 W_{-1}^2 W_0 - r^3 \times 2 \times z^2 W_{-1}^3 \\
s^3 \times 1 \times z^1 W_{-1}^3 &= 1 \times z^1 W_1^3 - 3r \times 1 \times z^1 W_1^2 W_0 + 3r^2 \times 1 \times z^1 W_0^2 W_1 - r^3 \times 1 \times z^1 W_0^3
\end{aligned}
$$

If we add the above equations side by side, we get

$$
\begin{aligned}
s^3(\sum_{k=1}^{n} kz^k W_{-k}^3) &= ((-n-1)z^{n+1} W_{-n+1}^3 + (-n-2)z^{n+2} W_{-n}^3 + z^1 W_1^3 + 2 \times z^2 W_0^3 \\
&\quad +z^2 \sum_{k=1}^{n} kz^k W_{-k}^3 + 2z^2 \sum_{k=1}^{n} z^k W_{-k}^3) - 3r((-n-1)z^{n+1} W_{-n+1}^2 W_{-n} \\
&\quad +z^1 W_1^2 W_0 + z^1 \sum_{k=1}^{n} kz^k W_{-k+1}^2 W_{-k} + z^1 \sum_{k=1}^{n} z^k W_{-k+1}^2 W_{-k}) \\
&\quad +3r^2((-n-1)z^{n+1} W_{-n}^2 W_{-n+1} + z^1 W_0^2 W_1 + z^1 \sum_{k=1}^{n} kz^k W_{-k}^2 W_{-k+1} \\
&\quad +z^1 \sum_{k=1}^{n} z^k W_{-k}^2 W_{-k+1}) - r^3((-n-1)z^{n+1} W_{-n}^3 + z^1 W_0^3 \\
&\quad +z^1 \sum_{k=1}^{n} kz^k W_{-k}^3 + z^1 \sum_{k=1}^{n} z^k W_{-k}^3).
\end{aligned} \tag{38}
$$

Next we calculate $\sum_{k=1}^{n} kz^k W_{-k+1}^2 W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s} W_{-n+1} + \frac{1}{s} W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$sW_{-n+1}^2 W_{-n} = W_{-n+1}^2 W_{-n+2} - rW_{-n+1}^3$$

and so

$$
\begin{aligned}
s \times n \times z^n W_{-n+1}^2 W_{-n} &= nz^n W_{-n+1}^2 W_{-n+2} - r \times n \times z^n W_{-n+1}^3 \\
s(n-1)z^{n-1} W_{-n+2}^2 W_{-n+1} &= (n-1)z^{n-1} W_{-n+2}^2 W_{-n+3} - r(n-1)z^{n-1} W_{-n+2}^3 \\
s(n-2)z^{n-2} W_{-n+3}^2 W_{-n+2} &= (n-2)z^{n-2} W_{-n+3}^2 W_{-n+4} - r(n-2)z^{n-2} W_{-n+3}^3 \\
&\vdots \\
s \times 3 \times z^3 W_{-2}^2 W_{-3} &= 3z^3 W_{-2}^2 W_{-1} - r \times 3 \times z^3 W_{-2}^3 \\
s \times 2 \times z^2 W_{-1}^2 W_{-2} &= 2 \times z^2 W_{-1}^2 W_0 - r \times 2 \times z^2 W_{-1}^3 \\
s \times 1 \times z^1 W_0^2 W_{-1} &= 1 \times z^1 W_0^2 W_1 - r \times 1 \times z^1 W_0^3
\end{aligned}
$$

If we add the above equations side by side, we get

$$s\sum_{k=1}^{n}kz^{k}W_{-k+1}^{2}W_{-k}=(-(n+1)z^{n+1}W_{-n}^{2}W_{-n+1}+z^{1}W_{0}^{2}W_{1}+z^{1}\sum_{k=1}^{n}kz^{k}W_{-k}^{2}W_{-k+1} \\ +z^{1}\sum_{k=1}^{n}z^{k}W_{-k}^{2}W_{-k+1})-r(-(n+1)z^{n+1}W_{-n}^{3}+z^{1}W_{0}^{3} \\ +z^{1}\sum_{k=1}^{n}kz^{k}W_{-k}^{3}+z^{1}\sum_{k=1}^{n}z^{k}W_{-k}^{3}). \quad (39)$$

Next we calculate $\sum_{k=1}^{n}z^{k}W_{-k+1}^{2}W_{-k}$. Again using the recurrence relation

$$W_{-n+2}=rW_{-n+1}+sW_{-n}$$

i.e.

$$sW_{-n}=W_{-n+2}-rW_{-n+1}$$

we obtain

$$s^{2}W_{-n}^{2} = W_{-n+2}^{2}-2rW_{-n+2}W_{-n+1}+r^{2}W_{-n+1}^{2} \\ \Rightarrow s^{2}W_{-n}^{2}W_{-n+1}=W_{-n+2}^{2}W_{-n+1}-2rW_{-n+1}^{2}W_{-n+2}+r^{2}W_{-n+1}^{3}$$

and so

$$s^{2}\times n\times z^{n}W_{-n}^{2}W_{-n+1}=nz^{n}W_{-n+2}^{2}W_{-n+1}-2r\times n\times z^{n}W_{-n+1}^{2}W_{-n+2}+r^{2}\times n\times z^{n}W_{-n+1}^{3} \\ s^{2}(n-1)z^{n-1}W_{-n+1}^{2}W_{-n+2}=(n-1)z^{n-1}W_{-n+3}^{2}W_{-n+2}-2r(n-1)z^{n-1}W_{-n+2}^{2}W_{-n+3} \\ +r^{2}(n-1)z^{n-1}W_{-n+2}^{3} \\ s^{2}(n-2)z^{n-2}W_{-n+2}^{2}W_{-n+3}=(n-2)z^{n-2}W_{-n+4}^{2}W_{-n+3}-2r(n-2)z^{n-2}W_{-n+3}^{2}W_{-n+4} \\ +r^{2}(n-2)z^{n-2}W_{-n+3}^{3} \\ \vdots \\ s^{2}\times 3\times z^{3}W_{-3}^{2}W_{-2}=3\times z^{3}W_{-1}^{2}W_{-2}-2r\times 3\times z^{3}W_{-2}^{2}W_{-1}+r^{2}\times 3\times z^{3}W_{-2}^{3} \\ s^{2}\times 2\times z^{2}W_{-2}^{2}W_{-1}=2\times z^{2}W_{0}^{2}W_{-1}-2r\times 2\times z^{2}W_{-1}^{2}W_{0}+r^{2}\times 2\times z^{2}W_{-1}^{3} \\ s^{2}\times 1\times z^{1}W_{-1}^{2}W_{0}=1\times z^{1}W_{1}^{2}W_{0}-2r\times 1\times z^{1}W_{0}^{2}W_{1}+r^{2}\times 1\times z^{1}W_{0}^{3}$$

If we add the above equations side by side, we get

$$s^{2}\sum_{k=1}^{n}kz^{k}W_{-k}^{2}W_{-k+1}=(-(n+1)z^{n+1}W_{-n+1}^{2}W_{-n}+z^{1}W_{1}^{2}W_{0}+z^{1}\sum_{k=1}^{n}kz^{k}W_{-k+1}^{2}W_{-k} \\ +z^{1}\sum_{k=1}^{n}z^{k}W_{-k+1}^{2}W_{-k})-2r(-(n+1)z^{n+1}W_{-n}^{2}W_{-n+1}+z^{1}W_{0}^{2}W_{1} \\ +z^{1}\sum_{k=1}^{n}kz^{k}W_{-k}^{2}W_{-k+1}+z^{1}\sum_{k=1}^{n}z^{k}W_{-k}^{2}W_{-k+1})+r^{2}(-(n+1)z^{n+1}W_{-n}^{3} \\ +z^{1}W_{0}^{3}+z^{1}\sum_{k=1}^{n}kz^{k}W_{-k}^{3}+z^{1}\sum_{k=1}^{n}z^{k}W_{-k}^{3}). \quad (40)$$

Then, using Theorem 2.6 and solving the system (38)-(39)-(40), the required results of (a) (i), (b) (i) and (c) (i) of Theorem 3.6 follow.

References

[1] Andrica, D., Bagdasar, O., Recurrent Sequences Key Results, Applications, and Problems, Springer, 2020.
[2] Djordjević, G.B., Milovanović, G.V., Special Classes of Plynomials, University of Niš, Faculty of Technology, Leskovac, 2014. http://www.mi.sanu.ac.rs/~gvm/Teze/Special%20Classes%20of%20Polynomials.pdf
[3] Frei, G., Binary Lucas and Fibonacci Polynomials, I, Math. Nadir. 96, 83-112, 1980.
[4] Flórez, R., McAnally, N., Mukherjee, A., Identities for the Generalized Fibonacci Polynomial, Integers, 18B, 2018.

[5] He,T.X., Peter J.-S. Shiue, P.J.S., On Sequences of Numbers and Polynomials Defined by Linear Recurrence Relations of Order 2, International Journal of Mathematics and Mathematical Sciences, Volume 2009, Article ID 709386, 21 pages, doi:10.1155/2009/709386.

[6] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[7] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 1 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), Second Edition, John Wiley&Sons, New York, 2018.

[8] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 2 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), John Wiley&Sons, New York, 2019.

[9] Soykan, Y., Generalized Fibonacci Numbers: Sum Formulas, Journal of Advances in Mathematics and Computer Science, 35(1), 89-104, 2020, DOI: 10.9734/JAMCS/2020/v35i130241.

[10] Soykan, Y., Horadam Numbers: Sum Formulas $\sum_{k=0}^{n} kx^{k}W_{k}$ and $\sum_{k=1}^{n} kx^{k}W_{-k}$, Asian Basic and Applied Research Journal, 5(3), 6-24, 2022. Article no.ABAARJ.838

[11] Soykan, Y., Horadam Numbers: Sum of the Squares of Terms of Sequence, Int. J. Adv. Appl. Math. and Mech. 7(4), 34-50, 2020.

[12] Soykan, Y., A Study on the Sums of Squares of Generalized Fibonacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kx^{k}W_{k}^{2}$ and $\sum_{k=1}^{n} kx^{k}W_{-k}^{2}$, Asian Journal of Advanced Research and Reports, 12(1), 44-67, 2020. DOI:

[13] Soykan, Y., A Study on Generalized Fibonacci Numbers: Sum Formulas $\sum_{k=0}^{n} kx^{k}W_{k}^{3}$ and $\sum_{k=1}^{n} kx^{k}W_{-k}^{3}$ for the Cubes of Terms, Earthline Journal of Mathematical Sciences, 4(2), 297-331, 2020. https://doi.org/10.34198/ejms.4220.297331

[14] Soykan, Y., A Study On Sums of Cubes of Generalized Fibonacci Numbers: Closed Formulas of $\sum_{k=0}^{n} x^{k}W_{k}^{3}$ and $\sum_{k=1}^{n} x^{k}W_{-k}^{3}$, Int. J. Adv. Appl. Math. and Mech. 8(3), 9-23, 2021.

[15] Soykan, Y., On Generalized Fibonacci Polynomials: Horadam Polynomials, Earthline Journal of Mathematical Sciences, accepted.

[16] Soykan, Y., Generalized Fibonacci Numbers: Sum Formulas, Minel Yayın, 2022. https://www.minelyayin.com/generalized-fibonacci-numbers-sum-formulas-51

[17] Vajda, S,. Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications, Dover Publications Inc.; 2008.

[18] Wang, J., Some New Results for the (p,q)-Fibonacci and Lucas Polynomials, Advances in Difference Equations, 2014. http://www.advancesindifferenceequations.com/content/2014/1/64

[19] Wang, W., Wang, H., Generalized-Humbert Polynomials via Generalized Fibonacci Polynomials, Applied Mathematics and Computation 307, 204–216, 2017. https://doi.org/10.1016/j.amc.2017.02.050.