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Fundamentals and Recent Studies of Finsler Geometry

Review Article

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Abstract: The history of development of Finsler geometry can be divided into different periods. The main aim of this work is to concentrate on the basic concepts and definitions of Finsler geometry. In addition, we will shed light on various recent studies that are in connection to it.

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1. Introduction

Finsler geometry is a kind of differential geometry. It usually considers as a generalization of Riemannian geometry. In fact, Riemann, in his epoc-making lecture in 1854, already suggested a possibility of studying a geometry more general than Riemannian geometry, but he said the geometrical meanings of quantities appearing in such a generalized space will not be clear and it can't produce any contribution to the geometry. Furthermore, Mathematicins neglected the study of such spaces for more than 60 years. In his age of twenty four years, Finsler took up the problem related to the space equipped with the metric function which was mentioned by Riemann. In 1918, he submitted his epoc-making thesis to Gotingen university. He studied this geometry from the stand point of a geometrization of the calculus of variations. This thesis draws the attention to the most of the mathematicians working in geometry. So, Finsler (1894-1970) was considered the originator of the Finsler geometry.

In 1927, Taylor gave the name Finsler space to the manifold equipped with this generalized metric.

In 1934, Cartan introduced a system of axioms to give uniquely a Finsler connection from the fundamental function. In the same line, Randers drew the attention to several physicists towards Finsler geometry.

In 1951, Rund introduced a new concept of parallelism considering Finsler geometry as locally Minkowskian.

Later on Mokoto Matsumoto devoted his effort to such approach and contributed much to this field. He wrote a monograph "The theory of Finsler connections" and circulated it among the mathematicians working in the field.

The Finsler geometry has many applications in various fields of Physics and Biology such as the theory of relativity, thermodynamics, optics, ecology, evolution and developmental biology. Several mathematicians contributed to the study and improvement of Finsler geometry.

The historical studies about development stages for Finsler geometry have been introduced by Matsumoto [37] and Won [14]. In this paper, we will elaborate the discussion to conclude thirty studies in terms to it.

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2. Basic Concepts of Finsler Geometry

In this section, we introduce some important concepts and definitions in Finsler geometry [1–3, 6, 25, 47].

2.1. Finsler Space

Let R be a region of an $n-$dimensional space X_n, which is completely covered by a coordinates system, such that each point P of R is represented by $n-$tuples $(x^1, x^2, \ldots, x^n)$ called coordinates of P, these coordinates will be taken as real throughout. A set of points of R whose coordinates are functions of a single real parameter t is regarded as a curve C of X_n. Thus, these parametric equations

$$x^i = x^i(t), \quad i = 1, ..., n \tag{1}$$

represent a curve C of X_n. Let us assume that the curve $C : x^i = x^i(t)$ be at least class C^1. The entities

$$y^i = \dot{x}^i = \frac{dx^i}{dt}$$

Definition 2.1.
An $n-$dimensional space X_n equipped with a function $F(x, y)$ which denoted by $F_n = (X_n, F(x, y))$ called a Finsler space if the function $F(x, y)$ satisfies the following three conditions [13, 31]

Condition (i) The function $F(x, y)$ is positively homogeneous of degree one in y^i, i.e.

$$F(x, ky) = kF(x, y),$$

where k is some positive scalar.

Condition (ii) The function $F(x, y)$ is positive unless all y^i vanish simultaneously, i.e. $F(x, y) > 0$, with $\sum_i \left(y^i\right)^2 \neq 0$.

Condition (iii) The quadratic form

$$\{\dot{\partial}_i \dot{\partial}_j F^2(x,\ y)\}\, \xi^i \xi^j, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i}$$

is assumed to be positive definite for all variable ξ^i.

The function $F(x, y)$ is called the fundamental function or the metric function of the Finsler space F_n [25, 47].

2.2. The Metric Tensor, Tangent Space, $(h)hv-$Torsion Tensor and Generalized Christoffel's Symbols

Let us consider a set of n^2 quantities $g_{ij}(x, y)$ defined by [25]

$$g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2(x, y). \tag{2}$$

The quantities g_{ij} constitute the components of a covariant tensor of types (0,2). This tensor is called the metric tensor of the space F_n. It is obvious from (2) and due to the homogeneity of $F(x, y)$ in y^i, then the metric tensor $g_{ij}(x, y)$ is positively homogeneous of degree zero in y^i and symmetric in its indices.

By Euler's theorem on homogeneous functions, the derivatives of $F(x, y)$ implies the following:

$$\begin{cases} a)\ g_{ij}(x, y)\, y^i y^j = F^2(x, y) \\ b)\ y^i \dot{\partial}_i F(x,\ y) = F(x,\ y) \\ c)\ y^i \dot{\partial}_i \dot{\partial}_j\ F(x,\ y) = 0 \\ d)\ det\left[\dot{\partial}_i \dot{\partial}_j F(x,\ y)\right] = 0, \end{cases}$$

where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

Since $x^i = x^i(t)$ a curve C in the Finser space F_n and let us consider a change of local coordinates represented by

$$\bar{x}^i = \bar{x}^i\left(x^j(t)\right). \tag{3}$$

Then, the components $y^i = \frac{dx^i}{dt}$ of the tangent vector to the curve C transformed according to

$$\bar{y}^i = \left(\partial_j \bar{x}^i\right) y^j, \qquad \partial_j = \frac{\partial}{\partial x^j},$$

A system of $n-$quantities X^i whose transformation law under (3) is analogous to that of y^i is called a contravariant vector attached to the point $P(x^i)$ of F_n. Such contravariant vectors attached to $P(x^i)$ constitute the elements of a vector space. This vector space is called the tangent space at $P(x^i)$ and denoted by $T_n(P)$ or $T_n(x^i)$.

Corresponding to each arbitrary contravariant vector y^i of the tangent space $T_n(P)$, there is a covariant vector y_i such that

$$y_i = g_{ij}(x,y)\, y^j.$$

The two sets of quantities g_{ij} and g^{ij} which are components of the metric tensor and associate metric tensor related by

$$g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Matsumoto [34] introduced the $(h)hv-$torsion tensor C_{ijk} that is positively homogeneous of degree -1 in y^i and defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2.$$

This tensor satisfies the following

$$C_{ijk} = g_{hj}C^h_{ik}, \quad C^h_{ik} = g^{hj}C_{ijk} \quad \text{and} \quad C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0,$$

where C^i_{jk} is called associate tensor of the $(h)hv-$torsion tensor C_{ijk}.

Let us define the generalized Christoffel symbols of the first and second kind in Riemannian geometry as [25]

$$\begin{cases} a)\ \gamma_{ijk} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik}\right) \\ b)\ \gamma^h_{ik} = g^{hj}\, \gamma_{ijk}, \end{cases}$$

respectively. Christoffel symbols of the first and second kind are positively homogeneous of degree zero in y^i and symmetric in the first and second lower indices.

2.3. Cartan's Connection

Cartan in his second postulate gave the expression for the variation of an arbitrary vector field X^i under the infinitesimal changes of its line-element (x, y) to $(x + dx, y + dy)$ by means of covariant or (absolute) differential. This variation is given by [15, 25]

$$DX^i = dX^i + X^j\left(C^i_{jk}dy^k + \Gamma^i_{jk}dx^k\right), \tag{4}$$

where

$$\Gamma^i_{jk} = \gamma^i_{jk} - C^i_{mk}G^m_j + g^{ih}C_{jkm}G^m_h.$$

Also

$$a)\ G^i = \frac{1}{2}\gamma^i_{jk}y^j y^k \quad \text{and} \quad b)\ G^i_j = \dot{\partial}_j G^i. \tag{5}$$

The function G^i is positively homogeneous of degree two in y^i. Eliminating dy^k from (4) and the absolute differential of l^j, Cartan deduces [15]

$$DX^i = F\, X^i|_k\, D\, l^k + X^i_{|k}\, dx^k + y^k\left(\dot{\partial}_k X^i\right)\frac{dF}{F},$$

where

$$X^i|_k = \dot{\partial}_k X^i + X^r C^i_{rk} \tag{6}$$

and

$$X^i_{|k} = \partial_k X^i - \left(\dot{\partial}_r X^i\right)G^r_k + X^r\Gamma^{*i}_{rk}. \tag{7}$$

This function Γ^{*i}_{rk} is defined by

$$\Gamma^{*i}_{rk} = \Gamma^i_{rk} - C^i_{mr}\,\Gamma^m_{sk}\, y^s. \tag{8}$$

These functions Γ^{*i}_{rk} and G^r_k are connected by

$$G^r_k = \Gamma^{*r}_{sk}\, y^s.$$

The functions Γ^{*i}_{rk} which defined by (8) is called Cartan's connection parameter, this is symmetric in its lower indices and positively homogeneous of degree zero in y^i.

The equations (6) and (7) introduce two processes of covariant differentiation which are called $v-$covariant differentiation (Cartan's first kind covariant differentiation) and $h-$covariant differentiation (Cartan's second kind covariant differentiation), respectively. So $X^i|_k$ and $X^i_{|k}$ are $v-$covariant derivative and $h-$covariant derivative of the vector field X^i, respectively.

For an arbitrary vector field X^i, the commutation formula for the operators of partial differentiation with respect to y^j and $h-$covariant differentiation is given by

$$\dot\partial_j\left(X^i_{|k}\right) - \left(\dot\partial_j X^i\right)_{|k} = X^r\left(\dot\partial_j\Gamma^{*i}_{rk}\right) - \left(\dot\partial_r X^i\right)P^r_{jk},$$

where

$$P^r_{jk} = \left(\dot\partial_j\Gamma^{*r}_{hk}\right)y^h = \Gamma^{*r}_{jhk}y^h.$$

2.4. Berwald Connection

Let us consider a connection parameter G^i_{jk} of Berwald which connected with Cartan connection parameter Γ^{*i}_{jk} by

$$G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|h}\, y^h.$$

The function G^i_{jk} is positively homogeneous of degree zero in y^i and satisfies

$$G^i_{jk} = \dot\partial_j\dot\partial_k G^i,$$

where G^i is given by (5). Since G^i is positively homogeneous of degree two in y^i, then G^i_{kh} and G^i_h are homogeneous of degree zero and one in y^i, respectively. Obviously

$$G^i_{jk}y^j = \dot\partial_k G^i = G^i_k,\quad \dot\partial_k G^i_h = G^i_{kh} \quad\text{and}\quad G^i_k y^k = G^i_{kh}\, y^h y^k = 2G^i.$$

Analogous to Cartan's $h-$covariant differentiation, Berwald is defined covariant derivative for his connection parameter G^i_{jk}. Thus, Berwald's covariant derivative $\mathfrak{B}_k$ of an arbitrary tensor field T^i_j with respect to x^k is given by [25]

$$\mathfrak{B}_k T^i_j = \partial_k T^i_j + T^r_j G^i_{rk} - \left(\dot\partial_r T^i_j\right)G^r_k - T^i_r G^r_{jk}. \tag{9}$$

The processes of Berwald's covariant differentiation and the partial differentiation commute according to

$$\left(\dot\partial_k\mathfrak{B}_h - \mathfrak{B}_h\dot\partial_k\right)T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}$$

for an arbitrary tensor field T^i_j.

2.5. Cartan's Curvature Tensors

The $h-$covariant derivative of an arbitrary vector field X^i is given by (7). Taking the $h-$covariant derivative of (7) with respect to x^h, we have

$$X^i_{|k|h} = \partial_h\left(X^i_{|k}\right) - \left(X^i_{|r}\right)\Gamma^{*r}_{kh} + \left(X^r_{|k}\right)\Gamma^{*i}_{rh} - \dot\partial_r\left(X^i_{|k}\right)\Gamma^{*r}_{h\ell}y^\ell.$$

Using (7) in above equation and taking skew-symmetric part with respect to the indices k and h, we get the commutation formula for $h-$ covariant derivative of an arbitrary vector field X^i which is given by [25]

$$X^i_{|k|h} - X^i_{|h|k} = X^r K^i_{rkh} - \left(\dot\partial_r X^i\right)K^r_{skh}\, y^s,$$

where

$$K^i_{rkh} = \partial_h\Gamma^{*i}_{kr} + \left(\dot\partial_\ell\Gamma^{*i}_{rh}\right)G^\ell_k + \Gamma^{*i}_{mh}\Gamma^{*m}_{kr} - h/k^*.$$

The tensor K^i_{rkh} as defined above is called Cartan's fourth curvature tensor which is positively homogeneous of degree zero in y^i and skew-symmetric in its last two lower indices k and h.

We have the commutation formula [25]

$$X^i_{|k|h} - X^i_{|h|k} = R^i_{jkh} X^j - X^i|_j K^i_{rkh} y^r,$$

where

$$R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + \left(\dot{\partial}_\ell \Gamma^{*i}_{jh}\right) G^\ell_k + G^i_{jm}\left(\partial_h G^m_k - G^m_{h\ell} G^\ell_k\right) + \Gamma^{*i}_{mh}\Gamma^{*m}_{jk} - h/k.$$

The tensor R^i_{jkh} is called $h-$curvature tensor (Cartan's third curvature tensor), this tensor is positively homogeneous of degree zero in y^i and skew-symmetric in its last two lower indices k and h.

Also, Cartan's third curvature tensor R^i_{jkh} and its associative curvature tensor R_{ijkh} which satisfy the following identities are known as Bianchi identity [22, 25]

$$R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + y^m\left(R^s_{mkh} P^r_{ijs} + R^s_{mjk} P^r_{ihs} + R^s_{mhj} P^r_{iks}\right) = 0$$

where P^i_{jkh} is called $hv-$curvature tensor (Cartan's second curvature tensor) which is defined by

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} P^r_{kh} - C^i_{jh|k}$$

or equivalent by

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} C^r_{kh|s} y^s - C^i_{jh|k}$$

or

$$P^i_{jkh} = C^i_{kh|j} - g^{ir} C_{jkh|r} + C^r_{jk} P^i_{rh} - P^r_{jh} C^i_{rk}. \tag{10}$$

Cartan's secend curvature tensor P^i_{jkh} is positively homogeneous of degree -1 in y^i.

In view of (10), the curvature tensor P^i_{jkh} satisfies the following

$$P^i_{jkh} - P^i_{jhk} = -S^i_{jkh|r}\, y^r.$$

Also, we have the following commutation formula [25]

$$X^i|_k|_h - X^i|_h|_k = S^i_{jkh} X^j - X^i|_j S^j_{kh},$$

where S^i_{jkh} is called $v-$curvature tensor (Cartan's first curvature tensor) and defined by

$$S^i_{jkh} = C^i_{rk} C^r_{jh} - C^i_{rh} C^r_{jk}.$$

The tensor S^i_{jkh} is positively homogeneous of degree -2 in y^i and skew-symmetric in its last two lower indices k and h.

2.6. Berwald Curvature Tensor

Taking $\mathfrak{B}-$covariant derivative of (9) with respect to x^h and considered an arbitrary tensor X^i instead of T^i_j, we get

$$\mathfrak{B}_k \mathfrak{B}_h X^i = \partial_k\left(\mathfrak{B}_h X^i\right) - \left(\partial_s \mathfrak{B}_h X^i\right) G^s_k + \left(\mathfrak{B}_h X^r\right) G^i_{rk} - \left(\mathfrak{B}_r X^i\right) G^r_{hk}.$$

By putting the expression for $(\mathfrak{B}_h X^i)$ from (9) in above equation and taking skew-symmetric part with respect to the indices k and h, we get the commutation formula for Berwald's covariant differentiation as follows

$$\mathfrak{B}_h \mathfrak{B}_k X^i - \mathfrak{B}_k \mathfrak{B}_h X^i = X^r H^i_{rkh} - H^r_{kh}\left(\dot{\partial}_r X^i\right), \tag{11}$$

where the tensors H^i_{jkh} and H^i_{kh} are defined by

$$H^i_{jkh} = \partial_h G^i_{jk} + G^s_{jk} G^i_{sh} + G^i_{sjh} G^s_k - h/k$$

and

$$H^i_{kh} = \partial_h G^i_k + G^r_k G^i_{rh} - h/k,$$

respectively. The formula (11) is called generalized Ricci identity or Ricci commutation formula. Matsumoto called the tensors H^i_{jkh} and H^i_{kh} as Berwald curvature tensor and Berwald torsion tensor, respectively. These tensors H^i_{jkh} and H^i_{kh} are skew-symmetric in last two lower indices and positively homogeneous of degree zero and one in y^i, respectively.

2.7. Projective Curvature Tensor

Let $F_n = (X_n, F)$ and $\bar{F}_n = \left(X_n, \bar{F}\right)$ be two Finsler spaces on a common underlying space X_n. Let us consider a transformation $F_n \to \bar{F}_n$ such that every geodesic of F_n goes to a geodesic of $\bar{F}_n$ and the inverse is also true, then this transformation is called projective transformation or projective change [25]. A transformation $F_n \to \bar{F}_n$ is a projective transformation if and only if

$$\bar{G}^i = G^i + p y^i, \tag{12}$$

where p is a scalar which is positively homogeneous of degree one in y^i. Differentiating (12) partially with respect to y^h, we get

$$\bar{G}^i_h = G^i_h + y^i p_h + \delta^i_h p.$$

Berwald's connection parameter G^i_{jk} under the projective change (12) is given by

$$\bar{G}^i_{kh} = G^i_{kh} + \delta^i_k p_h + \delta^i_h p_k + y^i p_{kh},$$

where p_h and p_{kh} are the directional derivatives of p. Berwald deduced a tensor [25]

$$W^i_h = H^i_h - H\delta^i_h - \left[\left(\dot{\partial}_r H_h - \dot{\partial}_h H\right) y^i \right] / (n+1),$$

which remains invariant under a projective transformation. This tensor is called projective deviation tensor. In view of the homogeneity of H^i_h in y^i, then the tensor W^i_h is positively homogeneous of degree two in y^i. Similarly to Berwald's tensor, the following projective tensors are defined as

$$\begin{aligned} W^i_{kh} &= H^i_{kh} + \frac{y^i}{n+1}\left(H_{kh} - H_{hk}\right) + \frac{\delta^i_k}{n^2-1}\left(nH_h + y^r H_{hr}\right) \\ &\quad - \frac{\delta^i_h}{n^2-1}\left(nH_k + y^r H_{kr}\right) \end{aligned}$$

$$\begin{aligned} W^i_{jkh} &= H^i_{jkh} + \frac{\delta^i_j}{n+1}\left(H_{kh} - H_{hk}\right)\frac{y^i}{n+1}\left(\dot{\partial}_j H_{kh} - \dot{\partial}_j H_{hk}\right) \\ &+ \frac{\delta^i_k}{n^2-1}\left(nH_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{n^2-1}\left(nH_{jk} - H_{kj} + y^r \dot{\partial}_j H_{kr}\right), \end{aligned}$$

which also remain invariant under a projective transformation. The tensors W^i_{jkh} and W^i_{kh} are known as projective curvature tensor (Wely's projective curvature tensor) and projective torsion tensor (Wely's projective torsion tensor), respectively [25].

2.8. Special Spaces of Finsler space

Definition 2.2.
A Finsler space F_n is called a *Berwald space* if the Berwald connection coefficients are linear, i.e. G^i_{jk} are functions of position only. A Finsler space is Berwald space if and only if [21, 25]

$$G^i_{jkh} = 0 \quad \text{and} \quad C_{ijk|h} = 0.$$

Berwald himself called a Berwald space an *affinely connected space.*

Definition 2.3.
A Finsler space F_n is called a *Landsberg space* if the Berwald connection on it is h−metrical, i.e. $\mathfrak{B}_k g_{ij} = 0$. A Landsberg space is characterized by the following condition [16, 27, 50]

$$y_r G^r_{ijk} = -2C_{ijk|h} y^h = -2P_{ijk} = 0.$$

Various authors are denoted the tensor $C_{ijk|h} y^h$ by P_{ijk} as Izumi [23] and Izumi and Yoshida [24].

Definition 2.4.
A Finsler space F_n is called a *P2−like space* if the Cartan's secend curvature tensor P^i_{jkh} is characterized by the condition [1, 33]

$$P^i_{jkh} = \varphi_j C^i_{kh} - \varphi^i C_{jkh},$$

where φ_j and φ^i are non - zero covariant and contravariant vectors field, respectively.

Definition 2.5.
A Finsler space F_n is called a P^*-*Finsler space* if the $(v)hv-$torsion tensor P^i_{kh} is characterized by the condition [23, 41]

$$P^i_{kh} = \varphi C^i_{kh}, \quad \varphi \neq 0,$$

where $P^i_{jkh} y^j = P^i_{kh} = C^i_{kh|s} y^s$.

Definition 2.6.
A Finsler space F_n is called a $P-$*reducible space* if the associate tensor P_{jkh} of $(v)hv-$torsion tensor P^i_{kh} is characterized by one of the following conditions [40, 41, 44]

$$P_{jkh} = \lambda C_{jkh} + \varphi\Big(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\Big),$$

where λ and φ are scalar vectors positively homogeneous of degree one in y^j and h_{jk} is the angular metric tensor.

$$P_{jkh} = \frac{1}{(n+1)}\Big(h_{jk}P_h + h_{kh}P_j + h_{hj}P_k\Big),$$

where $P_{jkh} = C_{jkh|m} y^m, \quad P^i_{ik} = P_k$ and $h_{ij} = g_{ij} - l_i l_j$.

Definition 2.7.
A Finsler space F_n $(n \geq 2)$ with $C^2 = C_j C^j \neq 0$, it is called a $C2-$*like space* if the $(h)hv-$torsion tensor C_{jkh} can be written in the form [38, 55]

$$C_{jkh} = C_j C_k C_h / C^2,$$

where $C_j = g^{kh} C_{jkh}$.

Definition 2.8.
A Finsler space F_n is called a $C-$*reducible space* if the $(h)hv-$torsion tensor C_{jkh} is characterized by the condition [35, 36, 54]

$$C_{jkh} = \frac{1}{(n+1)}\Big(h_{jk}C_h + h_{kh}C_j + h_{hj}C_k\Big),$$

where $h_{jk} = g_{jk} - l_j l_k$ is an angular metric tensor.

Definition 2.9.
A Finsler metric F_n is called a *semi*$-C-$*reducible* if the $(h)hv-$torsion tensor C_{jkh} is given by [11, 38]

$$C_{jkh} = \Big[\frac{p}{1+n}\Big(h_{jk}I_h + h_{kh}I_j + h_{hj}I_k\Big) + \frac{q}{\|\mathbf{I}\|^2} I_j I_k I_h\Big],$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on F_n and $\|\mathbf{I}\|^2 = I^i I_i$.

Definition 2.10.
A Finsler metric F_n is called a $C3-$*like space* if the $(h)hv-$torsion tensor C_{jkh} is given by [45, 51, 53]

$$C_{jkh} = \Big[\Big(A_j h_{kh} + A_k h_{hj} + A_h h_{jk}\Big) + \Big(B_j I_k I_h + I_j B_k I_h + I_j I_k B_h\Big)\Big],$$

where $A_i = A_i(x, y)$ and $B_i = B_i(x, y)$ are $y-$homogeneous scalar functions on F_n of degree -1 and 1, respectively.

3. Literature Review

In introduction we shed light on the historical studies of Finsler geometry development in the twentieth century. In this part, we will provide thirty recent studies which are in the third millennium begin from 2009 until now.

In 2009, Aveesh et al. [52] modified the Rund's $h-$curvature tensor K^i_{ljk} to special form by using some special Finsler spaces such as $C-$reducible, $R3-$like Finsler spaces. Also they gave definition for $C2-$like space and $C-$reducible space.

In 2009, Zlatanovic and Mincic [32] obtained several identities for some curvature tensors (the antisymmetry with respect of two indices, the cyclic symmetry, the symmetry with respect of pairs of indices) in generalized Finsler space.

In 2010, Tiwari [12] discussed special Finsler spaces with semi$-T-$ condition and found the condition for a Landsberg space with semi$-T-$condition to vanish $h-$curvature tensor R_{ijkh}. Furthermore, the condition for a Finsler space with semi$-T-$condition to be locally Minkowski is obtained.

In 2011, Pandey et al. [43] introduced a Finsler space whose Berwald curvature tensor H^i_{jkh} satisfies the following condition

$$\mathfrak{B}_m H^i_{jkh} = \lambda_m H^i_{jkh} + \mu_m\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right), \quad P^i_{jkh} \neq 0, \tag{13}$$

where $\mathfrak{B}_m$ is Berwald's covariant differential operator with respect to x^m, λ_m and μ_m are called recurrence vectors. This space is called a generalized $H-$recurrent Finsler space. The Ricci tensor H_{kh}, curvature vector H_k and the scalar curvature H of such space are non-vanishing. Also under some certain conditions, the generalized $H-$recurrent Finsler space became a Landsberg space.

In 2011, Dwivedi [40] proved that every $C-$reducible Finsler space is $P-$reducible and converse is not necessarily true. Further, we have obtained the role of $P-$reducible Finsler spaces in different special Finsler spaces.

In 2012, Pandey and Saxena [42] established that an infinitesimal transformation in Finsler space is Lie recurrence if and only if the normal projective curvature tensor is Lie recurrent.

In 2013, Saxena and Swaroop [44] introduced a Finsler space whose $v(hv)-$torsion tensor P_{ijk} is given by

$$P_{ijk} = \lambda C_{ijk} + \varphi\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right), \tag{14}$$

where λ and φ are scalar vectors positively homogeneous of degree one in y^j and call such a Finsler space as $P-$reducible Finsler space. They proved that a $P-$Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater than three. Also we have worked out the role of $P-$reducibility condition in special Finsler spaces.

In 2014, Zafar and Musavvir [10] discussed some properties of W–curvature tensor. They developed the relationships between the divergences of W, projective, conformal, conharmonic and concircular curvature tensors. And they introduced a symmetry property of spacetime of general relativity which known as $W-$collineation and defined through the vanishing of Lie derivative of $W-$curvature tensor with respect to a vector field.

In 2015, Chethana and Narasimhamurthy [11] showed that every semi$-C$ $-$reducible manifold with $C-$reducible metric reduces to a Landsberg manifold. Also they proved that there is no existence of $C2-$like Randers metric.

In 2016, Qasem [17] discussed Berwald curvature tensor H^i_{jkh} which are generalized birecurrent tensor in sense of Berwald. He studied some properties of a generalized $H-$birecurrent space by using Berwald's covariant derivative of the second order for the $(h)v-$torsion tensor H^i_{jk} and deviation tensor H^i_j. In addition, some conditions have been pointed out which reduce a generalized $H-$birecurrent space $F_n (n > 2)$ into Landsberg space.

In 2016, Qasem and Abdallah [18] studied the generalized $\mathfrak{B}R-$recurrent space. this paper concentrated on the necessary and sufficient condition for some tensors which satisfy the generalized recurrence property in the sense of Berwald. They obtained the necessary and sufficient condition for Berwald curvature tensor H^i_{jkh}, its associative H_{jpkh} and Cartan's fourth curvature tensor to be generalized recurrent. And they proved that tensor $(H_{hk} - H_{kh})$ and $H-$Ricci tensor H_{kh} are non-vanishing. Also they concluded that the torsion tensor K^i_{jk}, deviation tensor K^i_h, $K-$Ricci tensor K_{jk}, curvature vectors K_k, R_j and curvature scalar H behave as recurrent.

In 2016, Qasem and Hadi [20] introduced a Finsler space which Cartan's third curvature tensor R^i_{jkh} satisfies the generalized birecurrence property in sense of Berwald. This space is called a generalized $\mathfrak{B}R-$birecurrent space. They proved that the curvature tensor H^i_{jkh}, torsion tensor H^i_{kh}, the deviation tensor H^i_h, Ricci tensors H_{jk}, curvature vector H_k and scalar curvature tensor H of such space are non-vanishing. Furthermore, they deduced that generalized $\mathfrak{B}R-$birecurrent space is necessarily Landsberg space. The necessary and sufficient condition for some tensors that are generalized birecurrent have been introduced. Certain identities belong to this space have been obtained.

In 2017, Kesarwani and Pandey [48] discussed the concept of directional recurrence of Cartan curvature tensor and obtained certain properties of such space.

In 2018, Crampin [30] proved that a Landsberg space whose mean Berwald curvature has vanished trace is a Berwald space.

In 2018, Zamanzadeh et al. [49] introduced the generalized $P-$reducible Finsler manifolds. They proved that every compact $GP-$reducible manifold with positive or negative character is a Randers manifold. Also they obtained the relation between the norm of Cartan torsion, mean Cartan torsion, Landsberg and mean Landsberg curvatures of the class of $GP-$reducible manifolds. In the end, they showed that every $GP-$reducible manifold admitting a concurrent vector field reduces to a weakly Landsberg manifold.

In 2018, Rawat and Chauhan [29] studied the decomposition of curvature tensor fields R^h_{ijk} in terms of two non-zero vectors and a tensor field in Einstein Sasakian recurrent space of first order. Further, several theorem have been established and proved.

In 2019, Abdallah et al. [1] concentrated on a Finsler space which Cartan's second curvature tensor P^i_{jkh} satisfies the generalized recurrence property in sense of Berwald. This space and tensor are called a generalized $\mathfrak{B}P-$recurrent space and generalized $\mathfrak{B}-$recurrent tensor, repectively. Several theorems have been established and proved in the mentioned space. They obtained Berwald's covariant derivative of first order for some tensors and we get some tensors are non-vanishing. Additionally, the necessary and sufficient condition for various tensors that are generalized recurrent have been studied. Also we showed that the behavior of some tensors as recurrent in the mentioned space. Certain identities belong to this space have been obtained.

In 2019, Abdallah et al. [2] deduced that the generalized $\mathfrak{B}P-$recurrent space can't be an affinely connected space and a Landsberg space. In addition, they extended the generalized $\mathfrak{B}P-$recurrent space by using the properties of $P2-$like space to get new space that called a $P2-$like generalized $\mathfrak{B}P-$recurrent space.

In 2020, Wosoughi [26] discussed a new special form in Finsler space and obtained the condition for Finsler space to be a Landsberg space.

In 2020, Beizavi [46] introduced a definition for semi$-C-$reducible space and $C3-$like space. Further, he studied the relationship between $C3-$like metric with $C-$reducible metric, semi$-C-$reducible metric and $C2-$like metric.

In 2020, Qasem et al. [19] introduced a Finsler space F_n whose Cartan's third curvature tensor R^i_{jkh} satisfies the following condition

$$R^i_{jkh|l|m|n} = c_{lmn} R^i_{jkh} + d_{lmn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right), \quad R^i_{jkh} \neq 0, \tag{15}$$

where $c_{lmn} = a_{l|mn} + a_{lm}\lambda_n$ and $d_{lmn} = a_{lm}\mu_n + d_{l|m|n}$ are non - zero covariant tensor field of third order. This space is called a neralized R^h-trirecurrent Finsler space. In addition, they obtained some tensors satisfy the generalized trirecurrence property. They studied the relation between Cartan's third curvature tensor R^i_{jkh} and Berwald curvature tensor H^i_{jkh}.

In 2020, Al – Qashbari [9] defined a Finsler space F_n for Weyl's projective curvature tensor W^i_{jkh} which satisfies the generalized birecurrence property with respect to Cartan's connection. This space is called a generalized W^h-birecurrent space and denoted briefly by GW^h-BRF_n. Further, he has obtained the necessary and sufficient condition for some tensors in this space. Several theorems of generalized W^h-birecurrent space have been obtained.

In 2020, Saleem [7] discussed the generalized recurrence property for normal projective curvature tensor N^i_{jkh} in sense of Cartan. A Finsler space F_n for which the normal projective curvature tensor N^i_{jkh} satisfies the following condition

$$N^i_{jkh|m} = \lambda_m N^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right), \quad N^i_{jkh} \neq 0, \tag{16}$$

is called a generalized $N_{|m}-$recurrent space. He obtained the curvature vector H_k, curvature scalar H and Ricci tensor N_{jk} are non-vanishing. Also he proved that the Weyl's projective curvature tensor is generalized recurrent tensor in above mentioned space.

In 2021, Mandal [28] introduced some Fundamental Formulas related to the torsion and curvature tensors. And he obtained the relation between them. He generalized the concept of recurrent Finsler connection by taking $h-$connection by applying $h-$covariant derivative of ϕ^P_{ij} as recurrent. Such connection is called generalized $h-$recurrent Finsler connection. Furthermore, the relation between curvature tensors of Cartan's connection and generalized $h-$recurrent Finsler connection has been established.

In 2021, Abdallah et al. [3] introduced a Finsler space F_n which Cartan's second curvature tensor P^i_{jkh} satisfies the following condition

$$\begin{aligned}\mathfrak{B}_l\mathfrak{B}_m P^i_{jkh} &= a_{lm} P^i_{jkh} + b_{lm}\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right)\\ &\quad -2y^t\mu_m\mathfrak{B}_t\left(\delta^i_j C_{khl} - \delta^i_k C_{jhl}\right),\end{aligned} \tag{17}$$

where $a_{lm} = \mathfrak{B}_l\lambda_m + \lambda_m\lambda_l$ and $b_{lm} = \lambda_m\mu_l + \mathfrak{B}_l\mu_m$ are non - zero covariant tensors field of second order known as recurrence tensors. These space and tensor are called generalized $\mathfrak{B}P-$birecurrent space and generalized $\mathfrak{B}-$birecurrent tensor, respectively. The relationship between the covariant derivative in sense of Berwald and $h-$covariant derivative in sense of Cartan has been obtained. Furthermore, the necessary and sufficient condition for some tensors that are generalized birecurrent have been introduced. We proved that the behavior of some tensors are as birecurrent.

In 2021, Bisht and Neg [39] studied decomposition of normal projective curvature tensor fields in Finsler manifolds. Some theorems have been established and proved related to it.

In 2021, Gangopadhyay and Tiwari [45] introduced the class of Finsler metrics that called $C3-$like metrics which satisfies the un-normal and normal Ricci flow equation and proved that such metrics are Einstein. Further, they obtained that $C3-$like Finsler metric may be considered as a generalization of $C-$reducible, semi$-C-$reducible and $C2-$like Finsler metrics.

In 2022, Saleem and Abdallah [8] discussed a Finsler space which the curvature tensor U^i_{jkh} satisfies the recurrence property in sense of Cartan. The relationship between the curvature tensor U^i_{jkh} and Douglas tensor D^i_{jkh} have been studied. The necessary and sufficient condition for some tensors to be recurrent were obtained. In the end, the recurrence property in a projection on indicatrix with respect to Cartan connection has been discussed.

In 2022, Abdallah et al. [4] applied three decomposable of Cartan's second curvature tensor to prove that Cartan's second curvature tensor in affinely connected space is symmetric in first and second indices of their decomposable.

In 2022, Abdallah et al. [5] discussed the decomposition of Cartan's second curvature tensor P^i_{jkh} in two spaces, generalized $\mathfrak{B}P-$recurrent space and generalized $\mathfrak{B}P-$birecurrent space. They obtained different tensors which satisfy the recurrence and birecurrece property under the decomposition. Also we proved the decomposition for different tensors are non-vanishing. In addition, some examples have been discussed under the decomposition in $G(\mathfrak{B}P)-RF_n$ and $G(\mathfrak{B}P)-BRF_n$.

4. Conclusion

This paper discussed the important definitions of Finsler geometry with eliminating of such provided definition. Furthermore, thirty previous studies of Finsler geometry had been shed light on. In the future these spaces can be generalized and practical applications can be found as well.

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