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Generalized Richard Numbers

Research Article

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Abstract: In this paper, we introduce and investigate the generalized Richard sequences and we deal with, in detail, two special cases, namely, Richard and Richard-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Richard, Richard-Lucas and Padovan, Perrin, adjusted Padovan numbers.

MSC: 11B37 • 11B39 • 11B83

Keywords: Richard numbers • Richard-Lucas numbers • Padovan numbers • Perrin numbers • adjusted Padovan numbers

1. Introduction

Padovan sequence $\{P_n\}_{n\geq 0}$ (OEIS: A000931, [10]), Perrin (Padovan-Lucas) sequence $\{E_n\}_{n\geq 0}$ (OEIS: A001608, [10]) and adjusted Padovan sequence $\{U_n\}_{n\geq 0}$ (a variant of the sequence $\{P_n\}$) are defined, respectively, by the third-order recurrence relations

$$P_{n+3} = P_{n+1} + P_n, \quad P_0 = 1, P_1 = 1, P_2 = 1, \tag{1}$$
$$E_{n+3} = E_{n+1} + E_n, \quad E_0 = 3, E_1 = 0, E_2 = 2, \tag{2}$$
$$U_{n+3} = U_{n+1} + U_n, \quad U_0 = 0, U_1 = 1, U_2 = 0. \tag{3}$$

The sequences $\{P_n\}_{n\geq 0}$, $\{E_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} P_{-n} &= -P_{-(n-1)} + P_{-(n-3)}, \\ E_{-n} &= -E_{-(n-1)} + E_{-(n-3)}, \\ U_{-n} &= -U_{-(n-1)} + U_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1)-(3) hold for all integer n. For more details on the generalized Padovan numbers, see for example [1, 2, 8, 16].

Now, we define two sequences related to Padovan, Perrin and adjusted Padovan numbers. Richard and Richard-Lucas numbers are defined as

$$R_n = R_{n-2} + R_{n-3} + 1, \quad \text{with } R_0 = 0, R_1 = 1, R_2 = 1, \quad n \geq 3,$$

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and

$$Q_n = Q_{n-2} + Q_{n-3} - 1, \quad \text{with } Q_0 = 4, Q_1 = 1, Q_2 = 3, \ n \geq 3,$$

respectively. The first few values of Richard and Richard-Lucas numbers are

$$0, 1, 1, 2, 3, 4, 6, 8, 11, 15, 20, 27, 36, 48, ...$$

and

$$4, 1, 3, 4, 3, 6, 6, 8, 11, 13, 18, 23, 30, 40, ...$$

respectively. The sequences $\{R_n\}$ and $\{Q_n\}$ satisfy the following fourth order linear recurrences:

$$\begin{aligned} R_n &= R_{n-1} + R_{n-2} - R_{n-4}, \quad R_0 = 0, R_1 = 1, R_2 = 1, R_3 = 2, \quad n \geq 4, \\ Q_n &= Q_{n-1} + Q_{n-2} - Q_{n-4}, \quad Q_0 = 4, Q_1 = 1, Q_2 = 3, Q_3 = 4, \quad n \geq 4. \end{aligned}$$

There are close relations between Richard, Richard-Lucas and Padovan, Perrin, adjusted Padovan numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} R_n &= U_{n+2} + U_{n+1} + U_n - 1, \\ Q_n &= -U_{n+2} + 3U_{n+1} + U_n + 1, \\ 23R_n &= 10E_{n+2} + 8E_{n+1} + E_n - 23, \\ Q_n &= E_n + 1, \\ R_n &= P_{n+2} - 1, \\ Q_n &= -3P_{n+2} + 2P_{n+1} + 4P_n + 1, \end{aligned}$$

and

$$\begin{aligned} U_{n+1} &= R_{n+1} - R_n, \\ 23U_n &= 9Q_{n+2} - 2Q_{n+1} - 6Q_n - 1, \\ E_n &= R_{n+3} - 2R_{n+2} + 3R_{n+1} - 2R_n, \\ E_n &= Q_{n+3} - Q_{n+1}, \\ P_n &= R_{n+3} - R_{n+2}, \\ 23P_n &= 7Q_{n+2} + Q_{n+1} + 3Q_n - 11. \end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Richard, Richard-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized (r, s, t, u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \ n \geq 4 \tag{4}$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [3, 6, 7, 9, 12, 14, 15, 18, 19]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $u \neq 0$. Therefore, recurrence (4) holds for all integers n.

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \tag{5}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= r, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\ \alpha\beta\gamma\delta &= -u. \end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.1.
(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) Binet's formula of generalized Tetranacci numbers is

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \quad (6)$$

where

$$\begin{aligned} p_1 &= W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0, \\ p_2 &= W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0, \\ p_3 &= W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0, \\ p_4 &= W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0. \end{aligned}$$

Usually, it is customary to choose $\alpha,\beta,\gamma,\delta$ so that the Equ. (5) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (5) for non-negative integers is valid for all integers n (see [4]).

Next, we consider two special cases of the generalized (r,s,t,u) sequence $\{W_n\}$ which we call them (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas sequences. (r,s,t,u)-Fibonacci sequence $\{G_n\}_{n\geq 0}$ and (r,s,t,u)-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3}+sG_{n+2}+tG_{n+1}+uG_n, \quad (7) \\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, \\ H_{n+4} &= rH_{n+3}+sH_{n+2}+tH_{n+1}+uH_n, \quad (8) \\ H_0 &= 4, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)}-\frac{s}{u}G_{-(n-2)}-\frac{r}{u}G_{-(n-3)}+\frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)}-\frac{s}{u}H_{-(n-2)}-\frac{r}{u}H_{-(n-3)}+\frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (7) and (8) hold for all integers n.

For all integers n, (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas numbers (using initial conditions in (7) or (8)) can be expressed using Binet's formulas as in the following corollary.

Corollary 1.1.
(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) Binet's formula of (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas numbers are

$$G_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

and

$$H_n = \alpha^n+\beta^n+\gamma^n+\delta^n,$$

respectively.

Proof. Take $W_n = G_n$ and $W_n = H_n$ in Theorem 1.1, respectively. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 1.1.
Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized (r,s,t,u) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0+(W_1-rW_0)z+(W_2-rW_1-sW_0)z^2+(W_3-rW_2-sW_1-tW_0)z^3}{1-rz-sz^2-tz^3-uz^4}. \quad (9)$$

Proof. For a proof, see Soykan [12], Lemma 1. □

The following theorem presents Simson's formula of generalized (r,s,t,u) sequence (generalized Tetranacci sequence) $\{W_n\}$.

Theorem 1.2 (Simson's Formula of Generalized (r,s,t,u) Numbers).
For all integers n, we have

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{10}$$

Proof. (10) is given in Soykan [11]. □

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.3.
For $n \in \mathbb{Z}$, for the generalized Tetranacci sequence (or generalized (r,s,t,u)-sequence or 4-step Fibonacci sequence) we have the following:

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n}+6H_nW_{2n}-3H_n^2W_n+3H_{2n}W_n+W_0H_n^3+2W_0H_{3n}-3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n}-H_nW_{2n}+\frac{1}{2}(H_n^2-H_{2n})W_n-\frac{1}{6}(H_n^3+2H_{3n}-3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [13], Theorem 1. □

Using Theorem 1.3, we have the following corollary, see Soykan [13], Corollary 4.

Corollary 1.2.
For $n \in \mathbb{Z}$, we have

(a) $2(-u)^{n+4}G_{-n} = -(3ru^2+t^3-3stu)^2G_n^3-(2su-t^2)^2G_{n+3}^2G_n-(-rt^2-tu+2rsu)^2G_{n+2}^2G_n-(-st^2+2s^2u+4u^2+rtu)^2G_{n+1}^2G_n+2(3ru^2+t^3-3stu)((-2su+t^2)G_{n+3}+(-rt^2-tu+2rsu)G_{n+2}+(-st^2+2s^2u+4u^2+rtu)G_{n+1})G_n^2+2(2su-t^2)(-rt^2-tu+2rsu)G_{n+3}G_{n+2}G_n+2(2su-t^2)(-st^2+2s^2u+4u^2+rtu)G_{n+3}G_{n+1}G_n-2(-st^2+2s^2u+4u^2+rtu)(-rt^2-tu+2rsu)G_{n+2}G_{n+1}G_n-2G_{3n}u^4+u^2(-2su+t^2)G_{2n+3}G_n+u^2(-rt^2-tu+2rsu)G_{2n+2}G_n+u^2(-st^2+2s^2u+4u^2+rtu)G_{2n+1}G_n-2u^2(2su-t^2)G_{2n}G_{n+3}+2u^2(-rt^2-tu+2rsu)G_{2n}G_{n+2}+2u^2(-st^2+2s^2u+4u^2+rtu)G_{2n}G_{n+1}-3u^2(3ru^2+t^3-3stu)G_{2n}G_n.$

(b) $H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3+2H_{3n}-3H_{2n}H_n\right).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0=0$ and $H_0=4$ in Theorem 1.3,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n}+6H_nG_{2n}-3H_n^2G_n+3H_{2n}G_n), \tag{11}$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3+2H_{3n}-3H_{2n}H_n\right), \tag{12}$$

respectively.

If we define the square matrix A of order 4 as

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n+tG_{n-1}+uG_{n-2} & tG_n+uG_{n-1} & uG_n \\ G_n & sG_{n-1}+tG_{n-2}+uG_{n-3} & tG_{n-1}+uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2}+tG_{n-3}+uG_{n-4} & tG_{n-2}+uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3}+tG_{n-4}+uG_{n-5} & tG_{n-3}+uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & sW_n+tW_{n-1}+uW_{n-2} & tW_n+uW_{n-1} & uW_n \\ W_n & sW_{n-1}+tW_{n-2}+uW_{n-3} & tW_{n-1}+uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2}+tW_{n-3}+uW_{n-4} & tW_{n-2}+uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3}+tW_{n-4}+uW_{n-5} & tW_{n-3}+uW_{n-4} & uW_{n-3} \end{pmatrix}$$

then we get the following Theorem.

Theorem 1.4.
For all integers m, n, we have

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

(b) $U_1 A^n = A^n U_1$.

(c) $U_{n+m} = U_n B_m = B_m U_n$.

Proof. For the proof, see Soykan [12], Theorem 19. □

Theorem 1.5.
For all integers m, n, we have

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \tag{13}$$

Proof. For the proof, see Soykan [12], Theorem 20. □

2. Generalized Richard Sequence

In this paper, we consider the case $r = 1, s = 1, t = 0, u = -1$. A generalized Richard sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = W_{n-1} + W_{n-2} - W_{n-4} \tag{14}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (14) holds for all integers n.

Characteristic equation of $\{W_n\}$ is

$$z^4 - z^3 - z^2 + 1 = (z^3 - z - 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned} \alpha &= \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} = 1.32471795724, \\ \beta &= \omega\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}, \\ \gamma &= \omega^2\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}, \\ \delta &= 1, \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= 1, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -1, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\ \alpha\beta\gamma\delta &= 1. \end{aligned}$$

Table 1. A few generalized Richard numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$W_1+W_2-W_3$
2	W_2	$W_0+W_1-W_2$
3	W_3	$W_0+W_2-W_3$
4	$W_2-W_0+W_3$	$2W_1-W_3$
5	$W_2-W_1-W_0+2W_3$	$2W_0-W_2$
6	$W_2-W_1-2W_0+3W_3$	$W_1+2W_2-2W_3$
7	$2W_2-2W_1-3W_0+4W_3$	$W_0+2W_1-2W_2$
8	$2W_2-3W_1-4W_0+6W_3$	$2W_0-W_1+W_2-W_3$
9	$3W_2-4W_1-6W_0+8W_3$	$3W_1-W_0+W_2-2W_3$
10	$4W_2-6W_1-8W_0+11W_3$	$3W_0-3W_2+W_3$
11	$5W_2-8W_1-11W_0+15W_3$	$4W_2-3W_3$
12	$7W_2-11W_1-15W_0+20W_3$	$4W_1-3W_2$
13	$9W_2-15W_1-20W_0+27W_3$	$4W_0-3W_1$

Note also that

$$\begin{aligned}\alpha+\beta+\gamma &= 0,\\ \alpha\beta+\alpha\gamma+\beta\gamma &= -1,\\ \alpha\beta\gamma &= 1.\end{aligned}$$

The first few generalized Richard numbers with positive subscript and negative subscript are given in the following Table 1.

Note that the sequences $\{R_n\}$ and $\{Q_n\}$ which are defined in the section Introduction, are the special cases of the generalized Richard sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$ in this section as well. Richard sequence $\{R_n\}_{n\geq 0}$ and Richard-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}R_n &= R_{n-1}+R_{n-2}-R_{n-4}, \quad R_0=0, R_1=1, R_2=1, R_3=2, \quad n\geq 4,\\ Q_n &= Q_{n-1}+Q_{n-2}-Q_{n-4}, \quad Q_0=4, Q_1=1, Q_2=3, Q_3=4, \quad n\geq 4.\end{aligned}$$

The sequences $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}R_{-n} &= R_{-(n-2)}+R_{-(n-3)}-R_{-(n-4)}\\ Q_{-n} &= Q_{-(n-2)}+Q_{-(n-3)}-Q_{-(n-4)}\end{aligned}$$

for $n=1,2,3,...$ respectively.

Next, we present the first few values of the Richard and Richard-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
R_n	0	1	1	2	3	4	6	8	11	15	20	27	36	48
R_{-n}	0	0	0	−1	0	−1	−1	0	−2	0	−1	−2	1	−3
Q_n	4	1	3	4	3	6	6	8	11	13	18	23	30	40
Q_{-n}	4	0	2	3	−2	5	−1	0	6	−6	7	0	−5	13

Theorem 1.1 can be used to obtain the Binet formula of generalized Richard numbers. Using these (the above) roots and the recurrence relation, Binet's formula of generalized Richard numbers can be given as follows:

Theorem 2.1.
(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta=1$) For all integers n, Binet's formula of generalized Richard numbers is

$$\begin{aligned}W_n &= \frac{(\alpha W_3-\alpha(1-\alpha)W_2+(-\alpha^2+1)W_1-W_0)\alpha^n}{2\alpha^2+\alpha-3}\\ &+\frac{(\beta W_3-\beta(1-\beta)W_2+(-\beta^2+1)W_1-W_0)\beta^n}{2\beta^2+\beta-3}\\ &+\frac{(\gamma W_3-\gamma(1-\gamma)W_2+(-\gamma^2+1)W_1-W_0)\gamma^n}{2\gamma^2+\gamma-3}-W_3+W_1+W_0.\end{aligned}$$

Richard and Richard-Lucas numbers can be expressed using Binet's formulas as follows:

Corollary 2.1.
(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formulas of Richard and Richard-Lucas numbers are

$$
\begin{aligned}
R_n &= \frac{(\alpha+1)\alpha^n}{2\alpha^2+\alpha-3} + \frac{(\beta+1)\beta^n}{2\beta^2+\beta-3} + \frac{(\gamma+1)\gamma^n}{2\gamma^2+\gamma-3} - 1 \\
&= \frac{\alpha^{n+3}}{2\alpha^2+\alpha-3} + \frac{\beta^{n+3}}{2\beta^2+\beta-3} + \frac{\gamma^{n+3}}{2\gamma^2+\gamma-3} - 1, \\
Q_n &= \alpha^n + \beta^n + \gamma^n + 1,
\end{aligned}
$$

respectively.

Note that for all integers n, Padovan, Perrin and adjusted Padovan numbers can be expressed using Binet's formulas as

$$
\begin{aligned}
P_n &= \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)}, \\
E_n &= \alpha^n + \beta^n + \gamma^n, \\
U_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},
\end{aligned}
$$

respectively, see Soykan [16] for more details. So, by using Binet's formulas of Richard, Richard-Lucas and Padovan, Perrin, adjusted Padovan numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

Lemma 2.1.
For all integers n, the following equalities (identities) are true:

(a)

- $U_{n+3} = R_{n+3} - R_{n+2}$.
- $U_n = R_{n+3} - R_{n+2} - R_{n+1} + R_n$.
- $R_{n+4} = 3U_{n+2} + 4U_{n+1} + 2U_n - 1$.
- $R_n = U_{n+2} + U_{n+1} + U_n - 1$.
- $U_n = -R_{n+2} + 2R_n + 1$.
- $U_{n+1} = R_{n+1} - R_n$.

(b)

- $23U_{n+3} = 11Q_{n+3} + 7Q_{n+2} - 10Q_{n+1} - 8Q_n$.
- $23U_n = Q_{n+3} + 9Q_{n+2} - 3Q_{n+1} - 7Q_n$.
- $Q_{n+4} = 3U_{n+2} + 2U_{n+1} + 2U_n + 1$.
- $Q_n = -U_{n+2} + 3U_{n+1} + U_n + 1$.
- $23U_n = 9Q_{n+2} - 2Q_{n+1} - 6Q_n - 1$.
- $Q_{n+1} + 3Q_n = 9U_{n+1} + 2U_n + 4$.

(c)

- $E_{n+3} = 2R_{n+2} + R_{n+1} - 3R_n$.
- $E_n = R_{n+3} - 2R_{n+2} + 3R_{n+1} - 2R_n$.
- $23R_{n+4} = 19E_{n+2} + 29E_{n+1} + 18E_n - 23$.
- $23R_n = 10E_{n+2} + 8E_{n+1} + E_n - 23$.
- $E_n = -2R_{n+2} + 4R_{n+1} - R_n + 1$.
- $5R_{n+1} - 4R_n = E_{n+1} + 2E_n - 1$.

(d)

- $E_{n+3} = 2Q_{n+3} - Q_{n+1} - Q_n$.
- $E_n = Q_{n+3} - Q_{n+1}$.
- $Q_{n+4} = E_{n+2} + E_{n+1} + 1$.
- $Q_n = E_n + 1$.

(e)

- $P_{n+3} = R_{n+3} - R_n$.
- $P_n = R_{n+3} - R_{n+2}$.
- $R_{n+4} = P_{n+2} + 2P_{n+1} + P_n - 1$.
- $R_n = P_{n+2} - 1$.
- $P_n = -R_{n+2} + R_{n+1} + R_n + 1$.
- $R_{n+1} = P_{n+1} + P_n - 1$.

(f)

- $23P_{n+3} = 29Q_{n+3} + 8Q_{n+2} - 18Q_{n+1} - 19Q_n$.
- $23P_n = 11Q_{n+3} + 7Q_{n+2} - 10Q_{n+1} - 8Q_n$.
- $Q_{n+4} = 3P_{n+2} - P_n + 1$.
- $Q_n = -3P_{n+2} + 2P_{n+1} + 4P_n + 1$.
- $23P_n = 7Q_{n+2} + Q_{n+1} + 3Q_n - 11$.
- $3Q_{n+1} + 2Q_n = 7P_{n+1} - P_n + 5$.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 2.2.
Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Richard sequence $\{W_n\}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - W_0)z + (W_2 - W_1 - W_0)z^2 + (W_3 - W_2 - W_1)z^3}{1 - z - z^2 + z^4}.$$

Proof. Take $r = 1, s = 1, t = 0, u = -1$ in Lemma 1.1.
The previous lemma gives the following results as particular examples.

Corollary 2.2.
Generating functions of Richard and Richard-Lucas numbers are

$$\sum_{n=0}^{\infty} R_n z^n = \frac{z}{1 - z - z^2 + z^4} = \frac{z}{(1-z)(1-z^2-z^3)},$$
$$\sum_{n=0}^{\infty} Q_n z^n = \frac{4 - 3z - 2z^2}{1 - z - z^2 + z^4} = \frac{4 - 3z - 2z^2}{(1-z)(1-z^2-z^3)},$$

respectively.

3. Simson Formulas

Now, we present Simson's formula of generalized Richard numbers.

Theorem 3.1 (Simson's Formula of Generalized Richard Numbers).
For all integers n, we have

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = W_3^4+W_1^4+W_0^4+2W_2^2W_3^2+3W_1^2W_3^2+2W_1^2W_2^2+3W_0^2W_2^2-(2W_1+3W_2)W_3^3-(W_0+W_1-W_3)W_2^3+(W_0-W_2-3W_3)W_1^3-(2W_2+W_3)W_0^3+(4W_0W_2-3W_0W_1-W_0^2+4W_1W_2)W_3^2+(5W_0W_1-5W_0W_3-4W_1W_3)W_2^2+(W_3-2W_2-W_0)W_0W_1^2+(4W_1W_3-3W_1W_2+W_2W_3)W_0^2-W_0W_1W_2W_3.$$

Proof. Take $r=1, s=1, t=0, u=-1$ in Theorem 1.2. □
The previous theorem gives the following results as particular examples.

Corollary 3.1.
For all integers n, the Simson's formulas of Richard and Richard-Lucas numbers are given as

$$\begin{vmatrix} R_{n+3} & R_{n+2} & R_{n+1} & R_n \\ R_{n+2} & R_{n+1} & R_n & R_{n-1} \\ R_{n+1} & R_n & R_{n-1} & R_{n-2} \\ R_n & R_{n-1} & R_{n-2} & R_{n-3} \end{vmatrix} = 1,$$

$$\begin{vmatrix} Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \end{vmatrix} = -23.$$

respectively.

4. Some Identities

In this section, we obtain some identities of Richard and Richard-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{R_n\}$.

Lemma 4.1.
The following equalities are true:

(a) $W_n=(2W_1-W_3)R_{n+5}+(W_0-2W_1+W_2)R_{n+4}+(2W_3-2W_2-W_1)R_{n+3}+(W_2-2W_0)R_{n+2}.$

(b) $W_n=(W_0+W_2-W_3)R_{n+4}+(W_1-2W_2+W_3)R_{n+3}+(W_2-2W_0)R_{n+2}+(W_3-2W_1)R_{n+1}.$

(c) $W_n=(W_0+W_1-W_2)R_{n+3}+(2W_2-W_0-W_3)R_{n+2}+(W_3-2W_1)R_{n+1}+(W_3-W_2-W_0)R_n.$

(d) $W_n=(W_1+W_2-W_3)R_{n+2}+(W_0-W_1-W_2+W_3)R_{n+1}+(W_3-W_2-W_0)R_n+(W_2-W_1-W_0)R_{n-1}.$

(e) $W_n=W_0R_{n+1}+(W_1-W_0)R_n+(W_2-W_1-W_0)R_{n-1}+(W_3-W_2-W_1)R_{n-2}.$

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n=a\times R_{n+5}+b\times R_{n+4}+c\times R_{n+3}+d\times R_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a\times R_5+b\times R_4+c\times R_3+d\times R_2 \\ W_1 &= a\times R_6+b\times R_5+c\times R_4+d\times R_3 \\ W_2 &= a\times R_7+b\times R_6+c\times R_5+d\times R_4 \\ W_3 &= a\times R_8+b\times R_7+c\times R_6+d\times R_5 \end{aligned}$$

we find that $a=2W_1-W_3, b=W_0-2W_1+W_2, c=2W_3-2W_2-W_1, d=W_2-2W_0$. The other equalities can be proved similarly. □
Note that all the identities in the above Lemma can be proved by induction as well.
Next, we present a few basic relations between $\{W_n\}$ and $\{Q_n\}$.

Lemma 4.2.
The following equalities are true:

(a) $23W_n = -(24W_0+30W_1+2W_2-33W_3)Q_{n+5}-(9W_0-6W_1-5W_2+2W_3)Q_{n+4}+(26W_0+21W_1+6W_2-30W_3)Q_{n+3}+(30W_0+26W_1-9W_2-24W_3)Q_{n+2}$.

(b) $23W_n = -(33W_0+24W_1-3W_2-31W_3)Q_{n+4}+(2W_0-9W_1+4W_2+3W_3)Q_{n+3}+(30W_0+26W_1-9W_2-24W_3)Q_{n+2}+(24W_0+30W_1+2W_2-33W_3)Q_{n+1}$.

(c) $23W_n = -(31W_0+33W_1-7W_2-34W_3)Q_{n+3}-(3W_0-2W_1+6W_2-7W_3)Q_{n+2}+(24W_0+30W_1+2W_2-33W_3)Q_{n+1}+(33W_0+24W_1-3W_2-31W_3)Q_n$.

(d) $23W_n = -(34W_0+31W_1-W_2-41W_3)Q_{n+2}-(7W_0+3W_1-9W_2-W_3)Q_{n+1}+(33W_0+24W_1-3W_2-31W_3)Q_n+(31W_0+33W_1-7W_2-34W_3)Q_{n-1}$.

(e) $23W_n = -(41W_0+34W_1-10W_2-42W_3)Q_{n+1}-(W_0+7W_1+2W_2-10W_3)Q_n+(31W_0+33W_1-7W_2-34W_3)Q_{n-1}+(34W_0+31W_1-W_2-41W_3)Q_{n-2}$.

Now, we give a few basic relations between $\{R_n\}$ and $\{Q_n\}$.

Lemma 4.3.
The following equalities are true:

$$
\begin{aligned}
23R_n &= 34Q_{n+5}+7Q_{n+4}-33Q_{n+3}-31Q_{n+2},\\
23R_n &= 41Q_{n+4}+Q_{n+3}-31Q_{n+2}-34Q_{n+1},\\
23R_n &= 42Q_{n+3}+10Q_{n+2}-34Q_{n+1}-41Q_n,\\
23R_n &= 52Q_{n+2}+8Q_{n+1}-41Q_n-42Q_{n-1},\\
23R_n &= 60Q_{n+1}+11Q_n-42Q_{n-1}-52Q_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
Q_n &= -2R_{n+5}+5R_{n+4}+R_{n+3}-5R_{n+2},\\
Q_n &= 3R_{n+4}-R_{n+3}-5R_{n+2}+2R_{n+1},\\
Q_n &= 2R_{n+3}-2R_{n+2}+2R_{n+1}-3R_n,\\
Q_n &= 4R_{n+1}-3R_n-2R_{n-1}.
\end{aligned}
$$

5. Relations Between Special Numbers

In this section, we present identities on Richard, Richard-Lucas numbers and Padovan, Perrin, adjusted Padovan numbers. We know from Lemma 2.1 that

$$
\begin{aligned}
R_n &= P_{n+2}-1,\\
Q_n &= -3P_{n+2}+2P_{n+1}+4P_n+1,
\end{aligned}
$$

Note also that from Lemma 4.1 and Lemma 4.2, we have the formulas of W_n as

$$
\begin{aligned}
W_n &= (W_0+W_1-W_2)R_{n+3}+(2W_2-W_0-W_3)R_{n+2}+(W_3-2W_1)R_{n+1}+(W_3-W_2-W_0)R_n,\\
23W_n &= -(31W_0+33W_1-7W_2-34W_3)Q_{n+3}-(3W_0-2W_1+6W_2-7W_3)Q_{n+2}\\
&\quad +(24W_0+30W_1+2W_2-33W_3)Q_{n+1}+(33W_0+24W_1-3W_2-31W_3)Q_n.
\end{aligned}
$$

Using the above identities, we obtain relation of generalized Richard numbers in the following forms (in terms of Padovan numbers):

Lemma 5.1.
For all integers n, we have the following identities:

(a) $W_n = (W_0+W_1-W_2)P_{n+5}+(2W_2-W_0-W_3)P_{n+4}+(W_3-2W_1)P_{n+3}+(W_3-W_2-W_0)P_{n+2}-W_3+W_1+W_0$.

(b) $W_n = (W_1-W_0)P_{n+2}+(W_2-W_1)P_{n+1}+(W_3-W_2-W_1+W_0)P_n-W_3+W_1+W_0$.

6. On the Recurrence Properties of Generalized Richard Sequence

Taking $r = 1, s = 1, t = 0, u = -1$ in Theorem 1.3, we obtain the following Proposition.

Proposition 6.1.
For $n \in \mathbb{Z}$, generalized Richard numbers (the case $r = 1, s = 1, t = 0, u = -1$) have the following identity:

$$W_{-n} = \frac{1}{6}(-6W_{3n} + 6Q_n W_{2n} - 3Q_n^2 W_n + 3Q_{2n} W_n + W_0 Q_n^3 + 2W_0 Q_{3n} - 3W_0 Q_n Q_{2n}).$$

From the above Proposition 6.1 (or by taking $G_n = R_n$ and $H_n = Q_n$ in (11) and (12) respectively), we have the following corollary which gives the connection between the special cases of generalized Richard sequence at the positive index and the negative index: for Richard and Richard-Lucas numbers: take $W_n = R_n$ with $R_0 = 0, R_1 = 1, R_2 = 1, R_3 = 2$ and take $W_n = Q_n$ with $Q_0 = 4, Q_1 = 1, Q_2 = 3, Q_3 = 4$, respectively. Note that in this case $H_n = Q_n$.

Corollary 6.1.
For $n \in \mathbb{Z}$, we have the following recurrence relations:

(a) *Richard sequence:*

$$R_{-n} = \frac{1}{6}(-6R_{3n} + 6Q_n R_{2n} - 3Q_n^2 R_n + 3Q_{2n} R_n).$$

(b) *Richard-Lucas sequence:*

$$Q_{-n} = \frac{1}{6}\left(Q_n^3 + 2Q_{3n} - 3Q_{2n} Q_n\right).$$

We can also present the formulas of R_{-n} and Q_{-n} in the following forms.

Corollary 6.2.
For $n \in \mathbb{Z}$, we have the following recurrence relations:

(a) $R_{-n} = \frac{1}{6}(-6R_{3n} + 6(2R_{n+3} - 2R_{n+2} + 2R_{n+1} - 3R_n)R_{2n} - 3(2R_{n+3} - 2R_{n+2} + 2R_{n+1} - 3R_n)^2 R_n + 3(2R_{2n+3} - 2R_{2n+2} + 2R_{2n+1} - 3R_{2n})R_n)$.

(b) $R_{-n} = U_n U_{n-2} - 3U_n U_{n+1} - 3U_n U_{n-1} + U_n U_{n+2} - U_n^2 + U_{2n} + U_{n-1} U_{n+1} - 3U_{n-1} U_{n-2} + U_{2n-2} + U_{2n-4} - U_{n-1}^2 - U_{n-2}^2 - 1$.

(c) $Q_{-n} = \frac{1}{2}(E_n^2 - E_{2n} + 2)$.

(d) $R_{-n} = \frac{1}{2}(9P_n^2 + 4P_{n-1}^2 + 8P_{n-2}^2 + 3P_{2n-2} - 2P_{2n-3} - 2P_{2n-4} - 12P_n P_{n-1} - 18P_n P_{n-2} + 12P_{n-1} P_{n-2} - 2)$.

Proof.

(a) By using the identity $Q_n = 2R_{n+3} - 2R_{n+2} + 2R_{n+1} - 3R_n$ and Corollary 6.1, (or by using Corollary 1.2 (a)), we get (a).

(b) Since $R_n = U_{n+2} + U_{n+1} + U_n - 1$ and $U_{-n} = -U_n^2 + U_{2n} + U_{n+2} U_n - 3U_{n+1} U_n$ (see, for example Soykan [17]), we get (b).

(c) Since $Q_n = E_n + 1$ and $E_{-n} = \frac{1}{2}(E_n^2 - E_{2n})$ (see, for example Soykan [17]), we obtain (c).

(d) Since

$$R_n = P_{n+2} - 1,$$

and

$$P_{-n} = \frac{1}{2}(9P_{n+2}^2 + 4P_{n+1}^2 + 8P_n^2 + 3P_{2n+2} - 2P_{2n+1} - 2P_{2n} - 12P_{n+2} P_{n+1} - 18P_{n+2} P_n + 12P_{n+1} P_n),$$

(see, for example Soykan [17]), we get (d). □

7. Sums

The following Corollary gives sum formulas of Padovan numbers.

Corollary 7.1.
For $n \geq 0$, Padovan numbers have the following properties:

(a) $\sum_{k=0}^{n} P_k = P_{n+3} + P_{n+2} - 2.$

(b) $\sum_{k=0}^{n} P_{2k} = P_{2n+1} + P_{2n} - 1.$

(c) $\sum_{k=0}^{n} P_{2k+1} = P_{2n+2} + P_{2n+1} - 1.$

Proof. It is given in Soykan [16]. □
The following Corollary presents sum formulas of Richard and Richard-Lucas numbers.

Corollary 7.2.
For $n \geq 0$, Richard and Richard-Lucas numbers have the following properties (in terms of Padovan numbers):

1.

(a) $\sum_{k=0}^{n} R_k = 2P_{n+2} + 2P_{n+1} + P_n - n - 5.$
(b) $\sum_{k=0}^{n} R_{2k} = P_{2n+2} + P_{2n+1} + P_{2n} - n - 3.$
(c) $\sum_{k=0}^{n} R_{2k+1} = P_{2n+2} + 2P_{2n+1} + P_{2n} - n - 3.$

2.

(a) $\sum_{k=0}^{n} Q_k = 2P_{n+1} + 3P_n + n - 1.$
(b) $\sum_{k=0}^{n} Q_{2k} = -P_{2n+2} + 3P_{2n+1} + P_{2n} + n + 1.$
(c) $\sum_{k=0}^{n} Q_{2k+1} = 3P_{2n+2} - P_{2n} + n - 1.$

Proof. The proof follows from Corollary 7.1 and the identities

$$\begin{aligned} R_n &= P_{n+2} - 1 \\ Q_n &= -3P_{n+2} + 2P_{n+1} + 4P_n + 1. \quad \square \end{aligned}$$

8. Matrices and Identities Related With Generalized Richard Numbers

If we define the square matrix A of order 4 as

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} R_{n+1} & R_n - R_{n-2} & -R_{n-1} & -R_n \\ R_n & R_{n-1} - R_{n-3} & -R_{n-2} & -R_{n-1} \\ R_{n-1} & R_{n-2} - R_{n-4} & -R_{n-3} & -R_{n-2} \\ R_{n-2} & R_{n-3} - R_{n-5} & -R_{n-4} & -R_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & W_n - W_{n-2} & -W_{n-1} & -W_n \\ W_n & W_{n-1} - W_{n-3} & -W_{n-2} & -W_{n-1} \\ W_{n-1} & W_{n-2} - W_{n-4} & -W_{n-3} & -W_{n-2} \\ W_{n-2} & W_{n-3} - W_{n-5} & -W_{n-4} & -W_{n-3} \end{pmatrix}$$

then we get the following Theorem.

Theorem 8.1.
For all integers m, n, we have

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} R_{n+1} & R_n - R_{n-2} & -R_{n-1} & -R_n \\ R_n & R_{n-1} - R_{n-3} & -R_{n-2} & -R_{n-1} \\ R_{n-1} & R_{n-2} - R_{n-4} & -R_{n-3} & -R_{n-2} \\ R_{n-2} & R_{n-3} - R_{n-5} & -R_{n-4} & -R_{n-3} \end{pmatrix}.$$

(b) $U_1 A^n = A^n U_1$.

(c) $U_{n+m} = U_n B_m = B_m U_n$.

Proof. Take $r = 1, s = 1, t = 0, u = -1$ in Theorem 1.4. □
Using the above last Theorem and the identity

$$R_n = P_{n+2} - 1$$

we obtain the following identity for Padovan numbers.

Corollary 8.1.
For all integers n, we have the following formula for Padovan numbers:

$$A^n = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{n+3} - 1 & P_{n+2} - P_n & -P_{n+1} + 1 & -P_{n+2} + 1 \\ P_{n+2} - 1 & P_{n+1} - P_{n-1} & -P_n + 1 & -P_{n+1} + 1 \\ P_{n+1} - 1 & P_n - P_{n-2} & -P_{n-1} + 1 & -P_n + 1 \\ P_n - 1 & P_{n-1} - P_{n-3} & -P_{n-2} + 1 & -P_{n-1} + 1 \end{pmatrix}.$$

Next, we present an identity for W_{n+m}.

Theorem 8.2.
For all integers m, n, we have

$$W_{n+m} = W_n R_{m+1} + W_{n-1}(R_m - R_{m-2}) - W_{n-2} R_{m-1} - W_{n-3} R_m. \quad (15)$$

Proof. Take $r = 1, s = 1, t = 0, u = -1$ in Theorem 1.5.. □
As particular cases of the above theorem, we give identities for R_{n+m} and Q_{n+m}.

Corollary 8.2.
For all integers m, n, we have

$$\begin{aligned} R_{n+m} &= R_n R_{m+1} + R_{n-1}(R_m - R_{m-2}) - R_{n-2} R_{m-1} - R_{n-3} R_m. \\ Q_{n+m} &= Q_n R_{m+1} + Q_{n-1}(R_m - R_{m-2}) - Q_{n-2} R_{m-1} - Q_{n-3} R_m. \end{aligned}$$

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