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On Analytical Solutions of the Classical Gas Dynamics Equation

Research Article

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Abstract: We solve the classical gas dynamics equation using the semi analytic iterative method. Numerical examples are given and the results show the efficiency and accuracy of the method for finding analytical solutions to such equations.

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Keywords: Gas dynamics equation • Semi-analytic iterative method • Adomian decomposition method • Variational iteration method • Homotopy perturbation method

1. Introduction

Gas dynamics is a science belonging to the branch of fluid dynamics concerned with the study of the motion of gases and its effects on physical systems. This science is based on the principles of fluid mechanics and thermodynamics and arises from studies of gas flows which include choked flows in nozzles and valves, shock waves around jets, aerodynamic heating on atmospheric reentry vehicles and flow of gas fuel within a jet engine [1]. Gas dynamics equations are nonlinear partial differential equations (PDEs) based on the physical laws that exist in standard engineering practice such as conservation of mass, conservation of momentum, conservation of energy, and so on [2]. These mathematical models are particularly applicable to three types of nonlinear waves, namely, shock fronts, rarefactions and contact discontinuities [3]. Gas dynamics equations (GDEs) have generated much interest in the literature and have been solved by means of a variety of methods which have included, but not limited to, Laplace homotopy perturbation method (LHPM) [4, 5], fractional natural decomposition method (FNDM) [6], fractional reduced differential transform method (FRDTM) [3], new integral projected differential transform method (NIPDTM) [7] and homotopy perturbation method with natural transform (NHPM) [8].

This paper applies the semi analytic iterative method (SAIM), first proposed by Temimi and Ansari [10], to the solution of the classical gas dynamics equation. This method has been used for solving different types of linear and nonlinear ordinary differential equations, PDEs and higher-order integrodifferential equations [10–12]. More recently it has been applied to solution of the Korteweg-de Vries (KdV) equation [13]. To the best of the author's knowledge, this method has not been applied to the solution of the one-dimensional classical gas dynamics equation.

Consider the fractional nonlinear gas dynamics equation

$$D_t^{\alpha} u + u u_x - u(1-u) = f(x,t), \; t > 0, \; -\infty < x < \infty, \tag{1}$$

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for $0 < \alpha \leq 1$, subject to the initial condition

$$u(x,0) = h(x), \tag{2}$$

where t is the time, x is the spatial coordinate, $u(x,t)$ is the probability density function and α is a parameter describing the order of the time-fractional derivatives. In the case $\alpha = 1$, Eq. (1) reduces to the classical nonlinear gas dynamics equation (GDE)

$$u_t + uu_x - u(1-u) = f(x,t), \ \ t > 0, \tag{3}$$

which is the focus of this paper.

The rest of the paper is organized as follows: Section 2 describes the proposed method of solution. In Section 3 the results of numerical experiments based on a selection of test problems are presented and compared with the exact solution and solutions from previous methods used in the literature and Section 4 offers some conclusions.

2. Description of the Method

The semi analytic iterative method (SAIM) which we propose to use in the solution of gas dynamics equations was used by Yassein [12] to solve higher order integro-differential equations and by Yassein and Aswhad [13] to solve KdV equations. The method was also used by Kasumo [14, 15] in the solution of the Klein-Gordon, Korteweg-de Vries and Burgers equations. This method uses an iterative approach together with analytical computations to provide a solution of a modified reformulated linear problem. The SAIM was inspired by the homotopy analysis method (HAM) which is a general approximate analytical approach for obtaining convergent series solutions of strongly nonlinear problems [11]. The SAIM is an efficient, reliable and powerful iterative scheme that offers several advantages over existing methods such as Picard's successive approximations method (SAM), He's variational iteration method (VIM) and the Adomian decomposition method (ADM). It is very easy to implement since it avoids the calculation of Adomian polynomials for the nonlinear term in the ADM or Lagrange multipliers in the VIM, thus demanding less computational work [16]. In this paper we propose to use the SAIM to solve the classical GDE of the form (3), with the initial condition (2). Eq. (3) can be expressed as

$$Lu + Nu = f(x,t), \tag{4}$$

with the condition $B\left(u, \frac{\partial u}{\partial t}\right) = 0$, where $Lu = u_t$, $Nu = uu_x - u(1-u)$, $f(x,t)$ is the source term and B is the boundary operator. Assuming that $u_0(x,t)$ is an initial approximation of the solution $u(x,t)$, we take it to be the solution of the equation

$$L[u_0(x,t)] = 0, \tag{5}$$

with $B\left(u_0, \frac{\partial u_0}{\partial t}\right) = 0$. Note that

$$u_0(x,t) = u(x,0) = h(x).$$

To generate the next iteration to the solution, we solve the equation

$$L[u_1(x,t)] = -N[u_0(x,t)] + f(x,t) \tag{6}$$

with $B\left(u_1, \frac{\partial u_1}{\partial t}\right) = 0$. Continuing in this manner leads to a simple iterative procedure which is effectively the solution of a linear set of problems, i.e.,

$$L[u_{n+1}(x,t)] = -N[u_n(x,t)] + f(x,t), \tag{7}$$

with $B\left(u_{n+1}, \frac{\partial u_{n+1}}{\partial t}\right) = 0$, from which the general iterative relation for solving the standard GDE (3) is

$$u_{n+1}(x,t) = u_{n+1}(x,0) + L^{-1}\left\{-N[u_n(x,t)] + f(x,t)\right\}, \tag{8}$$

where $L^{-1} = \int_0^t (\cdot)\,ds$. Thus, the solution to the problem (3) with condition (2) is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Note that if $f(x,t) = 0$, then the GDE is homogeneous and is solved using the iterative scheme

$$u_{n+1}(x,t) = u_{n+1}(x,0) + L^{-1}\left\{-N[u_n(x,t)]\right\}. \tag{9}$$

3. Numerical Examples and Discussion

In this section, we illustrate the SAIM by considering three numerical examples of the classical GDE to demonstrate the applicability of the method, as well as to validate its reliability and efficiency. All the computations associated with these examples were performed using a Samsung Series 3 PC with an Intel Celeron CPU 847 at 1.10 GHz with 6.0 GB internal memory and 64-bit operating system (Windows 8). The figures were constructed using MATLAB R2016a. The results are presented in tables and figures accompanying the discussion.

Example 3.1.
Consider the homogeneous GDE [3, 7]:

$$u_t + uu_x - u(1-u) = 0, \tag{10}$$

subject to the initial condition

$$u(x,0) = \mathrm{e}^{-x}.$$

This GDE has exact solution $u(x,t) = \mathrm{e}^{t-x}$ which grows exponentially with time t. To solve (10) using the SAIM, we rewrite it in operator-theoretic form as

$$Lu + Nu = 0, \tag{11}$$

where $Lu = u_t$ and $Nu = uu_x - u(1-u)$. The initial approximation $u_0(x,t)$ is obtained by solving the equation

$$L[u_0(x,t)] = 0, \text{ with } u_0(x,0) = \mathrm{e}^{-x}. \tag{12}$$

Using the initial condition, the solution of the primary problem is

$$u_0(x,t) = u_0(x,0) = \mathrm{e}^{-x}.$$

The general recursive relation for solving (10) is

$$L[u_{n+1}(x,t)] = -N[u_n(x,t)], \text{ with } u_{n+1}(x,0) = \mathrm{e}^{-x}, \tag{13}$$

that is,

$$u_{n+1}(x,t) = u_{n+1}(x,0) + \int_0^t \left[-u_n u_{n_x} + u_n(1-u_n)\right] ds. \tag{14}$$

From the recursive relation, we have the approximations

$$\begin{aligned}
u_0(x,t) &= \mathrm{e}^{-x}\\
u_1(x,t) &= \mathrm{e}^{-x} - \int_0^t \left[u_0 u_{0_x} - u_0(1-u_0)\right] ds = \mathrm{e}^{-x}(1+t)\\
u_2(x,t) &= \mathrm{e}^{-x} - \int_0^t \left[u_1 u_{1_x} - u_1(1-u_1)\right] ds = \mathrm{e}^{-x}\left(1+t+\frac{t^2}{2}\right)\\
u_3(x,t) &= \mathrm{e}^{-x} - \int_0^t \left[u_2 u_{2_x} - u_2(1-u_2)\right] ds = \mathrm{e}^{-x}\left(1+t+\frac{t^2}{2}+\frac{t^3}{6}\right)
\end{aligned}$$

and so on. Thus, the solution is

$$u(x,t) = \mathrm{e}^{-x}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right) = \mathrm{e}^{-x}\mathrm{e}^{t} = \mathrm{e}^{t-x}$$

which is the exact solution of the GDE (10), the same result obtained by FRDTM [3], LHPM [4] and NHPM [8, 9]. The approximate solutions from the SAIM are close to the exact solutions at the seventh term and are shown in Table 1 and Fig. 1. The absolute errors are also given and it can be seen that they are reducing as x increases.

Table 1. Comparison of Exact and Approximate Solutions from SAIM for Example 3.1

x	t	$u(x,t)$	$u_{\mathrm{SAIM}}(x,t)$	$e = \lvert u - u_{\mathrm{SAIM}}\rvert$
0.0	1.00	2.718281828459050	2.718055555555560	0.000226272903490
0.1	1.00	2.459603111156950	2.459398370967180	0.000204740189770
0.2	1.00	2.225540928492470	2.225355671907790	0.000185256584680
0.3	1.00	2.013752707470480	2.013585080380720	0.000167627089760
0.4	1.00	1.822118800390510	1.821967125127430	0.000151675263080
0.5	1.00	1.648721270700130	1.648584029246700	0.000137241453430
0.6	1.00	1.491824697641270	1.491700516438900	0.000124181202370
0.7	1.00	1.349858807576000	1.349746443777480	0.000112363798520
0.8	1.00	1.221402758160170	1.221301087190840	0.000101670969330
0.9	1.00	1.105170918075650	1.105078922378270	0.000091995697380
1.0	1.00	1.000000000000000	0.999916758850712	0.000083241149288

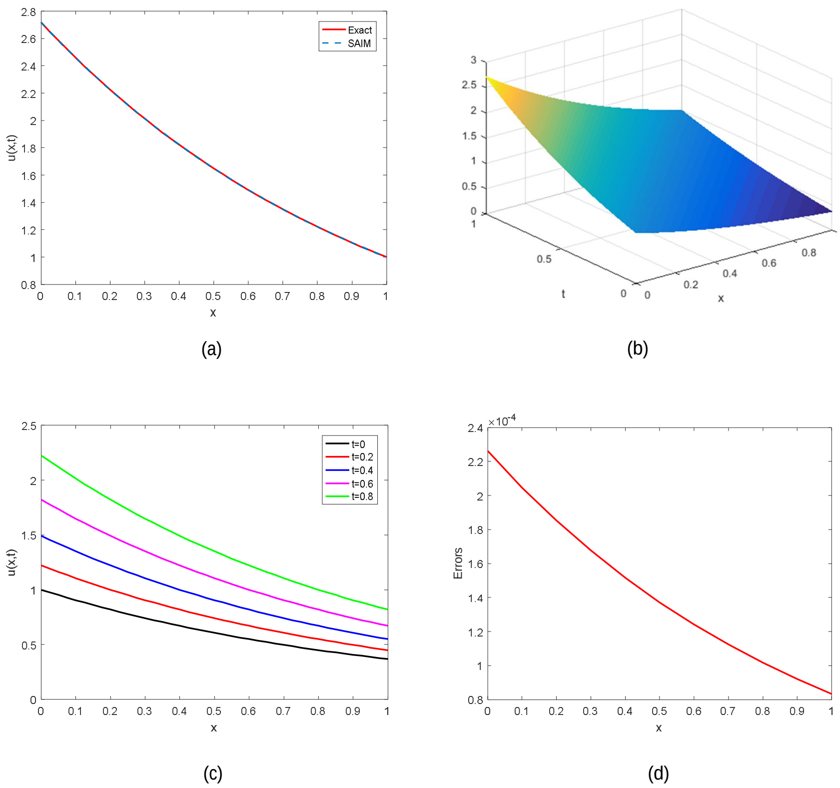

Fig. 1. (a) Comparison of exact and SAIM solutions for the GDE in Example 3.1 for $0 \le x, y \le 1$ and a fixed $t = 1$; (b) Space-time surface plot for $0 \le x, t \le 1$; (c) Comparison of SAIM solutions at different values of t; (d) Absolute errors between the exact and SAIM solutions

Example 3.2.
Consider the homogeneous GDE [3, 7]:

$$u_t + uu_x - u(1-u)\ln a = 0, \tag{15}$$

subject to the initial condition

$$u(x,0) = a^{-x}.$$

The exact solution of (15) is $u(x,t) = a^{t-x}$. Rewriting (15) as

$$Lu = -Nu,$$

where $Lu = u_t$, $Nu = uu_x - u(1-u)\ln a$, the general recursive relation is given by

$$L[u_{n+1}(x,t)] = -N[u_n(x,t)], \text{ with } u_{n+1}(x,0) = a^{-x}. \tag{16}$$

We use the iteration

$$u_{n+1}(x,t) = u_{n+1}(x,0) - \int_0^t \left[u_n u_{n_x} - u_n(1-u_n)\right] ds \tag{17}$$

to obtain the successive approximations

$$u_0(x,t) = a^{-x}$$
$$u_1(x,t) = a^{-x} - \int_0^t \left[u_0 u_{0_x} - u_0(1-u_0)\right] ds = a^{-x}(1 + t\ln a)$$
$$u_2(x,t) = a^{-x} - \int_0^t \left[u_1 u_{1_x} - u_1(1-u_1)\right] ds = a^{-x}\left(1 + t\ln a + \frac{(t\ln a)^2}{2}\right)$$
$$u_3(x,t) = a^{-x} + \int_0^t \left[u_2 u_{2_x} - u_2(1-u_2)\right] ds = a^{-x}\left(1 + t\ln a + \frac{(t\ln a)^2}{2} + \frac{(t\ln a)^3}{6}\right)$$

and so on, leading to the solution

$$u(x,t) = a^{-x}\left(1 + t\ln a + \frac{(t\ln a)^2}{2!} + \frac{(t\ln a)^3}{3!} \cdots + \right) = a^{-x}e^{t\ln a} = a^{t-x}$$

which is the exact solution also obtained using the FRDTM [3] and the NIPDTM [7]. Table 2 compares the approximate results for $t = 1$ up to the eighth term with the exact results. Fig. 2(a) shows the results for different values of $a = 10, 20, 30, 40$ for $0 \le x \le 1$ and $t = 1$ and Fig. 2(b) is a space-time surface plot of the SAIM solution at $a = 10$.

Table 2. Comparison of Exact and Approximate Solutions from SAIM for Example 3.2

x	t	$u(x,t)$	$u_{\text{SAIM}}(x,t)$	$e = \lvert u - u_{\text{SAIM}}\rvert$
0.0	1.00	10.000000000	9.973936032	0.026063968
0.1	1.00	7.943282347	7.922579002	0.020703345
0.2	1.00	6.309573445	6.293128193	0.016445252
0.3	1.00	5.011872336	4.998809408	0.013062928
0.4	1.00	3.981071706	3.970695453	0.010376253
0.5	1.00	3.162277660	3.154035510	0.008242150
0.6	1.00	2.511886432	2.505339459	0.006546973
0.7	1.00	1.995262315	1.990061870	0.005200445
0.8	1.00	1.584893192	1.580762332	0.004130860
0.9	1.00	1.258925412	1.255644153	0.003281259
1.0	1.00	1.000000000	0.997393603	0.002606397

Example 3.3.
Consider the nonhomogeneous GDE

$$u_t + uu_x - u(1-u) = -e^{t-x}, \tag{18}$$

subject to the initial condition

$$u(x,0) = 1 - e^{-x}$$

and having exact solution $u(x,t) = 1 - e^{t-x}$ [3, 17, 18].

In operator-theoretic form, Eq. (18) is expressed as

$$Lu + Nu = f(x,t),$$

where, $Lu = u_t$, $Nu = uu_x - u(1-u)$ and $f(x,t) = -e^{t-x}$. Since the primary problem $Lu_0 = 0$, with $u_0(x,0) = 1 - e^{-x}$, has a solution $u_0(x,t) = 1 - e^{-x}$, Eq. (18) can be solved using the general iterative scheme

$$u_{n+1}(x,t) = u_{n+1}(x,0) + \int_0^t \left[-u_n u_{n_x} + u_n(1-u_n) - e^{s-x}\right] ds.$$

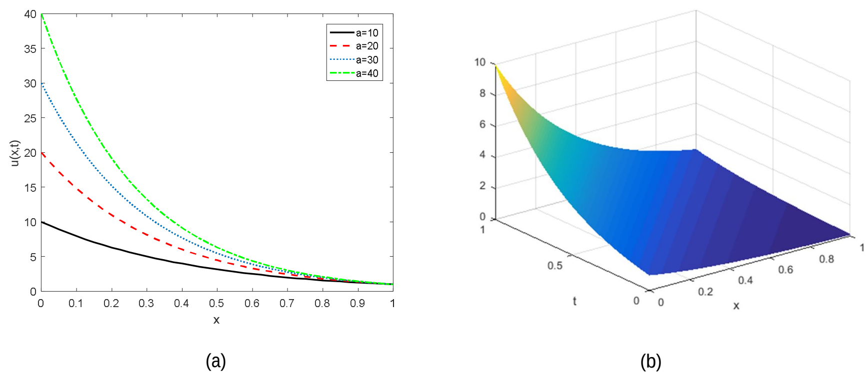

Fig. 2. (a) Comparison of exact and SAIM solutions for the GDE in Example 3.2 for $0 \le x \le 1$ and $t = 1$ at different values of a; (b) Space-time surface plot for $a = 10$ and $0 \le x, t \le 1$

Thus, the first four approximations are

$$u_0(x,t) = 1 - \mathrm{e}^{-x}$$

$$u_1(x,t) = 1 - \mathrm{e}^{-x} + \int_0^t \left[-u_0 u_{0_x} + u_0(1-u_0) - \mathrm{e}^{s-x}\right] ds = 1 - \mathrm{e}^{t-x}$$

$$u_2(x,t) = 1 - \mathrm{e}^{-x} + \int_0^t \left[-u_1 u_{1_x} + u_1(1-u_1) - \mathrm{e}^{s-x}\right] ds = 1 - \mathrm{e}^{t-x}$$

$$\vdots$$

$$u_{n+1}(x,t) = 1 - \mathrm{e}^{t-x},\ n \ge 1$$

Thus, the solution to (18) is

$$u(x,t) = 1 - \mathrm{e}^{t-x},$$

which is the exact solution for the given gas dynamics equation. This solution was also obtained using the FRDTM [3], NHPM [8, 9], q-HAM [17] and ADM [18]. Fig. 3 compares the results from the SAIM with the exact solution for $t = 1$ and $0 \le x \le 1$.

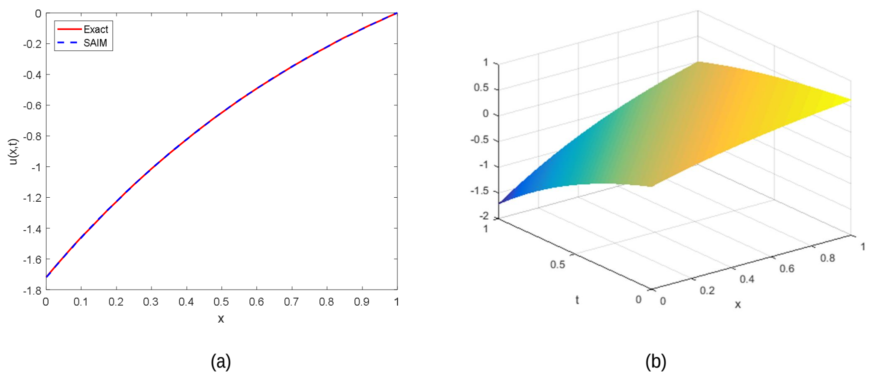

Fig. 3. (a) Comparison of exact and SAIM solutions for the GDE in Example 3.3 for $0 \le x \le 1$ and $t = 1$; (b) Space-time surface plot for $0 \le x, t \le 1$

4. Conclusion

This paper has used the semi analytic iterative method to obtain exact or closed-form solutions to the classical gas dynamics equation. The results show the ability of this method to produce exact to near-exact solutions to nonlinear GDEs and has confirmed the suitability of this method for solving these and other types of nonlinear PDEs.

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