Int. J. Adv. Appl. Math. and Mech. 10(4) (2023) 17 – 22 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com
IJAAMM
**International Journal of Advances in Applied Mathematics and Mechanics**

# Geometrical Motives Commutative Diagram to the derived category DQFT

**Research Article**

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**Received 11 March 2023; accepted (in revised version) 25 Mach 2023**

**Abstract:** The tensor structures of triangulated categories in derived categories of ètale sheaves with transfers are considered, taking the tensor product of categories $X \otimes Y = X \times Y$, in the category $\mathrm{Cor}_k$ (finite correspondences category) being this the product of the underlying schemes on $k$. Likewise, is constructed from a total tensor product on the category PSL($k$), the generalizations on derived categories using pre-sheaves and contravariant/covariant functors on additive categories to define the exactness of infinite sequences and resolution of spectral sequences of modules in triangulated derived categories of objects in $\Delta^n \times A^1$, for morphism of $A^1$ - homotopy. Then through a motives algebra which inherits the generalized tensor product is defined a triangulated category whose motivic cohomology is a hypercohomology from the category $Sm_k$,which has implications in the geometrical motives applied to a bundle of geometrical stacks to field theory. Then we can consider the motives in the hypercohomology to the category DQFT. The mean result will be the creation of theorem that incorporates a 2-simplicial decomposition of $\Delta^3 \times A^1$, in four triangular diagrams of derived categories from $Sm_k$, which come from a derived category of geometrical motives of DQFT.

**MSC:** 14A15 • 13D09 • 14F08 • 14F20 • 14F42

**Keywords:** ètale Cohomology • Derived QFT Category • Geometrical Motives • Motivic Cohomology • Tensor Structures • Total Tensor Product • Triangulated Derived Categories



## 1. Introduction

The principal goal is obtain commutative diagrams of derived categories of motives, considering that the problem on the determination of a diagram of schemes in quantum field theory no necessary is commutative, even the general problem results non-symmetric, and much of the evolved rings are not commutative, which carry us to work the tensor structure of derived categories from the category PSL($k$), to obtain the generalizations on derived categories using pre-sheaves and contravariant and covariant functors on additive categories to define the exactness of infinite sequences and resolution of spectral sequences, in a symmetrical context developed to obtain a motives cohomology. Likewise through a motives algebra which inherits the generalized tensor product is defined a triangulated category whose motivic cohomology [1-3] is a hypercohomology from the category $Sm_k$, which has implications in the geometrical motives applied to a bundle of geometrical stacks in field theory, in a way symmetric (for example from the geometrical Langlands ramification, after of work certain $\infty$ −algebras, the quantum field equations in this context has solutions in a dual space considering the category of vector spaces SpecSymm$T$ [4]), which can see it reflected in the algebraic context through commutatively between categories with these tensor structures [5]. Then this means that we can consider the motives in the hypercohomoloy to the category DQFT, where this hypercohomology

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is constructed from étale (or Lichtenbaum) motivic cohomology $H_L^{p,q}(X,\mathbb{Q})$, which is defined to be the étale hypercohomology of the complex $\mathbb{Q}(q)$). However considering of fact, that some context inside the quantum field theory requires a Zariski topology, finally our hypercohomology can be described to the category DQFT, through motives of a hypercohomology from the category $Sm_k$, defined as [2, 5]:

$$\mathrm{Hom}_{\mathrm{DM}^{eff}_{gm(k)}}(m(X),\mathbb{Q}(q)[p]) \cong H^{\bullet}_{Nis}(X,\mathbb{Q}(q)) = \mathbb{H}^{p,q}(X_{Nis},\mathbb{Q}(q)), \tag{1}$$

which comes from the hypercohomology

$$H_L^{p,q}(X,\mathbb{Q}) = \mathbb{H}^{p,q}(X,\mathbb{Q}), \tag{2}$$

The category PSL$(k)$, is Abelian [1] and therefore has enough injectives and projectives that can be used to create the conditions for the invariant presheaves of homotopy required to realization of the commutative diagrams in $A^1$ −homotopy of morphisms in the category $Sm_k$, of finite schemes $X$, and $Y$. Likewise from a simplicial complex and its correspondence with a corresponding diagram of $A^1$ −morphisms in a category, can be determined a general diagram that can be induced to the category DQFT, from a scheme of associated motives to a scheme $X$. A simplicial complex candidate can be $\Delta^3$, which has the correspondence (figure 1):

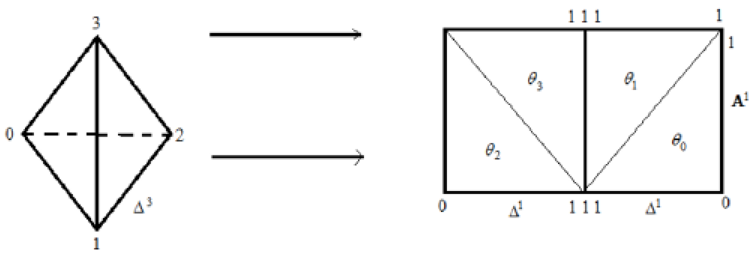


**Fig. 1.** 2-Simplicial decomposition of $\Delta^3 \times A^1$.

Likewise, we will be used this to obtain a general diagram that can be induced to the category DQFT, as mentioned before, from a scheme of associated motives to a scheme $X$, (which is the class $m(X)$ of $C_*\mathbb{Z}_{tr}(X)$, which is clearly modulus $A^1$ −homotopy in an approximate triangulated category $\mathrm{DM}^{eff,-}_{Nis}(k,R)$, [1] constructed from the derived category of PSL$(k)$.
As was shown in different previous works [5, 6], the geometrical motives required in our research are a result of embedding the derived category $\mathrm{DM}^{-}_{gm}(k,R)$, (geometrical motives category) in the category $\mathrm{DM}^{eff,-}_{\acute{e}t}(k,\mathbb{Z}/m)$, considering the category of smooth schemes on the field $k$.

## 2. Derived triangulated categories with structure by pre-sheaves $\otimes^L$, and $\otimes^{tr}_{L,\acute{e}t}$

The tensor product of the derived category of bounded above complexes of étale sheaves of $R$ −modules $\otimes^{tr}_{L,\acute{e}t}$, preserves quasi-isomorphisms [7, 8]. Also the category of bounded above complexes of étale sheaves of $R$ −modules with transfers is a tensor triangulated category [9, 10].

In particular, and by a motives algebra in the derived category of étale sheaves of $\mathbb{Z}/m$ −module with transfers, the operation

$$m \to m(1) = m \otimes^{tr}_{L,\acute{e}t} \mathbb{Z}/M(1), \tag{3}$$

is inversible. Then $\forall\ E, F$, are bounded above complexes of locally constant étale sheaves of $R$ −module $E \otimes^{tr}_{L,\acute{e}t} F$, is quasi-isomorphic to $E \otimes^{\mathbb{L}}_{R} F$, which is their total tensor product of complexes of étale sheaves of $R$ −modules. Indeed, we consider the morphism $f: E \to E'$, of bounded above complexes of presheaves of $R$ −modules with transfers. Then in particular for étale sheave we have $E_{\acute{e}t} \to E^{'}_{\acute{e}t}$, then we have

$$E \otimes^{tr}_{L,\acute{e}t} F \to E^{'} \otimes^{tr}_{L,\acute{e}t} F,$$

[1] *This category has the total tensor product inherited from the total tensor product of PSL$(k)$.*

is quasi-isomorphism of $F$. Now if $F$, is a locally complete étale sheaf of $R$ −modules then $E^{'} \otimes^{tr}_{L,ét} F, \rightarrow E \otimes^{tr}_{L,ét} F$, is a quasi-isomorphism for every étale sheaf with transfers $E$. But $\otimes \cong \otimes^{\mathbb{L}}$, in $\wp$, and using the a natural mapping of presheaves given by $\lambda : h_X \otimes^{tr}_{ét} h_Y \rightarrow h_{X \otimes^{tr}_{ét} Y}$, where every $h_{X_i} = R(X_i)$, having the right exactness of $\otimes_{R,}$, and $\otimes^{tr}_{ét}$, and being $E, F$, are bounded above complexes of locally constant étale sheaves of $R$ −module then $E \times^{tr}_{L,ét} F \rightarrow E \otimes^{\mathbb{L}}_{R} F$, is a quasi-isomorphism.
Similarly as with the étale sheaves, a presheaf with functors $F$, is a Nisnevich sheaf with transfers if its underlying presheaf is a Nisnevich sheaf on $Sm/k$. Clearly every étale sheaf with transfers is a Nisnevich sheaf with transfers. In motives with $\mathbb{Q}$ −coefficients with transfers we have the result:

Lemma 2. 1. Let $F$, be a Zariski sheaf of $\mathbb{Q}$ −modules with transfers. Then $F$, is also an étale sheaf with transfers.
*Proof.*[3].■

Then is deduced from theorem that characterizes the Nisnevich sheaves [2, 3 ,9] whose category $Sh_{Nis}(Cor_k)$, and the before lemma 3. 1, the following corollary.

Corollary 2. 1. If $F$, is a presheaf of $\mathbb{Q}$ −modules with transfers then $F_{Nis} = F_{ét}$.

For other side, the construction of a derived category as such $\mathrm{DM}^{eff.-}_{Nis}(k,R)$, is parallel to the construction of $\mathrm{DM}^{eff.-}_{ét}(k,R)$. If $k$ admits regularizations of singularities then $\mathrm{DM}^{eff.-}_{ét}(k,R)$, allows us to extend motivic cohomology to all schemes of finite type as a cdh, hypercohomology group.

If $\mathbb{Q} \subseteq R$, we will show that $\mathrm{DM}^{eff.-}_{Nis}(k,R)$, and $\mathrm{DM}^{eff.-}_{ét}(k,R)$, are equivalent. Likewise, $\mathrm{D}^- = D^-(Sh_{ét}(Cor_k,R))$, is a derived category which is a tensor triangulated category. The same is applicable in the Nisnevich topology for derived category $\mathrm{D}^-(Sh_{Nis}(Cor_k,R))$.

Likewise, $\forall\ C, D \in \wp$, and therefore in $\mathrm{Ch}^- R(\mathscr{A})$, we have:

$$C \otimes^{tr}_{L,Nis} D \cong (C \otimes^{tr}_{L} D)_{Nis}, \tag{4}$$

In particular the derived category $\mathrm{D}^-$, of bounded above complexes of Nisnevich sheaves with transfers is a tensor triangulated category under $\otimes^{tr}_{L,Nis}$. Then by the proposition that says that $h_X = R_{tr}(X)$, [3] is projective if

$$R_{tr}(X) \otimes^{tr} R_{tr}(Y) = R_{tr}(X \times Y), \tag{5}$$

Then we have in the motives context

$$m(X) \otimes^{tr}_{L,Nis} m(Y) = m(X \times Y), \tag{6}$$

Likewise, we can to define the category $\mathrm{DM}^{eff}_{gm}(k,R)$, to be the thick subcategory of $\mathrm{DM}^{eff.-}_{Nis}(k,R)$, generated by the motives $m(X)$, where $X$, is smooth over $k$. Objects in $\mathrm{DM}^{eff}_{gm}(k,R)$, are the effective geometric motives, which will be the objects that we require in our motivic cohomology, that we obtain for resolution of the decomposing of $X \times A^1$ in $A^1$ −homotopy of morphisms in the category $Sm_k$.

## 3. Fundamental Backgrounds

Under the consideration realized in the book chapter [11] and the motivic cohomology treatment given in [2, 3, 9, and 13] as the embedding theorem in $\mathrm{DM}^{eff}_{ét}(k)$, we can consider the following triangulated diagram:

$$\begin{array}{ll} Sm_k \rightarrow \mathrm{DM}^{eff}_{ét}(k) & \\ m \searrow \quad\ \ \downarrow Id, & \\ \quad \mathrm{DM}^{eff}_{ét}(k) & \end{array} \tag{7}$$

which has implications in the geometrical motives applied to a bundle of geometrical stacks in mathematical physics, as has been studied and showed in [8, 11, 12].

We consider the derived category to quantum field theory DQFT, as the characterized by the motives in a hypercohomology from the category $Sm_k$, [5, 6].

Theorem 3. 1 (F. Bulnes). Suppose that $\mathbb{M}$, is a complex Riemannian manifold with singularities. Let $X$, and $Y$, be smooth projective varieties in $\mathbb{M}$ [2]. We know that solutions of the field equations $d$da = 0, [4,-6] are given in a category Spec($Sm_k$), (see [4]). Solution context of the quantum field equations for $d$da = 0, are defined in hypercohomology on $\mathbb{Q}$ −coefficients fr m the category $Sm_k$, defined on a numerical field $k$, considering the derived tensor product $\otimes_{\acute{e}t}^{tr}$, of presheaves. Then the following tensor triangulated diagram is true and commutative:

$$\begin{array}{c} DQFT \\ i \swarrow \qquad \searrow F, \\ \mathrm{MD}_{gm}(\mathbb{Q}) \to MD(\mathscr{D}_y) \end{array} \tag{8}$$

*Proof.*[8].■

The category $\mathrm{DM}_{gm}^{eff}(k,R)$, has a tensor triangulated structure and the tensor product of its motives is $m(X)\otimes m(Y) = m(X\times Y)$. Remember that the triangulated category of geometrical motives $\mathrm{DM}_{gm}(k,R)$, is defined formally inverting the functor of the Tate objects, which are objects of a motivic category called Tannakian category [6].

We enunciate the following result important in the technical detail of the topologies required to DQFT.

Theorem 5. 2. If $\mathbb{Q}\subseteq R$, then

$$\omega : \mathrm{DM}_{Nis}^{eff.-}(k,R) \to \mathrm{DM}_{\acute{e}t}^{eff.-}(k,R), \tag{9}$$

is an equivalence of tensor triangulated categories.

*Proof.*[6].■

We want to apply the considerations of before sections to give a tensor triangulated category to a quantum version of motivic cohomology on étale Sheaves, from $\Delta^3$ −simplicial that shows the $A^1$ −homotopy in an approximate triangulated category $\mathrm{DM}_{Nis}^{eff.-}(k,R)$, which for every Nisnevich sheaf with transfers that is an étale sheaf with transfers, is a category $\mathrm{DM}_{\acute{e}t}^{eff.-}(k,R)$. The Nisnevich detail in the derived category is due to the importance in motivic homotopy theory of that the objects of interests are "spaces", which are simplicial sheaves of sets on the big Nisnevich site that is the category $Sm/k$.

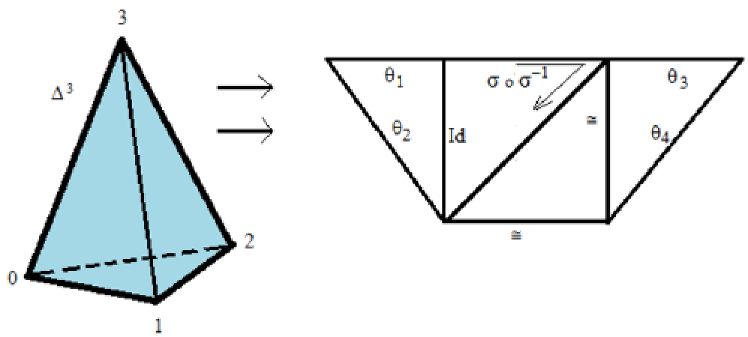


**Fig. 2.** 2-Simplicial decomposition of $\Delta^3\times A^1$, for DQFT.

In reality we consider two topologies for aspects of localization and covering.

We have the following commutative diagram in the geometrical motives context that are useful to link the derived category DQFT.

[2] *Singular projective varieties useful in quantization process of the complex Riemannian manifold. The quantization condition compact quantizable Käehler manifolds can be embedded into projective space.*

## 4. Result

Lemma 4. 1. The following diagram is commutative

$$Sm_k \xrightarrow{i'} \mathrm{DM}^{eff}_{gm}(k) \xrightarrow{\sigma} \mathrm{DM}_{gm}(k) \xleftarrow{i} DQFT \tag{10}$$

$$m \searrow \quad \updownarrow Id \quad \sigma \swarrow \quad \updownarrow\cong \quad \swarrow F,$$

$$\mathrm{DM}^{eff}_{gm}(k) \xrightarrow{\cong} \mathrm{DM}(\mathscr{D}_Y)$$

We consider the following aspects before of the demonstration.
We say that a diagram in $Cor_k$, is homotopy commutative if every pair of composites $f, g : X \to Y$, with the same source and target are $A^1$ −homotopic. Any homotopy invariant presheaf with transfers identifies $A^1$ −homotopic maps, and converts a homotopy commutative diagram into a commutative diagram.

*Proof.* Then we apply the Theorem 21. 6, and the lemma 21. 7 pages 177, 178 [3]. The strong fact is consider that all category in the diagram (10) which is $Cor_k$. is homotopy commutative, being every pair $(i, f)$, with source and target $A^1$ −homotopic mappings. Likewise, $i' = i$, from $m : Sm_k \to \mathrm{DM}^{eff}_{-}(k)$ and the embedding $\sigma$, is a specific mapping which is fitting into a homotopy commutative diagram in $Cor_k$. Further the composition DQFT $\xrightarrow{i} \mathrm{DM}_{gm}(k) \xrightarrow{\sigma} \mathrm{DM}^{eff}_{gm}(k)$, is zero (see lemma 21.9 [3, 13]).
The details of this demonstration can be obtained considering the proposition 11. 15, applied to the exterior triangles of diagram (10). Also results very helpful the fact of the singular homology [14] to start $Cor_k/A^1$ −homotopy. ■

## 5. Applications

This help us to have a quantum field theory of simplicial geometry and construct a model crystallographic Universe on the simplicial frameworks and establish morphism of homotopy commutative relations which can induce to a hypercohomology to the solution of some field equations and gravitational aspects, at least in mocroscopic level. For example, some of the field theories as the Schwinger-Dyson equation in three-dimensional simplicial quantum gravity, novedous triangle relations and absense of Tachyons in Liouville string field theory [15], where could be contained in a derived category of form $\mathrm{DM}(\mathscr{D}_Y)$, or the diagrams of the Polyakov string theory [16] can be used the simplicial geometry and its decomposition in trangulated diagrams of schemes belonging to the category $Sm_k$, and morphisms between schemes of the category $Cor_k$, all with the total tensor product on the category $\mathrm{PSL}(k)$, to obtain the generalizations on derived categories using pre-seaves and contravariant and covariant functors on additive categories to define the exactness of infinite sequences and resolution of spectral sequences. From a point of tensor structure, the advantages from the studies of tensor triangulated category to a quantum version considering a motivic cohomology on étale sheaves is the factorization algebras in quantum field theory, where is necessary consider the combined observation measures from many components with an commutative property for their diagrams between their derived categories. Also the development of the called homotopy quantum field theory, takes elements of our morphisms in homotopy and the characterization of a total tensor product between multiplicity modules in interacting process of many particles or fields.

**Fig. 3.** For space-time topological objects in physics and biology, Michel Planat, Marcelo M. Amaral, David Chester, Klee Irwin propose a type of algebraic processing based on schemes in which the discrimination of singularities within objects is based on the space-time-spin group $SL(2,\mathbb{C})$ [17, 18]. Such topological objects possess an homotopy structure encoded in their fundamental group and the related $SL(2,\mathbb{C})$, multivariate polynomial character variety contains a plethora of singularities somehow analogous to the frequency spectrum in time structures[17].

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