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Sums and Generating Functions of Generalized Fibonacci Polynomials via Matrix Methods

Research Article

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Abstract: In this paper, we study on the generalized Fibonacci polynomials and we deal with two special cases namely, $(r,s)-$Fibonacci and $(r,s)-$Fibonacci-Lucas polynomials. We present sum formulas, generating functions, Simson's formulas for these polynomial sequences via matrix methods. Moreover, we evaluate the infinite sums of special cases of $(r,s)-$Fibonacci and $(r,s)-$Fibonacci-Lucas polynomials.

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1. Introduction and Preliminaries: Generalized Fibonacci Polynomials

The generalized Fibonacci polynomials (or Horadam polynomials or x-Horadam numbers or generalized $(r(x),s(x))$-polynomials or $(r(x),s(x))$ Horadam polynomials or 2-step Fibonacci polynomials)

$$\{W_n(W_0,W_1;r(x),s(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n(x)=r(x)W_{n-1}(x)+s(x)W_{n-2}(x),\qquad W_0(x)=a(x),W_1(x)=b(x),\quad n\geq 2 \tag{1}$$

where $W_0(x),W_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x)\neq 0$, $s(x)\neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x)=-\frac{r(x)}{s(x)}W_{-(n-1)}(x)+\frac{1}{s(x)}W_{-(n-2)}(x)$$

for $n=1,2,3,...$ when $s(x)\neq 0$. Therefore, recurrence (1) holds for all integers n. Note that $W_{-n}(x)$ need not to be a polynomial in the ordinary sense. For more details on generalized Fibonacci (Horadam) polynomials, see [10]. For some references on special cases of Horadam polynomials see [3–5, 9, 16, 17] for papers and [1, 2, 6–8, 11, 15] for books.

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Binet's formula of generalized Fibonacci (Horadam) polynomials can be calculated using its characteristic equation (the quadratic equation, polynomial) which is given as

$$y^2 - r(x)y - s(x) = 0. \tag{2}$$

The roots of characteristic equation are

$$\alpha(x) := \alpha = \frac{r(x) + \sqrt{r^2(x) + 4s(x)}}{2}, \quad \beta(x) := \beta = \frac{r(x) - \sqrt{r^2(x) + 4s(x)}}{2}, \tag{3}$$

Now, we define two special cases of the polynomials $W_n(x)$. $(r(x), s(x))$-Fibonacci polynomials $\{G_n(0,1; r(x), s(x))\}_{n\geq 0}$ (or shortly $G_n(x)$) and $(r(x), s(x))$-Lucas polynomials $\{H_n(2, r(x); r(x), s(x))\}_{n\geq 0}$ (or shortly $H_n(x)$) are defined, respectively, by the second-order recurrence relations

$$G_{n+2}(x) = r(x)G_{n+1} + s(x)G_n(x), \quad G_0(x) = 0, G_1(x) = 1, \tag{4}$$
$$H_{n+2}(x) = r(x)H_{n+1} + s(x)H_n(x), \quad H_0(x) = 2, H_1(x) = r(x). \tag{5}$$

The (sequences of polinomials) $\{G_n(x)\}_{n\geq 0}$ and $\{H_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n}(x) &= -\frac{r(x)}{s(x)}G_{-(n-1)}(x) + \frac{1}{s(x)}G_{-(n-2)}(x), \\ H_{-n}(x) &= -\frac{r(x)}{s(x)}H_{-(n-1)}(x) + \frac{1}{s(x)}H_{-(n-2)}(x), \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (4) and (5) hold for all integers n.

NOTE: For the sake of simplicity throughout the rest of the paper, we use

$$W_n, r, s, W_0, W_1, \alpha, \beta, G_n, H_n, G_0, G_1, H_0, H_1$$

instead of

$$W_n(x), r(x), s(x), W_0(x), W_1(x), \alpha(x), \beta(x), G_n(x), H_n(x), G_0(x), G_1(x), H_0(x), H_1(x),$$

respectively, unless otherwise stated. . For example, we write

$$W_n = rW_{n-1} + sW_{n-2}, \quad W_0 = a, W_1 = b, \quad n \geq 2$$

for the equation (1).

Using the roots α, β and recurrence relation (1), Binet's formula can be given as follows:

Theorem 1.1 ([10], Theorem 2).
The general term of the generalized Fibonacci (Horadam) polynomials W_n can be presented by the following Binet's formula:

$$\begin{aligned} W_n &= \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n, & \text{if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1}, & \text{if } \alpha = \beta \text{ (Single Root Case)} \end{cases} \\ &= \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n, & \text{if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \frac{r}{2}(n-1)W_0)\left(\frac{r}{2}\right)^{n-1}, & \text{if } \alpha = \beta \text{ (Single Root Case)} \end{cases}. \end{aligned} \tag{6}$$

2. Simson's Formulas of Fibonacci Polynomials

The following theorem gives Simson's formula of the generalized Fibonacci polynomials $\{W_n\}$.

Theorem 2.1 (Simson Formula of Generalized Fibonacci (Horadam) Polynomials [10], Theorem 5).
For all integers n, we have

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = (-1)^n s^n \begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix}. \tag{7}$$

Note that (7) can be written as

$$\begin{vmatrix} W_{n+m+1} & W_{n+m} \\ W_{n+m} & W_{n+m-1} \end{vmatrix} = (-1)^{n+m} s^{n+m} \begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix}$$

for all integers n, m.

Note also that

$$\begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix} = \begin{vmatrix} W_1 & W_0 \\ W_0 & \frac{1}{s}(W_1 - rW_0) \end{vmatrix}$$

since

$$W_{-1} = \frac{1}{s}(W_1 - rW_0).$$

We define

$$\Lambda_W(n) = W_{n+1}^2 - sW_n^2 - rW_nW_{n+1}.$$

Then

$$\Lambda_W(0) = W_1^2 - sW_0^2 - rW_0W_1. \tag{8}$$

Simson's formulas of W_n, G_n H_n can be given in the following forms.

Lemma 2.1.
For all integers n, we have

(a) $\Lambda_W(n) = (-1)^n s^n \Lambda_W(0)$, *i.e.,*

$$(W_{n+1}^2 - sW_n^2 - rW_nW_{n+1}) = (-1)^n s^n (W_1^2 - sW_0^2 - rW_0W_1). \tag{9}$$

(b) $\Lambda_G(n) = (-1)^n s^n \Lambda_G(0) = (-1)^n s^n$, *i.e.,*

$$(G_{n+1}^2 - sG_n^2 - rG_nG_{n+1}) = (-1)^n s^n. \tag{10}$$

(c) $\Lambda_H(n) = (-1)^n s^n \Lambda_H(0) = (r^2+4s)(-1)^{n+1} s^n$, *i.e.,*

$$(H_{n+1}^2 - sH_n^2 - rH_nH_{n+1}) = (r^2+4s)(-1)^{n+1} s^n. \tag{11}$$

Proof.

(a) Note that (7) can be written in the following form:

$$\begin{aligned} \begin{vmatrix} W_{n+1} & W_n \\ W_n & \frac{1}{s}(W_{n+1} - rW_n) \end{vmatrix} &= (-1)^n s^n \begin{vmatrix} W_1 & W_0 \\ W_0 & \frac{1}{s}(W_1 - rW_0) \end{vmatrix} \\ &= (-1)^n s^n \times \frac{1}{s}(W_1^2 - sW_0^2 - rW_0W_1) \\ &= (-1)^n s^{n-1}(W_1^2 - sW_0^2 - rW_0W_1) \\ &= (-1)^n s^{n-1} \Lambda_W(0) \end{aligned}$$

since

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2}, \\ W_{n+1} &= rW_n + sW_{n-1} \Rightarrow W_{n-1} = \frac{1}{s}(W_{n+1} - rW_n), \\ W_{-1} &= \frac{1}{s}(W_1 - rW_0). \end{aligned}$$

Note also that

$$\begin{aligned} \begin{vmatrix} W_{n+1} & W_n \\ W_n & \frac{1}{s}(W_{n+1} - rW_n) \end{vmatrix} &= \frac{1}{s}(W_{n+1}^2 - sW_n^2 - rW_nW_{n+1}) \\ &\Rightarrow \\ \frac{1}{s}(W_{n+1}^2 - sW_n^2 - rW_nW_{n+1}) &= (-1)^n s^{n-1} \Lambda_W(0) \\ &\Rightarrow \\ (W_{n+1}^2 - sW_n^2 - rW_nW_{n+1}) &= (-1)^n s^n (W_1^2 - sW_0^2 - rW_0W_1). \end{aligned}$$

So we get the identity (9).

(b) Since

$$\Lambda_G(0) = G_1^2 - sG_0^2 - rG_0G_1 = 1,$$

we get the identity (10).

(c) Since

$$\Lambda_H(0) = H_1^2 - sH_0^2 - rH_0H_1 = -(r^2+4s),$$

we get the identity (11). □

We will use the identities (9), (10) and (11) in the following sections.

3. Generalized Fibonacci Polynomials by Matrix Methods

In this section, we present matrix representations of the sequences W_n, G_n and H_n. We also introduce Simson matrix and investigate its properties.

3.1. Matrix Representations of the Sequences W_n, G_n and H_n

In the next theorem, we find the Binet's formulas of W_n, G_n and H_n by matrix method with other results.

Theorem 3.1.
For all integers m, n, we have

(a)

(i) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}.$$

(ii) *if $r^2+4s \neq 0$, i.e., $\alpha \neq \beta$ then*

$$\begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix}\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} H_{n+1} & sH_n \\ H_n & sH_{n-1} \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix}^{-1}\begin{pmatrix} H_{n+1} & sH_n \\ H_n & sH_{n-1} \end{pmatrix},$$

that is,

$$\begin{aligned} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n &= \frac{1}{r^2+4s}\begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix}\begin{pmatrix} H_{n+1} & sH_n \\ H_n & sH_{n-1} \end{pmatrix} \\ &= \frac{1}{r^2+4s}\begin{pmatrix} rH_{n+1}+2sH_n & rsH_n+2s^2H_{n-1} \\ 2H_{n+1}-rH_n & 2sH_n-rsH_{n-1} \end{pmatrix}, \end{aligned}$$

and if $r^2+4s=0$, i.e., $\alpha=\beta$ then

$$\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \frac{1}{2\alpha}(n+1)H_{n+1} & -\frac{1}{2}\alpha nH_n \\ \frac{1}{2}\frac{n}{\alpha}H_n & -\frac{1}{2}\alpha(n-1)H_{n-1} \end{pmatrix}.$$

(iii)

$$\begin{aligned} &(-W_1^2+sW_0^2+rW_1W_0)\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \\ &= \begin{pmatrix} (-W_1+rW_0)W_{n+1}+sW_0W_n & s(W_0W_{n+1}-W_1W_n) \\ W_0W_{n+1}-W_1W_n & -W_1W_{n+1}+(rW_1+sW_0)W_n \end{pmatrix} \end{aligned}$$

i.e.,

$$\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \frac{\begin{pmatrix} (W_1 - rW_0)W_{n+1} - sW_0W_n & -s(W_0W_{n+1} - W_1W_n) \\ -W_0W_{n+1} + W_1W_n & W_1W_{n+1} - (rW_1 + sW_0)W_n \end{pmatrix}}{W_1^2 - sW_0^2 - rW_1W_0}$$

$$= \begin{pmatrix} -\dfrac{W_{n+1}(W_1 - rW_0) - sW_0W_n}{sW_0^2 + rW_0W_1 - W_1^2} & s\dfrac{W_0W_{n+1} - W_1W_n}{sW_0^2 + rW_0W_1 - W_1^2} \\ \dfrac{W_0W_{n+1} - W_1W_n}{sW_0^2 + rW_0W_1 - W_1^2} & -\dfrac{W_1W_{n+1} - W_n(rW_1 + sW_0)}{sW_0^2 + rW_0W_1 - W_1^2} \end{pmatrix}$$

where

$$-W_1^2 + sW_0^2 + rW_0W_1 = \begin{vmatrix} W_0 & W_1 \\ W_1 & rW_1 + sW_0 \end{vmatrix} = \begin{vmatrix} W_0 & W_1 \\ W_1 & W_2 \end{vmatrix}.$$

(b)

$$A^n = G_nA + sG_{n-1}I,$$

i.e.,

$$A^n = G_nA + (G_{n+1} - rG_n)I,$$

that is

$$A^n = G_{n+1}I + G_n(A - rI),$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof.

(a) Proof are given in [10].

(b) Proof can be given by mathematical induction. But, we present a direct proof as follows. By using (a) (i), we get

$$\begin{aligned} G_nA + sG_{n-1}I &= G_n\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} + sG_{n-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} rG_n + sG_{n-1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix} \\ &= A^n. \end{aligned}$$

Note that since $sG_{n-1} = (G_{n+1} - rG_n)$, we obtain

$$A^n = G_nA + (G_{n+1} - rG_n)I.$$

Then, after rearranging we get

$$A^n = G_{n+1}I + G_n(A - rI). \quad \square$$

We need the following theorem.

Theorem 3.2 ((Honsberger's Identity)[10]).
For all integers m, n we have

$$W_{n+m} = W_nG_{m+1} + sW_{n-1}G_m. \tag{12}$$

Now, we present some consequences of the last theorem.

Corollary 3.1.
For all integers m, n, j, we have the following properties:

$$W_{mn+j} = W_{j+1}G_{mn} + sW_jG_{mn-1}, \quad (13)$$
$$G_{mn+j} = G_{j+1}G_{mn} + sG_jG_{mn-1}, \quad (14)$$
$$H_{mn+j} = H_{j+1}G_{mn} + sH_jG_{mn-1}, \quad (15)$$

i.e.,

$$W_{mn+j} = W_jG_{mn+1} + (W_{j+1} - rW_j)G_{mn}, \quad (16)$$
$$G_{mn+j} = G_jG_{mn+1} + (G_{j+1} - rG_j)G_{mn}, \quad (17)$$
$$H_{mn+j} = H_jG_{mn+1} + (H_{j+1} - rH_j)G_{mn}. \quad (18)$$

Proof. If we make the following changes

$$n \Leftrightarrow a$$
$$m \Leftrightarrow b$$

in (12), i.e.,

$$W_{n+m} = W_nG_{m+1} + sW_{n-1}G_m,$$

we get

$$W_{a+b} = W_aG_{b+1} + sW_{a-1}G_b \quad (19)$$

Now, if we make the following changes

$$a \Leftrightarrow j+1$$
$$b \Leftrightarrow mn-1$$

in (19), we obtain

$$W_{mn+j} = W_{j+1}G_{mn} + sW_jG_{mn-1}. \quad (20)$$

It follows that

$$W_{mn+j} = W_jG_{mn+1} + (W_{j+1} - rW_j)G_{mn} \quad (21)$$

since

$$G_{mn+1} = rG_{mn} + sG_{mn-1}$$

i.e.,

$$sG_{mn-1} = G_{mn+1} - rG_{mn}.$$

To complete the proof, set $W_n = G_n$ and $W_n = H_n$ in identities (20) and (21), respectively. □

We also need the following Remark in the sequel.

Remark 3.1.
(13) can be written it the following forms:

$$W_{mn+j+1} = W_{j+1}G_{mn+1} + sW_jG_{mn}, \quad (22)$$
$$W_{mn+j} = W_jG_{mn+1} + sW_{j-1}G_{mn}, \quad (23)$$
$$W_{mn+j-1} = W_jG_{mn} + sW_{j-1}G_{mn-1}, \quad (24)$$

For (22), first replace j with $j+1$ and then use the identities (by definition)

$$sG_{mn-1} = G_{mn+1} - rG_{mn},$$
$$W_{j+2} = rW_{j+1} + sW_j,$$

in (13).

For (23), replace

$$sG_{mn-1} = G_{mn+1} - rG_{mn},$$
$$W_{j+1} = rW_j + sW_{j-1},$$

in (13).

For (24), just replace j with $j-1$ in (13).

3.2. Simson Matrix and its Properties

For $n \in \mathbb{Z}$, we define

$$f_W(n) = \begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence W_n. Similarly, as special cases of W_n, Simson matrices of the sequence G_n and H_n are

$$f_G(n) = \begin{pmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{pmatrix},$$

and

$$f_H(n) = \begin{pmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{pmatrix},$$

respectively.

Note that $f_W(n), f_G(n)$ and $f_H(n)$ can be written, respectively, as

$$\begin{aligned} f_W(n) &= \begin{pmatrix} W_{n+1} & W_n \\ W_n & \frac{1}{s}(W_{n+1} - rW_n) \end{pmatrix}, \\ f_G(n) &= \begin{pmatrix} G_{n+1} & G_n \\ G_n & \frac{1}{s}(G_{n+1} - rG_n) \end{pmatrix}, \\ f_H(n) &= \begin{pmatrix} H_{n+1} & H_n \\ H_n & \frac{1}{s}(H_{n+1} - rH_n) \end{pmatrix}, \end{aligned}$$

since

$$\begin{aligned} W_{n-1} &= \frac{1}{s}(W_{n+1} - rW_n), \\ G_{n-1} &= \frac{1}{s}(G_{n+1} - rG_n), \\ H_{n-1} &= \frac{1}{s}(H_{n+1} - rH_n). \end{aligned}$$

Lemma 3.1.
For all integers n, m and j, the followings hold.

(a) $f_W(n) = rf_W(n-1) + sf_W(n-2)$, *i.e.,*

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = r\begin{pmatrix} W_n & W_{n-1} \\ W_{n-1} & W_{n-2} \end{pmatrix} + s\begin{pmatrix} W_{n-1} & W_{n-2} \\ W_{n-2} & W_{n-3} \end{pmatrix}.$$

(b) $f_W(n) = Af_W(n-1)$ *and* $f_W(n) = A^n f_W(0)$, *i.e.,*

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}\begin{pmatrix} W_n & W_{n-1} \\ W_{n-1} & W_{n-2} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{pmatrix}.$$

(c) $f_W(n+m) = A^n f_W(m)$ *and* $f_W(n+m) = A^m f_W(n)$ *i.e.,*

$$\begin{pmatrix} W_{n+m+1} & W_{n+m} \\ W_{n+m} & W_{n+m-1} \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_{m+1} & W_m \\ W_m & W_{m-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+m+1} & W_{n+m} \\ W_{n+m} & W_{n+m-1} \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix},$$

and $f_W(n) = A^m f_W(n-m)$, *i.e.,*

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n-m+1} & W_{n-m} \\ W_{n-m} & W_{n-m-1} \end{pmatrix}.$$

(d)

$$f_W(mn+j) = A^{mn} f_W(j)$$

and

$$f_W(mn+j) = (G_n A + sG_{n-1} I)^m f_W(j),$$

i.e.,

$$f_W(mn+j) = (G_n A + (G_{n+1} - rG_n) I)^m f_W(j).$$

(e)

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix} \begin{pmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+m+1} & W_{n+m} \\ W_{n+m} & W_{n+m-1} \end{pmatrix} = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix} \begin{pmatrix} W_{m+1} & W_m \\ W_m & W_{m-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \begin{pmatrix} G_{m+1} & sG_m \\ G_m & sG_{m-1} \end{pmatrix} \begin{pmatrix} W_{n-m+1} & W_{n-m} \\ W_{n-m} & W_{n-m-1} \end{pmatrix}.$$

(f)

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \frac{\begin{pmatrix} (W_1 - rW_0)W_{n+1} - sW_0 W_n & -s(W_0 W_{n+1} - W_1 W_n) \\ -W_0 W_{n+1} + W_1 W_n & W_1 W_{n+1} - (rW_1 + sW_0)W_n \end{pmatrix}}{W_1^2 - sW_0^2 - rW_1 W_0} \begin{pmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+m+1} & W_{n+m} \\ W_{n+m} & W_{n+m-1} \end{pmatrix} = \frac{\begin{pmatrix} (W_1 - rW_0)W_{n+1} - sW_0 W_n & -s(W_0 W_{n+1} - W_1 W_n) \\ -W_0 W_{n+1} + W_1 W_n & W_1 W_{n+1} - (rW_1 + sW_0)W_n \end{pmatrix}}{W_1^2 - sW_0^2 - rW_1 W_0} \begin{pmatrix} W_{m+1} & W_m \\ W_m & W_{m-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \frac{\begin{pmatrix} (W_1 - rW_0)W_{m+1} - sW_0 W_m & -s(W_0 W_{m+1} - W_1 W_m) \\ -W_0 W_{m+1} + W_1 W_m & W_1 W_{m+1} - (rW_1 + sW_0)W_m \end{pmatrix}}{W_1^2 - sW_0^2 - rW_1 W_0} \begin{pmatrix} W_{n-m+1} & W_{n-m} \\ W_{n-m} & W_{n-m-1} \end{pmatrix}.$$

(g) *if* $r^2 + 4s \neq 0$, *i.e.,* $\alpha \neq \beta$ *then*

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \frac{1}{r^2 + 4s} \begin{pmatrix} rH_{n+1} + 2sH_n & rsH_n + 2s^2 H_{n-1} \\ 2H_{n+1} - rH_n & 2sH_n - rsH_{n-1} \end{pmatrix} \begin{pmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+m+1} & W_{n+m} \\ W_{n+m} & W_{n+m-1} \end{pmatrix} = \frac{1}{r^2 + 4s} \begin{pmatrix} rH_{n+1} + 2sH_n & rsH_n + 2s^2 H_{n-1} \\ 2H_{n+1} - rH_n & 2sH_n - rsH_{n-1} \end{pmatrix} \begin{pmatrix} W_{m+1} & W_m \\ W_m & W_{m-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \frac{1}{r^2 + 4s} \begin{pmatrix} rH_{m+1} + 2sH_m & rsH_m + 2s^2 H_{m-1} \\ 2H_{m+1} - rH_m & 2sH_m - rsH_{m-1} \end{pmatrix} \begin{pmatrix} W_{n-m+1} & W_{n-m} \\ W_{n-m} & W_{n-m-1} \end{pmatrix},$$

and if $r^2 + 4s = 0$, *i.e.,* $\alpha = \beta$ *then*

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\alpha}(n+1) H_{n+1} & -\frac{1}{2}\alpha n H_n \\ \frac{1}{2}\frac{n}{\alpha} H_n & -\frac{1}{2}\alpha(n-1)H_{n-1} \end{pmatrix} \begin{pmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+m+1} & W_{n+m} \\ W_{n+m} & W_{n+m-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\alpha}(n+1) H_{n+1} & -\frac{1}{2}\alpha n H_n \\ \frac{1}{2}\frac{n}{\alpha} H_n & -\frac{1}{2}\alpha(n-1)H_{n-1} \end{pmatrix} \begin{pmatrix} W_{m+1} & W_m \\ W_m & W_{m-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\alpha}(m+1) H_{m+1} & -\frac{1}{2}\alpha m H_m \\ \frac{1}{2}\frac{m}{\alpha} H_m & -\frac{1}{2}\alpha(m-1)H_{m-1} \end{pmatrix} \begin{pmatrix} W_{n-m+1} & W_{n-m} \\ W_{n-m} & W_{n-m-1} \end{pmatrix}.$$

Proof.

(a) Use (1), i.e. $W_n = rW_{n-1} + sW_{n-2}$.

(b) By using the definition of W_n, i.e., $W_n = rW_{n-1} + sW_{n-2}$, we get

$$\begin{aligned} Af_W(n-1) &= \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}\begin{pmatrix} W_n & W_{n-1} \\ W_{n-1} & W_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} rW_n + sW_{n-1} & rW_{n-1} + sW_{n-2} \\ W_n & W_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix} \\ &= f_W(n). \end{aligned}$$

Now, it follows that $f_W(n) = A^n f_W(0)$.

(c) By using (b) we get

$$f_W(n+m) = A^{n+m} f_W(0) = A^n A^m f_W(0) = A^n f_W(m).$$

By interchanging m and n in $f_W(n+m) = A^n f_W(m)$, we get $f_W(n+m) = A^m f_W(n)$.

Then it follows that

$$\begin{aligned} f_W(n+m) &= A^n f_W(m) \Leftrightarrow A^{-n} f_W(n+m) = f_W(m) \\ &\Leftrightarrow A^m f_W(-m+n) = f_W(n) \Leftrightarrow f_W(n) = A^m f_W(n-m). \end{aligned}$$

(d) By using Theorem 3.1 (b), i.e.,

$$G_n A + (G_{n+1} - rG_n)I = A^n$$

and the identities in (13), (22), (23), (24) i.e.,

$$\begin{aligned} W_{mn+j} &= W_{j+1}G_{mn} + sW_j G_{mn-1}, \\ W_{mn+j+1} &= W_{j+1}G_{mn+1} + sW_j G_{mn}, \\ W_{mn+j} &= W_j G_{mn+1} + sW_{j-1}G_{mn}, \\ W_{mn+j-1} &= W_j G_{mn} + sW_{j-1}G_{mn-1}, \end{aligned}$$

we get

$$\begin{aligned} (G_n A + (G_{n+1} - rG_n)I)^m f_W(j) &= A^{mn} f_W(j) \\ &= \begin{pmatrix} G_{mn+1} & sG_{mn} \\ G_{mn} & sG_{mn-1} \end{pmatrix}\begin{pmatrix} W_{j+1} & W_j \\ W_j & W_{j-1} \end{pmatrix} \\ &= \begin{pmatrix} W_{j+1}G_{mn+1} + sW_j G_{mn} & W_j G_{mn+1} + sW_{j-1}G_{mn} \\ W_{j+1}G_{mn} + sW_j G_{mn-1} & W_j G_{mn} + sW_{j-1}G_{mn-1} \end{pmatrix} \\ &= \begin{pmatrix} W_{mn+j+1} & W_{mn+j} \\ W_{mn+j} & W_{mn+j-1} \end{pmatrix} \\ &= f_W(mn+j). \end{aligned}$$

since

$$A^n = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}.$$

Note that $A^{mn} f_W(j) = f_W(mn+j)$ also follows from the identity $f_W(n+m) = A^n f_W(m)$ which is given in (c), by replacing n and m by mn and j respectively in $f_W(n+m) = A^n f_W(m)$.

(e) Use (b), (c) and Theorem 3.1 (a) which states that

$$A^n = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}.$$

(f) Use (b), (c) and Theorem 3.1 (a) (iii).

(g) Use (b), (c) and Theorem 3.1 (a) (ii). □

Taking the determinant of both sides of the identities given in Lemma 3.1, we obtain the following Theorem.

Theorem 3.3.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_W(n+m)) = (-s)^n \det(f_W(m)),$$

and

$$\det(f_W(n)) = (-s)^m \det(f_W(n-m)),$$

i.e.,

$$\begin{aligned} \begin{vmatrix} W_{n+m+1} & W_{n+m} \\ W_{n+m} & W_{n+m-1} \end{vmatrix} &= (-s)^n \begin{vmatrix} W_{m+1} & W_m \\ W_m & W_{m-1} \end{vmatrix} \\ &= (-s)^m \begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix}, \end{aligned}$$

and

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = (-s)^m \begin{vmatrix} W_{n-m+1} & W_{n-m} \\ W_{n-m} & W_{n-m-1} \end{vmatrix}.$$

(b) *(see Theorem 2.1) Simson's (or Cassini's) Identity:*

$$\det(f_W(n)) = (-s)^n \det(f_W(0)),$$

i.e.,

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = (-1)^n s^n \begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix}.$$

Proof.

(a) Taking the determinant of both sides of the identities

$$\begin{aligned} f_W(n+m) &= A^n f_W(m) \\ &= A^m f_W(n) \end{aligned}$$

and

$$f_W(n) = A^m f_W(n-m)$$

which are given in Lemma 3.1 (c), we get the required results.

(b) Take $m = 0$ in $\det(f_W(n+m)) = (-s)^n \det(f_W(m))$ in (a) or take the determinant of both sides of the identity $f_W(n) = A^n f_W(0)$ which is given in Lemma 3.1 (b). □

Remark 3.2.
To prove the second matrix identity in Lemma 3.1 (d), we used a consequence (Corollary 3.1) of Honsberger's Identity (Theorem 3.2). However, firstly, the second matrix identity in Lemma 3.1 (d) can be proved by induction and then Honsberger's Identity, i.e.,

$$W_{n+m} = W_n G_{m+1} + sW_{n-1} G_m$$

can be obtained just comparing the linear combination of the 2rd row and 1st column entries of the matrices.

From the last Theorem, we have the following Corollary which gives determinantal formulas of (r, s)-Fibonacci polynomials (take $W_n = G_n$ with $G_0 = 0, G_1 = 1$).

Corollary 3.2.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_G(n+m)) = (-s)^n \det(f_G(m)),$$

and

$$\det(f_G(n)) = (-s)^m \det(f_G(n-m)),$$

i.e.,

$$\begin{vmatrix} G_{n+m+1} & G_{n+m} \\ G_{n+m} & G_{n+m-1} \end{vmatrix} = (-s)^n \begin{vmatrix} G_{m+1} & G_m \\ G_m & G_{m-1} \end{vmatrix} = (-s)^m \begin{vmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{vmatrix},$$

and

$$\begin{vmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{vmatrix} = (-s)^m \begin{vmatrix} G_{n-m+1} & G_{n-m} \\ G_{n-m} & G_{n-m-1} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_G(n)) = (-s)^n \det(f_G(0)),$$

i.e.,

$$\begin{vmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{vmatrix} = (-1)^n s^{n-1}.$$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = r$ in the last Theorem, we have the following Corollary which gives determinantal formulas of (r,s)-Fibonacci-Lucas polynomials.

Corollary 3.3.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_H(n+m)) = (-s)^n \det(f_H(m)),$$

and

$$\det(f_H(n)) = (-s)^m \det(f_H(n-m)),$$

i.e.,

$$\begin{vmatrix} H_{n+m+1} & H_{n+m} \\ H_{n+m} & H_{n+m-1} \end{vmatrix} = (-s)^n \begin{vmatrix} H_{m+1} & H_m \\ H_m & H_{m-1} \end{vmatrix} = (-s)^m \begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix},$$

and

$$\begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} = (-s)^m \begin{vmatrix} H_{n-m+1} & H_{n-m} \\ H_{n-m} & H_{n-m-1} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_H(n)) = (-s)^n \det(f_H(0)),$$

i.e.,

$$\begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} = (r^2+4s)(-1)^{n+1} s^{n-1}.$$

4. The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Fibonacci Polynomials via Matrix Methods

From now on, through the paper, we suppose that z is a real or complex number. So we may assume that z is a scalar value (real or complex) number or function in $x \in \mathbb{R}$, for example $z = 3$, $z = 11 - 2i$, $z = \cos x$, $z = e^x$, $z = 2 + 5x$, $z = e^{ix} = \cos x + i \sin x$, $z = 5x^4 - 2x + 7$ for $x \in \mathbb{R}$. We also suppose that $z \neq 0$ if necessary (needed).

In this section, we give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Fibonacci polynomials via matrix methods. First, we need to present matrix notations to use linear algebra (matrix) method. Suppose that M is a $n \times n$ matrix. Then

$$MAdj(M) = \det(M)I$$

where I is the identity matrix and $Adj(M)$ is the adjugate of M. The *adjugate* or *classical adjoint* of a square matrix M is the transpose of the matrix of cofactors of M. The i, j *cofactor* M_{ij} of M is the scalar $(-1)^{i+j} \det M\left(i|j\right)$, where $M\left(i|j\right)$ denotes the matrix that you obtain from M by removing the ith row and jth column. Since,

$$\det(I - M)I = (I - M)Adj(I - M)$$

and

$$\left(\sum_{k=0}^{n} M^k\right)(I - M) = (I - M^{n+1})$$

for any square matrix M, we get

$$\left(\sum_{k=0}^{n} M^k\right)\det(I - M) = (I - M^{n+1})Adj(I - M). \tag{25}$$

Note also that

$$\begin{aligned} f_W(j) &= \begin{pmatrix} W_{j+1} & W_j \\ W_j & W_{j-1} \end{pmatrix} \\ &= \begin{pmatrix} W_{j+1} & W_j \\ W_j & \frac{1}{s}(W_{j+1} - rW_j) \end{pmatrix}. \end{aligned}$$

If

$$\begin{aligned} M &= zA^m = z\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^m \\ &= z\begin{pmatrix} G_{m+1} & sG_m \\ G_m & sG_{m-1} \end{pmatrix}, \end{aligned}$$

then we have

$$\det(I - zA^m)\left(\sum_{k=0}^{n} z^k A^{mk}\right) = (I - z^{n+1}A^{mn+m})Adj(I - zA^m),$$

and then, since $f_W(mk + j) = A^{mk} f_W(j)$ by Lemma 3.1 (d), we get

$$\begin{aligned} \det(I - zA^m)\left(\sum_{k=0}^{n} z^k f_W(mk + j)\right) &= \det(I - zA^m)\left(\sum_{k=0}^{n} z^k A^{mk}\right) f_W(j) \\ &= (I - z^{n+1}A^{mn+m})Adj(I - zA^m) f_W(j). \end{aligned}$$

4.1. The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Fibonacci Polynomials in Terms of Generalized Fibonacci Polynomials

By using Theorem 3.1 (a) (iii), we can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Fibonacci polynomials via matrix methods (in terms of elements of the sequence of generalized Fibonacci polynomials).

Theorem 4.1.
For all integers m and j, we have the following sum formulas.

(a) *If* $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4}{z^2\Gamma_1 + z\Gamma_2 + \Gamma_3} = \frac{\Theta_W(z)}{\Gamma_W(z)} \tag{26}$$

where

$\Theta_W(z) = z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4 = z^{n+2}((W_jW_{m+1} - W_{j+1}W_m)W_{mn+m+1} + ((W_{j+1} - rW_j)W_{m+1} - sW_jW_m)W_{mn+m}) + z^{n+1}((W_0W_{j+1} - W_1W_j)W_{mn+m+1} + (-W_1W_{j+1} + (rW_1 + sW_0)W_j)W_{m+mn}) + z((-W_0W_{j+1} + (-W_1 + rW_0)W_j)W_{m+1} + (W_1W_{j+1} + sW_0W_j)W_m) + (W_1^2 - sW_0^2 - rW_0W_1)W_j$

$z^{n+2}\Theta_1 = z^{n+2}((W_jW_{m+1} - W_{j+1}W_m)W_{mn+m+1} + ((W_{j+1} - rW_j)W_{m+1} - sW_jW_m)W_{mn+m})$

$z^{n+1}\Theta_2 = z^{n+1}((W_0W_{j+1} - W_1W_j)W_{mn+m+1} + (-W_1W_{j+1} + (rW_1 + sW_0)W_j)W_{m+mn})$

$z\Theta_3 = z((-W_0W_{j+1} + (-W_1 + rW_0)W_j)W_{m+1} + (W_1W_{j+1} + sW_0W_j)W_m)$

$\Theta_4 = (W_1^2 - sW_0^2 - rW_0W_1)W_j$

i.e.,

$\Theta_W(z) = z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4 = z^{n+2}(-1)(W_1^2 - sW_0^2 - rW_0W_1)((-H_mG_j + G_{m+j})W_{mn+m+1} + (-H_m(G_{j+1} - rG_j) + G_{m+j+1} - rG_{m+j})W_{mn+m}) + z^{n+1}(-1)(W_1^2 - sW_0^2 - rW_0W_1)(G_jW_{mn+m+1} + (G_{j+1} - rG_j)W_{m+mn}) + z(W_1^2 - sW_0^2 - rW_0W_1)(G_mW_{j+1} - G_{m+1}W_j) + (W_1^2 - sW_0^2 - rW_0W_1)W_j$

$z^{n+2}\Theta_1 = z^{n+2}(-1)(W_1^2 - sW_0^2 - rW_0W_1)((-H_mG_j + G_{m+j})W_{mn+m+1} + (-H_m(G_{j+1} - rG_j) + G_{m+j+1} - rG_{m+j})W_{mn+m})$

$z^{n+1}\Theta_2 = z^{n+1}(-1)(W_1^2 - sW_0^2 - rW_0W_1)(G_jW_{mn+m+1} + (G_{j+1} - rG_j)W_{m+mn})$

$z\Theta_3 = z(W_1^2 - sW_0^2 - rW_0W_1)(G_mW_{j+1} - G_{m+1}W_j)$

$\Theta_4 = (W_1^2 - sW_0^2 - rW_0W_1)W_j$

and

$\Gamma_W(z) = z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = z^2(-1)^m s^m(W_1^2 - sW_0^2 - rW_0W_1) + z(-1)H_m(W_1^2 - sW_0^2 - rW_0W_1) + (W_1^2 - sW_0^2 - rW_0W_1)$

$z^2\Gamma_1 = z^2(W_{m+1}^2 - sW_m^2 - rW_mW_{m+1})$

$z\Gamma_2 = z((-2W_1 + rW_0)W_{m+1} + (rW_1 + 2sW_0)W_m)$

$\Gamma_3 = W_1^2 - sW_0^2 - rW_0W_1$

i.e.,

$\Gamma_W(z) = z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = (z^2(-1)^m s^m + z(-1)H_m + 1)(W_1^2 - sW_0^2 - rW_0W_1)$

$z^2\Gamma_1 = z^2(-1)^m s^m(W_1^2 - sW_0^2 - rW_0W_1)$

$z\Gamma_2 = z(-1)H_m(W_1^2 - sW_0^2 - rW_0W_1)$

$\Gamma_3 = W_1^2 - sW_0^2 - rW_0W_1$

(b) *If* $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = u(z-a)(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $z = a$ *or* $z = b$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+2)z^{n+1}\Theta_1 + (n+1)z^n\Theta_2 + \Theta_3}{2z\Gamma_1 + \Gamma_2}.$$

(c) *If* $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = u(z-a)^2 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $z = a$*, then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+2)(n+1)z^n\Theta_1 + (n+1)nz^{n-1}\Theta_2}{2\Gamma_1}.$$

Proof. Note that

$$\Lambda_W(0) = W_1^2 - sW_0^2 - rW_1W_0.$$

(a) We set, by using Theorem 3.1 (a) (iii),

$$\begin{aligned} M &= zA^m = z\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^m = z\begin{pmatrix} G_{m+1} & sG_m \\ G_m & sG_{m-1} \end{pmatrix} \\ &= z\begin{pmatrix} G_{m+1} & sG_m \\ G_m & G_{m+1} - rG_m \end{pmatrix} \\ &= \frac{z\begin{pmatrix} (W_1 - rW_0)W_{m+1} - sW_0W_m & -s(W_0W_{m+1} - W_1W_m) \\ -W_0W_{m+1} + W_1W_m & W_1W_{m+1} - (rW_1 + sW_0)W_m \end{pmatrix}}{W_1^2 - sW_0^2 - rW_1W_0} \end{aligned}$$

in (25). Then we get

$$I - zA^m = \begin{pmatrix} z\dfrac{W_{m+1}(W_1 - rW_0) - sW_0W_m}{sW_0^2 + rW_0W_1 - W_1^2} + 1 & -sz\dfrac{W_0W_{m+1} - W_1W_m}{sW_0^2 + rW_0W_1 - W_1^2} \\ -z\dfrac{W_0W_{m+1} - W_1W_m}{sW_0^2 + rW_0W_1 - W_1^2} & z\dfrac{W_1W_{m+1} - W_m(rW_1 + sW_0)}{sW_0^2 + rW_0W_1 - W_1^2} + 1 \end{pmatrix}$$

After some calculations, we see that

$$\begin{aligned} \det(I - zA^m) &= \frac{1}{W_1^2 - sW_0^2 - rW_1W_0}\Gamma_W(z) \\ &= \frac{1}{W_1^2 - sW_0^2 - rW_1W_0}(z^2\Gamma_1 + z\Gamma_2 + \Gamma_3) \end{aligned}$$

where $\Gamma_W(z) = z^2\Gamma_1 + z\Gamma_2 + \Gamma_3, z^2\Gamma_1, z\Gamma_1, \Gamma_3$ are as in the statement of (a) of Theorem, i.e.,
$\Gamma_W(z) = z^2\Gamma_1 + z\Gamma_2 + \Gamma_3,$
$z^2\Gamma_1 = z^2(W_{m+1}^2 - sW_m^2 - rW_mW_{m+1}),$
$z\Gamma_2 = z((-2W_1 + rW_0)W_{m+1} + (rW_1 + 2sW_0)W_m),$
$\Gamma_3 = W_1^2 - sW_0^2 - rW_0W_1,$
By Lemma 2.1 (a) we know that

$$W_{m+1}^2 - sW_m^2 - rW_mW_{m+1} = (-1)^m s^m (W_1^2 - sW_0^2 - rW_0W_1)$$

so that

$$z^2\Gamma_1 = z^2(-1)^m s^m (W_1^2 - sW_0^2 - rW_0W_1).$$

Since

$$(-2W_1 + rW_0)W_{m+1} + (rW_1 + 2sW_0)W_m = -H_m(W_1^2 - sW_0^2 - rW_0W_1)$$

we get

$$z\Gamma_2 = z(-1)H_m(W_1^2 - sW_0^2 - rW_0W_1).$$

Therefore,

$$\begin{aligned} \det(I - zA^m) &= \frac{1}{W_1^2 - sW_0^2 - rW_1W_0}(z^2(W_{m+1}^2 - sW_m^2 - rW_mW_{m+1}) \\ &\quad + z((-2W_1 + rW_0)W_{m+1} + (rW_1 + 2sW_0)W_m) + W_1^2 - sW_0^2 - rW_0W_1) \\ &= \frac{(z^2(-1)^m s^m + z(-1)H_m + 1)(W_1^2 - sW_0^2 - rW_0W_1)}{W_1^2 - sW_0^2 - rW_1W_0} \\ &= \frac{z^2\Gamma_1 + z\Gamma_2 + \Gamma_3}{W_1^2 - sW_0^2 - rW_1W_0} \\ &= z^2(-1)^m s^m + z(-1)H_m + 1. \end{aligned}$$

Now,

$$Adj(I - zA^m) = \begin{pmatrix} z\dfrac{W_1W_{m+1} - W_m(rW_1 + sW_0)}{sW_0^2 + rW_0W_1 - W_1^2} + 1 & sz\dfrac{W_0W_{m+1} - W_1W_m}{sW_0^2 + rW_0W_1 - W_1^2} \\ z\dfrac{W_0W_{m+1} - W_1W_m}{sW_0^2 + rW_0W_1 - W_1^2} & z\dfrac{W_{m+1}(W_1 - rW_0) - sW_0W_m}{sW_0^2 + rW_0W_1 - W_1^2} + 1 \end{pmatrix}$$

and

$$I - z^{n+1}A^{mn+m} = \begin{pmatrix} z^{n+1}\dfrac{(W_1 - rW_0)W_{m+mn+1} - sW_0W_{m+mn}}{sW_0^2 + rW_0W_1 - W_1^2} + 1 & -sz^{n+1}\dfrac{W_0W_{m+mn+1} - W_1W_{m+mn}}{sW_0^2 + rW_0W_1 - W_1^2} \\ -z^{n+1}\dfrac{W_0W_{m+mn+1} - W_1W_{m+mn}}{sW_0^2 + rW_0W_1 - W_1^2} & 1 - z^{n+1}\dfrac{(rW_1 + sW_0)W_{m+mn} - W_1W_{m+mn+1}}{sW_0^2 + rW_0W_1 - W_1^2} \end{pmatrix}$$

and

$$f_W(mk + j) = \begin{pmatrix} W_{mk+j+1} & W_{mk+j} \\ W_{mk+j} & \dfrac{1}{s}(W_{mk+j+1} - rW_{mk+j}) \end{pmatrix}.$$

Then, the 2nd row and 1st column entry of matrix $(I - z^{n+1}A^{mn+m})Adj(I - zA^m)f_W(j)$ is equal to

$$\frac{1}{W_1^2 - sW_0^2 - rW_1W_0}(z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4)$$

where $\Theta_W(z) = z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4, z^{n+2}\Theta_1, z^{n+1}\Theta_2, z\Theta_3, \Theta_4$, are as in the statement of (a) of Theorem. Note that we have used d'Ocagne's identity, i.e.,

$$W_nW_{m+1} - W_{n+1}W_m = -(W_1^2 - sW_0^2 - rW_0W_1)(-H_mG_n + G_{m+n})$$

and the identities

$$\begin{aligned}(-W_1^2 + sW_0^2 + rW_1W_0)G_n &= W_0W_{n+1} - W_1W_n \\ s(-W_1^2 + sW_0^2 + rW_1W_0)G_n &= -W_1W_{n+2} + (rW_1 + sW_0)W_{n+1}\end{aligned}$$

when computing and simplfying $z^{n+2}\Theta_1, z^{n+1}\Theta_2$ and $z\Theta_3$.

Note also that, since $\det(I - zA^m)(\sum_{k=0}^{n} z^k f_W(mk + j) = (I - z^{n+1}A^{mn+m})Adj(I - zA^m)f_W(j)$, i.e., matrices over the either side is equal, the 2nd row and 1st column entry of matrix $\det(I - zA^m)(\sum_{k=0}^{n} z^k f_W(mk + j))$ is equal to the 2nd row and 1st column entry of matrix $(I - z^{n+1}A^{mn+m})Adj(I - zA^m)f_W(j)$. So, to complete the proof, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices. Then, we get

$$\begin{aligned}&\frac{1}{W_1^2 - sW_0^2 - rW_1W_0}(z^2\Gamma_1 + z\Gamma_2 + \Gamma_3)\sum_{k=0}^{n} z^kW_{mk+j} \\ =\ &\frac{1}{W_1^2 - sW_0^2 - rW_1W_0}(z^{n+2}\Theta_1 + z^{n+3}\Theta_2 + z\Theta_3 + \Theta_4)\end{aligned}$$

and so

$$\begin{aligned}\sum_{k=0}^{n} z^kW_{mk+j} &= \frac{z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4}{z^2\Gamma_1 + z\Gamma_2 + \Gamma_3} \\ &= \frac{\Theta_W(z)}{\Gamma_W(z)}.\end{aligned}$$

(b) We use (26). For $z = a$ or $z = b$ or $z = c$, the right hand side of the above sum formula (26) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned}\sum_{k=0}^{n} a^kW_k &= \left.\frac{\frac{d}{dz}(z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4)}{\frac{d}{dz}(z^2\Gamma_1 + z\Gamma_2 + \Gamma_3)}\right|_{z=a} \\ &= \left.\frac{(n+2)z^{n+1}\Theta_1 + (n+1)z^n\Theta_2 + \Theta_3}{2z\Gamma_1 + \Gamma_2}\right|_{z=a}, \\ \sum_{k=0}^{n} b^kW_k &= \left.\frac{(n+2)z^{n+1}\Theta_1 + (n+1)z^n\Theta_2 + \Theta_3}{2z\Gamma_1 + \Gamma_2}\right|_{z=b}.\end{aligned}$$

(c) We use (26). For $z = a$, the right hand side of the above sum formula (26) is an indeterminate form. Now, we can use L'Hospital rule (two times). Then we get (c) by using

$$\sum_{k=0}^{n} a^kW_k = \left.\frac{\frac{d^2}{dz^2}(z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4)}{\frac{d^2}{dz^2}(z^2\Gamma_1 + z\Gamma_2 + \Gamma_3)}\right|_{z=a} . \square$$

Now, we consider special cases of Theorem 4.1.

Theorem 4.2.
For all integers m and j, we have the following sum formulas.

(a) $(m = 1, j = 0)$.

(i) *If* $z^2(-1)s + z(-1)r + 1 \neq 0$, *i.e., if* $z \neq \frac{1}{2s}(-r - \sqrt{r^2 + 4s})$, $z \neq \frac{1}{2s}(-r + \sqrt{r^2 + 4s})$ *then*

$$\sum_{k=0}^{n} z^kW_k = \frac{z^{n+2}(-1)sW_n + z^{n+1}(-1)W_{n+1} + z(W_1 - rW_0) + W_0}{z^2(-1)s + z(-1)r + 1}.$$

(ii) *If* $z^2(-1)s+z(-1)r+1=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2s}(-r-\sqrt{r^2+4s})$ *or* $z=\frac{1}{2s}(-r+\sqrt{r^2+4s})$ *provided that* $s\neq -\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{(n+2)z^{n+1}(-1)sW_n+(n+1)z^n(-1)W_{n+1}+(W_1-rW_0)}{2z(-1)s+(-1)r}.$$

(iii) *If* $z^2(-1)s+z(-1)r+1=(z-\frac{2}{r})^2=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=-\frac{r}{2s}=\frac{2}{r}$, $s=-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{(n+2)(n+1)z^n(-1)sW_n+(n+1)nz^{n-1}(-1)W_{n+1}}{2(-1)s}.$$

(b) $(m=2, j=0)$.

(i) *If* $z^2s^2+z(-1)(r^2+2s)+1\neq 0$, *i.e., if* $z\neq \frac{1}{2s^2}((r^2+2s)+r\sqrt{r^2+4s})$, $z\neq \frac{1}{2s^2}((r^2+2s)-r\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{z^{n+2}s^2W_{2n}+z^{n+1}(-1)(rW_{2n+1}+sW_{2n})+z(rW_1-(r^2+s)W_0)+W_0}{z^2s^2+z(-1)(r^2+2s)+1}.$$

(ii) *If* $z^2s^2+z(-1)(r^2+2s)+1=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2s^2}((r^2+2s)+r\sqrt{r^2+4s})$ *or* $z=\frac{1}{2s^2}((r^2+2s)-r\sqrt{r^2+4s})$ *provided that* $s\neq -\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{(n+2)z^{n+1}s^2W_{2n}+(n+1)z^n(-1)(rW_{2n+1}+sW_{2n})+(rW_1-(r^2+s)W_0)}{2zs^2+(-1)(r^2+2s)}.$$

(iii) *If* $z^2s^2+z(-1)(r^2+2s)+1=(z-\frac{4}{r^2})^2=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=\frac{r^2+2s}{2s^2}=\frac{4}{r^2}$, $s=-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{(n+2)(n+1)z^n s^2W_{2n}+(n+1)nz^{n-1}(-1)(rW_{2n+1}+sW_{2n})}{2s^2}.$$

(c) $(m=2, j=1)$.

(i) *If* $z^2s^2+z(-1)(r^2+2s)+1\neq 0$, *i.e., if* $z\neq \frac{1}{2s^2}((r^2+2s)+r\sqrt{r^2+4s})$, $z\neq \frac{1}{2s^2}((r^2+2s)-r\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{z^{n+2}s^2W_{2n+1}+z^{n+1}(-1)((r^2+s)W_{2n+1}+srW_{2n})+z(-1)s(W_1-rW_0)+W_1}{z^2s^2+z(-1)(r^2+2s)+1}.$$

(ii) *If* $z^2s^2+z(-1)(r^2+2s)+1=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2s^2}((r^2+2s)+r\sqrt{r^2+4s})$ *or* $z=\frac{1}{2s^2}((r^2+2s)-r\sqrt{r^2+4s})$ *provided that* $s\neq -\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{(n+2)z^{n+1}s^2W_{2n+1}+(n+1)z^n(-1)((r^2+s)W_{2n+1}+srW_{2n})+(-1)s(W_1-rW_0)}{2zs^2+(-1)(r^2+2s)}.$$

(iii) *If* $z^2s^2+z(-1)(r^2+2s)+1=(z-\frac{4}{r^2})^2=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=\frac{r^2+2s}{2s^2}=\frac{4}{r^2}$, $s=-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{(n+2)(n+1)z^n s^2W_{2n+1}+(n+1)nz^{n-1}(-1)((r^2+s)W_{2n+1}+srW_{2n})}{2s^2}.$$

(d) $(m=-1, j=0)$.

(i) *If* $z^2(-1)+zr+s\neq 0$, *i.e., if* $z\neq \frac{1}{2}(r+\sqrt{r^2+4s})$, $z\neq \frac{1}{2}(r-\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{z^{n+2}(-1)W_{-n}+z^{n+1}(-1)sW_{-n-1}+zW_1+sW_0}{z^2(-1)+zr+s}.$$

(ii) *If* $z^2(-1)+zr+s=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2}(r+\sqrt{r^2+4s})$ *or* $z=\frac{1}{2}(r-\sqrt{r^2+4s})$ *provided that* $s\neq -\dfrac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{(n+2)z^{n+1}(-1)W_{-n}+(n+1)z^n(-1)sW_{-n-1}+W_1}{2z(-1)+r}.$$

(iii) *If* $z^2(-1)+zr+s=(z-\dfrac{r}{2})^2=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=\dfrac{r}{2}$, $s=-\dfrac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{(n+2)(n+1)z^n(-1)W_{-n}+(n+1)nz^{n-1}(-1)sW_{-n-1}}{2(-1)}.$$

(e) $(m=-2, j=0)$.

(i) *If* $z^2+z(-1)(r^2+2s)+s^2\neq 0$, *i.e., if* $z\neq \frac{1}{2}((r^2+2s)+r\sqrt{r^2+4s})$, $z\neq \frac{1}{2}((r^2+2s)-r\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{z^{n+2}W_{-2n}+z^{n+1}(-1)s(W_{-2n}-rW_{-2n-1})+z(-1)(rW_1+sW_0)+s^2W_0}{z^2+z(-1)(r^2+2s)+s^2}.$$

(ii) *If* $z^2+z(-1)(r^2+2s)+s^2=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2}((r^2+2s)+r\sqrt{r^2+4s})$ *or* $z=\frac{1}{2}((r^2+2s)-r\sqrt{r^2+4s})$ *provided that* $s\neq -\dfrac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{(n+2)z^{n+1}W_{-2n}+(n+1)z^n(-1)s(W_{-2n}-rW_{-2n-1})+(-1)(rW_1+sW_0)}{2z+(-1)(r^2+2s)}.$$

(iii) *If* $z^2+z(-1)(r^2+2s)+s^2=(z-\dfrac{r^2}{4})^2=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=\dfrac{r^2+2s}{2}=\dfrac{r^2}{4}$, $s=-\dfrac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{(n+2)(n+1)z^nW_{-2n}+(n+1)nz^{n-1}(-1)s(W_{-2n}-rW_{-2n-1})}{2}.$$

(f) $(m=-2, j=1)$.

(i) *If* $z^2+z(-1)(r^2+2s)+s^2\neq 0$, *i.e., if* $z\neq \frac{1}{2}((r^2+2s)+r\sqrt{r^2+4s})$, $z\neq \frac{1}{2}((r^2+2s)-r\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{z^{n+2}(rW_{-2n}+sW_{-2n-1})+z^{n+1}(-1)s^2W_{-2n-1}+z(-1)((r^2+s)W_1+rsW_0)+s^2W_1}{z^2+z(-1)(r^2+2s)+s^2}.$$

(ii) *If* $z^2+z(-1)(r^2+2s)+s^2=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2}((r^2+2s)+r\sqrt{r^2+4s})$ *or* $z=\frac{1}{2}((r^2+2s)-r\sqrt{r^2+4s})$ *provided that* $s\neq -\dfrac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{(n+2)z^{n+1}(rW_{-2n}+sW_{-2n-1})+(n+1)z^n(-1)s^2W_{-2n-1}+(-1)((r^2+s)W_1+rsW_0)}{2z+(-1)(r^2+2s)}.$$

(iii) *If* $z^2+z(-1)(r^2+2s)+s^2=(z-\dfrac{r^2}{4})^2=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=\dfrac{r^2+2s}{2}=\dfrac{r^2}{4}$, $s=-\dfrac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{(n+2)(n+1)z^n(rW_{-2n}+sW_{-2n-1})+(n+1)nz^{n-1}(-1)s^2W_{-2n-1}}{2}.$$

4.2. The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Fibonacci Polynomials in Terms of Generalized Fibonacci Polynomials and (r,s)-Fibonacci Polynomials

By using Theorem 3.1 (a) (i), we can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Fiibonacci polynomials via matrix methods (in terms of elements of the sequence of generalized Fibonacci polynomials and (r,s)-Fibonacci polynomials).

Theorem 4.3.
For all integers m and j, we have the following sum formulas.

(a) *If* $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 \neq 0$ *then*

$$\begin{aligned}\sum_{k=0}^{n} z^k W_{mk+j} &= \frac{z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4}{z^2\Gamma_1 + z\Gamma_2 + \Gamma_3} \\ &= \frac{\Theta_W(z)}{\Gamma_W(z)}\end{aligned}$$

where

$\Theta_W(z) = z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4 = z^{n+2}((W_jG_{m+1} - W_{j+1}G_m)G_{m+mn+1} + ((W_{j+1} - rW_j)G_{m+1} - sW_jG_m)G_{m+mn}) + z^{n+1}(-W_jG_{m+mn+1} + (rW_j - W_{j+1})G_{m+mn}) + z(W_{j+1}G_m - W_jG_{m+1}) + W_j,$

$z^{n+2}\Theta_1 = z^{n+2}((W_jG_{m+1} - W_{j+1}G_m)G_{m+mn+1} + ((W_{j+1} - rW_j)G_{m+1} - sW_jG_m)G_{m+mn}),$

$z^{n+1}\Theta_2 = z^{n+1}(-W_jG_{m+mn+1} + (rW_j - W_{j+1})G_{m+mn}),$

$z\Theta_3 = z(W_{j+1}G_m - W_jG_{m+1}),$

$\Theta_4 = W_j,$

and

$\Gamma_W(z) = z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = z^2(-1)^m s^m + z(rG_m - 2G_{m+1}) + 1,$

$z^2\Gamma_1 = z^2(-1)^m s^m,$

$z\Gamma_2 = z(rG_m - 2G_{m+1}),$

$\Gamma_3 = 1.$

(b) *If* $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = u(z-a)(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+2)z^{n+1}\Theta_1 + (n+1)z^n\Theta_2 + \Theta_3}{2z\Gamma_1 + \Gamma_2}.$$

(c) *If* $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = u(z-a)^2 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+2)(n+1)z^n\Theta_1 + (n+1)nz^{n-1}\Theta_2}{2\Gamma_1}.$$

Proof. We only prove (a). The proof of (b) and (c) are as in Theorem 4.1 (b) and (c), respectively.
Proof of (a). We use the same method as in Theorem 4.1 by setting

$$\begin{aligned}M &= zA^m = z\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^m = z\begin{pmatrix} G_{m+1} & sG_m \\ G_m & sG_{m-1} \end{pmatrix} \\ &= z\begin{pmatrix} G_{m+1} & sG_m \\ G_m & G_{m+1} - rG_m \end{pmatrix}\end{aligned}$$

in (25). Then we get

$$\begin{aligned}I - zA^m &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} zG_{m+1} & szG_m \\ zG_m & -z(rG_m - G_{m+1}) \end{pmatrix} \\ &= \begin{pmatrix} 1 - zG_{m+1} & -szG_m \\ -zG_m & z(rG_m - G_{m+1}) + 1 \end{pmatrix}.\end{aligned}$$

After some calculations, we see that

$$\det(I - zA^m) = z^2(G_{m+1}^2 - sG_m^2 - rG_mG_{m+1}) + z(rG_m - 2G_{m+1}) + 1.$$

By Lemma 2.1 (b) we know that

$$(G_{m+1}^2 - sG_m^2 - rG_mG_{m+1}) = (-1)^m s^m.$$

So

$$\begin{aligned}\det(I - zA^m) &= z^2(-1)^m s^m + z(rG_m - 2G_{m+1}) + 1 \\ &= \Gamma_W(z)\end{aligned}$$

where $\Gamma_W(z) = z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4, z^3\Gamma_1, z^2\Gamma_2, z\Gamma_3, \Gamma_4$ are as in the statement of (a) of Theorem.

Now,

$$Adj(I-zA^m) = \begin{pmatrix} z(rG_m - G_{m+1})+1 & szG_m \\ zG_m & 1-zG_{m+1} \end{pmatrix}$$

and

$$I - z^{n+1}A^{mn+m} = \begin{pmatrix} 1-z^{n+1}G_{m+mn+1} & -sz^{n+1}G_{m+mn} \\ -z^{n+1}G_{m+mn} & z^{n+1}(rG_{m+mn}-G_{m+mn+1})+1 \end{pmatrix}$$

and

$$f_W(mk+j) = \begin{pmatrix} W_{mk+j+1} & W_{mk+j} \\ W_{mk+j} & \frac{1}{s}(W_{mk+j+1}-rW_{mk+j}) \end{pmatrix}.$$

Then, the 2nd row and 1st column entry of matrix $(I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$ is equal to

$$z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4$$

where $\Theta_W(z) = z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4, z^{n+2}\Theta_1, z^{n+1}\Theta_2, z\Theta_3, \Theta_4$, are as in the statement of (a) of Theorem.

Note also that, since $\det(I-zA^m)(\sum_{k=0}^{n} z^k f_W(mk+j) = (I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$, i.e., matrices over the either side is equal, the 2nd row and 1st column entry of matrix of matrix $\det(I-zA^m)(\sum_{k=0}^{n} z^k f_W(mk+j))$ is equal to the 2nd row and 1st column entry of matrix $(I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$. So, to complete the proof, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices. Then, we get

$$(z^2\Gamma_1 + z\Gamma_2 + \Gamma_3)\sum_{k=0}^{n} z^k W_{mk+j} = (z^{n+2}\Theta_1 + z^{n+3}\Theta_2 + z\Theta_3 + \Theta_4)$$

and so

$$\begin{aligned} \sum_{k=0}^{n} z^k W_{mk+j} &= \frac{z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4}{z^2\Gamma_1 + z\Gamma_2 + \Gamma_3} \\ &= \frac{\Theta_W(z)}{\Gamma_W(z)}. \ \square \end{aligned}$$

5. Generating Function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of Generalized Fibonacci Polynomials

In this section, we present generating function of the sequence W_{mn+j} and its special cases.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Fibonacci polynomials and (r,s)-Fibonacci and (r,s)-Fibonacci-Lucas polynomials).

Lemma 5.1.

Assume that $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}\}$. Suppose that $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$ is the ordinary generating function of the generalized Fibonacci (sequence of) polynomials $\{W_{mn+j}\}$. Then, $\sum_{n=0}^{\infty} W_{mn+j}z^n$ is given by

$$\begin{aligned} \sum_{n=0}^{\infty} W_{mn+j}z^n &= \frac{z\Theta_3 + \Theta_4}{z^2\Gamma_1 + z\Gamma_2 + \Gamma_3} \\ &= \frac{z(G_m W_{j+1} - G_{m+1}W_j) + W_j}{z^2(-1)^m s^m + z(-1)H_m + 1} \\ &= \frac{z(G_m W_{j+1} - G_{m+1}W_j) + W_j}{z^2(-1)^m s^m + z(rG_m - 2G_{m+1}) + 1} \end{aligned}$$

where (as in Theorem 4.1 (a))

$z\Theta_3 = z((-W_0W_{j+1} + (-W_1 + rW_0)W_j)W_{m+1} + (W_1W_{j+1} + sW_0W_j)W_m)$,
$\Theta_4 = (W_1^2 - sW_0^2 - rW_0W_1)W_j$,
and
$z^2\Gamma_1 = z^2(W_{m+1}^2 - sW_m^2 - rW_mW_{m+1})$,
$z\Gamma_2 = z((-2W_1 + rW_0)W_{m+1} + (rW_1 + 2sW_0)W_m)$,
$\Gamma_3 = (W_1^2 - sW_0^2 - rW_0W_1)$.

Proof. Use Theorem 4.1 (a) and Theorem 1.1. □

Now, we consider special cases of Lemma 5.1.

Corollary 5.1.
The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z|<\min\{|\alpha|^{-1}, |\beta|^{-1}\})$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0+(W_1-rW_0)z}{1-rz-sz^2}.$$

(b) $(m=2, j=0, |z|<\min\{|\alpha|^{-2}, |\beta|^{-2}\})$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{W_0+(rW_1-(r^2+s)W_0)z}{1-(r^2+2s)z+s^2z^2}.$$

(c) $(m=2, j=1, |z|<\min\{|\alpha|^{-2}, |\beta|^{-2}\})$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{W_1-s(W_1-rW_0)z}{1-(r^2+2s)z+s^2z^2}.$$

(d) $(m=-1, j=0, |z|<\min\{|\alpha|, |\beta|\})$.

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{sW_0+W_1z}{s+rz-z^2}.$$

(e) $(m=-2, j=0, |z|<\min\{|\alpha|^{2}, |\beta|^{2}\})$.

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{s^2W_0-(rW_1+sW_0)z}{s^2-(r^2+2s)z+z^2}.$$

(f) $(m=-2, j=1, |z|<\min\{|\alpha|^{2}, |\beta|^{2}\})$.

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{s^2W_1-((r^2+s)W_1+rsW_0)z}{s^2-(r^2+2s)z+z^2}.$$

Proof. Use Lemma 5.1 (or Theorem 4.2). □

Lemma 5.1 gives the following results as particular examples (generating functions of (r,s)-Fibonacci and (r,s)-Fibonacci-Lucas polynomials).

Corollary 5.2.
Assume that $|z|<\min\{|\alpha|^{-m}, |\beta|^{-m}\}$. Generating functions of (r,s)-Fibonacci and (r,s)-Fibonacci-Lucas polynomials are given, respectively, as follows:

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} G_{mn+j} z^n &= \frac{z(G_mG_{j+1}-G_{m+1}G_j)+G_j}{z^2(G_{m+1}^2-sG_m^2-rG_mG_{m+1})+z(rG_m-2G_{m+1})+1}\\ &= \frac{z(G_mG_{j+1}-G_{m+1}G_j)+G_j}{z^2(-1)^ms^m+z(-1)H_m+1}\\ &= \frac{z(G_mG_{j+1}-G_{m+1}G_j)+G_j}{z^2(-1)^ms^m+z(rG_m-2G_{m+1})+1}.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{n=0}^{\infty} H_{mn+j} z^n &= \frac{z((-2H_{j+1}+rH_j)H_{m+1}+(rH_{j+1}+2sH_j)H_m)-(r^2+4s)H_j}{z^2(H_{m+1}^2-sH_m^2-rH_mH_{m+1})+z(r^2+4s)H_m-(r^2+4s)}\\ &= \frac{z(G_mH_{j+1}-G_{m+1}H_j)+H_j}{z^2(-1)^ms^m+z(-1)H_m+1}\\ &= \frac{z(G_mH_{j+1}-G_{m+1}H_j)+H_j}{z^2(-1)^ms^m+z(rG_m-2G_{m+1})+1}.\end{aligned}$$

Proof. In Lemma 5.1, take $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = r$, respectively. □

Now, we consider special cases of Corollay 5.1 (or Corollary 5.2).

Corollary 5.3.
The ordinary generating functions of the sequences G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1} *and* H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_n z^n &= \frac{z}{1 - rz - sz^2},\\ \sum_{n=0}^{\infty} H_n z^n &= \frac{2 - rz}{1 - rz - sz^2}.\end{aligned}$$

(b) $(m = 2, j = 0, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n} z^n &= \frac{rz}{1 - (r^2 + 2s)z + s^2z^2},\\ \sum_{n=0}^{\infty} H_{2n} z^n &= \frac{2 - (r^2 + 2s)z}{1 - (r^2 + 2s)z + s^2z^2}.\end{aligned}$$

(c) $(m = 2, j = 1, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n+1} z^n &= \frac{1 - sz}{1 - (r^2 + 2s)z + s^2z^2},\\ \sum_{n=0}^{\infty} H_{2n+1} z^n &= \frac{r + rsz}{1 - (r^2 + 2s)z + s^2z^2}.\end{aligned}$$

(d) $(m = -1, j = 0, |z| < \min\{|\alpha|, |\beta|\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-n} z^n &= \frac{z}{s + rz - z^2},\\ \sum_{n=0}^{\infty} H_{-n} z^n &= \frac{2s + rz}{s + rz - z^2}.\end{aligned}$$

(e) $(m = -2, j = 0, |z| < \min\{|\alpha|^2, |\beta|^2\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n} z^n &= \frac{-rz}{s^2 - (r^2 + 2s)z + z^2},\\ \sum_{n=0}^{\infty} H_{-2n} z^n &= \frac{2s^2 - (r^2 + 2s)z}{s^2 - (r^2 + 2s)z + z^2}.\end{aligned}$$

(f) $(m = -2, j = 1, |z| < \min\{|\alpha|^2, |\beta|^2\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n+1} z^n &= \frac{s^2 - (r^2 + s)z}{s^2 - (r^2 + 2s)z + z^2},\\ \sum_{n=0}^{\infty} H_{-2n+1} z^n &= \frac{rs^2 - r(r^2 + 3s)z}{s^2 - (r^2 + 2s)z + z^2}.\end{aligned}$$

Proof. Use Corollay 5.1 (or Corollary 5.2). □

6. Special Cases of Generating Function of Generalized Fibonacci Polynomials

In this section, we present special cases of the ordinary generating function of generalized Fibonacci polynomials.

6.1. Generating Function of Generalized Fibonacci Numbers

In this subsection, we consider the case $r = 1, s = 1$. A generalized Fibonacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = W_{n-1} + W_{n-2}, \tag{27}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (27) holds for all integer n. The Binet formula of generalized Fibonacci numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n \tag{28}$$

where α and β are the roots of the quadratic equation $x^2 - x - 1 = 0$. Moreover

$$\alpha = \frac{1+\sqrt{5}}{2}$$
$$\beta = \frac{1-\sqrt{5}}{2}$$

So

$$W_n = \frac{(W_1 - \beta W_0)\left(\frac{1+\sqrt{5}}{2}\right)^n - (W_1 - \alpha W_0)\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Now, we define two special cases of the sequence $\{W_n\}$. Fibonacci sequence $\{F_n\}_{n\geq 0}$ and Lucas sequence $\{L_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1, \tag{29}$$
$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, L_1 = 1, \tag{30}$$

The sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$F_{-n} = F_{-(n-1)} + F_{-(n-2)},$$
$$L_{-n} = L_{-(n-1)} + L_{-(n-2)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (29)-(30) hold for all integer n.For all integers n, Fibonacci and Lucas numbers can be expressed using Binet's formulas as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$L_n = \alpha^n + \beta^n,$$

respectively. Note that here, $G_n = F_n$ and $H_n = L_n$.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j} z^n$ of the generalized Fibonacci numbers.

Lemma 6.1.
Assume that $|z| < \min\{\left|\frac{1+\sqrt{5}}{2}\right|^{-m}, \left|\frac{1-\sqrt{5}}{2}\right|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j} z^n$ *is the ordinary generating function of the generalized Fibonacci numbers* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j} z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j} z^n = \frac{\rho_1}{\rho_2} = \frac{z(F_m W_{j+1} - F_{m+1} W_j) + W_j}{z^2(-1)^m + z(-1)L_m + 1}$$
$$= \frac{z(F_m W_{j+1} - F_{m+1} W_j) + W_j}{z^2(-1)^m + z(F_m - 2F_{m+1}) + 1}$$

where
$\rho_1 = z((-W_0 W_{j+1} + (-W_1 + W_0)W_j)W_{m+1} + (W_1 W_{j+1} + W_0 W_j)W_m) + (W_1^2 - W_0^2 - W_0 W_1)W_j,$
$\rho_2 = z^2(W_{m+1}^2 - W_m^2 - W_m W_{m+1}) + z((-2W_1 + W_0)W_{m+1} + (W_1 + 2W_0)W_m) + (W_1^2 - W_0^2 - W_0 W_1).$

Proof. Set $r = 1, s = 1, G_n = F_n$ and $H_n = L_n$ in Lemma 5.1. □
Now, we consider special cases of the last Lemma.

Corollary 6.1.
The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:

(a) $(m = 1, j = 0, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-1} \simeq 0.618033)$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - W_0)z}{1 - z - z^2}.$$

(b) $(m = 2, j = 0, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-2} \simeq 0.381966)$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{W_0 + (W_1 - 2W_0)z}{1 - 3z + z^2}.$$

(c) $(m = 2, j = 1, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-2} \simeq 0.381966)$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{W_1 - (W_1 - W_0)z}{1 - 3z + z^2}.$$

(d) $(m = -1, j = 0, |z| < \left|\frac{1-\sqrt{5}}{2}\right| \simeq 0.618033)$.

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{W_0 + W_1 z}{1 + z - z^2}.$$

(e) $(m = -2, j = 0, |z| < \left|\frac{1-\sqrt{5}}{2}\right|^{2} \simeq 0.381966)$.

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{W_0 - (W_1 + W_0)z}{1 - 3z + z^2}.$$

(f) $(m = -2, j = 1, |z| < \left|\frac{1-\sqrt{5}}{2}\right|^{2} \simeq 0.381966)$.

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{W_1 - (2W_1 + W_0)z}{1 - 3z + z^2}.$$

The last Lemma gives the following results as particular examples (generating functions of Fibonacci and Fibonacci-Lucas numbers).

Corollary 6.2.
Assume that $|z| < \min\{\left|\frac{1+\sqrt{5}}{2}\right|^{-m}, \left|\frac{1-\sqrt{5}}{2}\right|^{-m}\}$. Generating functions of Fibonacci and Fibonacci-Lucas numbers are given, respectively, as follows:

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} F_{mn+j} z^n &= \frac{z(F_m F_{j+1} - F_{m+1} F_j) + F_j}{z^2(F_{m+1}^2 - F_m^2 - F_m F_{m+1}) + z(F_m - 2F_{m+1}) + 1} \\ &= \frac{z(F_m F_{j+1} - F_{m+1} F_j) + F_j}{z^2(-1)^m + z(-1)L_m + 1} \\ &= \frac{z(F_m F_{j+1} - F_{m+1} F_j) + F_j}{z^2(-1)^m + z(F_m - 2F_{m+1}) + 1}.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{n=0}^{\infty} L_{mn+j} z^n &= \frac{z((-2L_{j+1} + L_j)L_{m+1} + (L_{j+1} + 2L_j)L_m) - 5L_j}{z^2(L_{m+1}^2 - L_m^2 - L_m L_{m+1}) + 5zL_m - 5} \\ &= \frac{z(F_m L_{j+1} - F_{m+1} L_j) + L_j}{z^2(-1)^m + z(-1)L_m + 1} \\ &= \frac{z(F_m L_{j+1} - F_{m+1} L_j) + L_j}{z^2(-1)^m + z(F_m - 2F_{m+1}) + 1}.\end{aligned}$$

Now, we consider special cases of the last two corollaries.

Corollary 6.3.
The ordinary generating functions of the sequences F_n, F_{2n}, F_{2n+1}, F_{-n}, F_{-2n}, F_{-2n+1} and L_n, L_{2n}, L_{2n+1}, L_{-n}, L_{-2n}, L_{-2n+1} are given as follows:

(a) $\left(m = 1, j = 0, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-1} \simeq 0.618033\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} F_n z^n &= \frac{z}{1-z-z^2}, \\ \sum_{n=0}^{\infty} L_n z^n &= \frac{2-z}{1-z-z^2}. \end{aligned}$$

(b) $\left(m = 2, j = 0, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-2} \simeq 0.381966\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} F_{2n} z^n &= \frac{z}{1-3z+z^2}, \\ \sum_{n=0}^{\infty} L_{2n} z^n &= \frac{2-3z}{1-3z+z^2}. \end{aligned}$$

(c) $\left(m = 2, j = 1, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-2} \simeq 0.381966\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} F_{2n+1} z^n &= \frac{1-z}{1-3z+z^2}, \\ \sum_{n=0}^{\infty} L_{2n+1} z^n &= \frac{1+z}{1-3z+z^2}. \end{aligned}$$

(d) $\left(m = -1, j = 0, |z| < \left|\frac{1-\sqrt{5}}{2}\right| \simeq 0.618033\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} F_{-n} z^n &= \frac{z}{1+z-z^2}, \\ \sum_{n=0}^{\infty} L_{-n} z^n &= \frac{2+z}{1+z-z^2}. \end{aligned}$$

(e) $\left(m = -2, j = 0, |z| < \left|\frac{1-\sqrt{5}}{2}\right|^{2} \simeq 0.381966\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} F_{-2n} z^n &= \frac{-z}{1-3z+z^2}, \\ \sum_{n=0}^{\infty} L_{-2n} z^n &= \frac{2-3z}{1-3z+z^2}. \end{aligned}$$

(f) $\left(m = -2, j = 1, |z| < \left|\frac{1-\sqrt{5}}{2}\right|^{2} \simeq 0.381966\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} F_{-2n+1} z^n &= \frac{1-2z}{1-3z+z^2}, \\ \sum_{n=0}^{\infty} L_{-2n+1} z^n &= \frac{1-4z}{1-3z+z^2}. \end{aligned}$$

From the last corollary, we obtain the following results for Fibonacci and Fibonacci-Lucas numbers.

Corollary 6.4.
Infinite sums of F_n, F_{2n}, F_{2n+1}, F_{-n}, F_{-2n}, F_{-2n+1} and L_n, L_{2n}, L_{2n+1}, L_{-n}, L_{-2n}, L_{-2n+1} are given as follows:

(a) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{F_n}{2^n} = 2,$$
$$\sum_{n=0}^{\infty} \frac{L_n}{2^n} = 6.$$

(b) $z = \frac{1}{3}$.

$$\sum_{n=0}^{\infty} \frac{F_{2n}}{3^n} = 3,$$
$$\sum_{n=0}^{\infty} \frac{L_{2n}}{3^n} = 9.$$

(c) $z = \frac{1}{3}$.

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}}{3^n} = 6,$$
$$\sum_{n=0}^{\infty} \frac{L_{2n+1}}{3^n} = 12.$$

(d) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{F_{-n}}{2^n} = \frac{2}{5},$$
$$\sum_{n=0}^{\infty} \frac{L_{-n}}{2^n} = 2.$$

(e) $z = \frac{1}{3}$.

$$\sum_{n=0}^{\infty} \frac{F_{-2n}}{3^n} = -3,$$
$$\sum_{n=0}^{\infty} \frac{L_{-2n}}{3^n} = 9.$$

(f) $z = \frac{1}{3}$.

$$\sum_{n=0}^{\infty} \frac{F_{-2n+1}}{3^n} = 3,$$
$$\sum_{n=0}^{\infty} \frac{L_{-2n+1}}{3^n} = -3.$$

6.2. Generating Function of Generalized Pell Numbers

In this subsection, we consider the case $r = 2, s = 1$. A generalized Pell sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 2W_{n-1} + W_{n-2}, \tag{31}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -2W_{-(n-1)} + W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (31) holds for all integer n.

The Binet formula of generalized Pell numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n \tag{32}$$

where α and β are the roots of the quadratic equation $x^2-2x-1=0$. Moreover

$$\begin{aligned}\alpha &= 1+\sqrt{2},\\ \beta &= 1-\sqrt{2}.\end{aligned}$$

So

$$W_n=\frac{(W_1-\beta W_0)(1+\sqrt{2})^n-(W_1-\alpha W_0)(1-\sqrt{2})^n}{2\sqrt{2}}.$$

Now, we define two special cases of the sequence $\{W_n\}$. Pell sequence $\{P_n\}_{n\geq 0}$ and Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$P_n = 2P_{n-1}+P_{n-2}, \quad P_0=1, P_1=0, \tag{33}$$

$$Q_n = 2Q_{n-1}+Q_{n-2}, \quad Q_0=2, Q_1=2, \tag{34}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}P_{-n} &= -2P_{-(n-1)}+P_{-(n-2)}\\ Q_{-n} &= -2Q_{-(n-1)}+Q_{-(n-2)}\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (33)-(34) hold for all integer n.

For all integers n, Pell and Pell-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned}P_n &= \frac{\alpha^n-\beta^n}{\alpha-\beta},\\ Q_n &= \alpha^n+\beta^n,\end{aligned}$$

respectively. Here, $G_n=P_n$ and $H_n=Q_n$.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of the generalized Pell numbers $\{W_{mn+j}\}$.

Lemma 6.2.

Assume that $|z|<\min\{\left|1+\sqrt{2}\right|^{-m},\left|1-\sqrt{2}\right|^{-m}\}$. Suppose that $f_{W_{mn+j}}(z)=\sum_{n=0}^{\infty} W_{mn+j}z^n$ is the ordinary generating function of the generalized Pell numbers $\{W_{mn+j}\}$. Then, $\sum_{n=0}^{\infty} W_{mn+j}z^n$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} W_{mn+j}z^n &= \frac{\rho_1}{\rho_2}\\ &= \frac{z(P_mW_{j+1}-P_{m+1}W_j)+W_j}{z^2(-1)^m+z(-1)Q_m+1}\\ &= \frac{z(P_mW_{j+1}-P_{m+1}W_j)+W_j}{z^2(-1)^m+z(2P_m-2P_{m+1})+1}\end{aligned}$$

where

$\rho_1=z((-W_0W_{j+1}+(-W_1+2W_0)W_j)W_{m+1}+(W_1W_{j+1}+W_0W_j)W_m)+(W_1^2-W_0^2-2W_0W_1)W_j,$
$\rho_2=z^2(W_{m+1}^2-W_m^2-2W_mW_{m+1})+z((-2W_1+2W_0)W_{m+1}+(2W_1+2W_0)W_m)+(W_1^2-W_0^2-2W_0W_1).$

Proof. Set $r=2, s=1, G_n=P_n$ and $H_n=Q_n$ in Lemma 5.1. □

Now, we consider special cases of the last Lemma.

Corollary 6.5.

The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z|<\left|1+\sqrt{2}\right|^{-1}\simeq 0.414213)$.

$$\sum_{n=0}^{\infty} W_nz^n=\frac{W_0+(W_1-2W_0)z}{1-2z-z^2}.$$

(b) $(m=2, j=0, |z|<\left|1+\sqrt{2}\right|^{-2}\simeq 0.171572)$.

$$\sum_{n=0}^{\infty} W_{2n}z^n=\frac{W_0+(rW_1-5W_0)z}{1-6z+z^2}.$$

(c) $(m=2, j=1, |z|<\left|1+\sqrt{2}\right|^{-2}\simeq 0.171572)$.

$$\sum_{n=0}^{\infty} W_{2n+1}z^{n}=\frac{W_{1}-(W_{1}-2W_{0})z}{1-6z+z^{2}}.$$

(d) $(m=-1, j=0, |z|<\left|1-\sqrt{2}\right|\simeq 0.414213)$.

$$\sum_{n=0}^{\infty} W_{-n}z^{n}=\frac{W_{0}+W_{1}z}{1+2z-z^{2}}$$

(e) $(m=-2, j=0, |z|<\left|1-\sqrt{2}\right|^{2}\simeq 0.171572)$.

$$\sum_{n=0}^{\infty} W_{-2n}z^{n}=\frac{W_{0}-(2W_{1}+W_{0})z}{1-6z+z^{2}}.$$

(f) $(m=-2, j=1, |z|<\left|1-\sqrt{2}\right|^{2}\simeq 0.171572)$.

$$\sum_{n=0}^{\infty} W_{-2n+1}z^{n}=\frac{W_{1}-(5W_{1}+2W_{0})z}{1-6z+z^{2}}.$$

The last Lemma gives the following results as particular examples (generating functions of Pell and Pell-Lucas numbers).

Corollary 6.6.
Assume that $|z|<\min\{\left|1+\sqrt{2}\right|^{-m},\left|1-\sqrt{2}\right|^{-m}\}$. *Generating functions of Pell and Pell-Lucas numbers are given, respectively, as follows:*

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} P_{mn+j}z^{n} &= \frac{z(P_{m}P_{j+1}-P_{m+1}P_{j})+P_{j}}{z^{2}(P_{m+1}^{2}-P_{m}^{2}-2P_{m}P_{m+1})+z(2P_{m}-2P_{m+1})+1}\\ &= \frac{z(P_{m}P_{j+1}-P_{m+1}P_{j})+P_{j}}{z^{2}(-1)^{m}+z(-1)Q_{m}+1}\\ &= \frac{z(P_{m}P_{j+1}-P_{m+1}P_{j})+P_{j}}{z^{2}(-1)^{m}+z(2P_{m}-2P_{m+1})+1}.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{mn+j}z^{n} &= \frac{z((-2Q_{j+1}+2Q_{j})Q_{m+1}+(2Q_{j+1}+2Q_{j})Q_{m})-8Q_{j}}{z^{2}(Q_{m+1}^{2}-Q_{m}^{2}-2Q_{m}Q_{m+1})+8zQ_{m}-8}\\ &= \frac{z(P_{m}Q_{j+1}-P_{m+1}Q_{j})+Q_{j}}{z^{2}(-1)^{m}+z(-1)Q_{m}+1}\\ &= \frac{z(P_{m}Q_{j+1}-P_{m+1}Q_{j})+Q_{j}}{z^{2}(-1)^{m}+z(2P_{m}-2P_{m+1})+1}.\end{aligned}$$

Now, we consider special cases of the last two corollaries.

Corollary 6.7.
The ordinary generating functions of the sequences P_n, P_{2n}, P_{2n+1}, P_{-n}, P_{-2n}, P_{-2n+1} *and* Q_n, Q_{2n}, Q_{2n+1}, Q_{-n}, Q_{-2n}, Q_{-2n+1} *are given as follows:*

(a) $(m=1, j=0, |z|<\left|1+\sqrt{2}\right|^{-1}\simeq 0.414213)$.

$$\begin{aligned}\sum_{n=0}^{\infty} P_{n}z^{n} &= \frac{z}{1-2z-z^{2}},\\ \sum_{n=0}^{\infty} Q_{n}z^{n} &= \frac{2-2z}{1-2z-z^{2}}.\end{aligned}$$

(b) $(m=2, j=0, |z| < \left|1+\sqrt{2}\right|^{-2} \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} P_{2n} z^n = \frac{2z}{1-6z+z^2},$$
$$\sum_{n=0}^{\infty} Q_{2n} z^n = \frac{2-6z}{1-6z+z^2}.$$

(c) $(m=2, j=1, |z| < \left|1+\sqrt{2}\right|^{-2} \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} P_{2n+1} z^n = \frac{1-z}{1-6z+z^2},$$
$$\sum_{n=0}^{\infty} Q_{2n+1} z^n = \frac{2+2z}{1-6z+z^2}.$$

(d) $(m=-1, j=0, |z| < \left|1-\sqrt{2}\right| \simeq 0.414213)$.

$$\sum_{n=0}^{\infty} P_{-n} z^n = \frac{z}{1+2z-z^2},$$
$$\sum_{n=0}^{\infty} Q_{-n} z^n = \frac{2+2z}{1+2z-z^2}.$$

(e) $(m=-2, j=0, |z| < \left|1-\sqrt{2}\right|^{2} \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} P_{-2n} z^n = \frac{-2z}{1-6z+z^2},$$
$$\sum_{n=0}^{\infty} Q_{-2n} z^n = \frac{2-6z}{1-6z+z^2}.$$

(f) $(m=-2, j=1, |z| < \left|1-\sqrt{2}\right|^{2} \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} P_{-2n+1} z^n = \frac{1-5z}{1-6z+z^2},$$
$$\sum_{n=0}^{\infty} Q_{-2n+1} z^n = \frac{2-14z}{1-6z+z^2}.$$

From the last corollary, we obtain the following results for Pell and Pell-Lucas numbers.

Corollary 6.8.
Infinite sums of P_n, P_{2n}, P_{2n+1}, P_{-n}, P_{-2n}, P_{-2n+1} and Q_n, Q_{2n}, Q_{2n+1}, Q_{-n}, Q_{-2n}, Q_{-2n+1} are given as follows:

(a) $z = \frac{1}{3}$.

$$\sum_{n=0}^{\infty} \frac{P_n}{3^n} = \frac{3}{2},$$
$$\sum_{n=0}^{\infty} \frac{Q_n}{3^n} = 6.$$

(b) $z = \frac{1}{6}$.

$$\sum_{n=0}^{\infty} \frac{P_{2n}}{6^n} = 12,$$
$$\sum_{n=0}^{\infty} \frac{Q_{2n}}{6^n} = 36.$$

(c) $z = \frac{1}{6}$.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{P_{2n+1}}{6^n} &= 30, \\ \sum_{n=0}^{\infty} \frac{Q_{2n+1}}{6^n} &= 84. \end{aligned}$$

(d) $z = \frac{1}{3}$.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{P_{-n}}{3^n} &= \frac{3}{14}, \\ \sum_{n=0}^{\infty} \frac{Q_{-n}}{3^n} &= \frac{12}{7}. \end{aligned}$$

(e) $z = \frac{1}{6}$.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{P_{-2n}}{6^n} &= -12, \\ \sum_{n=0}^{\infty} \frac{Q_{-2n}}{6^n} &= 36. \end{aligned}$$

(f) $z = \frac{1}{6}$.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{P_{-2n+1}}{6^n} &= 6 \\ \sum_{n=0}^{\infty} \frac{Q_{-2n+1}}{6^n} &= -12. \end{aligned}$$

6.3. Generating Function of Generalized Jacobsthal Numbers

In this subsection, we consider the case $r = 1, s = 2$. A generalized Jacobsthal sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = W_{n-1} + 2W_{n-2}, \tag{35}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (35) holds for all integer n.

The Binet formula of generalized Jacobsthal numbers can be written as

$$\begin{aligned} W_n &= \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n \\ &= \frac{p_1\alpha^n - p_2\beta^n}{\alpha - \beta} \end{aligned} \tag{36}$$

where α and β are the roots of the quadratic equation $x^2 - x - 2 = 0$ and

$$\begin{aligned} p_1 &= W_1 - \beta W_0 \\ p_2 &= W_1 - \alpha W_0. \end{aligned}$$

Moreover

$$\begin{aligned} \alpha &= 2, \\ \beta &= -1. \end{aligned}$$

So

$$W_n = \frac{(W_1 - \beta W_0) \times 2^n - (W_1 - \alpha W_0) \times (-1)^n}{3}.$$

Now, we define two special cases of the sequence $\{W_n\}$. Jacobsthal sequence $\{J_n\}_{n\geq 0}$ and Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, J_1 = 1, \tag{37}$$
$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2, j_1 = 1, \tag{38}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n} = -\frac{1}{2}J_{-(n-1)} + \frac{1}{2}J_{-(n-2)}$$
$$j_{-n} = -\frac{1}{2}j_{-(n-1)} + \frac{1}{2}j_{-(n-2)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (37)-(38) hold for all integer n.

For all integers n, Jacobsthal and Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{3},$$
$$j_n = \alpha^n + \beta^n,$$

respectively. Here, $G_n = J_n$ and $H_n = j_n$.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j} z^n$ of the generalized Jacobsthal numbers $\{W_{mn+j}\}$.

Lemma 6.3.
Assume that $|z| < \min\{2^{-m}, 1\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j} z^n$ *is the ordinary generating function of the generalized Jacobsthal numbers* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j} z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j} z^n = \frac{\rho_1}{\rho_2} = \frac{z(J_m W_{j+1} - J_{m+1} W_j) + W_j}{z^2(-1)^m 2^m + z(-1)j_m + 1} = \frac{z(J_m W_{j+1} - J_{m+1} W_j) + W_j}{z^2(-1)^m 2^m + z(J_m - 2J_{m+1}) + 1}$$

where

$\rho_1 = z((-W_0 W_{j+1} + (-W_1 + W_0)W_j)W_{m+1} + (W_1 W_{j+1} + 2W_0 W_j)W_m) + (W_1^2 - 2W_0^2 - W_0 W_1)W_j,$
$\rho_2 = z^2(W_{m+1}^2 - 2W_m^2 - W_m W_{m+1}) + z((-2W_1 + W_0)W_{m+1} + (W_1 + 4W_0)W_m) + (W_1^2 - 2W_0^2 - W_0 W_1).$

Proof. Set $r = 1, s = 2, G_n = J_n$ and $H_n = j_n$ in Lemma 5.1. □

Now, we consider special cases of the last Lemma.

Corollary 6.9.
The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < \frac{1}{2})$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - W_0)z}{1 - z - 2z^2}.$$

(b) $(m = 2, j = 0, |z| < \frac{1}{4})$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{W_0 + (W_1 - 3W_0)z}{1 - 5z + 4z^2}.$$

(c) $(m = 2, j = 1, |z| < \frac{1}{4})$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{W_1 - 2(W_1 - W_0)z}{1 - 5z + 4z^2}.$$

(d) $(m = -1, j = 0, |z| < 1)$.

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{2W_0 + W_1 z}{2 + z - z^2}.$$

(e) $(m = -2, j = 0, |z| < 1)$.

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{4W_0 - (W_1 + 2W_0)z}{4 - 5z + z^2}.$$

(f) $(m = -2, j = 1, |z| < 1)$.

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{4W_1 - (3W_1 + 2W_0)z}{4 - 5z + z^2}.$$

The last Lemma gives the following results as particular examples (generating functions of Jacobsthal and Jacobsthal-Lucas numbers).

Corollary 6.10.
Assume that $|z| < \min\{2^{-m}, 1\}$. *Generating functions of Jacobsthal and Jacobsthal-Lucas numbers are given, respectively, as follows:*

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} J_{mn+j} z^n &= \frac{z(J_m J_{j+1} - J_{m+1} J_j) + J_j}{z^2(J_{m+1}^2 - 2J_m^2 - J_m J_{m+1}) + z(J_m - 2J_{m+1}) + 1} \\ &= \frac{z(J_m J_{j+1} - J_{m+1} J_j) + J_j}{z^2(-1)^m 2^m + z(-1) j_m + 1} \\ &= \frac{z(J_m J_{j+1} - J_{m+1} J_j) + J_j}{z^2(-1)^m 2^m + z(J_m - 2J_{m+1}) + 1}.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{n=0}^{\infty} j_{mn+j} z^n &= \frac{z((-2j_{j+1} + j_j) j_{m+1} + (j_{j+1} + 4j_j) j_m) - 9j_j}{z^2(j_{m+1}^2 - 2j_m^2 - j_m j_{m+1}) + 9z j_m - 9} \\ &= \frac{z(J_m j_{j+1} - J_{m+1} j_j) + j_j}{z^2(-1)^m 2^m + z(-1) j_m + 1} \\ &= \frac{z(J_m j_{j+1} - J_{m+1} j_j) + j_j}{z^2(-1)^m 2^m + z(J_m - 2J_{m+1}) + 1}.\end{aligned}$$

Now, we consider special cases of the last two corollaries.

Corollary 6.11.
The ordinary generating functions of the sequences $J_n, J_{2n}, J_{2n+1}, J_{-n}, J_{-2n}, J_{-2n+1}$ *and* $j_n, j_{2n}, j_{2n+1}, j_{-n}, j_{-2n}, j_{-2n+1}$ *are given as follows:*

(a) $(m = 1, j = 0, |z| < \dfrac{1}{2})$.

$$\begin{aligned}\sum_{n=0}^{\infty} J_n z^n &= \frac{z}{1 - z - 2z^2}, \\ \sum_{n=0}^{\infty} j_n z^n &= \frac{2 - z}{1 - z - 2z^2}.\end{aligned}$$

(b) $(m=2, j=0, |z|<\frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} J_{2n}z^n &= \frac{z}{1-5z+4z^2},\\ \sum_{n=0}^{\infty} j_{2n}z^n &= \frac{2-5z}{1-5z+4z^2}.\end{aligned}$$

(c) $(m=2, j=1, |z|<\frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} J_{2n+1}z^n &= \frac{1-2z}{1-5z+4z^2},\\ \sum_{n=0}^{\infty} j_{2n+1}z^n &= = \frac{1+2z}{1-5z+4z^2}.\end{aligned}$$

(d) $(m=-1, j=0, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} J_{-n}z^n &= \frac{z}{2+z-z^2},\\ \sum_{n=0}^{\infty} j_{-n}z^n &= \frac{4+z}{2+z-z^2}.\end{aligned}$$

(e) $(m=-2, j=0, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} J_{-2n}z^n &= \frac{-z}{4-5z+z^2},\\ \sum_{n=0}^{\infty} j_{-2n}z^n &= \frac{8-5z}{4-5z+z^2}.\end{aligned}$$

(f) $(m=-2, j=1, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} J_{-2n+1}z^n &= \frac{4-3z}{4-5z+z^2},\\ \sum_{n=0}^{\infty} j_{-2n+1}z^n &= \frac{4-7z}{4-5z+z^2}.\end{aligned}$$

From the last corollary, we obtain the following results for Jacobsthal and Jacobsthal-Lucas numbers.

Corollary 6.12.
Infinite sums of $J_n, J_{2n}, J_{2n+1}, J_{-n}, J_{-2n}, J_{-2n+1}$ and $j_n, j_{2n}, j_{2n+1}, j_{-n}, j_{-2n}, j_{-2n+1}$ are given as follows:

(a) $z=\frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{J_n}{3^n} &= \frac{3}{4},\\ \sum_{n=0}^{\infty} \frac{j_n}{3^n} &= \frac{15}{4}.\end{aligned}$$

(b) $z=\frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{J_{2n}}{5^n} &= \frac{5}{4},\\ \sum_{n=0}^{\infty} \frac{j_{2n}}{5^n} &= \frac{25}{4}.\end{aligned}$$

(c) $z=\frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{J_{2n+1}}{5^n} &= \frac{15}{4},\\ \sum_{n=0}^{\infty}\frac{j_{2n+1}}{5^n} &= \frac{35}{4}.\end{aligned}$$

(d) $z=\frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{J_{-n}}{2^n} &= \frac{2}{9},\\ \sum_{n=0}^{\infty}\frac{j_{-n}}{2^n} &= 2.\end{aligned}$$

(e) $z=\frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{J_{-2n}}{2^n} &= -\frac{2}{7},\\ \sum_{n=0}^{\infty}\frac{j_{-2n}}{2^n} &= \frac{22}{7}.\end{aligned}$$

(f) $z=\frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{J_{-2n+1}}{2^n} &= \frac{10}{7},\\ \sum_{n=0}^{\infty}\frac{j_{-2n+1}}{2^n} &= \frac{2}{7}.\end{aligned}$$

6.4. Generating Function of Generalized Mersenne Numbers

In this subsection, we consider the case $r=3, s=-2$. A generalized Mersenne sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 3W_{n-1}-2W_{n-2} \tag{39}$$

with the initial values $W_0=c_0, W_1=c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=\frac{3}{2}W_{-(n-1)}-\frac{1}{2}W_{-(n-2)}$$

for $n=1,2,3,....$ Therefore, recurrence (39) holds for all integer n. For more information on generalized Mersenne numbers, see Soykan [12].

The Binet formula of generalized Mersenne numbers can be written as

$$W_n=\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n$$

where α and β are the roots of the quadratic equation $x^2-3x+2=0$. Moreover

$$\begin{aligned}\alpha &= 2\\ \beta &= 1\end{aligned}$$

So

$$W_n=(W_1-W_0)2^n-(W_1-2W_0). \tag{40}$$

Now, we define two special cases of the sequence $\{W_n\}$. Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$M_n = 3M_{n-1}-2M_{n-2}, \quad M_0=0, M_1=1, \tag{41}$$
$$H_n = 3H_{n-1}-2H_{n-2}, \quad H_0=2, H_1=3, \tag{42}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)}, \\ H_{-n} &= \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}, \end{aligned}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (41)-(42) hold for all integer n.

For all integers n, Mersenne and Mersenne-Lucas can be expressed using Binet's formulas as

$$\begin{aligned} M_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)} = 2^n - 1, \\ H_n &= \alpha^n + \beta^n = 2^n + 1, \end{aligned}$$

respectively. Here, $G_n = M_n$ and $H_n := H_n$.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j} z^n$ of the generalized Mersenne numbers $\{W_{mn+j}\}$.

Lemma 6.4.
Assume that $|z| < \min\{2^{-m}, 1\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j} z^n$ *is the ordinary generating function of the generalized Mersenne numbers* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j} z^n$ *is given by*

$$\begin{aligned} \sum_{n=0}^{\infty} W_{mn+j} z^n &= \frac{\rho_1}{\rho_2} \\ &= \frac{z(M_m W_{j+1} - M_{m+1} W_j) + W_j}{z^2 2^m + z(-1)H_m + 1} \\ &= \frac{z(M_m W_{j+1} - M_{m+1} W_j) + W_j}{z^2 2^m + z(3M_m - 2M_{m+1}) + 1} \end{aligned}$$

where

$\rho_1 = z((-W_0 W_{j+1} + (-W_1 + 3W_0)W_j)W_{m+1} + (W_1 W_{j+1} - 2W_0 W_j)W_m) + (W_1^2 + 2W_0^2 - 3W_0 W_1)W_j,$
$\rho_2 = z^2(W_{m+1}^2 + 2W_m^2 - 3W_m W_{m+1}) + z((-2W_1 + 3W_0)W_{m+1} + (3W_1 - 4W_0)W_m) + (W_1^2 + 2W_0^2 - 3W_0 W_1).$

Proof. Set $r = 3, s = -2, G_n = M_n$ and $H_n := H_n$ in Lemma 5.1. □

Now, we consider special cases of the last Lemma.

Corollary 6.13.
The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < \frac{1}{2})$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 3W_0)z}{1 - 3z + 2z^2}.$$

(b) $(m = 2, j = 0, |z| < \frac{1}{4})$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{W_0 + (3W_1 - 7W_0)z}{1 - 5z + 4z^2}.$$

(c) $(m = 2, j = 1, |z| < \frac{1}{4})$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{W_1 + 2(W_1 - 3W_0)z}{1 - 5z + 4z^2}.$$

(d) $(m = -1, j = 0, |z| < 1)$.

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{-2W_0 + W_1 z}{-2 + 3z - z^2}.$$

(e) $(m=-2, j=0, |z|<1)$.

$$\sum_{n=0}^{\infty} W_{-2n}z^n = \frac{4W_0-(3W_1-2W_0)z}{4-5z+z^2}.$$

(f) $(m=-2, j=1, |z|<1)$.

$$\sum_{n=0}^{\infty} W_{-2n+1}z^n = \frac{4W_1-(7W_1-6W_0)z}{4-5z+z^2}.$$

The last Lemma gives the following results as particular examples (generating functions of Mersenne and Mersenne-Lucas numbers).

Corollary 6.14.
Assume that $|z|<\min\{2^{-m},1\}$. *Generating functions of Mersenne and Mersenne-Lucas numbers are given, respectively, as follows:*

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} M_{mn+j}z^n &= \frac{z(M_mM_{j+1}-M_{m+1}M_j)+M_j}{z^2(M_{m+1}^2+2M_m^2-3M_mM_{m+1})+z(3M_m-2M_{m+1})+1}\\ &= \frac{z(M_mM_{j+1}-M_{m+1}M_j)+M_j}{z^22^m+z(-1)H_m+1}\\ &= \frac{z(M_mM_{j+1}-M_{m+1}M_j)+M_j}{z^22^m+z(3M_m-2M_{m+1})+1}.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{n=0}^{\infty} H_{mn+j}z^n &= \frac{z((-2H_{j+1}+3H_j)H_{m+1}+(3H_{j+1}-4H_j)H_m)-H_j}{z^2(H_{m+1}^2+2H_m^2-3H_mH_{m+1})+zH_m-1}\\ &= \frac{z(M_mH_{j+1}-M_{m+1}H_j)+H_j}{z^22^m+z(-1)H_m+1}\\ &= \frac{z(M_mH_{j+1}-M_{m+1}H_j)+H_j}{z^22^m+z(3M_m-2M_{m+1})+1}.\end{aligned}$$

Now, we consider special cases of the last two corollaries.

Corollary 6.15.
The ordinary generating functions of the sequences M_n, M_{2n}, M_{2n+1}, M_{-n}, M_{-2n}, M_{-2n+1} *and* H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} *are given as follows:*

(a) $(m=1, j=0, |z|<\frac{1}{2})$.

$$\begin{aligned}\sum_{n=0}^{\infty} M_nz^n &= \frac{z}{1-3z+2z^2},\\ \sum_{n=0}^{\infty} H_nz^n &= \frac{2-3z}{1-3z+2z^2}.\end{aligned}$$

(b) $(m=2, j=0, |z|<\frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} M_{2n}z^n &= \frac{3z}{1-5z+4z^2},\\ \sum_{n=0}^{\infty} H_{2n}z^n &= \frac{2-5z}{1-5z+4z^2}.\end{aligned}$$

(c) $(m=2, j=1, |z|<\frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} M_{2n+1}z^n &= = \frac{1+2z}{1-5z+4z^2},\\ \sum_{n=0}^{\infty} H_{2n+1}z^n &= = \frac{3-6z}{1-5z+4z^2}.\end{aligned}$$

(d) $((m=-1, j=0, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} M_{-n}z^n &= \frac{z}{-2+3z-z^2},\\ \sum_{n=0}^{\infty} H_{-n}z^n &= \frac{-4+3z}{-2+3z-z^2}.\end{aligned}$$

(e) $(m=-2, j=0, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} M_{-2n}z^n &= \frac{-3z}{4-5z+z^2},\\ \sum_{n=0}^{\infty} H_{-2n}z^n &= \frac{8-5z}{4-5z+z^2}.\end{aligned}$$

(f) $(m=-2, j=1, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} M_{-2n+1}z^n &= \frac{4-7z}{4-5z+z^2},\\ \sum_{n=0}^{\infty} H_{-2n+1}z^n &= \frac{12-9z}{4-5z+z^2}.\end{aligned}$$

From the last corollary, we obtain the following results for Mersenne and Mersenne-Lucas numbers.

Corollary 6.16.
Infinite sums of M_n, M_{2n}, M_{2n+1}, M_{-n}, M_{-2n}, M_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} are given as follows:

(a) $z=\frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{M_n}{3^n} &= \frac{3}{2},\\ \sum_{n=0}^{\infty} \frac{H_n}{3^n} &= \frac{9}{2}.\end{aligned}$$

(b) $z=\frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{M_{2n}}{5^n} &= \frac{15}{4},\\ \sum_{n=0}^{\infty} \frac{H_{2n}}{5^n} &= \frac{25}{4}.\end{aligned}$$

(c) $z=\frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{M_{2n+1}}{5^n} &= \frac{35}{4},\\ \sum_{n=0}^{\infty} \frac{H_{2n+1}}{5^n} &= \frac{45}{4}.\end{aligned}$$

(d) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{M_{-n}}{2^n} &= -\frac{2}{3}, \\ \sum_{n=0}^{\infty} \frac{H_{-n}}{2^n} &= \frac{10}{3}.\end{aligned}$$

(e) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{M_{-2n}}{2^n} &= -\frac{6}{7}, \\ \sum_{n=0}^{\infty} \frac{H_{-2n}}{2^n} &= \frac{22}{7}.\end{aligned}$$

(f) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{M_{-2n+1}}{2^n} &= \frac{2}{7}, \\ \sum_{n=0}^{\infty} \frac{H_{-2n+1}}{2^n} &= \frac{30}{7}.\end{aligned}$$

6.5. Generating Function of Generalized balancing Numbers

In this subsection, we consider the case $r = 6, s = -1$. A generalized balancing sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 6W_{n-1} - W_{n-2} \tag{43}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 6W_{-(n-1)} - W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (43) holds for all integer n. For more information on generalized balancing numbers, see Soykan [13].

The Binet formula of generalized balancing numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n$$

where α and β are the roots of the quadratic equation $x^2 - 6x + 1 = 0$.Moreover

$$\begin{aligned}\alpha &= 3 + 2\sqrt{2}, \\ \beta &= 3 - 2\sqrt{2}.\end{aligned}$$

So

$$W_n = \frac{W_1 - (3 - 2\sqrt{2})W_0}{4\sqrt{2}}(3 + 2\sqrt{2})^n - \frac{W_1 - (3 + 2\sqrt{2})W_0}{4\sqrt{2}}(3 - 2\sqrt{2})^n. \tag{44}$$

Now, we define three special cases of the sequence $\{W_n\}$. balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$\begin{aligned}B_n &= 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1, &(45)\\ H_n &= 6H_{n-1} - H_{n-2}, \quad H_0 = 2, H_1 = 6, &(46)\\ C_n &= 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3. &(47)\end{aligned}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)} - B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)} - H_{-(n-2)}, \\ C_{-n} &= 6C_{-(n-1)} - C_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (45)-(47) hold for all integer n.

For all integers n, balancing, modified Lucas-balancing and Lucas-balancing numbers can be expressed using Binet's formulas as

$$\begin{aligned} B_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}, \\ H_n &= \alpha^n + \beta^n, \\ C_n &= \frac{\alpha^n + \beta^n}{2}, \end{aligned}$$

respectively. Here, $G_n = B_n$ and $H_n := H_n$.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j} z^n$ of the generalized balancing numbers $\{W_{mn+j}\}$.

Lemma 6.5.
Assume that $|z| < \min\{|3+2\sqrt{2}|^{-m}, |3-2\sqrt{2}|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j} z^n$ *is the ordinary generating function of the generalized balancing numbers* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j} z^n$ *is given by*

$$\begin{aligned} \sum_{n=0}^{\infty} W_{mn+j} z^n &= \frac{\rho_1}{\rho_2} \\ &= \frac{z(B_m W_{j+1} - B_{m+1} W_j) + W_j}{z^2 + z(-1)H_m + 1} \\ &= \frac{z(B_m W_{j+1} - B_{m+1} W_j) + W_j}{z^2 + z(6B_m - 2B_{m+1}) + 1} \end{aligned}$$

where
$\rho_1 = z((-W_0 W_{j+1} + (-W_1 + 6W_0) W_j) W_{m+1} + (W_1 W_{j+1} - W_0 W_j) W_m) + (W_1^2 + W_0^2 - 6W_0 W_1) W_j,$
$\rho_2 = z^2(W_{m+1}^2 + W_m^2 - 6W_m W_{m+1}) + z((-2W_1 + 6W_0) W_{m+1} + (6W_1 - 2W_0) W_m) + (W_1^2 + W_0^2 - 6W_0 W_1).$

Proof. Set $r = 6, s = -1, G_n = B_n$ and $H_n := H_n$ in Lemma 5.1. □

Now, we consider special cases of the last Lemma.

Corollary 6.17.
The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < |3+2\sqrt{2}|^{-1} \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 6W_0) z}{1 - 6z + z^2}.$$

(b) $(m = 2, j = 0, |z| < |3+2\sqrt{2}|^{-2} \simeq 0.029437)$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{W_0 + (6W_1 - 35W_0) z}{1 - 34z + z^2}.$$

(c) $(m = 2, j = 1, |z| < |3+2\sqrt{2}|^{-2} \simeq 0.029437)$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{W_1 + (W_1 - 6W_0) z}{1 - 34z + z^2}.$$

(d) $(m = -1, j = 0, |z| < |3-2\sqrt{2}| \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{-W_0 + W_1 z}{-1 + 6z - z^2}.$$

(e) $(m=-2, j=0, |z|<\left|3-2\sqrt{2}\right|^{2}\simeq 0.029437)$.

$$\sum_{n=0}^{\infty} W_{-2n}z^{n}=\frac{W_{0}-(6W_{1}-W_{0})z}{1-34z+z^{2}}.$$

(f) $(m=-2, j=1, |z|<\left|3-2\sqrt{2}\right|^{2}\simeq 0.029437)$.

$$\sum_{n=0}^{\infty} W_{-2n+1}z^{n}=\frac{W_{1}-(35W_{1}-6W_{0})z}{1-34z+z^{2}}.$$

The last Lemma gives the following results as particular examples (generating functions of balancing, modified Lucas balancing, Lucas-balancing numbers).

Corollary 6.18.
Assume that $|z|<\min\{\left|3+2\sqrt{2}\right|^{-m},\left|3-2\sqrt{2}\right|^{-m}\}$. *Generating functions of balancing, modified Lucas balancing, Lucas-balancing numbers are given, respectively, as follows:*

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} B_{mn+j}z^{n} &= \frac{z(B_{m}B_{j+1}-B_{m+1}B_{j})+B_{j}}{z^{2}(B_{m+1}^{2}+B_{m}^{2}-6B_{m}B_{m+1})+z(6B_{m}-2B_{m+1})+1}\\ &= \frac{z(B_{m}B_{j+1}-B_{m+1}B_{j})+B_{j}}{z^{2}+z(-1)H_{m}+1}\\ &= \frac{z(B_{m}B_{j+1}-B_{m+1}B_{j})+B_{j}}{z^{2}+z(6B_{m}-2B_{m+1})+1}.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{n=0}^{\infty} H_{mn+j}z^{n} &= \frac{z((-2H_{j+1}+6H_{j})H_{m+1}+(6H_{j+1}-2H_{j})H_{m})-32H_{j}}{z^{2}(H_{m+1}^{2}+H_{m}^{2}-6H_{m}H_{m+1})+32zH_{m}-32}\\ &= \frac{z(B_{m}H_{j+1}-B_{m+1}H_{j})+H_{j}}{z^{2}+z(-1)H_{m}+1}\\ &= \frac{z(B_{m}H_{j+1}-B_{m+1}H_{j})+H_{j}}{z^{2}+z(6B_{m}-2B_{m+1})+1}.\end{aligned}$$

(c)

$$\begin{aligned}\sum_{n=0}^{\infty} C_{mn+j}z^{n} &= \frac{z((-C_{j+1}+3C_{j})C_{m+1}+(3C_{j+1}-C_{j})C_{m})-8C_{j}}{z^{2}(C_{m+1}^{2}+C_{m}^{2}-6C_{m}C_{m+1})+16zC_{m}-8}\\ &= \frac{z(B_{m}C_{j+1}-B_{m+1}C_{j})+C_{j}}{z^{2}+z(-1)H_{m}+1}\\ &= \frac{z(B_{m}C_{j+1}-B_{m+1}C_{j})+C_{j}}{z^{2}+z(6B_{m}-2B_{m+1})+1}.\end{aligned}$$

Now, we consider special cases of the last two corollaries.

Corollary 6.19.
The ordinary generating functions of the sequences B_{n}, B_{2n}, B_{2n+1}, B_{-n}, B_{-2n}, B_{-2n+1} *and* H_{n}, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} *and* C_{n}, C_{2n}, C_{2n+1}, C_{-n}, C_{-2n}, C_{-2n+1} *are given as follows:*

(a) $(m=1, j=0, |z|<\left|3+2\sqrt{2}\right|^{-1}\simeq 0.171572)$.

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n}z^{n} &= \frac{z}{1-6z+z^{2}},\\ \sum_{n=0}^{\infty} H_{n}z^{n} &= \frac{2-6z}{1-6z+z^{2}},\\ \sum_{n=0}^{\infty} C_{n}z^{n} &= \frac{1-3z}{1-6z+z^{2}}.\end{aligned}$$

(b) $(m=2, j=0, |z|<\left|3+2\sqrt{2}\right|^{-2}\simeq 0.029437)$.

$$\begin{aligned}\sum_{n=0}^{\infty} B_{2n}z^n &= \frac{6z}{1-34z+z^2},\\ \sum_{n=0}^{\infty} H_{2n}z^n &= \frac{2-34z}{1-34z+z^2},\\ \sum_{n=0}^{\infty} C_{2n}z^n &= \frac{1-17z}{1-34z+z^2}.\end{aligned}$$

(c) $(m=2, j=1, |z|<\left|3+2\sqrt{2}\right|^{-2}\simeq 0.029437)$.

$$\begin{aligned}\sum_{n=0}^{\infty} B_{2n+1}z^n &= \frac{1+z}{1-34z+z^2},\\ \sum_{n=0}^{\infty} H_{2n+1}z^n &= \frac{6-6z}{1-34z+z^2},\\ \sum_{n=0}^{\infty} C_{2n+1}z^n &= \frac{3-3z}{1-34z+z^2}.\end{aligned}$$

(d) $(m=-1, j=0, |z|<\left|3-2\sqrt{2}\right|\simeq 0.171572)$.

$$\begin{aligned}\sum_{n=0}^{\infty} B_{-n}z^n &= \frac{z}{-1+6z-z^2},\\ \sum_{n=0}^{\infty} H_{-n}z^n &= \frac{-2+6z}{-1+6z-z^2},\\ \sum_{n=0}^{\infty} C_{-n}z^n &= \frac{-1+3z}{-1+6z-z^2}.\end{aligned}$$

(e) $(m=-2, j=0, |z|<\left|3-2\sqrt{2}\right|^{2}\simeq 0.029437)$.

$$\begin{aligned}\sum_{n=0}^{\infty} B_{-2n}z^n &= \frac{-6z}{1-34z+z^2},\\ \sum_{n=0}^{\infty} H_{-2n}z^n &= \frac{2-34z}{1-34z+z^2},\\ \sum_{n=0}^{\infty} C_{-2n}z^n &= \frac{1-17z}{1-34z+z^2}.\end{aligned}$$

(f) $(m=-2, j=1, |z|<\left|3-2\sqrt{2}\right|^{2}\simeq 0.029437)$.

$$\begin{aligned}\sum_{n=0}^{\infty} B_{-2n+1}z^n &= \frac{1-35z}{1-34z+z^2},\\ \sum_{n=0}^{\infty} H_{-2n+1}z^n &= \frac{6-198z}{1-34z+z^2},\\ \sum_{n=0}^{\infty} C_{-2n+1}z^n &= \frac{3-99z}{1-34z+z^2}.\end{aligned}$$

From the last corollary, we obtain the following results for balancing, modified Lucas balancing, Lucas-balancing numbers.

Corollary 6.20.

Infinite sums of B_n, B_{2n}, B_{2n+1}, B_{-n}, B_{-2n}, B_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} and C_n, C_{2n}, C_{2n+1}, C_{-n}, C_{-2n}, C_{-2n+1} are given as follows:

(a) $z=\frac{1}{6}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{B_n}{6^{n+1}} &= 1,\\ \sum_{n=0}^{\infty} \frac{H_n}{6^{n+2}} &= 1,\\ \sum_{n=0}^{\infty} \frac{C_n}{6^{n+1}} &= 3.\end{aligned}$$

(b) $z = \frac{1}{36}$.

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{36^n} = \frac{216}{73},$$
$$\sum_{n=0}^{\infty} \frac{H_{2n}}{36^n} = \frac{1368}{73},$$
$$\sum_{n=0}^{\infty} \frac{C_{2n}}{36^n} = \frac{684}{73}.$$

(c) $z = \frac{1}{36}$.

$$\sum_{n=0}^{\infty} \frac{B_{2n+1}}{36^n} = \frac{1332}{73},$$
$$\sum_{n=0}^{\infty} \frac{H_{2n+1}}{36^n} = \frac{7560}{73},$$
$$\sum_{n=0}^{\infty} \frac{C_{2n+1}}{36^n} = \frac{3780}{73}.$$

(d) $z = \frac{1}{6}$.

$$\sum_{n=0}^{\infty} \frac{B_{-n}}{6^{n+1}} = -1,$$
$$\sum_{n=0}^{\infty} \frac{H_{-n}}{6^{n+2}} = 1,$$
$$\sum_{n=0}^{\infty} \frac{C_{-n}}{6^{n+1}} = 3.$$

(e) $z = \frac{1}{36}$.

$$\sum_{n=0}^{\infty} \frac{B_{-2n}}{36^n} = -\frac{216}{73},$$
$$\sum_{n=0}^{\infty} \frac{H_{-2n}}{36^n} = \frac{1368}{73},$$
$$\sum_{n=0}^{\infty} \frac{C_{-2n}}{36^n} = \frac{684}{73}.$$

(f) $z = \frac{1}{36}$.

$$\sum_{n=0}^{\infty} \frac{B_{-2n+1}}{36^n} = \frac{36}{73},$$
$$\sum_{n=0}^{\infty} \frac{H_{-2n+1}}{36^n} = \frac{648}{73},$$
$$\sum_{n=0}^{\infty} \frac{C_{-2n+1}}{36^n} = \frac{324}{73}.$$

6.6. Generating Function of Generalized Oresme Numbers

In this subsection, we consider the case $r = 1, s = -\frac{1}{4}$. A generalized Oresme sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} \tag{48}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 4W_{-(n-1)} - 4W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (48) holds for all integer n. For more information on generalized Oresme numbers, see Soykan [14].

Binet formula of generalized Oresme numbers can be given as

$$W_n = (D_1 + D_2 n)\alpha^n \tag{49}$$

where

$$\begin{array}{rcl} D_1 &=& W_0, \\ D_2 &=& \dfrac{1}{\alpha}(W_1 - \alpha W_0). \end{array}$$

i.e.,

$$W_n = (W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)\, n)\alpha^n$$

Here, $\alpha = \beta = \frac{1}{2}$ are the roots of the quadratic equation

$$x^2 - x + \frac{1}{4} = 0. \tag{50}$$

i.e. the roots of characteristic equation (50) are equal. Note that

$$W_n = (W_0 + 2\left(W_1 - \frac{1}{2}W_0\right) n) \times \frac{1}{2^n}.$$

Now, we define three special cases of the sequence $\{W_n\}$. Modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = G_{n+1} - \frac{1}{4}G_n, \quad G_0 = 0, G_1 = 1, \tag{51}$$

$$H_{n+2} = H_{n+1} - \frac{1}{4}H_n, \quad H_0 = 2, H_1 = 1, \tag{52}$$

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, \quad O_0 = 0, O_1 = \frac{1}{2}. \tag{53}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{array}{rcl} G_{-n} &=& 4G_{-(n-1)} - 4G_{-(n-2)}, \\ H_{-n} &=& 4H_{-(n-1)} - 4H_{-(n-2)}, \\ O_{-n} &=& 4O_{-(n-1)} - 4O_{-(n-2)}, \end{array}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (51)-(53) hold for all integer n.

For all integers n, modified Oresme, Oresme-Lucas and Oresme numbers can be expressed using Binet's formulas as

$$\begin{array}{rcl} G_n &=& n\alpha^{n-1} = \dfrac{n}{2^{n-1}}, \\ H_n &=& 2\alpha^n = \dfrac{1}{2^{n-1}}, \\ O_n &=& n\alpha^n = \dfrac{n}{2^n}, \end{array}$$

respectively. Here, $G_n := G_n$ and $H_n := H_n$.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_{mn+j} z^n$ of the generalized Oresme numbers $\{W_{mn+j}\}$.

Lemma 6.6.
Assume that $|z| < 2^m$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j} z^n$ *is the ordinary generating function of the generalized Oresme numbers* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j} z^n$ *is given by*

$$\begin{aligned}\sum_{n=0}^{\infty} W_{mn+j} z^n &= \frac{\rho_1}{\rho_2} \\ &= \frac{z(G_m W_{j+1} - G_{m+1} W_j) + W_j}{z^2 2^{-2m} + z(-1)H_m + 1} \\ &= \frac{z(G_m W_{j+1} - G_{m+1} W_j) + W_j}{z^2 2^{-2m} + z(G_m - 2G_{m+1}) + 1}\end{aligned}$$

where

$$\rho_1 = z((-W_0 W_{j+1} + (-W_1 + W_0) W_j) W_{m+1} + (W_1 W_{j+1} - \frac{1}{4} W_0 W_j) W_m) + (W_1^2 + \frac{1}{4} W_0^2 - W_0 W_1) W_j,$$

$$\rho_2 = z^2 (W_{m+1}^2 + \frac{1}{4} W_m^2 - W_m W_{m+1}) + z((-2W_1 + W_0) W_{m+1} + (W_1 - \frac{1}{2} W_0) W_m) + (W_1^2 + \frac{1}{4} W_0^2 - W_0 W_1).$$

Proof. Set $r = 1, s = -\frac{1}{4}, G_n := G_n$ and $H_n := H_n$ in Lemma 5.1. □
Now, we consider special cases of the last Lemma.

Corollary 6.21.
The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < 2)$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{4W_0 + 4(W_1 - W_0)z}{4 - 4z + z^2}.$$

(b) $(m = 2, j = 0, |z| < 4)$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{16W_0 + (16W_1 - 12W_0)z}{16 - 8z + z^2}.$$

(c) $(m = 2, j = 1, |z| < 4)$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{16W_1 + 4(W_1 - W_0)z}{16 - 8z + z^2}.$$

(d) $(m = -1, j = 0, |z| < \frac{1}{2})$.

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{-W_0 + 4W_1 z}{-1 + 4z - 4z^2}.$$

(e) $(m = -2, j = 0, |z| < \frac{1}{4})$.

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{W_0 - 4(4W_1 - W_0)z}{1 - 8z + 16z^2}.$$

(f) $(m = -2, j = 1, |z| < \frac{1}{4})$.

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{W_1 - 4(3W_1 - W_0)z}{1 - 8z + 16z^2}.$$

The last Lemma üstteki lemma gives the following results as particular examples (generating functions of modified Oresme, Oresme-Lucas and Oresme numbers).

Corollary 6.22.
Assume that $|z| < 2^m$. *Generating functions of modified Oresme, Oresme-Lucas and Oresme numbers are given, respectively, as follows:*

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} G_{mn+j}z^n &= \frac{z(G_mG_{j+1}-G_{m+1}G_j)+G_j}{z^2(G_{m+1}^2+\frac{1}{4}G_m^2-G_mG_{m+1})+z(G_m-2G_{m+1})+1}\\ &= \frac{z(G_mG_{j+1}-G_{m+1}G_j)+G_j}{z^2 2^{-2m}+z(-1)H_m+1}\\ &= \frac{z(G_mG_{j+1}-G_{m+1}G_j)+G_j}{z^2 2^{-2m}+z(G_m-2G_{m+1})+1}.\end{aligned}$$

(b)

$$\begin{aligned}\sum_{n=0}^{\infty} H_{mn+j}z^n &= \frac{z(G_mH_{j+1}-G_{m+1}H_j)+H_j}{z^2 2^{-2m}+z(-1)H_m+1}\\ &= \frac{z(G_mH_{j+1}-G_{m+1}H_j)+H_j}{z^2 2^{-2m}+z(G_m-2G_{m+1})+1}.\end{aligned}$$

(c)

$$\begin{aligned}\sum_{n=0}^{\infty} O_{mn+j}z^n &= \frac{2(O_mO_{j+1}-O_jO_{m+1})z+O_j}{z^2(4O_{m+1}^2+O_m^2-4O_mO_{m+1})+2\left(O_m-2O_{m+1}\right)z+1}\\ &= \frac{z(G_mO_{j+1}-G_{m+1}O_j)+O_j}{z^2 2^{-2m}+z(-1)H_m+1}\\ &= \frac{z(G_mO_{j+1}-G_{m+1}O_j)+O_j}{z^2 2^{-2m}+z(G_m-2G_{m+1})+1}.\end{aligned}$$

Now, we consider special cases of the last two corollaries.

Corollary 6.23.
The ordinary generating functions of the sequences G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} and O_n, O_{2n}, O_{2n+1}, O_{-n}, O_{-2n}, O_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z|<2)$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_nz^n &= \frac{4z}{4-4z+z^2},\\ \sum_{n=0}^{\infty} H_nz^n &= \frac{8-4z}{4-4z+z^2},\\ \sum_{n=0}^{\infty} O_nz^n &= \frac{2z}{4-4z+z^2}.\end{aligned}$$

(b) $(m=2, j=0, |z|<4)$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n}z^n &= \frac{16z}{16-8z+z^2},\\ \sum_{n=0}^{\infty} H_{2n}z^n &= \frac{32-8z}{16-8z+z^2},\\ \sum_{n=0}^{\infty} O_{2n}z^n &= \frac{8z}{16-8z+z^2},\end{aligned}$$

(c) $(m=2, j=1, |z|<4)$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n+1}z^n &= \frac{16+4z}{16-8z+z^2},\\ \sum_{n=0}^{\infty} H_{2n+1}z^n &= \frac{16-4z}{16-8z+z^2},\\ \sum_{n=0}^{\infty} O_{2n+1}z^n &= \frac{8+2z}{16-8z+z^2}.\end{aligned}$$

(d) $(m = -1, j = 0, |z| < \frac{1}{2})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-n} z^n &= \frac{4z}{-1+4z-4z^2},\\ \sum_{n=0}^{\infty} H_{-n} z^n &= \frac{-2+4z}{-1+4z-4z^2},\\ \sum_{n=0}^{\infty} O_{-n} z^n &= \frac{2z}{-1+4z-4z^2}.\end{aligned}$$

(e) $(m = -2, j = 0, |z| < \frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n} z^n &= \frac{-16z}{1-8z+16z^2},\\ \sum_{n=0}^{\infty} H_{-2n} z^n &= \frac{2-8z}{1-8z+16z^2},\\ \sum_{n=0}^{\infty} O_{-2n} z^n &= \frac{-8z}{1-8z+16z^2}.\end{aligned}$$

(f) $(m = -2, j = 1, |z| < \frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n+1} z^n &= \frac{1-12z}{1-8z+16z^2},\\ \sum_{n=0}^{\infty} H_{-2n+1} z^n &= \frac{1-4z}{1-8z+16z^2},\\ \sum_{n=0}^{\infty} O_{-2n+1} z^n &= \frac{1-12z}{2-16z+32z^2}.\end{aligned}$$

From the last corollary, we obtain the following results for modified Oresme, Oresme-Lucas and Oresme numbers.

Corollary 6.24.
Infinite sums of $G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1}$ *and* $H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1}$ *and* $O_n, O_{2n}, O_{2n+1}, O_{-n}, O_{-2n}, O_{-2n+1}$ *are given as follows:*

(a) $z = 1$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_n &= 4,\\ \sum_{n=0}^{\infty} H_n &= 4,\\ \sum_{n=0}^{\infty} O_n &= 2.\end{aligned}$$

(b) $z = 1$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n} &= \frac{16}{9},\\ \sum_{n=0}^{\infty} H_{2n} &= \frac{8}{3},\\ \sum_{n=0}^{\infty} O_{2n} &= \frac{8}{9}.\end{aligned}$$

(c) $z = 1$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n+1} &= \frac{20}{9},\\ \sum_{n=0}^{\infty} H_{2n+1} &= \frac{4}{3},\\ \sum_{n=0}^{\infty} O_{2n+1} &= \frac{10}{9}.\end{aligned}$$

(d) $z = \frac{1}{4}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{G_{-n}}{4^n} &= -4,\\ \sum_{n=0}^{\infty}\frac{H_{-n}}{4^n} &= 4,\\ \sum_{n=0}^{\infty}\frac{O_{-n}}{4^n} &= -2.\end{aligned}$$

(e) $z = \frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{G_{-2n}}{5^n} &= -80,\\ \sum_{n=0}^{\infty}\frac{H_{-2n}}{5^n} &= 10,\\ \sum_{n=0}^{\infty}\frac{O_{-2n}}{5^n} &= -40.\end{aligned}$$

(f) $z = \frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{G_{-2n+1}}{5^n} &= -35,\\ \sum_{n=0}^{\infty}\frac{H_{-2n+1}}{5^n} &= 5,\\ \sum_{n=0}^{\infty}\frac{O_{-2n+1}}{5^n} &= -\frac{35}{2}.\end{aligned}$$

The results given in the last Corollary can also be obtained by using Binet's formulas. For example, the results of (e) of the last corollary can be computed by using Binets's formulas

$$\begin{aligned}G_n &= n\alpha^{n-1} = \frac{n}{2^{n-1}},\\ H_n &= 2\alpha^n = \frac{1}{2^{n-1}},\\ O_n &= n\alpha^n = \frac{n}{2^n},\end{aligned}$$

as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{G_{-2n}}{5^n} &= \lim_{k\to\infty}\sum_{n=0}^{k}\frac{\left(\frac{-2n}{2^{-2n-1}}\right)}{5^n} = \lim_{k\to\infty}\left(16\times\left(\frac{4}{5}\right)^k(k+5)-80\right) = -80,\\ \sum_{n=0}^{\infty}\frac{H_{-2n}}{5^n} &= \sum_{n=0}^{\infty}\frac{\frac{1}{2^{-2n-1}}}{5^n} = 10,\\ \sum_{n=0}^{\infty}\frac{O_{-2n}}{5^n} &= \sum_{n=0}^{\infty}\frac{\frac{G_{-2n}}{2}}{5^n} = -40.\end{aligned}$$

7. Some Remarks

When we defined the generalized Fibonacci polynomials in (1), we supposed that W_0, W_1 are arbitrary complex (or real) polynomials with real coefficients and r and s are polynomials with real coefficients with $s \neq 0$. However, if we take W_0, W_1, r and s are arbitrary complex or real functions (with real coefficients) and z are arbitrary complex or real number (function), then we can apply to the results obtained in the previous sections (when we check the proofs, we see that proofs work for these W_0, W_1, r, s and z). Now, we present some special cases of W_0, W_1, r, s and z as examples of functions.

7.1. The Case $r = x + \sin x$, $s = x + \cos x$, $x \in \mathbb{R}$

If we set

$$\begin{aligned} r &= x + \sin x, \\ s &= x + \cos x, \\ x &\in \mathbb{R}, \end{aligned}$$

in (1) then we get,

$$W_{n+2} = (x + \sin x)W_{n+1} + (x + \cos x)W_n$$

with $W_0 = a(x)$, $W_1 = b(x)$ and

$$\begin{aligned} G_{n+2} &= (x + \sin x)G_{n+1} + (x + \cos x)G_n, \\ G_0 &= 0, G_1 = 1, \\ H_{n+2} &= (x + \sin x)H_{n+1} + (x + \cos x)H_n, \\ H_0 &= 2, H_1 = x + \sin x. \end{aligned}$$

We can apply to the results of previous sections for the functions r, s, z. For example, for all integers n, we get, by Theorem 3.1 (a) (i),

$$\begin{pmatrix} x + \sin x & x + \cos x \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & (x + \cos x)G_n \\ G_n & (x + \cos x)G_{n-1} \end{pmatrix},$$

for all $x \in \mathbb{R}$.

We now apply to Theorem 4.2 for the case $r = x + \sin x$, $s = x + \cos x$, $x \in \mathbb{R}$.

Theorem 7.1.
For all integers m and j, we have the following sum formulas. If $z^2(-1)(x + \cos x) + z(-1)(x + \sin x) + 1 \neq 0$, i.e., if $z \neq \frac{1}{2(x+\cos x)}(-(x + \sin x) - \sqrt{(x + \sin x)^2 + 4(x + \cos x)})$, $z \neq \frac{1}{2(x+\cos x)}(-(x + \sin x) + \sqrt{(x + \sin x)^2 + 4(x + \cos x)})$ then

(a) $(m = 1, j = 0)$.

$$\sum_{k=0}^{n} z^k W_k = \frac{z^{n+2}(-1)(x + \cos x)W_n + z^{n+1}(-1)W_{n+1} + z(W_1 - rW_0) + W_0}{z^2(-1)(x + \cos x) + z(-1)(x + \sin x) + 1}$$

(b) $(m = 1, j = 0)$.

$$\sum_{k=0}^{n} z^k G_k = \frac{z^{n+2}(-1)(x + \cos x)G_n + z^{n+1}(-1)G_{n+1} + z}{z^2(-1)(x + \cos x) + z(-1)(x + \sin x) + 1}$$

(c) $(m = 1, j = 0)$.

$$\sum_{k=0}^{n} z^k H_k = \frac{z^{n+2}(-1)(x + \cos x)H_n + z^{n+1}(-1)H_{n+1} - z(x + \sin x) + 2}{z^2(-1)(x + \cos x) + z(-1)(x + \sin x) + 1}$$

If we set $z = 1$ in the last theorem, we get the following Corollary.

Corollary 7.1.
For all integers m and j, we have the following sum formulas. If $1 - \cos x - \sin x - 2x \neq 0$, then

(a) $(m = 1, j = 0)$.

$$\sum_{k=0}^{n} W_k = \frac{W_{n+1} + (x + \cos x)W_n - (W_1 - rW_0) - W_0}{-1 + \cos x + \sin x + 2x}.$$

(b) $(m = 1, j = 0)$.

$$\sum_{k=0}^{n} G_k = \frac{G_{n+1} + (x + \cos x)G_n - 1}{-1 + \cos x + \sin x + 2x}.$$

(c) $(m=1, j=0)$.

$$\sum_{k=0}^{n} H_k = \frac{H_{n+1}+(x+\cos x)H_n+(x+\sin x)-2}{-1+\cos x+\sin x+2x}.$$

If we set $z=e^{2ix}=\cos 2x+i\sin 2x$ (for $x\in\mathbb{R}$) in the last theorem, we get the following Corollary.

Corollary 7.2.
For all integers m and j, we have the following sum formulas. If $e^{4ix}(-1)(x+\cos x)+e^{2ix}(-1)(x+\sin x)+1\neq 0$, *then*

(a) $(m=1, j=0)$.

$$\sum_{k=0}^{n} e^{2ikx}W_k = \frac{e^{2i(n+2)x}(-1)(x+\cos x)W_n+e^{2i(n+1)x}(-1)W_{n+1}+e^{2ix}(W_1-rW_0)+W_0}{e^{4ix}(-1)(x+\cos x)+e^{2ix}(-1)(x+\sin x)+1}.$$

(b) $(m=1, j=0)$.

$$\sum_{k=0}^{n} e^{2ikx}G_k = \frac{e^{2i(n+2)x}(-1)(x+\cos x)G_n+e^{2i(n+1)x}(-1)G_{n+1}+e^{2ix}}{e^{4ix}(-1)(x+\cos x)+e^{2ix}(-1)(x+\sin x)+1}.$$

(c) $(m=1, j=0)$.

$$\sum_{k=0}^{n} e^{2ikx}H_k = \frac{e^{2i(n+2)x}(-1)(x+\cos x)H_n+e^{2i(n+1)x}(-1)H_{n+1}-e^{2ix}(x+\sin x)+2}{e^{4ix}(-1)(x+\cos x)+e^{2ix}(-1)(x+\sin x)+1}.$$

7.2. The Case $r=1$, $s=1$

For the case $r=1$, $s=1$,we now apply to Corollary 6.3 for specific z.

Corollary 7.3.
We have the following infinite sum formulas for $F_n, F_{2n}, F_{2n+1}, F_{-n}, F_{-2n}, F_{-2n+1}$ *and* $L_n, L_{2n}, L_{2n+1}, L_{-n}, L_{-2n}, L_{-2n+1}$:

(a) $(m=1, j=0, z=\frac{\cos x}{3}, |z|=\left|\frac{\cos x}{3}\right|<\left|\frac{1+\sqrt{5}}{2}\right|^{-1}\simeq 0.618033)$. *We can define two functions* $f,g:\mathbb{R}\to\mathbb{R}$ *by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\left(\frac{\cos x}{3}\right)^n F_n = \frac{6\cos x}{17-\cos 2x-6\cos x},\\ g(x) &= \sum_{n=0}^{\infty}\left(\frac{\cos x}{3}\right)^n L_n = \frac{36-6\cos x}{17-\cos 2x-6\cos x}.\end{aligned}$$

(b) $(m=2, j=0, z=\frac{\sin x-\cos x}{4}, |z|=\left|\frac{\sin x-\cos x}{4}\right|<\left|\frac{1+\sqrt{5}}{2}\right|^{-2}\simeq 0.381966)$. *We can define two functions* $f,g:\mathbb{R}\to\mathbb{R}$ *by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\left(\frac{\sin x-\cos x}{4}\right)^n F_{2n} = \frac{4(\sin x-\cos x)}{17+12(\cos x-\sin x)-2\cos x\sin x},\\ g(x) &= \sum_{n=0}^{\infty}\left(\frac{\sin x-\cos x}{4}\right)^n L_{2n} = \frac{32+12\cos x-12\sin x}{17+12(\cos x-\sin x)-2\cos x\sin x}.\end{aligned}$$

(c) $(m=2, j=1, z=\frac{2}{7+x^2}, |z|=\left|\frac{2}{7+x^2}\right|<\left|\frac{1+\sqrt{5}}{2}\right|^{-2}\simeq 0.381966)$. *We can define two functions* $f,g:\mathbb{R}\to\mathbb{R}$ *by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\left(\frac{2}{7+x^2}\right)^n F_{2n+1} = \frac{x^4+12x^2+35}{x^4+8x^2+11},\\ g(x) &= \sum_{n=0}^{\infty}\left(\frac{2}{7+x^2}\right)^n L_{2n+1} = \frac{x^4+16x^2+63}{x^4+8x^2+11}.\end{aligned}$$

(d) $(m = -1, j = 0, z = \frac{1}{4+e^{ix}} = \frac{1}{4+\cos x + i\sin x}, |z| = \frac{1}{\sqrt{17+8\cos x}} < \left|\frac{1-\sqrt{5}}{2}\right| \simeq 0.618033)$. *We can define two functions $f, g : \mathbb{R} \to \mathbb{C}$ by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \left(\frac{1}{4+e^{ix}}\right)^n F_{-n} = \frac{4+e^{ix}}{19+e^{2ix}+9e^{ix}}, \\ g(x) &= \sum_{n=0}^{\infty} \left(\frac{1}{4+e^{ix}}\right)^n L_{-n} = \frac{36+2e^{2ix}+17e^{ix}}{19+e^{2ix}+9e^{ix}}. \end{aligned}$$

(e) $(m = -2, j = 0, z = \frac{x}{5+x^2+x^4}, |z| = \left|\frac{x}{5+x^2+x^4}\right| < \left|\frac{1-\sqrt{5}}{2}\right|^2 \simeq 0.381966)$. *We can define two functions $f, g : \mathbb{R} \to \mathbb{R}$ by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \left(\frac{x}{5+x^2+x^4}\right)^n F_{-2n} = \frac{-x(x^4+x^2+5)}{x^8+2x^6-3x^5+11x^4-3x^3+11x^2-15x+25}, \\ g(x) &= \sum_{n=0}^{\infty} \left(\frac{x}{5+x^2+x^4}\right)^n L_{-2n} = \frac{(x^4+x^2+5)(2x^4+2x^2-3x+10)}{x^8+2x^6-3x^5+11x^4-3x^3+11x^2-15x+25}. \end{aligned}$$

(f) $(m = -2, j = 1, z = \frac{\cos x}{5+13x^2}, |z| = \left|\frac{\cos x}{5+13x^2}\right| < \left|\frac{1-\sqrt{5}}{2}\right|^2 \simeq 0.381966)$. *We can define two functions $f, g : \mathbb{R} \to \mathbb{R}$ by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \left(\frac{\cos x}{5+13x^2}\right)^n F_{-2n+1} = \frac{2(13x^2+5)(13x^2+5-2\cos x)}{\cos 2x - 6(13x^2+5)\cos x + 338x^4 + 260x^2 + 51}, \\ g(x) &= \sum_{n=0}^{\infty} \left(\frac{\cos x}{5+13x^2}\right)^n L_{-2n+1} = \frac{2(13x^2+5)(13x^2+5-4\cos x)}{\cos 2x - 6(13x^2+5)\cos x + 338x^4 + 260x^2 + 51}. \end{aligned}$$

References

[1] Andrica, D., Bagdasar, O., Recurrent Sequences Key Results, Applications, and Problems, Springer, 2020.

[2] Djordjević, G.B., Milovanović, G.V., Special Classes of Plynomials, University of Niš, Faculty of Technology, Leskovac, 2014. http://www.mi.sanu.ac.rs/~gvm/Teze/Special%20Classes%20of%20Polynomials.pdf

[3] Frei, G., Binary Lucas and Fibonacci Polynomials, I, Math. Nadir. 96, 83-112, 1980.

[4] Flórez, R., McAnally, N., Mukherjee, A., Identities for the Generalized Fibonacci Polynomial, Integers, 18B, 2018.

[5] He,T.X., Peter J.-S. Shiue, P.J.S., On Sequences of Numbers and Polynomials Defined by Linear Recurrence Relations of Order 2, International Journal of Mathematics and Mathematical Sciences, Volume 2009, Article ID 709386, 21 pages, doi:10.1155/2009/709386.

[6] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[7] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 1 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), Second Edition, John Wiley&Sons, New York, 2018.

[8] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 2 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), John Wiley&Sons, New York, 2019.

[9] Özkan, E., Altun, İ., (2019) Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials, Communications in Algebra, 47(10), 2019. DOI: 10.1080/00927872.2019.1576186

[10] Soykan, Y., On Generalized Fibonacci Polynomials: Horadam Polynomials, Earthline Journal of Mathematical Sciences, 11(1), 23-114, 2023. https://doi.org/10.34198/ejms.11123.23114

[11] Soykan, Y., Generalized Fibonacci Numbers: Sum Formulas, Minel Yayın, 2022. https://www.minelyayin.com/generalized-fibonacci-numbers-sum-formulas-51

[12] Soykan, Y., A Study On Generalized Mersenne Numbers, Journal of Progressive Research in Mathematics, 18(3), 90-112, 2021.

[13] Soykan, Y., A Study on Generalized Balancing Numbers, Asian Journal of Advanced Research and Reports, 15(5), 78-100, 2021. DOI: 10.9734/AJARR/2021/v15i530401

[14] Soykan, Y., Generalized Oresme Numbers, Earthline Journal of Mathematical Sciences, 7(2), 333-367, 2021. https://doi.org/10.34198/ejms.7221.333367

[15] Vajda, S,. Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications, Dover Publications Inc.; 2008.

[16] Wang, J., Some New Results for the (p,q)-Fibonacci and Lucas Polynomials, Advances in Difference Equations, 2014. http://www.advancesindifferenceequations.com/content/2014/1/64

[17] Wang, W., Wang, H., Generalized-Humbert Polynomials via Generalized Fibonacci Polynomials, Applied Mathematics and Computation 307, 204–216, 2017. https://doi.org/10.1016/j.amc.2017.02.050.