Int. J. Adv. Appl. Math. and Mech. 10(4) (2023) 72 – 90 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

# On Generalized Padovan Numbers

Research Article

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**Received 13 March 2023; accepted (in revised version) 24 March 2023**

**Abstract:** In this paper, we investigate the generalized Padovan sequences and we deal with, in detail, four special cases, namely, Padovan, Perrin, Padovan-Perrin and modified Padovan sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

**MSC:** 11B37 • 11B39 • 11B83

**Keywords:** Padavon numbers • Perrin numbers • Padovan-Perrin numbers

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## 1. Introduction

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam numbers and generalized Tribonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; Tribonacci, Tribonacci-Lucas, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers. The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. The ratio of two consecutive Fibonacci numbers converges to the Golden section (ratio), $\alpha_F = \frac{1+\sqrt{5}}{2}$; which appears in modern research, particularly physics of the high energy particles or theoretical physics. Another example, the ratio of two consecutive Tribonacci numbers converges to the Tribonacci ratio, $\alpha_T = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}$. One last example, the ratio of two consecutive Padovan numbers converges to the Plastic ratio, $\alpha_P$ (which is given in (1) below), which have many applications to such as architecture, see [53]. For a short introduction to these three constants, see [62].

Padovan (Cordonnier) numbers, Perrin (Padovan-Lucas) numbers and Van der Laan numbers are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} P_{n+3} &= P_{n+1} + P_n, \quad P_0 = 1, P_1 = 1, P_2 = 1, \\ E_{n+3} &= E_{n+1} + E_n, \quad E_0 = 3, E_1 = 0, E_2 = 2, \\ R_{n+3} &= R_{n+1} + R_n, \quad R_0 = 1, R_1 = 0, R_2 = 1, \text{ or } R_0 = 0, R_1 = 1, R_2 = 0. \end{aligned}$$

For historical background issues on these particular cases of generalized Padovan sequences, see [68].

Edouard Lucas [51] in 1876 introduced the sequence $E_n$ (Perrin sequence, see for example [56]) and the sequence $E_n$ was also discussed by Lucas in 1878 (American Journal of Mathematics, vol 1, page 230ff), who noted that if $p$ is a prime then $p$ divides $E_n$. This is an immediate consequence of Fermat's Little Theorem, and as such is a necessary but

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not sufficient condition for primality (for a proof see [57]). Subsequently (1899) the same sequence was mentioned by R. Perrin [60]. There are some other papers on the sequence $E_n$ after Perrin's works, see [26, 40, 52]. The most extensive (published) treatment of this sequence was given in an excellent paper by Bill Adams and Dan Shanks in [2]. Shanks and Adams referred (called) to this as Perrin's sequence.

Originating and naming of $\{P_n\}$ is rather less straightforward. Termed by Steward [80] (and Broadhurst and Kreimer [9] with the initial condion $P_0 = 0, P_1 = 0, P_2 = 1$) as the Padovan numbers in honour of the contemporary architect Richard Padovan, these numbers seemingly have a more extensive origine. The sequence $\{P_n\}$ seems to have been first discovered in 1924 by a French architecture student, Gérard Cordonnier and independently, the $\{P_n\}$ were rediscovered by Dom Hans van der Laan (see [68]).

The characteristic equation associated with Padovan, Perrin, Van der Laan sequences is $x^3 - x - 1 = 0$ with roots $\alpha, \beta$ and $\gamma$ in which

$$\alpha = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \simeq 1.32471795724 \tag{1}$$

is called plastic number (or plastic ratio or plastic constant or silver number) and

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \lim_{n\to\infty} \frac{E_{n+1}}{E_n} = \alpha.$$

The plastic number is used in art and architecture. Richard Padovan studied on plastic number in Architecture and Mathematics in [58, 59]. The "plastic number" made popular by Richard Padovan. Padovan pointed out that the plastic number was invented by a French architectural student, Gérard Cordonnier, in 1924 and by a Dutch Benedictine monk-architect, Hans van der Laan, in 1928. Pastic number was originally studied by G. Cordonnier in 1924. However, Hans van der Laan was the first who explained how it relates to the human perception of differences in size between three-dimensional objects and demonstrated his discovery in architectural design. Laan's main premise was that the plastic number ratio is "truly aesthetic in the original Greek sense, i.e. that its concern is not 'beauty' but clarity of perception" (see [58]). Cordonnier described applications to architecture, using the name radiant number and in 1958 he gave a lecture tour that illustrated the use of the plastic number in many existing buildings and monuments. Marohnić and Strmećki [53] constructed the Plastic number in a heuristic way, explaining its relation to human perception in three-dimensional space through architectural style of Dom Hans van der Laan. Note that plastic number is a morphic number, see [1] for details. For more details on plastic number, see [5, 38, 79, 85, 87].

Recently, these sequences ($\{P_n\}$ ,$\{E_n\}$ ,$\{R_n\}$ ) have been studied extensively by many authors, see for instance [3, 8, 15, 27, 36, 46, 49, 66, 69, 73, 74, 80, 86, 89]. See also web pages [54–56] for Padovan numbers.

Kaygisiz and Bozkurt [43] defined k sequences of generalized order-k Perrrin numbers. Kaygisiz and Sahin [44] defined generalized Van der Laan and Perrin Polynomials, and generalizations of Van der Laan and Perrin Numbers.

Many researchers have studied matrix representations of number sequences. In [92] and [94], Yilmaz and Taskara developed the matrix sequences that represent Padovan and Perrin numbers. Şahin [64] defined and studied generalized Perrin and Cordonnier matrices using the associated polynomials of Perrin and Cordonnier numbers. Kaygisiz and Sahin [45] calculated terms of associated polynomials of Perrin and Cordonnier numbers by using determinants and permanents of various Hessenberg matrices. In [90] authors gave matrix representation of Perrin sequences. See also [66, 72–74] for Padovan Q-matrix and related matrices. In [16], Cereceda provided some determinantal representation of the Padovan numbers by using the Hessenberg matrices.

The Padovan numbers and their properties have been studied by some other authors too, see for example, [4, 6, 10, 18, 19, 21–23, 28–35, 39, 42, 63, 81, 82, 84, 91, 93, 96]

It is the aim of this paper to define and to explore some of the properties of generalized Padovan numbers and is to investigate, in details, four particular case, namely sequences of Padovan, Perrin, Padovan-Perrin and modified Padovan numbers $\{P_n\}$, $\{E_n\}$, $\{S_n\}$ and $\{A_n\}$, respectively. Before, we recall the generalized Tribonacci sequence and its some properties.

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{2}$$

where $W_0, W_1, W_2$ are arbitrary complex (or real) numbers and $r, s, t$ are real numbers.

This sequence has been studied by many authors, see for example [11, 12, 17, 24, 25, 50, 61, 65, 67, 77, 78, 88, 95].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (2) holds for all integer $n$.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{3}$$

whose roots are

$$\begin{aligned}\alpha &= \alpha(r,s,t)=\frac{r}{3}+A+B\\ \beta &= \beta(r,s,t)=\frac{r}{3}+\omega A+\omega^2 B\\ \gamma &= \gamma(r,s,t)=\frac{r}{3}+\omega^2 A+\omega B\end{aligned}$$

where

$$\begin{aligned}A &= \left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}+\sqrt{\Delta}\right)^{1/3},\ B=\left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}-\sqrt{\Delta}\right)^{1/3}\\ \Delta &= \Delta(r,s,t)=\frac{r^3t}{27}-\frac{r^2s^2}{108}+\frac{rst}{6}-\frac{s^3}{27}+\frac{t^2}{4},\ \ \omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3)\end{aligned}$$

Note that we have the following identities

$$\begin{aligned}\alpha+\beta+\gamma &= r,\\ \alpha\beta+\alpha\gamma+\beta\gamma &= -s,\\ \alpha\beta\gamma &= t.\end{aligned}$$

If $\Delta(r,s,t)>0$, then the Equ. (3) has one real ($\alpha$) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers $n$, using Binet's formula

$$W_n=\frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{4}$$

where

$$b_1=W_2-(\beta+\gamma)W_1+\beta\gamma W_0,\ b_2=W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0,\ b_3=W_2-(\alpha+\beta)W_1+\alpha\beta W_0.$$

Note that the Binet form of a sequence satisfying (3) for non-negative integers is valid for all integers $n$, for a proof of this result see [37]]. This result of Howard and Saidak [37] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r=0$, $s=t=1$ and in this case we write $V_n=W_n$. A generalized Padovan sequence $\{V_n\}_{n\geq 0}=\{V_n(V_0,V_1,V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n=V_{n-2}+V_{n-3} \tag{5}$$

with the initial values $V_0=c_0, V_1=c_1, V_2=c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n}=-V_{-(n-1)}+V_{-(n-3)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence (5) holds for all integer $n$.

(4) can be used to obtain Binet formula of generalized Padovan numbers. Binet formula of generalized padovan numbers can be given as

$$V_n=\frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1=V_2-(\beta+\gamma)V_1+\beta\gamma V_0,\ b_2=V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0,\ b_3=V_2-(\alpha+\beta)V_1+\alpha\beta V_0. \tag{6}$$

Here, $\alpha,\beta$ and $\gamma$ are the roots of the cubic equation $x^3-x-1=0$. Moreover

$$\begin{aligned}\alpha &= \left(\frac{1}{2}+\sqrt{\frac{23}{108}}\right)^{1/3}+\left(\frac{1}{2}-\sqrt{\frac{23}{108}}\right)^{1/3}=1.32471795724\\ \beta &= \omega\left(\frac{1}{2}+\sqrt{\frac{23}{108}}\right)^{1/3}+\omega^2\left(\frac{1}{2}-\sqrt{\frac{23}{108}}\right)^{1/3}\\ \gamma &= \omega^2\left(\frac{1}{2}+\sqrt{\frac{23}{108}}\right)^{1/3}+\omega\left(\frac{1}{2}-\sqrt{\frac{23}{108}}\right)^{1/3}\end{aligned}$$

**Table 1.** A few generalized Padovan numbers

| $n$ | $V_n$ | $V_{-n}$ |
|---|---|---|
| 0 | $V_0$ | ... |
| 1 | $V_1$ | $V_2 - V_0$ |
| 2 | $V_2$ | $-V_2 + V_1 + V_0$ |
| 3 | $V_1 + V_0$ | $V_2 - V_1$ |
| 4 | $V_2 + V_1$ | $V_1 - V_0$ |
| 5 | $V_2 + V_1 + V_0$ | $-V_2 + 2V_0$ |
| 6 | $V_2 + 2V_1 + V_0$ | $2V_2 - V_1 - 2V_0$ |
| 7 | $2V_2 + 2V_1 + V_0$ | $-2V_2 + 2V_1 + V_0$ |
| 8 | $2V_2 + 3V_1 + 2V_0$ | $V_2 - 2V_1 + V_0$ |

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 0, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

The first few generalized Padovan numbers with positive subscript and negative subscript are given in the following Table 1. Now we define four special cases of the sequence $\{V_n\}$. Padovan (Cordonnier) sequence$\{P_n\}_{n\geq 0}$, Perrin (Padovan-Lucas) sequence $\{E_n\}_{n\geq 0}$, Padovan-Perrin sequence $\{S_n\}_{n\geq 0}$ and modified Padovan sequence $\{A_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} P_{n+3} &= P_{n+1} + P_n, \quad P_0 = 1, P_1 = 1, P_2 = 1, \\ E_{n+3} &= E_{n+1} + E_n, \quad E_0 = 3, E_1 = 0, E_2 = 2, \\ S_{n+3} &= S_{n+1} + S_n, \quad S_0 = 0, S_1 = 0, S_2 = 1, \\ A_{n+3} &= A_{n+1} + A_n, \quad A_0 = 3, A_1 = 1, A_2 = 3. \end{aligned}$$

Note that the case $V_n = R_n$, $R_0 = 1, R_1 = 0, R_2 = 1$ (or $V_n = R_n$, $R_0 = 0, R_1 = 1, R_2 = 0$) is called the sequence of the Van der Laan numbers, in the literature.

The sequences $\{P_n\}_{n\geq 0}$, $\{E_n\}_{n\geq 0}$, $\{S_n\}_{n\geq 0}$ and $\{A_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} + P_{-(n-3)} \quad (7)$$
$$E_{-n} = -E_{-(n-1)} + E_{-(n-3)} \quad (8)$$
$$S_{-n} = -S_{-(n-1)} + S_{-(n-3)} \quad (9)$$
$$A_{-n} = -A_{-(n-1)} + A_{-(n-3)} \quad (10)$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (7), (8), (9) and (10) hold for all integer $n$.

Note that $P_n$ and $S_n$ are two variants of the same sequence in [71]. In fact, the following are basically all variants of the same sequence in [71] which $P_n$ and $S_n$ belong: A000931, A078027, A096231, A124745, A133034, A134816, A164001, A182097, A228361 and probably A020720 (however, each one has its own special features and deserves its own entry). $E_n$ is the sequence A001608 in [71]and $A_n$ is the sequence A276276 in [71].

Next, we present the first few values of the Padovan, Perrin, Padovan-Perrin and modified Padovan numbers with positive and negative subscripts: For all integers $n$, Padovan, Perrin, Padovan-Perrin and modified Padovan numbers (using initial conditions in (6)) can be expressed using Binet's formulas as

$$\begin{aligned} P_n &= \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)}, \\ E_n &= \alpha^n + \beta^n + \gamma^n, \\ S_n &= \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \\ A_n &= \frac{(3\alpha+1)\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(3\beta+1)\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(3\gamma+1)\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

**Table 2.** The first few values of the special third-order numbers with positive and negative subscripts.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P_n$ | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 | 12 | 16 | 21 | 28 |
| $P_{-n}$ | | 0 | 1 | 0 | 0 | 1 | −1 | 1 | 0 | −1 | 2 | −2 | 1 | 1 |
| $E_n$ | 3 | 0 | 2 | 3 | 2 | 5 | 5 | 7 | 10 | 12 | 17 | 22 | 29 | 39 |
| $E_{-n}$ | | −1 | 1 | 2 | −3 | 4 | −2 | −1 | 5 | −7 | 6 | −1 | −6 | 12 |
| $S_n$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 |
| $S_{-n}$ | | 1 | −1 | 1 | 0 | −1 | 2 | −2 | 1 | 1 | −3 | 4 | −3 | 0 |
| $A_n$ | 3 | 1 | 3 | 4 | 4 | 7 | 8 | 11 | 15 | 19 | 26 | 34 | 45 | 60 |
| $A_{-n}$ | | 0 | 1 | 2 | −2 | 3 | −1 | −1 | 4 | −5 | 4 | 0 | −5 | 9 |

respectively. Note that $P_n, S_n$ and $A_n$ can be written as

$$\begin{aligned}
P_n &= \frac{\alpha^{n+5}}{2\alpha+3}+\frac{\beta^{n+5}}{2\beta+3}+\frac{\gamma^{n+5}}{2\gamma+3},\\
S_n &= \frac{\alpha^{n+1}}{2\alpha+3}+\frac{\beta^{n+1}}{2\beta+3}+\frac{\gamma^{n+1}}{2\gamma+3},\\
A_n &= \frac{(3\alpha+1)\alpha^{n+2}}{2\alpha+3}+\frac{(3\beta+1)\beta^{n+2}}{2\beta+3}+\frac{(3\gamma+1)\gamma^{n+2}}{2\gamma+3}.
\end{aligned}$$

## 2. Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence $V_n$.

**Lemma 2.1.**
*Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Padovan sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+V_1x+(V_2-V_0)x^2}{1-x^2-x^3}. \tag{11}$$

Proof. Using the definition of generalized Padovan numbers, and substracting $x^2\sum_{n=0}^{\infty} V_n x^n$ and $x^3\sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned}
(1-x^2-x^3)\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - x^2\sum_{n=0}^{\infty} V_n x^n - x^3\sum_{n=0}^{\infty} V_n x^n\\
&= \sum_{n=0}^{\infty} V_n x^n - \sum_{n=0}^{\infty} V_n x^{n+2} - \sum_{n=0}^{\infty} V_n x^{n+3}\\
&= \sum_{n=0}^{\infty} V_n x^n - \sum_{n=2}^{\infty} V_{n-2} x^{n} - \sum_{n=3}^{\infty} V_{n-3} x^{n}\\
&= (V_0+V_1x+V_2x^2)-V_0x^2\\
&\quad + \sum_{n=3}^{\infty}(V_n-V_{n-2}-V_{n-3})x^n\\
&= V_0+V_1x+V_2x^2-V_0x^2\\
&= V_0+V_1x+(V_2-V_0)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+V_1x+(V_2-V_0)x^2}{1-x^2-x^3}.$$

The previous lemma gives the following results as particular examples.

**Corollary 2.1.**
*Generated functions of Padovan, Perrin, Padovan-Perrin and modified Padovan numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty} P_n x^n &= \frac{1+x}{1-x^2-x^3},\\ \sum_{n=0}^{\infty} E_n x^n &= \frac{3-x^2}{1-x^2-x^3},\\ \sum_{n=0}^{\infty} S_n x^n &= \frac{x^2}{1-x^2-x^3},\\ \sum_{n=0}^{\infty} A_n x^n &= \frac{3+x}{1-x^2-x^3},\end{aligned}$$

*respectively.*

## 3. Obtaining Binet Formula From Generating Function

We next find Binet formula of generalized Grahaml numbers $\{V_n\}$ by the use of generating function for $V_n$.

**Theorem 3.1.**
*(Binet formula of generalized Padovan numbers)*

$$V_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{12}$$

*where*

$$\begin{aligned} d_1 &= V_0\alpha^2 + V_1\alpha + (V_2 - V_0),\\ d_2 &= V_0\beta^2 + V_1\beta + (V_2 - V_0),\\ d_3 &= V_0\gamma^2 + V_1\gamma + (V_2 - V_0).\end{aligned}$$

*Proof.* Let

$$h(x) = 1 - x^2 - x^3.$$

Then for some $\alpha, \beta$ and $\gamma$ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

i.e.,

$$1 - x^2 - x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x) \tag{13}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$, ve $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives $\alpha, \beta$, and $\gamma$ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{1}{x^2} - \frac{1}{x^3} = 0.$$

This implies $x^3 - x - 1 = 0$. Now, by (11) and (13), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + V_1 x + (V_2 - V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{V_0 + V_1 x + (V_2 - V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)}. \tag{14}$$

So

$$V_0 + V_1 x + (V_2 - V_0)x^2 = A_1(1-\beta x)(1-\gamma x) + A_2(1-\alpha x)(1-\gamma x) + A_3(1-\alpha x)(1-\beta x).$$

If we consider $x=\frac{1}{\alpha}$, we get $V_0+V_1\frac{1}{\alpha}+(V_2-V_0)\frac{1}{\alpha^2}=A_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})$. This gives

$$A_1=\frac{\alpha^2(V_0+V_1\frac{1}{\alpha}+(V_2-V_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)}=\frac{V_0\alpha^2+V_1\alpha+(V_2-V_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$A_2=\frac{V_0\beta^2+V_1\beta+(V_2-V_0)}{(\beta-\alpha)(\beta-\gamma)},A_3=\frac{V_0\gamma^2+V_1\gamma+(V_2-V_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (14) can be written as

$$\sum_{n=0}^{\infty}V_nx^n=A_1(1-\alpha x)^{-1}+A_2(1-\beta x)^{-1}+A_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty}V_nx^n=A_1\sum_{n=0}^{\infty}\alpha^nx^n+A_2\sum_{n=0}^{\infty}\beta^nx^n+A_3\sum_{n=0}^{\infty}\gamma^nx^n=\sum_{n=0}^{\infty}(A_1\alpha^n+A_2\beta^n+A_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n$$

where

$$\begin{aligned}A_1&=\frac{V_0\alpha^2+V_1\alpha+(V_2-V_0)}{(\alpha-\beta)(\alpha-\gamma)},\\ A_2&=\frac{V_0\beta^2+V_1\beta+(V_2-V_0)}{(\beta-\alpha)(\beta-\gamma)}\\ A_3&=\frac{V_0\gamma^2+V_1\gamma+(V_2-V_0)}{(\gamma-\alpha)(\gamma-\beta)}.\end{aligned}$$

and then we get (12).

Note that from (6) and (12) we have

$$\begin{aligned}V_2-(\beta+\gamma)V_1+\beta\gamma V_0&=V_0\alpha^2+V_1\alpha+(V_2-V_0),\\ V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0&=V_0\beta^2+V_1\beta+(V_2-V_0),\\ V_2-(\alpha+\beta)V_1+\alpha\beta V_0&=V_0\gamma^2+V_1\gamma+(V_2-V_0).\end{aligned}$$

Next, using Theorem 3.1, we present the Binet formulas of Padovan, Perrin, Padovan-Perrin and modified Padovan sequences.

**Corollary 3.1.**
*Binet formulas of Padovan, Perrin, Padovan-Perrin and modified Padovan sequences are*

$$\begin{aligned}P_n&=\frac{(\alpha+1)\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(\beta+1)\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{(\gamma+1)\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},\\ E_n&=\alpha^n+\beta^n+\gamma^n,\\ S_n&=\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},\\ A_n&=\frac{(3\alpha+1)\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(3\beta+1)\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{(3\gamma+1)\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},\end{aligned}$$

*respectively.*

We can find Binet formulas by using matrix method with a similar technique which is given in [47]. Take $k=i=3$ in Corollary 3.1 in [47]. Let

$$\Lambda=\begin{pmatrix}\alpha^2&\alpha&1\\ \beta^2&\beta&1\\ \gamma^2&\gamma&1\end{pmatrix},\Lambda_1=\begin{pmatrix}\alpha^{n-1}&\alpha&1\\ \beta^{n-1}&\beta&1\\ \gamma^{n-1}&\gamma&1\end{pmatrix},$$

$$\Lambda_2=\begin{pmatrix}\alpha^2&\alpha^{n-1}&1\\ \beta^2&\beta^{n-1}&1\\ \gamma^2&\gamma^{n-1}&1\end{pmatrix},\Lambda_3=\begin{pmatrix}\alpha^2&\alpha&\alpha^{n-1}\\ \beta^2&\beta&\beta^{n-1}\\ \gamma^2&\gamma&\gamma^{n-1}\end{pmatrix}.$$

Then the Binet formula for Padovan numbers is

$$\begin{aligned} P_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} P_{4-j} \det(\Lambda_j) = \frac{1}{\Lambda}(P_3 \det(\Lambda_1) + P_2 \det(\Lambda_2) + P_1 \det(\Lambda_3)) \\ &= \frac{1}{\det(\Lambda)}(2\det(\Lambda_1) + \det(\Lambda_2) + \det(\Lambda_3)) \\ &= \left( 2 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}. \end{aligned}$$

Similarly, we obtain the Binet formula for Perrin, Padovan-Perrin and modified Padovan as

$$\begin{aligned} E_n &= \frac{1}{\Lambda}(E_3 \det(\Lambda_1) + E_2 \det(\Lambda_2) + E_1 \det(\Lambda_3)) \\ &= \left( 3 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 2 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} \right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}. \end{aligned}$$

and

$$\begin{aligned} S_n &= \frac{1}{\Lambda}(S_3 \det(\Lambda_1) + S_2 \det(\Lambda_2) + S_1 \det(\Lambda_3)) \\ &= \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}. \end{aligned}$$

and

$$\begin{aligned} A_n &= \frac{1}{\Lambda}(A_3 \det(\Lambda_1) + A_2 \det(\Lambda_2) + A_1 \det(\Lambda_3)) \\ &= \left( 4 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 3 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \end{aligned}$$

respectively.

## 4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Padovan sequence $\{V_n\}_{n\geq 0}$.

**Theorem 4.1 (Simson Formula of Generalized Padovan Numbers).**
*For all integers n, we have*

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{15}$$

*Proof.* (15) is given in Soykan [76].
The previous theorem gives the following results as particular examples.

**Corollary 4.1.**
*For all integers $n$, Simson formula of Padovan, Perrin, Padovan-Perrin and modified Padovan numbers are given as*

$$\begin{vmatrix} P_{n+2} & P_{n+1} & P_n \\ P_{n+1} & P_n & P_{n-1} \\ P_n & P_{n-1} & P_{n-2} \end{vmatrix} = -1$$

*and*

$$\begin{vmatrix} E_{n+2} & E_{n+1} & E_n \\ E_{n+1} & E_n & E_{n-1} \\ E_n & E_{n-1} & E_{n-2} \end{vmatrix} = -23$$

*and*

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = -1$$

*and*

$$\begin{vmatrix} A_{n+2} & A_{n+1} & A_n \\ A_{n+1} & A_n & A_{n-1} \\ A_n & A_{n-1} & A_{n-2} \end{vmatrix} = -19$$

*respectively.*

## 5. Some Identities

In this section, we obtain some identities of Padovan, Perrin, Padovan-Perrin and modified Padovan numbers. First, we can give a few basic relations between $\{P_n\}$ and $\{E_n\}$.

**Lemma 5.1.**
*The following equalities are true:*

$$\begin{aligned} E_n &= -2P_{n+4} + 4P_{n+3} - P_{n+2}, \qquad (16) \\ E_n &= 4P_{n+3} - 3P_{n+2} - 2P_{n+1}, \\ E_n &= -3P_{n+2} + 2P_{n+1} + 4P_n, \\ E_n &= 2P_{n+1} + P_n - 3P_{n-1}, \\ E_n &= P_n - P_{n-1} + 2P_{n-2}, \end{aligned}$$

*and*

$$\begin{aligned} 23P_n &= -2E_{n+4} + 3E_{n+3} + 9E_{n+2}, \\ 23P_n &= 3E_{n+3} + 7E_{n+2} - 2E_{n+1}, \\ 23P_n &= 7E_{n+2} + E_{n+1} + 3E_n, \\ 23P_n &= E_{n+1} + 10E_n + 7E_{n-1}, \\ 23P_n &= 10E_n + 8E_{n-1} + E_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers $n$. We prove (16). To show (16), writing

$$E_n = a \times P_{n+4} + b \times P_{n+3} + c \times P_{n+2}$$

and solving the system of equations

$$\begin{aligned} E_0 &= a \times P_4 + b \times P_3 + c \times P_2 \\ E_1 &= a \times P_5 + b \times P_4 + c \times P_3 \\ E_2 &= a \times P_6 + b \times P_5 + c \times P_4 \end{aligned}$$

we find that $a = -2, b = 4, c = -1$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{P_n\}$ and $\{S_n\}$.

**Lemma 5.2.**
*The following equalities are true:*

$$\begin{aligned} S_n &= 2P_{n+4} - P_{n+3} - 2P_{n+2}, \\ S_n &= -P_{n+3} + 2P_{n+1}, \\ S_n &= P_{n+1} - P_n, \\ S_n &= -P_n + P_{n-1} + P_{n-2}, \end{aligned}$$

*and*

$$\begin{aligned} P_n &= S_{n+4}, \\ P_n &= S_{n+2} + S_{n+1}, \\ P_n &= S_{n+1} + S_n + S_{n-1}, \\ P_n &= S_n + 2S_{n-1} + S_{n-2}. \end{aligned}$$

Now, we give a few basic relations between $\{P_n\}$ and $\{A_n\}$.

**Lemma 5.3.**
*The following equalities are true:*

$$\begin{aligned} A_n &= -P_{n+4} + 3P_{n+3} - P_{n+2}, \\ A_n &= 3P_{n+3} - 2P_{n+2} - P_{n+1}, \\ A_n &= -2P_{n+2} + 2P_{n+1} + 3P_n, \\ A_n &= 2P_{n+1} + P_n - 2P_{n-1}, \\ A_n &= P_n + 2P_{n-2}, \end{aligned}$$

*and*

$$\begin{aligned} 19P_n &= -3A_{n+4} + A_{n+3} + 9A_{n+2}, \\ 19P_n &= A_{n+3} + 6A_{n+2} - 3A_{n+1}, \\ 19P_n &= 6A_{n+2} - 2A_{n+1} + A_n, \\ 19P_n &= -2A_{n+1} + 7A_n + 6 \times A_{n-1}, \\ 19P_n &= 7A_n + 4A_{n-1} - 2A_{n-2}. \end{aligned}$$

Next, we present a few basic relations between $\{E_n\}$ and $\{S_n\}$.

**Lemma 5.4.**
*The following equalities are true*

$$\begin{aligned} 23S_n &= 5E_{n+4} + 4E_{n+3} - 11E_{n+2}, \\ 23S_n &= 4E_{n+3} - 6E_{n+2} + 5E_{n+1}, \\ 23S_n &= -6E_{n+2} + 9E_{n+1} + 4E_n, \\ 23S_n &= 9E_{n+1} - 2E_n - 6E_{n-1}, \\ 23S_n &= -2E_n + 3E_{n-1} + 9E_{n-2}. \end{aligned}$$

*and*

$$\begin{aligned} E_n &= S_{n+4} - S_{n+3} + 2S_{n+2}, \\ E_n &= -S_{n+3} + 3S_{n+2} + S_{n+1}, \\ E_n &= 3S_{n+2} - S_n, \\ E_n &= 2S_n + 3S_{n-1}. \end{aligned}$$

Next, we give a few basic relations between $\{A_n\}$ and $\{E_n\}$.

**Lemma 5.5.**
*The following equalities are true*

$$\begin{aligned}
19E_n &= -28A_{n+4}+22A_{n+3}+27A_{n+2},\\
19E_n &= 22A_{n+3}-A_{n+2}-28A_{n+1},\\
19E_n &= -A_{n+2}-6A_{n+1}+22A_n,\\
19E_n &= -6A_{n+1}+21A_n-A_{n-1},\\
19E_n &= 21A_n-7A_{n-1}-6A_{n-2},
\end{aligned}$$

*and*

$$\begin{aligned}
23A_n &= -14E_{n+4}+21E_{n+3}+17E_{n+2},\\
23A_n &= 21E_{n+3}+3E_{n+2}-14E_{n+1},\\
23A_n &= 3E_{n+2}+7E_{n+1}+21E_n,\\
23A_n &= 7E_{n+1}+24E_n+3E_{n-1},\\
23A_n &= 24E_n+10E_{n-1}+7E_{n-2}.
\end{aligned}$$

Now, we present a few basic relations between $\{S_n\}$ and $\{A_n\}$.

**Lemma 5.6.**
*The following equalities are true*

$$\begin{aligned}
A_n &= S_{n+4}+2S_{n+2},\\
A_n &= 3S_{n+2}+S_{n+1},\\
A_n &= 3S_{n+2}+S_{n+1},\\
A_n &= S_{n+1}+3S_n+3S_{n-1},\\
A_n &= 3S_n+4S_{n-1}+S_{n-2},
\end{aligned}$$

*and*

$$\begin{aligned}
19S_n &= 4A_{n+4}+5A_{n+3}-12A_{n+2},\\
19S_n &= 5A_{n+3}-8A_{n+2}+4A_{n+1},\\
19S_n &= -8A_{n+2}+9A_{n+1}+5A_n,\\
19S_n &= 9A_{n+1}-3A_n-8A_{n-1},\\
19S_n &= -3A_n+A_{n-1}+9A_{n-2}.
\end{aligned}$$

We now present a few special identities for the modified Padovan sequence $\{A_n\}$.

**Theorem 5.1.**
*(Catalan's identity) For all integers $n$ and $m$, the following identity holds*

$$\begin{aligned}
A_{n+m}A_{n-m}-A_n^2 &= (P_{n+m}+2P_{n+m-2})(P_{n-m}+2P_{n-m-2})-(P_n+2P_{n-2})^2\\
&= -P_n^2-4P_nP_{n-2}-4P_{n-2}^2+P_{m+n}P_{n-m}+2P_{m+n}P_{n-m-2}\\
&\quad +2P_{m+n-2}P_{n-m}+4P_{m+n-2}P_{n-m-2}.
\end{aligned}$$

Proof. We use the identity

$$A_n = P_n + 2P_{n-2}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified Padovan sequnce

**Corollary 5.1.**
*(Cassini's identity) For all integers numbers $n$ and $m$, the following identity holds*

$$\begin{aligned}
A_{n+1}A_{n-1}-A_n^2 &= (P_{n+1}+2P_{n+1-2})(P_{n-1}+2P_{n-1-2})-(P_n+2P_{n-2})^2\\
&= P_{n-1}P_{n+1}-P_n^2-4P_nP_{n-2}+4P_{n-1}P_{n-3}+2P_{n+1}P_{n-3}+2P_{n-1}^2-4P_{n-2}^2.
\end{aligned}$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $A_n = P_n + 2P_{n-2}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified Padovan sequence $\{A_n\}$.

**Theorem 5.2.**
*Let n and m be any integers. Then the following identities are true:*

**(a)** *(d'Ocagne's identity)*

$$A_{m+1}A_n - A_m A_{n+1} = (P_{m+1} + 2P_{m+1-2})(P_n + 2P_{n-2}) - (P_m + 2P_{m-2})(P_{n+1} + 2P_{n+1-2}).$$

**(b)** *(Gelin-Cesàro's identity)*

$$A_{n+2}A_{n+1}A_{n-1}A_{n-2} - A_n^4 = (P_{n+2}+2P_{n+2-2})(P_{n+1}+2P_{n+1-2})(P_{n-1}+2P_{n-1-2})(P_{n-2}+2P_{n-2-2}) - (P_n+2P_{n-2})^4.$$

**(c)** *(Melham's identity)*

$$\begin{aligned} A_{n+1}A_{n+2}A_{n+6} - A_{n+3}^3 &= (P_{n+1} + 2P_{n+1-2})(P_{n+2} + 2P_{n+2-2})(P_{n+6} + 2P_{n+6-2}) - (P_n + 2P_{n-2})^3 \\ &= (P_{n+1} + 2P_{n-1})(P_{n+2} + 2P_n)(P_{n+6} + 2P_{n+4}) - (P_n + 2P_{n-2})^3. \end{aligned}$$

Proof. Use the identity $A_n = P_n + 2P_{n-2}$.

## 6. Linear Sums

The following proposition presents some formulas of generalized Padovan numbers with positive subscripts.

**Proposition 6.1.**
*If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

**(a)** $\sum_{k=0}^{n} V_k = V_{n+3} + V_{n+2} - V_2 - V_1$.

**(b)** $\sum_{k=0}^{n} V_{2k} = V_{2n+1} + V_{2n} - V_1$.

**(c)** $\sum_{k=0}^{n} V_{2k+1} = V_{2n+2} + V_{2n+1} - V_2$.

Proof. Take $r = 0, s = 1, t = 1$ in Theorem 2.1 in [75].
As special cases of above proposition, we have the following four corollaries. First one presents some summing formulas of Padovan numbers (take $V_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

**Corollary 6.1.**
*For $n \geq 0$ we have the following formulas:*

**(a)** $\sum_{k=0}^{n} P_k = P_{n+3} + P_{n+2} - 2$.

**(b)** $\sum_{k=0}^{n} P_{2k} = P_{2n+1} + P_{2n} - 1$.

**(c)** $\sum_{k=0}^{n} P_{2k+1} = P_{2n+2} + P_{2n+1} - 1$.

Second one presents some summing formulas of Perrin (Padovan-Lucas) numbers (take $V_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$).

**Corollary 6.2.**
*For $n \geq 0$ we have the following formulas:*

**(a)** $\sum_{k=0}^{n} E_k = E_{n+3} + E_{n+2} - 2$.

**(b)** $\sum_{k=0}^{n} E_{2k} = E_{2n+1} + E_{2n}$.

**(c)** $\sum_{k=0}^{n} E_{2k+1} = E_{2n+2} + E_{2n+1} - 2$.

Third one presents some summing formulas of Padovan-Perrin numbers (take $V_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

**Corollary 6.3.**
*For $n \geq 0$ we have the following formulas:*

**(a)** $\sum_{k=0}^{n} S_k = S_{n+3} + S_{n+2} - 1.$

**(b)** $\sum_{k=0}^{n} S_{2k} = S_{2n+1} + S_{2n}.$

**(c)** $\sum_{k=0}^{n} S_{2k+1} = S_{2n+2} + S_{2n+1} - 1.$

Fourth one presents some summing formulas of modified Padovan numbers (take $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$).

**Corollary 6.4.**
*For $n \geq 0$ we have the following formulas:*

**(a)** $\sum_{k=0}^{n} A_k = A_{n+3} + A_{n+2} - 4.$

**(b)** $\sum_{k=0}^{n} A_{2k} = A_{2n+1} + A_{2n} - 1.$

**(c)** $\sum_{k=0}^{n} A_{2k+1} = A_{2n+2} + A_{2n+1} - 3.$

The following proposition presents some formulas of generalized Padovan numbers with negative subscripts.

**Proposition 6.2.**
*If $r = 0, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

**(a)** $\sum_{k=1}^{n} V_{-k} = -2V_{-n-1} - 2V_{-n-2} - V_{-n-3} + V_2 + V_1.$

**(b)** $\sum_{k=1}^{n} V_{-2k} = -V_{-2n+1} + V_1.$

**(c)** $\sum_{k=1}^{n} V_{-2k+1} = -V_{-2n} - V_{-2n-1} + V_2.$

Proof. Take $r = 0, s = 1, t = 1$ in Theorem 3.1 in [75].

From the above proposition, we have the following corollary which gives sum formulas of Padovan numbers (take $V_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

**Corollary 6.5.**
*For $n \geq 1$, Padovan numbers have the following properties.*

**(a)** $\sum_{k=1}^{n} P_{-k} = -2P_{-n-1} - 2P_{-n-2} - P_{-n-3} + 2.$

**(b)** $\sum_{k=1}^{n} P_{-2k} = -P_{-2n+1} + 1.$

**(c)** $\sum_{k=1}^{n} P_{-2k+1} = -P_{-2n} - P_{-2n-1} + 1.$

Taking $V_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Padovan-Lucas numbers.

**Corollary 6.6.**
*For $n \geq 1$, Perrin (Padovan-Lucas) numbers have the following properties.*

**(a)** $\sum_{k=1}^{n} E_{-k} = -2E_{-n-1} - 2E_{-n-2} - E_{-n-3} + 2.$

**(b)** $\sum_{k=1}^{n} E_{-2k} = -E_{-2n+1}.$

**(c)** $\sum_{k=1}^{n} E_{-2k+1} = -E_{-2n} - E_{-2n-1} + 2.$

From the above proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers (take $V_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

**Corollary 6.7.**
*For $n \geq 1$, Padovan-Perrin numbers have the following properties.*

**(a)** $\sum_{k=1}^{n} S_{-k} = -2S_{-n-1} - 2S_{-n-2} - S_{-n-3} + 1.$

**(b)** $\sum_{k=1}^{n} S_{-2k} = -S_{-2n+1}.$

**(c)** $\sum_{k=1}^{n} S_{-2k+1} = -S_{-2n} - S_{-2n-1} + 1.$

From the above proposition, we have the following corollary which gives sum formulas of modified Padovan numbers (take $V_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$).

**Corollary 6.8.**
*For $n \geq 1$, modified Padovan numbers have the following properties.*

**(a)** $\sum_{k=1}^{n} A_{-k} = -2A_{-n-1} - 2A_{-n-2} - A_{-n-3} + 4.$

**(b)** $\sum_{k=1}^{n} A_{-2k} = -A_{-2n+1} + 1.$

**(c)** $\sum_{k=1}^{n} A_{-2k+1} = -A_{-2n} - A_{-2n-1} + 3.$

## 7. Matrices related with Generalized Padovan numbers

Matrix formulation of $W_n$ can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{17}$$

For matrix formulation (17), see [41]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix $A$ of order 3 as:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. From (5) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{18}$$

and from (17) (or using (18) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = P$ in (18) we have

$$\begin{pmatrix} P_{n+2} \\ P_{n+1} \\ P_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{n+1} \\ P_n \\ P_{n-1} \end{pmatrix}. \tag{19}$$

We also define

$$B_n = \begin{pmatrix} P_{n-2} & P_{n-1} & P_{n-3} \\ P_{n-3} & P_{n-2} & P_{n-4} \\ P_{n-4} & P_{n-3} & P_{n-5} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n-2} & V_{n-1} & V_{n-3} \\ V_{n-3} & V_{n-2} & V_{n-4} \\ V_{n-4} & V_{n-3} & V_{n-5} \end{pmatrix}$$

**Theorem 7.1.**
*For all integer $m, n \geq 0$, we have*

**(a)** $B_n = A^n$

**(b)** $C_1 A^n = A^n C_1$

**(c)** $C_{n+m} = C_n B_m = B_m C_n$.

**Proof.**

**(a)** By expanding the vectors on the both sides of (19) to 3-colums and multiplying the obtained on the right-hand side by $A$, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

**(b)** Using (a) and definition of $C_1$, (b) follows.

**(c)** We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n-3} & V_{n-2} & V_{n-4} \\ V_{n-4} & V_{n-3} & V_{n-5} \\ V_{n-5} & V_{n-4} & V_{n-6} \end{pmatrix} \\ &= \begin{pmatrix} V_{n-4}+V_{n-5} & V_{n-3}+V_{n-4} & V_{n-5}+V_{n-6} \\ V_{n-3} & V_{n-2} & V_{n-4} \\ V_{n-4} & V_{n-3} & V_{n-5} \end{pmatrix} = \begin{pmatrix} V_{n-2} & V_{n-1} & V_{n-3} \\ V_{n-3} & V_{n-2} & V_{n-4} \\ V_{n-4} & V_{n-3} & V_{n-5} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of matrix $A^n$ can be given as

$$A^n = A^{n-2} + A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 1$$

for all integer $m$ and $n$.

**Theorem 7.2.**
*For $m, n \geq 0$ we have*

$$V_{n+m} = V_{n-2}P_m + V_{n-3}P_{m+1} + V_{n-4}P_{m-1}. \tag{20}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of $C_{n+m}$ is the product of row $C_n$ and a column $B_m$. From the last equation we say that an element of $C_{n+m}$ is the product of a row $C_n$ and column $B_m$. We just compare the linear combination of the 2nd row and 1st column entries of the matrices $C_{n+m}$ and $C_n B_m$. This completes the proof.

**Remark 7.1.**
By induction, it can be proved that for all integers $m, n \leq 0$, (20) holds. So for all integers $m, n$, (20) is true.

**Corollary 7.1.**
*For all integers $m, n$, we have*

$$P_{n+m} = P_{n-2}P_m + P_{n-3}P_{m+1} + P_{n-4}P_{m-1}, \tag{21}$$

$$E_{n+m} = E_{n-2}P_m + E_{n-3}P_{m+1} + E_{n-4}P_{m-1}, \tag{22}$$

$$S_{n+m} = S_{n-2}P_m + S_{n-3}P_{m+1} + S_{n-4}P_{m-1}, \tag{23}$$

$$A_{n+m} = A_{n-2}P_m + A_{n-3}P_{m+1} + A_{n-4}P_{m-1}. \tag{24}$$

## 8. Conclusions

Sequences have been fascinating topic for mathematicians for centuries and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. Sequences of integer number such as Fibonacci, Lucas, Pell, Jacobsthal are the most well-known second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci sequences are a source of many nice and interesting identities. A similar interpretation exists for Lucas sequence. For rich applications of these second order sequences in science and nature, one can see the citations in [48].

As a third order sequence, we introduce the generalized Padovan sequence (it's four special cases, namely, Padovan, Perrin, Padovan-Perrin and modified Padovan sequences) and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences.

Third order sequences have many applications. As mentioned in Introduction, the plastic number is used in art and architecture and Richard Padovan studied on plastic number in Architecture and Mathematics in [58, 59].

We now present some other applications of third order sequences.

- For the applications of Padovan numbers and Tribonacci numbers to coding theory see [70] and [7], respectively.
- For the application of Padovan numbers to Gaussian numbers, see [82].
- For the application of Pell-Padovan numbers to quaternions and groups see [83] and [20], respectively.
- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions see [14] and [13], respectively.
- For the application of Tribonacci numbers to special matrices, see [8].

As future works, we plan to study on the other third order and higher order generalized sequences.

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