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Review of Euler's Method of Solving 1st Order 1st Degree ODEs with Initial Condition and its Modification for Better Accuracy

Review Article

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Abstract: There are several methods to solve first-order ordinary differential equations, and Euler's method is one of the most fundamental. In this paper, we have discussed the basic concepts from the mathematical and geometrical viewpoint of Euler's method and some established modified Euler's methods, namely Modified Euler's method, Improved Euler's method, and Improved Modified Euler's method. Our goal is to analyze the geometrical representation of these methods and approach a new modification of Euler's method to achieve better accuracy. Finally, we have modified the existing method and introduced the Adjusted Improved Modified Euler's (AIME) method. Then, we apply the newly modified method to solve first-order ordinary differential equations to test the accuracy. The numerical and graphical results show that AIME outperforms the other methods, and the error is comparatively low among the methods considered. In this work, we use MATLAB for numerical and graphical calculations.

MSC: 34A45 • 65J22

Keywords: Ordinary Differential Equations (ODE) • Numerical Solution • Euler's Method • Modified Euler's Method • Improved Euler's Method

Literature Review

Usually, we solve the differential equations in two ways. One is the analytical method, and the other is the numerical method. In fact, in many cases, differential equations cannot be solved analytically [1]. Because of this, a numerical method is a powerful tool that helps us solve problems that are impossible or very difficult to solve in analytical ways.

Numerical methods typically provide approximate values of the solutions at different points in a given interval [2]. Renowned mathematician Leonhard Euler 1768 gave a simple and excellent method to solve the first-order differential equation [3] numerically. Euler's method is a simple and basic method to solve the first-order and first-degree ordinary differential equation. Although this method is prolonged, the error is much higher than the exact solution [4]. Later, many mathematicians change Euler's method to increase its acceleration and reduce the mistake significantly [5, 6]. One of the most popular methods is the Modified Euler's method. Then several modifications are done in the Improved Euler's method, Improved Modified Euler's method [6]. In [10] the author introduced a modification namely Enhanced Euler's Method by changing the slope at the inner functional evaluation of the original formula. This article briefly reviews the geometric interpretation of Euler's method and other modified rules then proposed a new modification to achieve better performance of the method.

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1. Introduction

Euler's method is a numerical method for solving ordinary differential equations (ODEs) with a given initial value. It is named after the famous mathematician Leonhard Euler who introduced it in the 18th century. The method approximates the solution of an ODE by using small steps and computing the values of the function at each step. To apply Euler's method,we first need to have an initial value problem in the form of a first-order Ordinary differential equation:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

where y is the unknown function, f is a given function, and y_0 is the initial value of y at x_0.

The basic idea of Euler's method is to approximate the value of y at some point $x+h$, where h is a small step size, by using the value of y at x and the derivative of y at x. The approximation formula is: $y(x+h) \approx y(x) + h * f(x, y)$. This formula uses the slope of the tangent line to the function y at x to estimate the value of y at $x+h$. We can repeat this process with the new value of y to approximate the solution at subsequent points.

2. Derivation of Euler's Method

Let us consider the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

With the initial condition $y(x_0) = y_0$
We have Taylor series

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \dots \tag{2}$$

From (1), I get $y' = f$ so that $y'' = f'$ and so on. Then (2) becomes

$$y(x+h) = y(x) + hf + \frac{h^2}{2!}f' + \frac{h^3}{3!}f'' + \dots \tag{3}$$

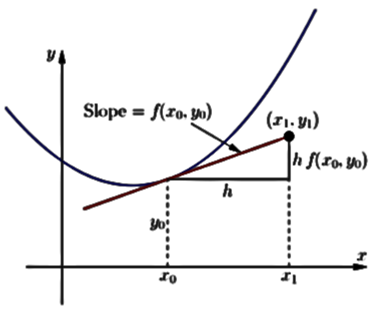
Where the derivatives f', f'' etc. are evaluated at the point $[x, y(x)]$.
If the step size h is taken sufficiently small, the terms containing h^2, h^3, etc. in (3) are still smaller and hence can be neglected to given an approximate solution $y(x+h) = y(x) + hf$.

In the first step, using $y(x_0) = y_0$ I get $y_1 = y_0 + hf(x_0, y_0)$, where $y(x_0+h) = y(x_1) = y_1$. Similarly, $y_2 = y_1 + hf(x_1, y_1)$ here $y(x_0+2h) = y(x_2) = y_2$. In geneal

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{4}$$

The successive values of $y_1, y_2, \dots$ at $x_1, x_2, \dots$ are obtained from (4) by putting $n = 0, 1, 2...$ This is called Euler's method or Euler-Cauchy method. It is also called first order method because we have taken terms only up to the term containing the first derivative $y'(x)$ in the Taylor series (2).

2.1. Graphical Representation of Euler's Method



Consider the first order first degree differential equation of the form $\frac{dy}{dx} = f(x, y)$ it $x_0 \le x \le x_n$ and $y(x_0) = y_0$. Divide x_0 to x_n into n equal steps i.e $h = (x_n - x_0)/n$. But $\tan\theta = \text{slopeat}(x_0, y_0)$. Tis gives $\tan\theta = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} = f(x_0, y_0)$. Now $\tan\theta = \frac{k}{h} = f(x_0, y_0)$ then $k = hf(x_0, y_0)$. So, 1st approximation to the solution at x_1 is $y_1 = y_0 + hf(x_0, y_0)$. Again, draw a tangent at (x_1, y_1), nd approximation is $y_2 = y_1 + hf(x_1, y_1)$ and so on. In geneal nth approximation is $y_{n+1} = y_n + hf(x_n, y_n)$.

3. Improved Euler's Method

The Euler method is a simple numerical method for solving ordinary differential equations (ODEs). HoIver, it is a first-order method and can produce errors that accumulate rapidly. Therefore, several improved versions of the Euler method have been developed. One such method is the Improved Euler method, also known as the Heun's method. It is a second-order method and provides more accurate solutions than the Euler method. The Improved Euler method approximates the solution of an ODE by using the following formula:

$$y_{n+1} = y_n + \left(\frac{h}{2}\right) * \left(f\left(x_n, y_n\right) + f\left(x_n + h, y_n + hf\left(x_n, y_n\right)\right)\right)$$

where y_n is the approximation of the solution at time x_n, y_{n+1} is the approximation of the solution at time $x_{n+1} = x_{n+1} + h$, h is the time step size, and f is the derivative of the solution with respect to time. The method works by using the slope at two points to estimate the value of the solution at the next point. First, it uses the slope at the current point, x_n, to estimate the value of the solution at x_{n+1}. Then, it uses this estimate to calculate the slope at $x_{n+1}/2$ and use this slope to estimate the value of the solution at x_{n+1}.

3.1. Derivation of Improved Euler's Method

In the improved Euler method, it starts from the initial value $\left(x_i, y_i\right)$ it is required to find an initial estimate of y_{i+1} by using the Euler's formula, $y_{i+1} = y_i + hf\left(x_i, y_i\right)$. But this formula is less accurate than the improved Euler's method so it is used as a predictor for an approximate value of y_{i+1}.
Now the value of y_{i+1}, is obtained by

$$y_{i+1} = y_i + \frac{h}{2}\left[f\left(x_i, y_i\right) + f\left(x_{i+1,} y_{i+1}\right)\right]$$

The value of y_{i+1} is corrected so the above formula is considered as the corrector formula. Now, to distinguish the two different values of y_{i+1}, obtained from the predictor and the corrector formula are respectively denoted by y^p_{i+1}, y^c_{i+1}. Thus, we have, Euler's predictor-corrector method as the predictor formula,

$$y_{i+1}{}^p = y_i + hf\left(x_i, y_i\right)$$

Now $\frac{dy}{dx} = f\left(x, y\right)$ then slopes $y_i{}' = f(x_i, y_i)$. and $y_{i+1}{}' = f(x_{i+1}, y_{i+1}{}^p)$.. Average of two Slope

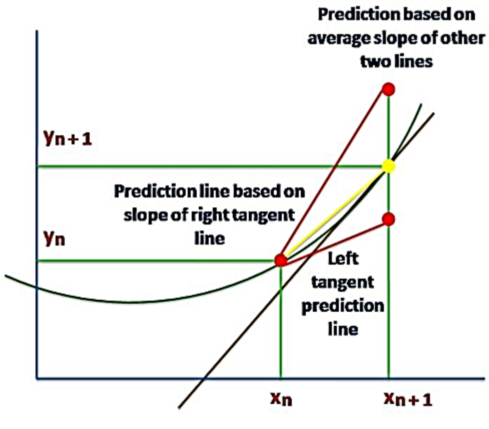
$$\bar{y}' = \frac{f\left(x_i, y_i\right) + f\left(x_{(i+1)}, y_{(i+1)}{}^p\right)}{2}$$

And the corrector formula,

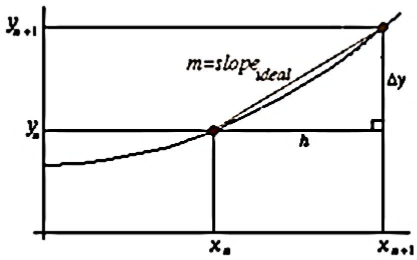
$$y_{i+1}{}^c = y_i + \frac{h}{2}\overline{y'} = y_i + \frac{h}{2}\left(f\left(x_i, y_i\right) + f\left(x_i + h, y_i + hf\left(x_i, y_i\right)\right)\right)$$

3.2. Graphical Representation of Improved Euler's Method

The numerical analysis would refer to Euler's method as a predictor algorithm, whereas Improve Euler's method is described as a predictor-corrector algorithm. If we consider the left end-point, which is simply the current point,



Euler's method can obtain the slope of the tangent line at this point: $Slope_{left} = f\left(x_i, y_i\right)$



Now for that right end-point that I've been concerning about so much. As we know, Euler gives a rough prediction of its location as being at the coordinates: $Slope_{right} = f\left(x_i + h, y_i + hf\left(x_i, y_i\right)\right)$

$$Slope_{ideal} = \frac{\left(Slope_{left} + Slope_{right}\right)}{2}$$

Using the principle that the slope of a line equates to the rise/run, we can find the coordinates at the end of the interval by using the following formula:

$$Slope_{ideal} = \frac{\Delta y}{h}$$
$$\Delta y = h * Slope_{ideal}$$

$$x_{i+1} = x_i + h, \quad y_{i+1} = y_i + \Delta y$$
$$y_{i+1} = y_i + h * Slope_{ideal}$$
$$y_{i+1} = y_i + \frac{h}{2}\left(f\left(x_i, y_i\right) + f\left(x_i + h, y_i + hf\left(x_i, y_i\right)\right)\right)$$

The accuracy of the Euler's method improves only linearly as the step size decreases, whereas the Improved Euler's method enhances the accuracy quadratically. The scheme can be compared with the implicit trapezoidal method but with $f\left(x_{i+1}, y_{i+1}\right)$ replaced by $f\left(x_{(i+1)}, {y^0}_{(i+1)}\right))$ in order to make it explicit. This method is an improved version of Euler's method. This method is an improved version of Euler's method. Since it attempts to correct the values of y_{i+1} (By Euler's method), it is classified as a one-step predictor-corrector method, where ${y^0}_{i+1}$ known as a predictor, and the final formula is known as a corrector.

4. Modified Euler's Method

The Modified Euler method is a numerical method for solving ordinary differential equations that uses a predictor-corrector approach to approximate the solution. The method improves on the basic Euler method by using an estimate of the derivative at the midpoint of the interval to correct the predicted value of the solution. The Modified Euler method is second-order accurate and has an error proportional to h^2.

4.1. Derivation of Modified Euler's Method

The Modified Euler method can be derived as follows:

The Modified Euler method can be derived as follows: Given an initial value problem of the form: $\frac{dy}{dx} = f\left(x, y\right), \quad y\left(x_0\right) = y_0$ where y is the solution we are trying to approximate, x is the independent variable (usually time), y is a known function, y_0 is the initial value of y at $x = x_0$. We want to approximate $y(x_n)$ for some value of $x_n = x_0 + n * h$, where h is the step size, n is the number of steps taken. The basic Euler method approximates $y(x_n)$ using the formula: $y_{n+1} = y_n + hf(x_n y_n)$ This formula uses the derivative $f(x_n, y_n)$ evaluated at the beginning of the interval to estimate the slope of the tangent line to the solution curve at x_n, and then uses this slope to extrapolate the solution to x_{n+1}. The Modified Euler method improves on the basic Euler method by using an estimate of the derivative at the midpoint of the interval to correct the predicted value. The Modified Euler method proceeds as follows:

Compute the predictor estimate of the solution at $x_n + \frac{h}{2}$, $\quad y^*\left(x_n + \frac{h}{2}\right) = y_n + \frac{h}{2} f\left(x_n, y_n\right)$
This estimate is obtained by using the derivative $f\left(x_n, y_n\right)$ evaluated at the beginning of the interval to approximate the slope of the tangent line to the solution curve at t_n, and then using this slope to extrapolate the solution to $x_n + \frac{h}{2}$.
Use the predictor estimate to compute an estimate of the derivative at $x_n + \frac{h}{2}$

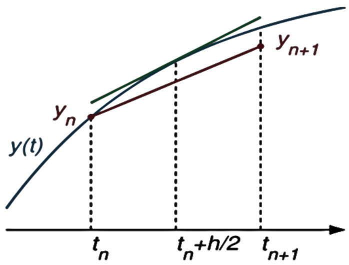
$$f^*\left(x_n + \frac{h}{2}, y^*\left(x_n + \frac{h}{2}\right)\right) = f\left(x_n + \frac{h}{2}, y^*\left(x_n + \frac{h}{2}\right)\right)$$

This estimate is obtained by evaluating the function f at $x_n + \frac{h}{2}$ and the predictor estimate of the solution $y^*\left(x_n + \frac{h}{2}\right)$. . Use the estimate of the derivative at $x_n + \frac{h}{2}$ to correct the predicted value of the solution at $x_n + \frac{h}{2}$:

$$y_{n+1} = y_n + hf\left(x_n + \tfrac{h}{2}, y^*\left(x_n + \tfrac{h}{2}\right)\right)$$
$$y_{n+1} = y_n + hf\left(x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2} f\left(x_n, y_n\right)\right)$$

This correction is obtained by using the corrected derivative $f^*\left(x_n + \frac{h}{2}, y^*\left(x_n + \frac{h}{2}\right)\right)$ to extrapolate the solution to $x_n + \frac{h}{2}$.

4.2. Graphical Representation of Modified Euler Method



By Euler method, Approximation at $x_n + h$ is, $y\left(x_n + h\right) = y_n + hf\left(x_n, y_n\right)$
So using this method I have,

$$y\left(x_n + \frac{h}{2}\right) = y_n + \frac{h}{2} f\left(x_n, y_n\right)$$

Predictor of solution,

$$y^*\left(x_n + \frac{h}{2}\right) = y_n + \frac{h}{2} f\left(x_n, y_n\right)$$

This gives

$$f^*\left(x_n + \frac{h}{2}, y^*\left(x_n + \frac{h}{2}\right)\right) = f\left(x_n + \frac{h}{2}, y^*\left(x_n + \frac{h}{2}\right)\right)$$

So, the corrector of solution

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y^*\left(x_n + \frac{h}{2}\right)\right)$$

Hence $x_n = t_n, \quad x_{n+\frac{h}{2}} = t_{n+\frac{h}{2}}, \quad x_{n+1} = t_{n+1}$

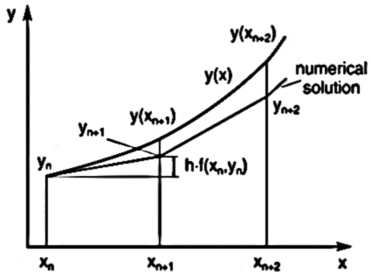
5. Improved Modified Euler's Method

The method we are attempting to improve upon is the *Modified Euler method*. What we are attempting to achieve, is an improvement on the Modified Euler method. I hope to achieve this, by inserting the forward Euler method, in place of y_n in the inner function evaluation of the Modified Euler method thus

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f\left(x_n, y_n + hf\left(x_n, y_n\right)\right)\right)$$

We can go on to rewrite it as $y_{n+1} = y_n + hk_2$, where

$$k_1 = f\left(x_n, y_n\right), \quad k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f\left(x_n, y_n + hk_1\right)\right)$$



6. Adjusted Improved Modified Euler's Method (AIME)

We know that Euler method, $y_{n+1} = y_n + hf(x_n, y_n)$
We have improved modified Euler Method, that is

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f\left(x_n + h, y_n + hf\left(x_n, y_n\right)\right)\right)$$

Consider, we make step size δ tends to zero and divide tangent by 2.

$$f\left(x_n + \delta, y(x_n + \delta\right) = f\left(x_n, y_n + k\right)$$
$$\tan\theta = \frac{dy}{dx} \to \frac{k}{h} = f\left(x_n, y_n\right) \to k = hf\left(x_n, y_n\right)$$

Since tangent divide by 2 then $k = \frac{h}{2} f\left(x_n, y_n\right)$

$$f\left(x_n + \delta, y\left(x_n + \delta\right)\right) = f\left(x_n, y_n + \frac{h}{2} f\left(x_n, y_n\right)\right)$$

So, Improved Modified Euler's equation becomes,

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f\left(x_n, y_n + \frac{h}{2} f\left(x_n, y_n\right)\right)\right)$$

This is our newly modified method, which we can call the Adjusted Improved Modified Euler's (AIME) method.

7. Numerical Experiment

In this section, we present the solution of initial value problem using the new method. We also present the results along with those obtained using the III-known methods. We solve the following first order ordinary differential equation $\frac{dy}{dx} = y - x^2 + 1,\ y(0) = 0.5$ at the interval $(0,2)$ and $h = 0.2$ by using Euler, Improved Euler, Modified Euler, Improved Modified Euler and our proposed Adjusted Improved Modified Euler (AIME). The results are presented in Figures below:

Table 1. Numerical Approximation of the Solution

x	Exact	Euler	ME	IE	IME	AIME
0	0.500000	0.500000	0.500000	0.500000	0.500000	0.500000
0.2	0.829299	0.800000	0.828000	0.826000	0.834000	0.828600
0.4	1.214088	1.152000	1.211360	1.206920	1.225856	1.212807
0.6	1.648941	1.550400	1.644659	1.637242	1.670608	1.647246
0.8	2.127230	1.988480	2.121284	2.110236	2.162184	2.125355
1.0	2.640859	2.458176	2.633167	2.617688	2.693153	2.639128
1.2	3.179942	2.949811	3.170463	3.149579	3.254419	3.178791
1.4	3.732400	3.451773	3.721165	3.693686	3.834849	3.732421
1.6	4.283484	3.950128	4.270622	4.235097	4.420816	4.285462
1.8	4.815176	4.428154	4.800959	4.755619	4.995638	4.820154
2.0	5.305472	4.865785	5.290369	5.233055	5.538901	5.314820

Since the solutions are relatively close there is compactness in the solution graph and it is difficult to distinguish the solution graphs by a single view. So, we calculate the absolute errors table for the methods and represent in a separate error graph.

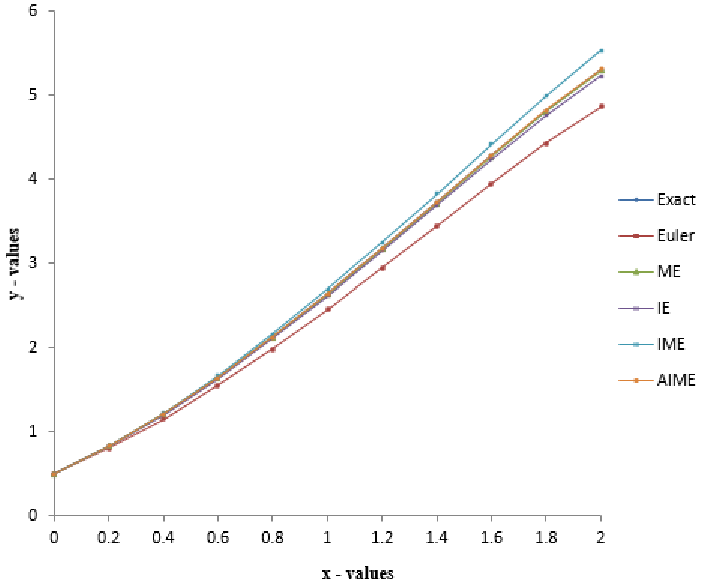


Fig. 1. Graphical Representation of the Solution

Table 2. Numerical Approximation of the Errors

Euler's Error	ME Error	IE Error	IME Error	AIME Error
0	0	0	0	0
0.029299	0.001299	0.003299	0.004701	0.000699
0.062088	0.002728	0.007168	0.011768	0.001281
0.098541	0.004282	0.011699	0.021667	0.001695
0.13875	0.005946	0.016994	0.034954	0.001875
0.182683	0.007692	0.023171	0.052294	0.001731
0.230131	0.009479	0.030363	0.074477	0.001151
0.280627	0.011235	0.038714	0.102449	2.1E-05
0.333356	0.012862	0.048387	0.137332	0.001978
0.387022	0.014217	0.059557	0.180462	0.004978
0.439687	0.015103	0.072417	0.233429	0.009348

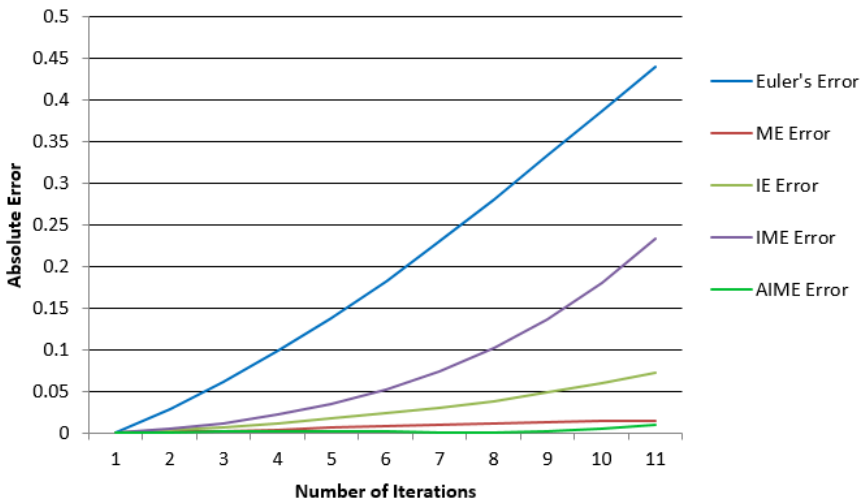


Fig. 2. Graphical Representation of the Errors

8. Result and Discussion

From the above table, we can see that the exact solution is 5.305472. Using Euler's Method, Modified Eulers method, Improved Euler's Method, Improved Modified Euler's Method, and Adjusted Improved Modified Euler's (AIME) method we obtain the corresponding approximations are 4.865785, 5.290369, 5.233055, 5.538901 and 5.314820 respectively. Comparing with exact solution, we can see that the AIME method gives the best approximation of the solution. Euler's method is is based on the idea of approximating the solution of an ODE using a linear approximation of the derivative. It is prone to numerical instability, especially when applied to stiff equations or equations with high frequency components. Improved Euler's method is also known as Heun's method or the explicit trapezoidal rule. It is an improvement over Euler's method, using a midpoint approximation for the derivative. This method is more accurate than Euler's method and less prone to numerical instability. Modified Euler's method is also known as the implicit trapezoidal rule. It is similar to Improved Euler's method, but it uses an implicit midpoint approximation for the derivative. This method is more accurate than Improved Euler's method and even less prone to numerical instability. Improved modified Euler's method is a combination of the improved Euler and modified Euler methods. It uses the midpoint approximation from Improved Euler's method to estimate the derivative at the beginning of the interval, and the implicit midpoint approximation from Modified Euler's method to estimate the derivative at the end of the interval. In the Adjusted Improved Modified Euler's (AIME) method, we used half tangent line at infinitesimal step size.

As we see that, Euler's, Modified Euler's, Improve Euler's, Improved Modified Euler's, Adjusted Improved Modified Euler's (AIME) which gives Approximation to the solution of first order first degree ODEs. But Among all those method, AIME method gives the best Approximation. So, we can say that, accuracy depends on how many times we can Improved or Modified the Euler's Method.

9. Conclusion

The purpose of this paper was to review the Euler's method which the most fundamental method to solve first-order ODE's and enhance its performance by simple modification. The common modified methods are also discussed with their geometrical point of view. After analyzing these methods, a simple modification has done to observe the performance and called the new method as Adjusted Improved Modified Euler's (AIME) method. Finally, some numerical experiments have done and we observed that the new Adjusted Improved Modified Euler's method (AIME) greatly outperforms the other Euler's methods.

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