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Interrelations between Horadam and Generalized Horadam-Leonardo Polynomials via Identities

Research Article

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Abstract: In this paper, we present identities between special cases of Horadam polynomials and special cases of generalized Horadam-Leonardo polynomials.

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1. Introduction: Generalized Fibonacci (Horadam) Polynomials and Generalized Horadam-Leonardo Polynomials

First, we recall the definition and some properties of Horadam polynomials and its two special cases. The generalized Fibonacci polynomials (or Horadam polynomials or x-Horadam numbers or generalized $(r(x), s(x))$-polynomials or $(r(x), s(x))$ Horadam polynomials or 2-step Fibonacci polynomials)

$$\{V_n(V_0(x), V_1(x); r(x), s(x))\}_{n\geq 0}$$

(or $\{V_n(x)\}_{n\geq 0}$ or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n(x) = r(x)V_{n-1}(x) + s(x)V_{n-2}(x), \quad V_0(x) = a(x), V_1(x) = b(x), \quad n \geq 2 \tag{1}$$

where $V_0(x), V_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x) \neq 0$, $s(x) \neq 0$.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n}(x) = -\frac{r(x)}{s(x)}V_{-(n-1)}(x) + \frac{1}{s(x)}V_{-(n-2)}(x)$$

for $n = 1, 2, 3, \ldots$ when $s(x) \neq 0$. Therefore, recurrence (1) holds for all integers n. Note that $V_{-n}(x)$ need not to be a polynomial in the ordinary sense.

For some references on special cases of second-order linear recurrence sequences of polynomials and numbers, see for instance [15, 16, 20, 50, 52, 70, 71] for papers and [1, 14, 24–26, 60, 68] for books.

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Binet's formula of generalized Fibonacci (Horadam) polynomials can be calculated using its characteristic equation (the quadratic equation, polynomial) which is given as

$$y^2 - r(x)y - s(x) = 0. \tag{2}$$

The roots of characteristic equation are

$$\alpha(x) := \alpha = \frac{r(x) + \sqrt{r^2(x) + 4s(x)}}{2}, \quad \beta(x) := \beta = \frac{r(x) - \sqrt{r^2(x) + 4s(x)}}{2}, \tag{3}$$

and the followings hold

$$\begin{aligned} \alpha + \beta &= r(x), \\ \alpha\beta &= -s(x), \\ (\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta = r^2(x) + 4s(x), \end{aligned}$$

If the roots α and β of characteristic equation (2) are distinct, i.e., $\alpha \neq \beta$ then $r^2(x) + 4s(x) \neq 0$ and if the roots α and β of characteristic equation (2) are equal, i.e., $\alpha = \beta$ then (2) can be written as

$$y^2 - r(x)y - s(x) = (y - \alpha)^2 = y^2 - 2\alpha y + \alpha^2 = 0$$

and, in this case,

$$\alpha = \frac{r(x)}{2}, \; r(x) = 2\alpha, \; s(x) = -\alpha^2 = -\frac{r^2(x)}{4}, \; r^2(x) + 4s(x) = 0.$$

Now, we define two special cases of the polynomials $V_n(x)$. $(r(x), s(x))$-Fibonacci polynomials $\{M_n(0,1; r(x), s(x))\}_{n\geq 0}$ (or shortly $M_n(x)$) and $(r(x), s(x))$-Lucas polynomials $\{N_n(2, r(x); r(x), s(x))\}_{n\geq 0}$ (or shortly $N_n(x)$) are defined, respectively, by the second-order recurrence relations

$$M_{n+2}(x) = r(x)M_{n+1} + s(x)M_n(x), \quad M_0(x) = 0, M_1(x) = 1, \tag{4}$$

$$N_{n+2}(x) = r(x)N_{n+1} + s(x)N_n(x), \quad N_0(x) = 2, N_1(x) = r(x). \tag{5}$$

The (sequences of polinomials) $\{M_n(x)\}_{n\geq 0}$ and $\{N_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n}(x) &= -\frac{r(x)}{s(x)}M_{-(n-1)}(x) + \frac{1}{s(x)}M_{-(n-2)}(x), \\ N_{-n}(x) &= -\frac{r(x)}{s(x)}N_{-(n-1)}(x) + \frac{1}{s(x)}N_{-(n-2)}(x), \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (4) and (5) hold for all integers n.

NOTE: For the sake of simplicity throughout the rest of the paper we use

$$V_n, r, s, V_0, V_1, \alpha, \beta, M_n, N_n, M_0, M_1, N_0, N_1$$

instead of

$$V_n(x), r(x), s(x), V_0(x), V_1(x), \alpha(x), \beta(x), M_n(x), N_n(x), M_0(x), M_1(x), N_0(x), N_1(x),$$

respectively. For example, we write

$$V_n = rV_{n-1} + sV_{n-2}, \quad V_0 = a, V_1 = b, \; n \geq 2$$

for the equation (1).

Using the roots α, β and recurrence relation (1), Binet's formula of V_n can be given as follows:

Theorem 1.1.

(a) *(Distinct Roots Case: $\alpha \neq \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$V_n = \frac{r_1\alpha^n}{\alpha - \beta} + \frac{r_2\beta^n}{\beta - \alpha} = \frac{r_1\alpha^n - r_2\beta^n}{\alpha - \beta} \tag{6}$$

where

$$r_1 = V_1 - \beta V_0, \; r_2 = V_1 - \alpha V_0.$$

(b) *(Single Root Case: $\alpha = \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$V_n = (D_1 + D_2 n)\alpha^n \tag{7}$$

where

$$\begin{aligned} D_1 &= V_0, \\ D_2 &= \frac{1}{\alpha}(V_1 - \alpha V_0). \end{aligned}$$

Note that Binet's formulas of M_n and N_n can be given, respectively, as follows:

$$M_n = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ n\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases},$$

$$N_n = \begin{cases} \alpha^n + \beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ 2\alpha^n & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.$$

Now, we define two sequences related to (r,s)-Fibonacci polynomials and (r,s)-Fibonacci-Lucas polynomials. For r,s satisfying Eq. (4) and (5), (r,s)-Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials are defined as

$$G_n(x) = rG_{n-1}(x) + sG_{n-2}(x) + 1 \text{ with } G_0(x) = 0, G_1(x) = 1,\ n \geq 2, \tag{8}$$

and

$$H_n(x) = rH_{n-1}(x) + sH_{n-2}(x) + (1 - s - r) \text{ with } H_0(x) = 3, H_1(x) = r + 1,\ n \geq 2, \tag{9}$$

respectively.

Note that $G_2(x) = r+1$ and $H_2(x) = r^2+2s+1$. The first few values of Horadam-Leonardo polynomials and Horadam-Leonardo-Lucas polynomials are

$$0, 1, r+1, r^2+r+s+1, r^3+r^2+r+2rs+s+1, \ldots$$

and

$$3, r+1, r^2+2s+1, r^3+3sr+1, r^4+4r^2s+2s^2+1, \ldots$$

respectively. Note also that from the equations (8) and (9), we get

$$\begin{aligned} sG_{n-3}(x) &= G_{n-1}(x) - rG_{n-2}(x) - 1, \\ sH_{n-3}(x) &= H_{n-1}(x) - rH_{n-2}(x) - (1 - s - r), \end{aligned}$$

and so the sequences $\{G_n(x)\}$ and $\{H_n(x)\}$ satisfy the following third order linear recurrences:

$$G_n(x) = (r+1)G_{n-1}(x) + (s-r)G_{n-2}(x) - sG_{n-3}(x), \tag{10}$$
$$H_n(x) = (r+1)H_{n-1}(x) + (s-r)H_{n-2}(x) - sH_{n-3}(x). \tag{11}$$

For more information on (r,s)-Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials, see [59].

Note that if we define a sequence of polynomials as

$$Y_n(x) = rY_{n-1}(x) + sY_{n-2}(x) + c(x), \quad \text{with} \quad Y_0(x) = d_1(x), Y_1(x) = d_2(x), \quad n \geq 2$$

where r,s satisfying Eq. (1) and $Y_0(x), Y_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $c(x)$ is a polynomial with real coefficients, then since

$$sY_{n-3}(x) = Y_{n-1}(x) - rY_{n-2}(x) - c(x),$$

we get

$$Y_n(x) = (r+1)Y_{n-1}(x) + (s-r)Y_{n-2}(x) - sY_{n-3}(x).$$

2. Generalized Horadam-Leonardo Polynomials

In this section, for r,s satisfying Eq. (1), we present a sequence and its two special cases, namely the generalized Horadam-Leonardo, (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials.

For r,s satisfying Eq. (1), generalized Horadam-Leonardo polynomials $\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1,W_2;r+1,s-r,-s)\}_{n\geq 0}$ (or shortly $\{W_n(x)\}_{n\geq 0}$) is defined by the third-order recurrence relation

$$W_n(x) = (r+1)W_{n-1}(x) + (s-r)W_{n-2}(x) - sW_{n-3}(x) \tag{12}$$

with the initial values $W_0(x) = c_0(x), W_1(x) = c_1(x), W_2(x) = c_2(x)$ not all being zero and $W_0(x), W_1(x), W_2(x)$ are arbitrary complex (or real) polynomials with real coefficients.

The sequence $\{W_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = \frac{s-r}{s}W_{-(n-1)}(x) + \frac{r+1}{s}W_{-(n-2)}(x) - \frac{1}{s}W_{-(n-3)}(x)$$

for $n = 1,2,3,\ldots$. Therefore, recurrence (12) holds for all integer n. Note that for $n \geq 1$, $W_{-n}(x)$ need not to be a polynomial in the ordinary sense. For more information on generalized Horadam-Leonardo polynomials, see [59].

Some special cases of generalized Horadam-Leonardo (sequence of) polynomials are given as follows (Table 1):

Table 1. A few special cases of generalized Horadam-Leonardo sequence

No	Sequences (Numbers)	r,s	Notation	References
1	Generalized Leonardo	$r=1, s=1$	$\{W_n(W_0,W_1,W_2;2,0,-1)\}$	[53]
2	Generalized John	$r=2, s=1$	$\{W_n(W_0,W_1,W_2;3,-1,-1)\}$	[54]
3	Generalized Ernst	$r=1, s=2$	$\{W_n(W_0,W_1,W_2;2,1,-2)\}$	[55]
4	Generalized Pisano	$r=1, s=-\frac{1}{4}$	$\{W_n(W_0,W_1,W_2;2,-\frac{5}{4},\frac{1}{4})\}$	[56]
5	Generalized Edouard	$r=6, s=-1$	$\{W_n(W_0,W_1,W_2;7,-7,1)\}$	[57]
6	Generalized Bigollo	$r=3, s=-2$	$\{W_n(W_0,W_1,W_2;4,-5,2)\}$	[58]

Generalized Horadam-Leonardo polynomial are special cases of generalized Tribonacci polynomials, for some references on generalized Tribonacci polynomials and its special cases, see for example [8, 28, 30, 32, 51].

Note that the sequences $\{G_n(x)\}$ and $\{H_n(x)\}$ which are defined in the section Introduction, are the special cases of the generalized Horadam-Leonardo (sequence of) polynomials $\{W_n(x)\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n(x)\}$ in this section as well. (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} G_n(x) &= (r+1)G_{n-1}(x) + (s-r)G_{n-2}(x) - sG_{n-3}(x), \\ G_0(x) &= 0, G_1(x) = 1, G_2(x) = r+1, \end{aligned} \tag{13}$$

and

$$\begin{aligned} H_n(x) &= (r+1)H_{n-1}(x) + (s-r)H_{n-2}(x) - sH_{n-3}(x), \\ H_0(x) &= 3, H_1(x) = r+1, H_2(x) = r^2+2s+1. \end{aligned} \tag{14}$$

The sequences $\{G_n(0,1,r+1;r+1,s-r,-s)\}_{n\geq 0}$ and $\{H_n(3,r+1,r^2+2s+1;r+1,s-r,-s)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n}(x) &= \frac{s-r}{s}G_{-(n-1)}(x) + \frac{r+1}{s}G_{-(n-2)}(x) - \frac{1}{s}G_{-(n-3)}(x), \\ H_{-n}(x) &= \frac{s-r}{s}H_{-(n-1)}(x) + \frac{r+1}{s}H_{-(n-2)}(x) - \frac{1}{s}H_{-(n-3)}(x), \end{aligned}$$

for $n = 1,2,3,\ldots$ respectively.

Remark 2.1.
For the sake of simplicity throughout the rest of the paper, we use

$$W_n, W_0, W_1, W_2, G_n, G_0, G_1, G_2, H_n, H_0, H_1, H_2,$$

instead of

$$W_n(x), W_0(x), W_1(x), W_2(x), G(x), G_0(x), G_1(x), G_2(x), H(x), H_0(x), H_1(x), H_2(x)$$

respectively, unless otherwise stated. For example, we write

$$W_n = (r+1)W_{n-1} + (s-r)W_{n-2} - sW_{n-3},\ W_0 = c_0, W_1 = c_1, W_2 = c_2,\ n \geq 3$$

for the equation (12). Also we write U_n, U_0, U_1, U_2 instead of $U_n(x)$ with initial conditions $U_0(x), U_1(x), U_2(x)$ for any subsequence $\{U_n(x)\}$ of $\{W_n\}$.

When r, s, W_0, W_1, W_2 are real numbers we call generalized Horadam-Leonardo, (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials as generalized Horadam-Leonardo, (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas numbers (sequences).

The characteristic equation (the cubic equation, auxiliary equation, polynomial) of W_n is given as

$$y^3 - (r+1)y^2 - (s-r)y + s = (y^2 - ry - s)(y-1) = 0. \tag{15}$$

The roots of characteristic equation are

$$\alpha = \frac{r+\sqrt{r^2+4s}}{2}, \quad \beta = \frac{r-\sqrt{r^2+4s}}{2}, \ \gamma = 1,$$

where α and β are as in (3).

We next present Binet's formula of generalized Horadam-Leonardo polynomials

Corollary 2.1 ([59]).
Binet's formula of generalized Horadam-Leonardo polynomials is given as follows according to the roots of characteristic equation (15):

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = 1$)*

$$\begin{aligned} W_n &= \frac{W_2-(\beta+1)W_1+\beta W_0}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{W_2-(\alpha+1)W_1+\alpha W_0}{(\beta-\alpha)(\beta-1)}\beta^n + \frac{W_2+(-(r+1)+1)W_1+(-s)W_0}{(-s)-(r+1)+2} \\ &= \frac{(\alpha W_2+\alpha(-(r+1)+\alpha)W_1+(-s)W_0)}{(r+1)\alpha^2+2((s-r)-r)\alpha+3(-s)}\alpha^n + \frac{(\beta W_2+\beta(-(r+1)+\beta)W_1+(-s)W_0)}{(r+1)\beta^2+2((s-r)-r)\beta+3(-s)}\beta^n \\ &\quad + \frac{W_2+(-(r+1)+1)W_1+(-s)W_0}{(r+1)+2((s-r)-r)+3(-s)}. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = 1$)*

$$W_n = \frac{1}{(1-(-s))^2}((W_2-2W_1+W_0)\alpha^n + (-W_2+2W_1+(-s)((-s)-2)W_0) + (1-(-s))(W_2-(1+(-s))W_1+(-s)W_0)n).$$

(c) *(Single Root Case: $\alpha = \beta = \gamma = 1 = \frac{(r+1)}{3}$)*

$$W_n = \frac{1}{2}(n(n-1)W_2 - 2n(n-2)W_1 + (n-1)(n-2)W_0).$$

Next, we present Binet's formulas of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials.

Corollary 2.2 ([59]).
For all integers n, Binet's formulas of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials are given as follows:

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = 1$)*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} + \frac{1}{(1-\alpha)(1-\beta)} \\ &= \frac{\alpha^{n+2}}{(r+1)\alpha^2+2(s-r)\alpha+3(-s)} + \frac{\beta^{n+2}}{(r+1)\beta^2+2(s-r)\beta+3(-s)} + \frac{1}{(r+1)+2(s-r)+3(-s)}, \\ H_n &= \alpha^n + \beta^n + 1. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = 1$)*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1} + ((1-\alpha)n - \alpha)}{(1-\alpha)^2}, \\ H_n &= \alpha^n + 2. \end{aligned}$$

(c) *(Single Root Case: $\alpha = \beta = \gamma = 1 = \frac{(r+1)}{3}$)*

$$\begin{aligned} G_n &= \frac{n(n+1)}{2}, \\ H_n &= 3. \end{aligned}$$

For more information on generalized Horadam-Leonardo polynomials, see Soykan [59]

3. Identities on Horadam and Generalized Horadam-Leonardo Polynomials

In this section, we present identities between Horadam, (r,s)-Horadam, (r,s)-Horadam-Lucas polynomials and generalized Horadam-Leonardo, (r,s)-Horadam-Leonardo, (r,s)-Horadam-Leonardo-Lucas polynomials

Since

$$N_n = \begin{cases} \alpha^n + \beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ 2\alpha^n & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}$$

and

$$H_n = \alpha^n + \beta^n + 1$$

we see that

$$N_n = H_n - 1$$

i.e.,

$$H_n = N_n + 1.$$

There are close relations between Horadam polynomials and generalized Horadam-Leonardo polynomials. For example, they satisfy the following interrelations:

Lemma 3.1.
The following equalities are true:

(a) $(W_2^2+(r-s+1)W_1^2-sW_0^2-(r+2)W_1W_2+rW_0W_2+(2s-r)W_0W_1)(W_2-rW_1-sW_0)(V_{n+2}+k) = \Lambda_1 W_{n+2}+\Lambda_2 W_{n+1}+\Lambda_3 W_n$

where

$\Lambda_1 = (rV_1+sV_0+k)W_2^2+((r^3+r^2+rs)V_1+rs(r+1)V_0+k(r-s+1))W_1^2-s((s+r^2)V_1+rsV_0+k)W_0^2-((r+s+2r^2)V_1+s(2r+1)V_0+k(r+2))W_1W_2+((r^2-rs+s)V_1+s(r-s)V_0+kr)W_0W_2-((r^3-r^2s-s^2)V_1+s(r^2-rs-s)V_0+k(r-2s))W_0W_1$,

$\Lambda_2 = -((r-s)V_1+sV_0+kr)W_2^2-r(r-s+1)(rV_1+sV_0+k)W_1^2+s((r^2-rs+s)V_1+s(r-s)V_0+kr)W_0^2+(r(2r-2s+1)V_1+s(2r-s+1)V_0+kr(r+2))W_1W_2-((r^2+s^2+s-2rs)V_1+s(r-2s)V_0+kr^2)W_0W_2+(r(r-s)^2V_1+s(r^2+s^2-s-2rs)V_0+kr(r-2s))W_0W_1$,

$\Lambda_3 = -s((V_1+k)W_2^2+(r(r+1)V_1+rsV_0+k(r-s+1))W_1^2-s(rV_1+sV_0+k)W_0^2-((2r+1)V_1+sV_0+k(r+2))W_1W_2+((r-s)V_1+sV_0+kr)W_0W_2+(-\left(r^2-rs-s\right)V_1+s(s-r)V_0+k(2s-r))W_0W_1)$,

and k *is a polynomial with real coefficients.*

(b) $(W_2^2+(r-s+1)W_1^2-sW_0^2-(r+2)W_1W_2+rW_0W_2+(2s-r)W_0W_1)(W_2-rW_1-sW_0)(V_n+k) = \Lambda_1 W_{n+2}+\Lambda_2 W_{n+1}+\Lambda_3 W_n$

where

$\Lambda_1 = (k+V_0)W_2^2+(rV_1+rV_0+k(r-s+1))W_1^2-s(k+V_1)W_0^2-(V_1+(r+1)V_0+k(2+r))W_1W_2+(V_1-sV_0+kr)W_0W_2-((r-s)V_1-sV_0+k(r-2s))W_0W_1$,

$\Lambda_2 = (V_1-(r+1)V_0-kr)W_2^2-r(r-s+1)(k+V_0)W_1^2+s(V_1-sV_0+kr)W_0^2+(-rV_1+(r^2+2r-s+1)V_0+kr(r+2))W_1W_2-((1+s)V_1-s(r+2)V_0+kr^2)W_0W_2+(rV_1-s(2r-s+1)V_0+kr(r-2s))W_0W_1$,

$\Lambda_3 = (-V_1+rV_0-ks)W_2^2-(rV_1-r(r-s)V_0+ks(r-s+1))W_1^2+s^2(k+V_0)W_0^2+((1+r)V_1-(r-s+r^2)V_0+ks(r+2))W_1W_2+s(V_1-(r+1)V_0-kr)W_0W_2+s(-V_1+(2r-s)V_0+k(r-2s))W_0W_1$,

and k *is a polynomial with real coefficients.*

(c)

(i) $(V_1^2-sV_0^2-rV_0V_1)(r+s-1)W_n = \Lambda_1 V_{n+1}+\Lambda_2 V_n+\Lambda_3$

where

$\Lambda_1 = (V_1-V_0)W_2+(-rV_1+(1-s)V_0)W_1+((r-1)V_1+sV_0)W_0$,

$\Lambda_2 = ((1-r)V_1-sV_0)W_2+((r^2+s-1)V_1+rsV_0)W_1-((r^2-r+s)V_1+s(r-1)V_0)W_0$,

$\Lambda_3 = -(V_1^2-sV_0^2-rV_0V_1)(W_2-rW_1-sW_0)$,

$\Lambda_3 = -(V_1^2-sV_0^2-rV_0V_1)(W_2-rW_1-sW_0)$.

(ii) $(V_1^2 - sV_0^2 - rV_0V_1)(r+s-1)W_{n+3} = \Lambda_1 V_{n+1} + \Lambda_2 V_n + \Lambda_3$
where
$\Lambda_1 = ((r^2+rs+s)V_1 - (r^3+r^2s+s^2+2rs)V_0)W_2 + (-(r^2-s^2+s)V_1 + r(r^2-s^2+2s)V_0)W_1 + s(-(r+s)V_1 + (r^2+rs+s)V_0)W_0$
$\Lambda_2 = s(((r+s)V_1 - (r^2+rs+s)V_0)W_2 + (-rV_1 + (r^2-s^2+s)V_0)W_1 + s(-V_1 + (r+s)V_0)W_0)$
$\Lambda_3 = -(V_1^2 - sV_0^2 - rV_0V_1)(W_2 - rW_1 - sW_0)$

Proof. Use Binet's formulas of Horadam polynomials and generalized Horadam-Leonardo polynomials or induction. □

The following Corollary present identities between (r,s)-Fibonacci polynomials and (r,s)-Horadam-Leonardo polynomials.

Corollary 3.1.
The following equalities are true:

(a)

(i) $M_{n+2} = (k+1)G_{n+2} - (kr+1)G_{n+1} - ksG_n - k$
where
k is any polynomial with real coefficients.

(ii) $M_{n+2} = G_{n+2} - G_{n+1}$

(iii) $M_{n+2} = 2G_{n+2} - (r+1)G_{n+1} - sG_n - 1$

(iv) $M_{n+2} = (r-1)G_{n+1} + sG_n + 1$

(b)

(i) $sM_n = (ks+1)G_{n+2} - (krs+r+1)G_{n+1} + (r-ks^2)G_n - ks$
where
k is any polynomial with real coefficients.

(ii) $sM_n = G_{n+2} - (r+1)G_{n+1} + rG_n$

(iii) $sM_n = (s+1)G_{n+2} - (rs+r+1)G_{n+1} + (r-s^2)G_n - s$

(iv) $sM_n = (-s+1)G_{n+2} - (-rs+r+1)G_{n+1} + (r+s^2)G_n + s$

(c)

(i) $(r+s-1)G_n = M_{n+1} + sM_n - 1$

(ii) $(r+s-1)G_{n+3} = (r^3+r^2s+s^2+2rs)M_{n+1} + s\left(s+rs+r^2\right)M_n - 1$

Proof. We use Lemma 3.1.

(a) For (i), $W_n = G_n$ with $G_0(x) = 0, G_1(x) = 1, G_2(x) = r+1$ and $V_n = M_n$ with $M_0(x) = 0, M_1(x) = 1$ in Lemma 3.1 (a). For (ii), (iii) and (iv), take $k = 0, k = 1, k = -1$, respectively, in (i).

(b) For (i), take $W_n = G_n$ with $G_0(x) = 0, G_1(x) = 1, G_2(x) = r+1$ and $V_n = M_n$ with $M_0(x) = 0, M_1(x) = 1$ in Lemma 3.1 (b). For (ii), (iii) and (iv), take $k = 0, k = 1, k = -1$, respectively, in (i).

(c)

(i) Take $W_n = G_n$ with $G_0(x) = 0, G_1(x) = 1, G_2(x) = r+1$ and $V_n = M_n$ with $M_0(x) = 0, M_1(x) = 1$ in Lemma 3.1 (c) (i).

(ii) Take $W_n = G_n$ with $G_0(x) = 0, G_1(x) = 1, G_2(x) = r+1$ and $V_n = M_n$ with $M_0(x) = 0, M_1(x) = 1$ in Lemma 3.1 (c) (ii). □

There are close relations between (r,s)-Fibonacci-Lucas polynomials and (r,s)-Horadam-Leonardo polynomials. For example, they satisfy the following interrelations:

Corollary 3.2.
The following equalities are true:

(a)

- **(i)** $N_{n+2} = (k+r)G_{n+2} + (2s-(k+1)r)G_{n+1} - s(k+2)G_n - k$
  where
  k is any polynomial with real coefficients.
- **(ii)** $N_{n+2} = rG_{n+2} + (2s-r)G_{n+1} - 2sG_n$
- **(iii)** $N_{n+2} = (1+r)G_{n+2} + (2s-2r)G_{n+1} - 3sG_n - 1$
- **(iv)** $N_{n+2} = (-1+r)G_{n+2} + 2sG_{n+1} - sG_n + 1$

(b)

- **(i)** $sN_n = (ks-r)G_{n+2} + (r^2+r+2s-krs)G_{n+1} - (r^2+2s+ks^2)G_n - ks$
  where
  k is any polynomial with real coefficients.
- **(ii)** $sN_n = -rG_{n+2} + (r^2+r+2s)G_{n+1} - (r^2+2s)G_n$
- **(iii)** $sN_n = (s-r)G_{n+2} + (r^2+r+2s-rs)G_{n+1} - (r^2+2s+s^2)G_n - s$
- **(iv)** $sN_n = -(s+r)G_{n+2} + (r^2+r+2s+rs)G_{n+1} - (r^2+2s-s^2)G_n + s$

(c)

- **(i)** $(r^2+4s)(r+s-1)G_n = (r+2s)N_{n+1} - s(r-2)N_n - (r^2+4s)$
- **(ii)** $(r^2+4s)(r+s-1)G_{n+3} = (r^4+r^3s+4r^2s+3rs^2+2s^2)N_{n+1} + s(r^3+r^2s+3rs+2s^2)N_n - (r^2+4s)$

Proof. We use Lemma 3.1.

(a) For (i), $W_n = G_n$ with $G_0(x) = 0, G_1(x) = 1, G_2(x) = r+1$ and $V_n = N_n$ with $N_0(x) = 2, N_1(x) = r$ in Lemma 3.1 (a). For (ii), (iii) and (iv), take $k = 0, k = 1, k = -1$, respectively, in (i).

(b) For (i), take $W_n = G_n$ with $G_0(x) = 0, G_1(x) = 1, G_2(x) = r+1$ and $V_n = N_n$ with $N_0(x) = 2, N_1(x) = r$ in Lemma 3.1 (b). For (ii), (iii) and (iv), take $k = 0, k = 1, k = -1$, respectively, in (i).

(c)

- **(i)** Take $W_n = G_n$ with $G_0(x) = 0, G_1(x) = 1, G_2(x) = r+1$ and $V_n = N_n$ with $N_0(x) = 2, N_1(x) = r$ in Lemma 3.1 (c) (i).
- **(ii)** Take $W_n = G_n$ with $G_0(x) = 0, G_1(x) = 1, G_2(x) = r+1$ and $V_n = N_n$ with $N_0(x) = 2, N_1(x) = r$ in Lemma 3.1 (c) (ii). □

The following Corollary present identities between (r,s)-Fibonacci polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials.

Corollary 3.3.
The following equalities are true:

(a)

- **(i)** $(r^2+4s)(r+s-1)M_{n+2} = (r^2+rs+2s-k(r^2+4s))H_{n+2} - (r^2+2s-2s^2-kr(r^2+4s))H_{n+1} + s(-r-2s+k(r^2+4s))H_n - k(r^2+4s)(r+s-1)$
  where
  k is any polynomial with real coefficients.
- **(ii)** $(r^2+4s)(r+s-1)M_{n+2} = (r^2+rs+2s)H_{n+2} - (r^2+2s-2s^2)H_{n+1} - s(r+2s)H_n$
- **(iii)** $(r^2+4s)(r+s-1)M_{n+2} = s(r-2)H_{n+2} + (r^3-r^2+4rs+2s^2-2s)H_{n+1} + s(r^2-r+2s)H_n - (r^2+4s)(r+s-1)$
- **(iv)** $(r^2+4s)(r+s-1)M_{n+2} = (2r^2+rs+6s)H_{n+2} - (r^3+r^2+4rs-2s^2+2s)H_{n+1} - s(r^2+r+6s)H_n + (r^2+4s)(r+s-1)$

(b)

- **(i)** $(r^2+4s)(r+s-1)M_n = -(r-2+k(r^2+4s))H_{n+2} + (r^2+2s-2+kr(r^2+4s))H_{n+1} + (-r^2+r-2s+ks(r^2+4s))H_n - k(r^2+4s)(r+s-1)$
  where
  k is any polynomial with real coefficients.

(ii) $(r^2+4s)(r+s-1)M_n = -(r-2)H_{n+2}+(r^2+2s-2)H_{n+1}+(-r^2+r-2s)H_n$

(iii) $(r^2+4s)(r+s-1)M_n = -(r^2+r+4s-2)H_{n+2}+(r^3+r^2+4sr+2s-2)H_{n+1}+(r^2s-r^2+r+4s^2-2s)H_n-(r^2+4s)(r+s-1)$

(iv) $(r^2+4s)(r+s-1)M_n = (r^2-r+4s+2)H_{n+2}+(-r^3+r^2-4sr+2s-2)H_{n+1}-(r^2s+r^2-r+4s^2+2s)H_n+(r^2+4s)(r+s-1)$

(c)

(i) $H_n = 2M_{n+1}-rM_n+1$

(ii) $H_{n+3} = r(r^2+3s)M_{n+1}+s(r^2+2s)M_n+1$

Proof. We use Lemma 3.1.

(a) For (i), $W_n = H_n$ with $H_0(x) = 3, H_1(x) = r+1, H_2(x) = r^2+2s+1$ and $V_n = M_n$ with $M_0(x) = 0, M_1(x) = 1$ in Lemma 3.1 (a). For (ii), (iii) and (iv), take $k = 0, k = 1, k = -1$, respectively, in (i).

(b) For (i), take $W_n = H_n$ with $H_0(x) = 3, H_1(x) = r+1, H_2(x) = r^2+2s+1$ and $V_n = M_n$ with $M_0(x) = 0, M_1(x) = 1$ in Lemma 3.1 (b). For (ii), (iii) and (iv), take $k = 0, k = 1, k = -1$, respectively, in (i).

(c)

(i) Take $W_n = H_n$ with $H_0(x) = 3, H_1(x) = r+1, H_2(x) = r^2+2s+1$ and $V_n = M_n$ with $M_0(x) = 0, M_1(x) = 1$ in Lemma 3.1 (c) (i).

(ii) Take $W_n = H_n$ with $H_0(x) = 3, H_1(x) = r+1, H_2(x) = r^2+2s+1$ and $V_n = M_n$ with $M_0(x) = 0, M_1(x) = 1$ in Lemma 3.1 (c) (ii). □

There are close relations between (r,s)-Fibonacci-Lucas polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials. For example, they satisfy the following interrelations:

Corollary 3.4.
The following equalities are true:

(a)

(i) $(r+s-1)N_{n+2} = (r+s-k)H_{n+2}+r(k-1)H_{n+1}+s(k-1)H_n-k(r+s-1)$
where
k is any polynomial with real coefficients.

(ii) $(r+s-1)N_{n+2} = (r+s)H_{n+2}-rH_{n+1}-sH_n$

(iii) $N_n = H_n-1$

(iv) $(r+s-1)N_{n+2} = (r+s+1)H_{n+2}-2rH_{n+1}-2sH_n+(r+s-1)$

(b)

(i) $(r+s-1)N_n = -(k-1)H_{n+2}+r(k-1)H_{n+1}+(r+ks-1)H_n-k(r+s-1)$
where
k is any polynomial with real coefficients.

(ii) $(r+s-1)N_n = H_{n+2}-rH_{n+1}+(r-1)H_n$

(iii) $N_n = H_n-1$

(iv) $(r+s-1)N_n = 2H_{n+2}-2rH_{n+1}+(r-s-1)H_n+(r+s-1)$

(c)

(i) $H_n = N_n+1$

(ii) $H_{n+3} = (r^2+s)N_{n+1}+rsN_n+1$

Proof. We use Lemma 3.1.

(a) For (i), $W_n = H_n$ with $H_0(x) = 3, H_1(x) = r+1, H_2(x) = r^2+2s+1$ and $V_n = N_n$ with $N_0(x) = 2, N_1(x) = r$ in Lemma 3.1 (a). For (ii), (iii) and (iv), take $k = 0, k = 1, k = -1$, respectively, in (i).

(b) For (i), take $W_n = H_n$ with $H_0(x) = 3, H_1(x) = r+1, H_2(x) = r^2+2s+1$ and $V_n = N_n$ with $N_0(x) = 2, N_1(x) = r$ in Lemma 3.1 (b). For (ii), (iii) and (iv), take $k = 0, k = 1, k = -1$, respectively, in (i).

(c)

(i) Take $W_n = H_n$ with $H_0(x) = 3, H_1(x) = r+1, H_2(x) = r^2+2s+1$ and $V_n = N_n$ with $N_0(x) = 2, N_1(x) = r$ in Lemma 3.1 (c) (i).

(ii) Take $W_n = H_n$ with $H_0(x) = 3, H_1(x) = r+1, H_2(x) = r^2+2s+1$ and $V_n = N_n$ with $N_0(x) = 2, N_1(x) = r$ in Lemma 3.1 (c) (ii). □

4. Conclusions

The Fibonacci and Lucas sequences which are special cases of Haradam sequence are sources of many nice and interesting identities. For rich applications of these second order sequences in science and nature, one can see the citations in [24–26].

In this study, we present identities between special cases of Horadam (generalized Fibonacci) polynomials and special cases of generalized Horadam-Leonardo polynomials.

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [2].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [5].
- For the application of Jacobsthal numbers to special matrices, see [69].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [19].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [65].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [66].
- For the applications of generalized Fibonacci numbers to binomial sums, see [64].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [35].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [36].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [11].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [31].
- For the application of higher order Jacobsthal numbers to quaternions, see [29].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [17].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [4].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [21].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [22].
- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see [27].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [62].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [9].

We now present some applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [7] and [6], respectively.

- For the application of Tribonacci numbers to special matrices, see [67].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [33] and [3], respectively.
- For the application of Pell-Padovan numbers to groups, see [10].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [18].
- For the application of Gaussian Tribonacci numbers to various graphs, see [61].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [12].
- For the application of Narayan numbers to finite groups see [23].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [37].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [38].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [39].
- For the application of generalized Tribonacci numbers to Sedenions, see [40].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [41].
- For the application of generalized Tribonacci numbers to circulant matrix, see [42].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [63].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [13].

Next, we now list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [43].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [**?**].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [45].
- For the application of generalized Tetranacci numbers to binomial transform, see [46].

We now present some applications of fifth order sequences.

- For the application of Pentanacci numbers to matrices, see [34].
- For the application of generalized Pentanacci numbers to quaternions, see [48].
- For the application of generalized Pentanacci numbers to binomial transform, see [47].

  We now present some applications of second order sequences of polynomials.
- For the application of generalized Fibonacci Polynomials to the summation formulas, see [49].
- For some applications of generalized Fibonacci Polynomials, see [50].

  We now present some applications of third order sequences of polynomials.
- For some applications of generalized Tribonacci Polynomials, see [51].

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