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# Age-structured three-patch model of malaria transmission dynamics in Chad

Research Article

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**Abstract:** In this article, we investigated an age-structured model of malaria transmission dynamics. The population was divided into four parts (susceptible, latent, infectious and recovered) in the first and third patch according to their epidemiological statuses. In the second patch, it was divided into three parts (susceptible, latent and infectious). We allowed migrations in all three patches. After determining the R0, the local and global stabilities of the disease-free equilibrium point were studied. We have shown the existence of an endemic equilibrium point. We ended our study with numerical simulations.

**MSC:** 92D30 • 93C20 • 97N40

**Keywords:** Age-Structured • Reproductive Number • Three-Patch • Malaria • Stability • Simulation



## 1. Introduction

Malaria, a vector-borne disease caused by the bites of infected female mosquitoes, is responsible for approximately 409,000 deaths per year worldwide according to the World Health Organization [1]. Its devastating effects mainly affect the sub-Saharian region where 94them, children under five years and pregnant women are the most vulnerable groups wich approximately 67of deaths. Although efforts of the government in different African countries are increasing every year, malaria remains one of the major public health problems faced by these countries. In the way to understand this helf problems mathematical models provide useful tools for the comprehension of dynamics of malaria transmission. Since the paper of Ross and MacDonald [2], various extensions of their models have been published. The model we propose here extends the ideas in [3] and [4]. The mathematical analysis follows the approaches of [3], [4], [5] and [6]. The paper is organise of follows: in section 2, we formulate the model. In section 3, we shows the existence and uniqueness of the solution. Section 4 deals with the analysis of the model by calculating the basic reproduction number after determining the steady state. In section 5, the existence of the endemic equilibrium point is examined. We finish our work in section 6 with the numerical simulations and concluate in section 7.

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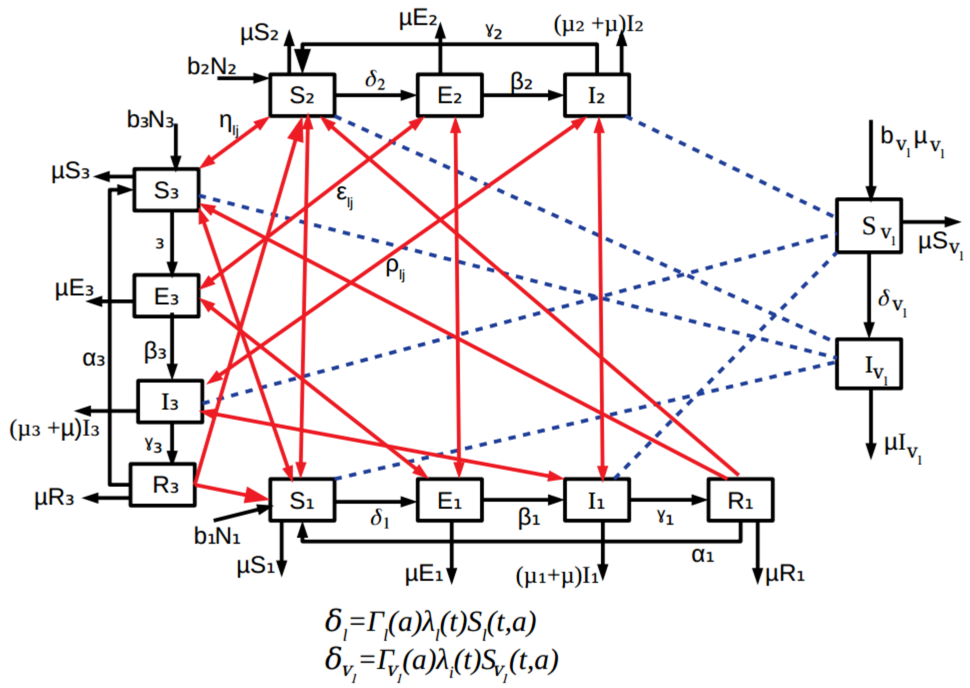


**Fig. 1.** Flow diagram of malaria transmission in three patches

## 2. Formulation of the model

This mathematical model will take into account the migrations in the three endemic areas, the effects of acquired immunity against the disease and the temporary nature of this immunity in the human population. We represent respectively by $S_l(t,a)$, $E_l(t,a)$, $I_l(t,a)$, $R_l(t,a)$ for Susceptible, Latent, Infectious, and Withdrawn humans. Similarly, we note respectively by $S_{v_l}(t,a)$ and $I_{v_l}(t,a)$ susceptible and infectious mosquitoes. The model parameters are:

• $N(t,a)$ and $N_v(t,a)$ represent total populations for humans and mosquitoes, respectively.

• $\Gamma_l(a)\lambda_l(t)$ and $\Gamma_{v_l}(a)\lambda_l(t)$ represent the infection forces for humans and mosquitoes respectively.

• $\Gamma_l(a)$ and $\Gamma_{v_l}(a)$ represent the probabilities of becoming infectious to humans and mosquitoes, respectively

• $\lambda_l(t)$ is the total number of infection sources

• $\varphi_l(a)$ is the pitting rate.

• $b_l(a)$ and $b_{v_l}(a)$ birth rates for humans and mosquitoes respectively

• $\mu(a)$ and $\mu_v(a)$ represent the natural mortality rates imposed on all compartments for humans and mosquitoes, respectively.

• $\mu_l(a)$ is the death rate due to the disease.

• $\beta_l(a)$ is the rate of passage from the latent class to that of the infectious.

• $\gamma_l(a)$ is the transition rate from the infectious class to that of the retired.

• $\alpha_l(a)$ is the transition rate from the class of withdrawn to that of susceptible.

• $\eta_l j$, $\epsilon_l j$, $\rho_l j$ (l,j=1,2,3 with $l \neq j$) denote respectively the migration coefficients of susceptible, latent and infectious. We note $\alpha_m$ that of the retired, $m = 1$ and $m = 3$.

•$\Lambda_l$ is the initial population at patch l.

We denote by $a_+$ the maximum age for the human population.

The general dynamics is shown in the figure as follows:

The black arrows indicate the passage from one class to another in the same patch, the red arrows represent the migrations of the human population in the different patches and the dotted lines indicate the possibility for the human or mosquito population to be infected.

This dynamics is governed by the following system of partial differential equations:

$$
\left\{
\begin{array}{rcl}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)S_1(t,a) &=& b_1(a)N_1(t,a)+\alpha_1R_1(t,a)+\alpha_3R_3(t,a)+\eta_{21}S_2(t,a)+\eta_{31}S_3(t,a)\\
&& -\left[\eta_{12}+\eta_{13}+\Gamma_1(a)\lambda_1(t)+\mu(a)\right]S_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)E_1(t,a) &=& \Gamma_1(a)\lambda_1(t)S_1(t,a)+\epsilon_{21}E_2(t,a)+\epsilon_{31}E_3(t,a)-\left[\epsilon_{12}+\epsilon_{13}+\beta_1(a)+\mu(a)\right]E_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_1(t,a) &=& \beta_1(a)E_1(t,a)+\rho_{21}I_2(t,a)+\rho_{31}I_3(t,a)-\left[\rho_{12}+\rho_{13}+\gamma_1(a)+\mu(a)+\mu_1(a)\right]I_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)R_1(t,a) &=& \gamma_1(a)I_1(t,a)-\left[3\alpha_1+\mu(a)\right]R_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)S_2(t,a) &=& b_2(a)N_2(t,a)+\alpha_1R_1(t,a)+\alpha_3R_3(t,a)+\eta_{12}S_1(t,a)+\eta_{32}S_3(t,a)+\gamma_2(a)I_2(t,a)\\
&& -\left[\eta_{21}+\eta_{23}+\Gamma_2(a)\lambda_2(t)+\mu(a)\right]S_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)E_2(t,a) &=& \Gamma_2(a)\lambda_2(t)S_2(t,a)+\epsilon_{12}E_1(t,a)+\epsilon_{32}E_3(t,a)-\left[\epsilon_{21}+\epsilon_{23}+\beta_2(a)+\mu(a)\right]E_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_2(t,a) &=& \beta_2(a)E_2(t,a)+\rho_{12}I_1(t,a)+\rho_{32}I_3(t,a)-\left[\rho_{21}+\rho_{23}+\gamma_2(a)+\mu(a)+\mu_2(a)\right]I_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)S_3(t,a) &=& b_3(a)N_3(t,a)+\alpha_1R_1(t,a)+\alpha_3R_3(t,a)+\eta_{13}S_1(t,a)+\eta_{23}S_2(t,a)\\
&& -\left[\eta_{31}+\eta_{32}+\Gamma_3(a)\lambda_3(t)+\mu(a)\right]S_3(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)E_3(t,a) &=& \Gamma_3(a)\lambda_3(t)S_3(t,a)+\epsilon_{13}E_1(t,a)+\epsilon_{23}E_3(t,a)-\left[\epsilon_{31}+\epsilon_{32}+\beta_3(a)+\mu(a)\right]E_3(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_3(t,a) &=& \beta_3(a)E_3(t,a)+\rho_{13}I_1(t,a)+\rho_{23}I_2(t,a)-\left[\rho_{31}+\rho_{32}+\gamma_3(a)+\mu(a)+\mu_3(a)\right]I_3(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)R_3(t,a) &=& \gamma_3(a)I_3(t,a)-\left[3\alpha_3+\mu(a)\right]R_3(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)S_{v_1}(t,a) &=& b_{v_1}(a)N_{v_1}(t,a)-\left[\Gamma_{v_1}(a)\lambda_1(t)+\mu_v(a)\right]S_{v_1}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_{v_1}(t,a) &=& \Gamma_{v_1}(a)\lambda_1(t)S_{v_1}(t,a)-\mu_v(a)I_{v_1}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)S_{v_2}(t,a) &=& b_{v_2}(a)N_{v_2}(t,a)-\left[\Gamma_{v_2}(a)\lambda_2(t)+\mu_v(a)\right]S_{v_2}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_{v_2}(t,a) &=& \Gamma_{v_2}(a)\lambda_2(t)S_{v_2}(t,a)-\mu_v(a)I_{v_2}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)S_{v_3}(t,a) &=& b_{v_3}(a)N_{v_3}(t,a)-\left[\Gamma_{v_3}(a)\lambda_3(t)+\mu_v(a)\right]S_{v_3}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_{v_3}(t,a) &=& \Gamma_{v_3}(a)\lambda_3(t)S_{v_3}(t,a)-\mu_v(a)I_{v_3}(t,a)
\end{array}
\right. \tag{1}
$$

The initial and boundary conditions of the model are:

$$
\left\{
\begin{array}{rcl}
S_l(t,0) &=& \int_{a_1}^{a_2} b_l(a)N_l(t,0)\,da\\
I_l(t,0) &=& E_l(t,0)=0\\
R_1(t,0) &=& R_3(t,0)=0\\
S_{v_l}(t,0) &=& N_{v_l}(t,0)\\
I_{v_l}(t,0) &=& 0
\end{array}
\right.
$$

and

$$
\left\{
\begin{array}{rcl}
S_l(0,a) &=& S_{0l}(a)\\
E_l(0,a) &=& E_{0l}(a)\\
I_l(0,a) &=& I_{0l}(a)\\
R_1(0,a) &=& R_{01}(a)\\
R_3(0,a) &=& R_{03}(a)\\
S_{v_l}(0,a) &=& S_{v_{0l}}(a) \quad with \quad l=1,2,3\\
I_{v_l}(0,a) &=& I_{v_{0l}}(a)
\end{array}
\right.
$$

Let $N(t,a)=N_1(t,a)+N_2(t,a)+N_3(t,a)$ and $N_l(t,a)=\pi_l N(t,a)$, with $\pi_1+\pi_2+\pi_3=1$.

$$
N(t,a)=\sum_{l=1}^{3}\left(S_l(t,a)+E_l(t,a)+I_l(t,a)\right)+R_1(t,a)+R_3(t,a), \qquad N_v(t,a)=\sum_{l=1}^{3}\left(S_{v_l}+I_{v_l}\right)
$$

and
$b(a)=\pi_1 b_1(a)+\pi_2 b_2(a)+\pi_3 b_3(a)$
Adding all equations in system [1], we get:

$$
\left\{
\begin{array}{rcl}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)N(t,a) &=& \left(b(a)-\mu(a)\right)N(t,a)-\sum_{l=1}^{3}\mu_l(a)I_l(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)N_v(t,a) &=& \left(b_v(a)-\mu_v(a)\right)N_v(t,a)
\end{array}
\right.
$$

The first equation of this system clearly shows that the total population is affected by the disease.

The infection strength used in this work is given as in [4]:

$$\delta_l = \Gamma_l(a)\lambda_l(t) = \Gamma_l(a)\int_0^{a_+}\varphi_l(a)\left(\frac{I_l(t,a)+\theta_l I_{v_l}(t,a)}{N_l(t,a)}\right)da$$

where $\theta_l = \frac{N_l(t,a)}{N_{v_l}(t,a)}$ with l=1,2,3.

For the purpose of the analysis, we introduce the following fractions:

$$\begin{cases} s_l(t,a) &= \frac{S_l(t,a)}{N(t,a)}\\ e_l(t,a) &= \frac{E_l(t,a)}{N(t,a)}\\ i_l(t,a) &= \frac{I_l(t,a)}{N(t,a)}\\ r_1(t,a) &= \frac{R_1(t,a)}{N(t,a)}\\ r_3(t,a) &= \frac{R_3(t,a)}{N(t,a)}\\ s_{v_l}(t,a) &= \frac{S_{v_l}(t,a)}{N_v(t,a)}\\ i_{v_l}(t,a) &= \frac{I_{v_l}(t,a)}{N_v(t,a)} \end{cases}.$$

We get the following normalized system:

$$\begin{cases}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s_1(t,a) &= \pi_1 b_1(a)+\alpha_1 r_1(t,a)+\alpha_3 r_3(t,a)+\eta_{21}s_2(t,a)+\eta_{31}s_3(t,a)\\
&\quad -\left[\eta_{12}+\eta_{13}+b(a)+\Gamma_1(a)\lambda_1(t)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]s_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)e_1(t,a) &= \Gamma_1(a)\lambda_1(t)s_1(t,a)+\epsilon_{21}e_2(t,a)+\epsilon_{31}e_3(t,a)\\
&\quad -\left[\epsilon_{12}+\epsilon_{13}+\beta_1(a)+b(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]e_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_1(t,a) &= \beta_1(a)e_1(t,a)+\rho_{21}i_2(t,a)+\rho_{31}i_3(t,a)\\
&\quad -\left[\rho_{12}+\rho_{13}+\gamma_1(a)+\mu_1(a)+b(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]i_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)r_1(t,a) &= \gamma_1(a)\,i_1(t,a)-\left[3\alpha_1+b(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]r_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s_2(t,a) &= \pi_2 b_2(a)+\alpha_1 r_1(t,a)+\alpha_3 r_3(t,a)+\eta_{12}s_1(t,a)+\eta_{32}s_3(t,a)+\gamma_2(a)\,i_2(t,a)\\
&\quad -\left[\eta_{21}+\eta_{23}+\Gamma_2(a)\lambda_2(t)+b(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]s_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)e_2(t,a) &= \Gamma_2(a)\lambda_2(t)s_2(t,a)+\epsilon_{12}e_1(t,a)+\epsilon_{32}e_3(t,a)\\
&\quad -\left[\epsilon_{21}+\epsilon_{23}+\beta_2(a)+b(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]e_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_2(t,a) &= \beta_2(a)e_2(t,a)+\rho_{12}i_1(t,a)+\rho_{32}i_3(t,a)\\
&\quad -\left[\rho_{21}+\rho_{23}+\mu_2(a)+b(a)+\gamma_2(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]i_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s_3(t,a) &= \pi_3 b_3(a)+\alpha_1 r_1(t,a)+\alpha_3 r_3(t,a)+\eta_{13}s_1(t,a)+\eta_{23}s_2(t,a)\\
&\quad -\left[\eta_{31}+\eta_{32}+b(a)+\Gamma_3(a)\lambda_3(t)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]s_3(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)e_3(t,a) &= \Gamma_3(a)\lambda_3(t)s_3(t,a)+\epsilon_{13}e_1(t,a)+\epsilon_{23}e_2(t,a)\\
&\quad -\left[\epsilon_{31}+\epsilon_{32}+\beta_3(a)+b(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]e_3(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_3(t,a) &= \beta_3(a)e_3(t,a)+\rho_{13}i_1(t,a)+\rho_{23}i_2(t,a)\\
&\quad -\left[\rho_{31}+\rho_{32}+\mu_3(a)+b(a)+\gamma_3(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]i_3(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)r_3(t,a) &= \gamma_3(a)\,i_3(t,a)-\left[3\alpha_3+b(a)-\sum_{l=1}^3\mu_l(a)\,i_l(t,a)\right]r_3(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s_{v_1}(t,a) &= b_{v_1}(a)-\left[\Gamma_{v_1}(a)\lambda_1(t)+\mu_v(a)\right]s_{v_1}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_{v_1}(t,a) &= \Gamma_{v_1}(a)\lambda_1(t)s_{v_1}(t,a)-\mu_v(a)\,i_{v_1}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s_{v_2}(t,a) &= b_{v_2}(a)-\left[\Gamma_{v_2}(a)\lambda_2(t)+\mu_v(a)\right]s_{v_2}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_{v_2}(t,a) &= \Gamma_{v_2}(a)\lambda_2(t)s_{v_2}(t,a)-\mu_v(a)\,i_{v_2}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s_{v_3}(t,a) &= b_{v_3}(a)-\left[\Gamma_{v_3}(a)\lambda_l(t)+\mu_v(a)\right]s_{v_3}(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_{v_3}(t,a) &= \Gamma_{v_3}(a)\lambda_3(t)s_{v_3}(t,a)-\mu_v(a)\,i_{v_3}(t,a)
\end{cases} \tag{2}$$

with the following boundary conditions:

$$s_l(t,0)=\Lambda_l;\, e_l(t,0)=i_l(t,0)=r_l(t,0)=0$$

with $\Lambda_1+\Lambda_2+\Lambda_3=1$

$\Lambda_l$ denotes the proportion of the initial population of patch l.

## 3. Formulation of the model into an abstract Cauchy problem

To show the existence and uniqueness of the solution of the system (2), we first transform it into a non-homogeneous abstract Cauchy problem that we solve by the theory of $C_0$-semi-groups.

Consider the Banach space X defined by $X = \left(L^1[0,a_+[\right)^{17}$, equipped with the norm:

$$\|\varphi\| = \sum_l\sum_j \|\varphi_{ij}\| + \sum_{l=1}^{3}\left(\left\|\varphi_{s_{v_l}}\right\| + \left\|\varphi_{i_{v_l}}\right\|\right),\ with\ (l,j) \in \{1,2,3\}\times\{1,2,3,4\}\setminus\{(2,4)\}$$

where

$$\varphi(a) = \left(\varphi_{11}(a), \varphi_{12}(a), \varphi_{13}(a), \varphi_{14}(a), \varphi_{21}(a), \varphi_{22}(a), \varphi_{23}(a), \varphi_{31}(a),\right.$$
$$\left.\varphi_{32}(a), \varphi_{33}(a), \varphi_{34}(a), \varphi_{s_{v_1}}(a), \varphi_{i_{v_1}}(a), \varphi_{s_{v_2}}(a), \varphi_{i_{v_2}}(a), \varphi_{s_{v_3}}(a), \varphi_{i_{v_3}}(a)\right)^T \in X$$

and $\|.\|$ is a norm on $L^1[0,a_+[$. Note

$$\Omega := \left\{(s_1,e_1,i_1,r_1,s_2,e_2,i_2,s_3,e_3,i_3,r_3,s_{v_l},i_{v_l}) \in X_+ :\right.$$
$$\left.0 \le s_1+e_1+i_1+r_1 \le 1, 0 \le s_2+e_2+i_2 \le 1, 0 \le s_3+e_3+i_3+r_3 \le 1, 0 \le s_{v_l}+i_{v_l} \le 1\right\}$$

with l=1,2,3.

The normalized system state space where$X_+ = \left(L^1_+[0,a_+[\right)^{17}$, and $L^1_+[0,a_+[$ denotes the positive cone of $L^1[0,a_+[$.

Let A be the linear operator defined by

$$(A\varphi)(a) = \left(A_{11}, A_{12}, A_{13}, A_{14}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}, A_{34}, A_{s_{v_1}}, A_{i_{v_1}}, A_{s_{v_2}}, A_{i_{v_2}}, A_{s_{v_3}}, A_{i_{v_3}}\right)^T$$

To determine the components $A_{ij}$, we neglect the terms of order two and those which are not multiplied by the terms $s_l$, $e_l$, $i_l$, $r_l$, $s_{v_l}$ and $i_{v_l}$ in the system (2), we thus obtain:

$$A_{11} = \left(-\frac{\partial\varphi_{11}}{\partial a} - (\eta_{12}+\eta_{13}+b(a))\varphi_{11}, 0, 0, \alpha_1\varphi_{14}, \eta_{21}\varphi_{21}, 0, 0, \eta_{31}\varphi_{31}, 0, 0, \alpha_3\varphi_{34}, 0,0,0,0,0,0\right)$$
$$A_{12} = \left(0, -\frac{\partial\varphi_{12}}{\partial a} - (\epsilon_{12}+\epsilon_{13}+b(a)+\beta_1(a))\varphi_{12}, 0,0,0, \epsilon_{21}\varphi_{22}, 0,0, \epsilon_{31}\varphi_{32}, 0,0,0,0,0,0,0,0\right)$$
$$A_{13} = \left(0, \beta_1(a)\varphi_{12}, -\frac{\partial\varphi_{13}}{\partial a} - (\rho_{12}+\rho_{13}+b(a)+\mu_1(a)+\gamma_1(a))\varphi_{13}, 0,0,0, \rho_{21}\varphi_{23}, 0,0, \rho_{31}\varphi_{33}, 0,0,0,0,0,0,0\right)$$
$$A_{14} = \left(0,0, \gamma_1(a)\varphi_{13}, -\frac{\partial\varphi_{14}}{\partial a} - (3\alpha_1+b(a))\varphi_{14}, 0,0,0,0,0,0,0,0,0,0,0,0,0\right)$$
$$A_{21} = \left(\eta_{12}\varphi_{11}, 0,0, \alpha_1\varphi_{14}, -\frac{\partial\varphi_{21}}{\partial a} - (\eta_{21}+\eta_{23}+b(a))\varphi_{21}, 0,0, \eta_{32}\varphi_{31}, 0,0, \alpha_3\varphi_{34}0,0,0,0,0,0\right)$$
$$A_{22} = \left(0, \epsilon_{12}\varphi_{12}, 0,0,0, -\frac{\partial\varphi_{22}}{\partial a} - (\epsilon_{21}+\epsilon_{23}+\beta_2(a)+b(a))\varphi_{22}, 0,0, \epsilon_{32}\varphi_{32}, 0,0,0,0,0,0,0,0\right)$$
$$A_{23} = \left(0,0, \rho_{12}\varphi_{13}, 0,0, \beta_2(a)\varphi_{22}, -\frac{\partial\varphi_{23}}{\partial a} - (\rho_{21}+\rho_{23}+\gamma_2(a)+\mu_2(a)+b(a))\varphi_{23}, 0,0, \rho_{32}\varphi_{33}, 0,0,0,0,0,0,0\right)$$
$$A_{31} = \left(\eta_{13}\varphi_{11}, 0,0, \alpha_1\varphi_{14}, \eta_{23}\varphi_{21}, 0,0, -\frac{\partial\varphi_{31}}{\partial a} - (\eta_{31}+\eta_{32}+b(a))\varphi_{31}, 0,0, \alpha_3\varphi_{34}, 0,0,0,0,0,0\right)$$
$$A_{32} = \left(0, \epsilon_{13}\varphi_{12}, 0,0,0, \epsilon_{23}\varphi_{22}, 0,0, -\frac{\partial\varphi_{32}}{\partial a} - (\epsilon_{31}+\epsilon_{32}+\beta_3(a)+b(a))\varphi_{32}, 0,0,0,0,0,0,0,0\right)$$
$$A_{33} = \left(0,0, \rho_{13}\varphi_{13}, 0,0,0, \rho_{23}\varphi_{23}, 0,0, -\frac{\partial\varphi_{33}}{\partial a} - (\rho_{31}+\rho_{32}+\mu_3(a)+\gamma_3(a)+b(a))\varphi_{33}, 0,0,0,0,0,0,0\right)$$
$$A_{34} = \left(0,0,0,0,0,0,0,0,0, \gamma_3(a)\varphi_{33}, -\frac{\partial\varphi_{34}}{\partial a} - (3\alpha_2+b(a))\varphi_{34}, 0,0,0,0,0,0\right)$$
$$A_{s_{v_1}} = \left(0,0,0,0,0,0,0,0,0,0,0, -\frac{\partial\varphi_{s_{v_1}}}{\partial a} - b_{v_1}(a)\varphi_{s_{v_1}}, 0,0,0,0,0\right)$$
$$A_{i_{v_1}} = \left(0,0,0,0,0,0,0,0,0,0,0,0, -\frac{\partial\varphi_{i_{v_1}}}{\partial a} - \mu_v(a)\varphi_{i_{v_1}}, 0,0,0,0\right)$$
$$A_{s_{v_2}} = \left(0,0,0,0,0,0,0,0,0,0,0,0,0, -\frac{\partial\varphi_{s_{v_2}}}{\partial a} - b_{v_2}(a)\varphi_{s_{v_2}}, 0,0,0\right)$$
$$A_{i_{v_1}} = \left(0,0,0,0,0,0,0,0,0,0,0,0,0,0, -\frac{\partial\varphi_{i_{v_2}}}{\partial a} - \mu_v(a)\varphi_{i_{v_2}}, 0,0\right)$$
$$A_{s_{v_3}} = \left(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0, -\frac{\partial\varphi_{s_{v_3}}}{\partial a} - b_{v_3}(a)\varphi_{s_{v_3}}, 0\right)$$
$$A_{i_{v_3}} = \left(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0, -\frac{\partial\varphi_{i_{v_3}}}{\partial a} - \mu_v(a)\varphi_{i_{v_3}}\right)$$

Let

$$\varphi(a) \in D(A)\ where D(A) = \left\{\begin{array}{c}\varphi \in X\setminus\varphi_{lj} \in AC[0,a_+[ :\\ \varphi(0) = (b_1(a),0,0,0,b_2(a),0,0,b_3(a),0,0,0,b_{v_1}(a),0,b_{v_2}(a),0,b_{v_3}(a),0)\end{array}\right\}$$

with $AC[0,a_+[$ the set of absolutely continuous functions on $[0,a_+[$.

We also define a nonlinear operator $F : X \to X$ by

$$
(F\varphi)(a) = \begin{pmatrix}
\pi_1 b_1(a) - \left[\frac{1}{\theta_1}(Q\varphi_{13})(a) + \theta_1\left(Q\varphi_{i_{v_1}}\right)(a) - \mu_1(a)\varphi_{13} - \mu_2(a)\varphi_{23} - \mu_3(a)\varphi_{33}\right]\varphi_{11} \\
\left[\frac{1}{\theta_1}(Q\varphi_{13})(a) + \theta_1\left(Q\varphi_{i_{v_1}}\right)(a)\right]\varphi_{11} + [\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23} + \mu_3(a)\varphi_{33}]\varphi_{12} \\
[\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23} + \mu_3(a)\varphi_{33}]\varphi_{13} \\
[\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23} + \mu_3(a)\varphi_{33}]\varphi_{14} \\
\pi_2 b_2(a) - \left[\frac{1}{\theta_2}(Q\varphi_{23})(a) + \theta_2\left(Q\varphi_{i_{v_2}}\right)(a) - \mu_1(a)\varphi_{13} - \mu_2(a)\varphi_{23} - \mu_3(a)\varphi_{33}\right]\varphi_{21} \\
\left[\frac{1}{\theta_2}(Q\varphi_{23})(a) + \theta_2\left(Q\varphi_{i_{v_2}}\right)(a)\right]\varphi_{21} + [\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23} + \mu_3(a)\varphi_{33}]\varphi_{22} \\
[\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23} + \mu_3(a)\varphi_{33}]\varphi_{23} \\
\pi_3 b_3(a) - \left(\frac{1}{\theta_3}(Q\varphi_{23})(a) + \theta_3\left(Q\varphi_{i_{v_3}}\right)(a) - \mu_1(a)\varphi_{13} - \mu_2(a)\varphi_{23} - \mu_3(a)\varphi_{33}\right)\varphi_{31} \\
\left[\frac{1}{\theta_3}(Q\varphi_{23})(a) + \theta_3\left(Q\varphi_{i_{v_3}}\right)(a)\right]\varphi_{31} + [\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23} + \mu_3(a)\varphi_{33}]\varphi_{32} \\
[\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23} + \mu_3(a)\varphi_{33}]\varphi_{33} \\
[\mu_1(a)\varphi_{13} + \mu_2(a)\varphi_{23} + \mu_3(a)\varphi_{33}]\varphi_{34} \\
b_{v_1} - \left[\frac{1}{\theta_l}(Q\varphi_{11})(a) + \theta_l\left(Q\varphi_{i_{v_1}}\right)(a)\right]\varphi_{s_{v_1}} \\
\left[\frac{1}{\theta_l}(Q\varphi_{11})(a) + \theta_l\left(Q\varphi_{i_{v_1}}\right)(a)\right]\varphi_{s_{v_1}} \\
b_{v_2} - \left[\frac{1}{\theta_l}(Q\varphi_{21})(a) + \theta_l\left(Q\varphi_{i_{v_2}}\right)(a)\right]\varphi_{s_{v_2}} \\
\left[\frac{1}{\theta_l}(Q\varphi_{21})(a) + \theta_l\left(Q\varphi_{i_{v_2}}\right)(a)\right]\varphi_{s_{v_2}} \\
b_{v_3} - \left[\frac{1}{\theta_l}(Q\varphi_{31})(a) + \theta_l\left(Q\varphi_{i_{v_3}}\right)(a)\right]\varphi_{s_{v_3}} \\
\left[\frac{1}{\theta_l}(Q\varphi_{31})(a) + \theta_l\left(Q\varphi_{i_{v_3}}\right)(a)\right]\varphi_{s_{v_3}}
\end{pmatrix}
$$

$$
(Q\varphi_{lj})(a) = \Gamma_l(a)\int_0^{a_+} \varphi_l(a)\varphi_{lj}(a)\,da and (l,j) \in \{1,2,3\} \times \{1,2,3,4\} \setminus \{2,4\}
$$

where

$$
u(t) = \begin{pmatrix} s_1(.,t), e_1(.,t), i_1(.,t), r_1(.,t), s_2(.,t), e_2(.,t), i_2(.,t), s_3(.,t), e_3(.,t), i_3(.,t), \\ r_3(.,t), s_{v_1}(.,t), i_{v_1}(.,t), s_{v_2}(.,t), i_{v_2}(.,t), s_{v_3}(.,t), i_{v_3}(.,t) \end{pmatrix}^T
$$

We can therefore rewrite the system (2) in this form:

$$
\begin{cases} \frac{d}{dt}u(t) &= Au(t) + F(u(t)) \\ u(0) &= u_0 \end{cases} \tag{3}
$$

with

$$
u_0(a) = \begin{pmatrix} s_{01}(a), e_{01}(a), i_{01}(a), r_{01}(a), s_{02}(a), e_{02}(a), i_{02}(a), s_{03}(a), e_{03}(a), \\ i_{03}(a), r_{03}(a), s_{v0_1}(a), i_{v01}(a), s_{v0_2}(a), i_{v_{02}}(a), s_{v0_3}(a), i_{v03}(a) \end{pmatrix}^T
$$

The following results give the properties of operators A and F:

Lemma 1. [[5], [7]] The operator A generates a $C_0$ -semigroup of the bounded linear operators $e^{tA}$ and the space $\Omega$ is positively invariant by $e^{tA}$ .

Lemma 2. The operator F is continuously Frechet differentiable on X.

**Theorem 3.1.**

*For each $u_0 \in X_+$, there are a maximal interval of existence $[0, t_{max})$ and a unique continuous mild solution $u(t, u_0) \in X_+$, $t \in [0, t_{max})$ for (3) such that :*

$$
u(t) = u_0 e^{tA} + \int_0^t e^{A(t-\tau)} F(u(\tau))\,d\tau.
$$

Proof. [[5], [6]]

## 4. Existence of local and global stabilities of the disease-free equilibrium

### 4.1. Determining the disease-free equilibrium point

To determine the disease-free equilibrium point, we consider the system (2). The regular state

$$\left(s_1(a), e_1(a), i_1(a), r_1(a), s_2(a), e_2(a), i_2(a), s_3(a), e_3(a), i_3(a), r_3(a), s_{v_1}(a), i_{v_1}(a), s_{v_2}(a), i_{v_2}(a), s_{v_3}(a), i_{v_3}(a)\right)$$

of the system (2) satisfying the following system of ordinary differential equations:

$$\left\{\begin{array}{lcl}
\frac{d}{da}s_1(a) &=& \pi_1 b_1(a)+\alpha_1 r_1(a)+\alpha_3 r_3(a)+\eta_{21}s_2(a)+\eta_{31}s_3(a) \\
&& -\left[\eta_{12}+\eta_{13}+b(a)+\Gamma_1(a)\widetilde{\lambda_1}(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]s_1(a) \\
\frac{d}{da}e_1(a) &=& \Gamma_1(a)\widetilde{\lambda_1}(a)\,s_1(a)+\epsilon_{21}e_2(a)+\epsilon_{31}e_3(a)-\left[\epsilon_{12}+\epsilon_{13}+\beta_1(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]e_1(a) \\
\frac{d}{da}i_1(a) &=& \beta_1(a)\,e_1(a)+\rho_{21}i_2(a)+\rho_{31}i_3(a)-\left[\gamma_1(a)+\rho_{12}+\rho_{13}+\mu_1(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]i_1(a) \\
\frac{d}{da}r_1(a) &=& \gamma_1(a)\,i_1(a)-\left[3\alpha_1+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]r_1(a) \\
\frac{d}{da}s_2(a) &=& \pi_2 b_2(a)+\alpha_1 r_1(a)+\alpha_3 r_3(a)+\eta_{12}s_2(a)+\eta_{32}s_3(a)+\gamma_2(a)\,i_2(a) \\
&& -\left[\eta_{21}+\eta_{23}+\Gamma_2(a)\widetilde{\lambda_2}(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]s_2(a) \\
\frac{d}{da}e_2(a) &=& \Gamma_2(a)\widetilde{\lambda_2}(a)\,s_2(a)+\epsilon_{12}e_1(a)+\epsilon_{32}e_3(a)-\left[\epsilon_{21}+\epsilon_{23}+\beta_2(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]e_2(a) \\
\frac{d}{da}i_2(a) &=& \beta_2(a)\,e_2(a)+\rho_{12}i_1(a)+\rho_{32}i_3(a)-\left[\rho_{21}+\rho_{23}+\mu_2(a)+b(a)+\gamma_2(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]i_2(a) \\
\frac{d}{da}s_3(a) &=& \pi_3 b_3(a)+\alpha_1 r_1(a)+\alpha_3 r_3(a)+\eta_{13}s_1(a)+\eta_{23}s_2(a) \\
&& -\left[\eta_{31}+\eta_{32}+\Gamma_3(a)\widetilde{\lambda_3}(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]s_3(a) \\
\frac{d}{da}e_3(a) &=& \Gamma_3(a)\widetilde{\lambda_3}(a)\,s_3(a)+\epsilon_{13}e_1(a)+\epsilon_{23}e_2(a)-\left[\epsilon_{31}+\epsilon_{32}+\beta_3(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]e_3(a) \\
\frac{d}{da}i_3(a) &=& \beta_3(a)\,e_3(a)+\rho_{13}i_1(a)+\rho_{23}i_2(a)-\left[\gamma_3(a)+\rho_{31}+\rho_{32}+\mu_3(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]i_3(a) \\
\frac{d}{da}r_3(a) &=& \gamma_3(a)\,i_3(a)-\left[3\alpha_3+b(a)-\sum_{l=1}^{3}\mu_l(a)\,i_l(a)\right]r_3(a) \\
\frac{d}{da}s_{v_1}(a) &=& b_{v_1}(a)-\left[\Gamma_{v_1}(a)\widetilde{\lambda_1}(a)+\mu_v(a)\right]s_{v_1}(a) \\
\frac{d}{da}i_{v_1}(a) &=& \Gamma_{v_1}(a)\widetilde{\lambda_1}(a)\,s_{v_1}(a)-\mu_v(a)\,i_{v_1}(a) \\
\frac{d}{da}s_{v_2}(a) &=& b_{v_2}(a)-\left[\Gamma_{v_2}(a)\widetilde{\lambda_2}(a)+\mu_v(a)\right]s_{v_2}(a) \\
\frac{d}{da}i_{v_2}(a) &=& \Gamma_{v_2}(a)\widetilde{\lambda_2}(a)\,s_{v_2}(a)-\mu_v(a)\,i_{v_2}(a) \\
\frac{d}{da}s_{v_3}(a) &=& b_{v_3}(a)-\left[\Gamma_{v_3}(a)\widetilde{\lambda_3}(a)+\mu_v(a)\right]s_{v_3}(a) \\
\frac{d}{da}i_{v_3}(a) &=& \Gamma_{v_3}(a)\widetilde{\lambda_3}(a)\,s_{v_3}(a)-\mu_v(a)\,i_{v_3}(a)
\end{array}\right. \tag{4}$$

where

$$\widetilde{\lambda_l}(a)=\int_0^{a_+}\varphi_l(a)\left(\frac{1}{\theta_l}i_l(a)+\theta_l i_{v_l}(a)\right)da$$

with the border conditions: $s_l(0)=\Lambda_l$, $e_l(0)=i_l(0)=r_l(0)=0$ where $\Lambda_l$ denotes the proportion of the initial population of patch l with l=1,2,3.

In the absence of the disease, the force of infection, the transition coefficients from the latent state to the infectious state, from the infectious state to the recovered state, from the recovered state to the susceptible state, of migrations for the latent, infectious, recovered, and the mortality rate linked to the disease are zero, we therefore obtain

$$\left\{\begin{array}{lcl}
\frac{d}{da}s_1(a) &=& \pi_1 b_1(a)+\eta_{21}s_2(a)+\eta_{31}s_3(a)-\left[\eta_{12}+\eta_{13}+b(a)\right]s_1(a) \\
\frac{d}{da}s_2(a) &=& \pi_2 b_2(a)+\eta_{12}s_1(a)+\eta_{32}s_3(a)-\left[\eta_{21}+\eta_{23}+b(a)\right]s_2(a) \\
\frac{d}{da}s_3(a) &=& \pi_3 b_3(a)+\eta_{13}s_1(a)+\eta_{23}s_2(a)-\left[\eta_{31}+\eta_{32}+b(a)\right]s_3(a) \\
\frac{d}{da}s_{v_l}(a) &=& b_{v_l}(a)-\mu_v(a)\,s_{v_l}(a)
\end{array}\right. \tag{5}$$

By integrating this system, we obtain:

$$\left\{\begin{array}{lcl}
s_1^0(a) &=& \Lambda_1 e^{-\int_0^{a_+}(\eta_{12}+\eta_{13}+b(\tau))d\tau}+\int_0^{a_+}e^{-\int_\eta^{a_+}(\eta_{12}+\eta_{13}+b(\tau))d\tau}\left(\pi_1 b_1(\eta)+\eta_{21}s_2^0(\eta)+\eta_{31}s_3^0(\eta)\right)d\eta \\
s_2^0(a) &=& \Lambda_2 e^{-\int_0^{a_+}(\eta_{21}+\eta_{23}+b(\tau))d\tau}+\int_0^{a_+}e^{-\int_\eta^{a_+}(\eta_{21}+\eta_{23}+b(\tau))d\tau}\left(\pi_2 b_2(\eta)+\eta_{12}s_1^0(\eta)+\eta_{32}s_3^0(\eta)\right)d\eta \\
s_3^0(a) &=& \Lambda_3 e^{-\int_0^{a_+}(\eta_{31}+\eta_{32}+b(\tau))d\tau}+\int_0^{a_+}e^{-\int_\eta^{a_+}(\eta_{31}+\eta_{32}+b(\tau))d\tau}\left(\pi_3 b_3(\eta)+\eta_{13}s_1^0(\eta)+\eta_{23}s_2^0(\eta)\right)d\eta \\
s_{v_l}^0(a) &=& \Lambda_{v_l}e^{-\int_0^{a_+}b_v(\tau)d\tau}+\int_0^{a_+}e^{-\int_\eta^{a_+}b_v(\tau)d\tau}b_{v_l}(\eta)\,d\eta
\end{array}\right. \tag{6}$$

where l=1,2,3.

### 4.2. Determination of reproduction number

To determine the basic reproduction number, we introduce a perturbation around the disease-free equilibrium:

$$\begin{cases} s_l(t,a) &= \overline{s_l}(t,a)+s_l^0(a) \\ e_l(t,a) &= \overline{e_l}(t,a)+e_l^0(a) \\ i_l(t,a) &= \overline{i_l}(t,a)+i_l^0(a) \\ r_1(t,a) &= \overline{r_1}(t,a)+r_1^0(a) \\ r_3(t,a) &= \overline{r_3}(t,a)+r_3^0(a) \\ s_{v_l}(t,a) &= \overline{s_{v_l}}(t,a)+s_{v_l}^0(a) \\ i_{v_l}(t,a) &= \overline{i_{v_l}}(t,a)+i_{v_l}^0(a) \end{cases} \tag{7}$$

where l=1,2,3. We obtain the following system of equations:

$$\begin{cases} \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{s_1}(t,a) &= \alpha_1\overline{r_1}(t,a)+\alpha_3\overline{r_3}(t,a)+\eta_{21}\overline{s_2}(t,a)+\eta_{31}\overline{s_3}(t,a) \\ & -\left(\eta_{12}+\eta_{13}+b(a)\right)\overline{s_1}(t,a)-\left[\Gamma_1(a)\overline{\lambda_1}(a)-\sum_{l=1}^3\mu_l(a)\overline{i_l}(t,a)\right]s_1^0(a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{e_1}(t,a) &= \Gamma_1(a)\overline{\lambda_1}(a)s_1^0(t)+\epsilon_{21}\overline{e_2}(t,a)+\epsilon_{31}\overline{e_3}(t,a)-\left[\epsilon_{12}+\epsilon_{13}+\beta_1(a)+b(a)\right]\overline{e_1}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_1(t,a) &= \beta_1(a)\overline{e_1}(t,a)+\rho_{21}\overline{i_2}(t,a)+\rho_{31}\overline{i_3}(t,a)-\left[\gamma_1(a)+\rho_{12}+\rho_{13}+\mu_1(a)+b(a)\right]\overline{i_1}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)r_1(t,a) &= \gamma_1(a)\overline{i_1}(t,a)-[3\alpha_1+b(a)]\overline{r_1}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{s_2}(t,a) &= \alpha_1\overline{r_1}(t,a)+\alpha_3\overline{r_3}(t,a)+\eta_{12}\overline{s_1}(t,a)+\eta_{32}\overline{s_3}(t,a)-\left(\eta_{21}+\eta_{23}+b(a)\right)\overline{s_2}(t,a) \\ & +\gamma_2(a)\overline{i_2}(t,a)-\left[\Gamma_2(a)\overline{\lambda_2}(a)-\sum_{l=1}^3\mu_l(a)\overline{i_l}(t,a)\right]s_2^0(a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{e_2}(t,a) &= \Gamma_2(a)\overline{\lambda_2}(a)s_2^0(t,a)+\epsilon_{12}\overline{e_1}(t,a)+\epsilon_{32}\overline{e_3}(t,a)-\left[\epsilon_{21}+\epsilon_{23}+\beta_2(a)+b(a)\right]e_2^0(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{i_2}(t,a) &= \beta_2(a)e_2^0(a)+\rho_{12}\overline{i_1}(t,a)+\rho_{32}\overline{i_3}(t,a)-\left[\rho_{21}+\rho_{23}+\mu_2(a)+\gamma_2(a)+b(a)\right]\overline{i_2}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{s_3}(t,a) &= \alpha_1\overline{r_1}(t,a)+\alpha_3\overline{r_3}(t,a)+\eta_{13}\overline{s_1}(t,a)+\eta_{23}\overline{s_2}(t,a)-\left(\eta_{31}+\eta_{32}+b(a)\right)\overline{s_3}(t,a) \\ & -\left[\Gamma_3(a)\overline{\lambda_3}(a)-\sum_{l=1}^3\mu_l(a)\overline{i_l}(t,a)\right]s_3^0(a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{e_3}(t,a) &= \Gamma_3(a)\overline{\lambda_3}(a)s_3^0(a)+\epsilon_{13}\overline{e_1}(t,a)+\epsilon_{23}\overline{e_2}(t,a)-\left[\epsilon_{31}+\epsilon_{32}+\beta_3(a)+b(a)\right]\overline{e_3}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{i_3}(t,a) &= \beta_3(a)\overline{e_3}(t,a)+\rho_{13}\overline{i_1}(t,a)+\rho_{23}\overline{i_2}(t,a)-\left[\gamma_3(a)+\rho_{31}+\rho_{32}+\mu_3(a)+b(a)\right]\overline{i_3}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{r_3}(t,a) &= \gamma_3(a)\overline{i_3}(t,a)-[3\alpha_3+b(a)]\overline{r_3}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{s_{v_1}}(t,a) &= b_{v_1}(a)-\mu_v(a)\overline{s_{v_1}}(t,a)-\Gamma_{v_1}(a)\overline{\lambda_1}(a)s_{v_1}^0(a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{i_{v_1}}(t,a) &= \Gamma_{v_1}(a)(a)\overline{\lambda_1}(a)s_{v_1}^0(a)-\mu_v(a)\overline{i_{v_1}}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{s_{v_2}}(t,a) &= b_{v_2}(a)-\mu_v(a)\overline{s_{v_2}}(t,a)-\Gamma_{v_2}(a)\overline{\lambda_2}(a)s_{v_2}^0(a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{i_{v_2}}(t,a) &= \Gamma_{v_2}(a)(a)\overline{\lambda_2}(a)s_{v_2}^0(a)-\mu_v(a)\overline{i_{v_2}}(t,a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{s_{v_3}}(t,a) &= b_{v_3}(a)-\mu_v(a)\overline{s_{v_3}}(t,a)-\Gamma_{v_3}(a)\overline{\lambda_3}(a)s_{v_3}^0(a) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\overline{i_{v_3}}(t,a) &= \Gamma_{v_3}(a)(a)\overline{\lambda_3}(a)s_{v_3}^0(a)-\mu_v(a)\overline{i_{v_3}}(t,a) \end{cases} \tag{8}$$

Consider exponential solutions of the form:

$$\begin{cases} \overline{s_l}(t,a)=\overline{s_l}(a)e^{\lambda t}; \quad \overline{e_l}(t,a)=\overline{e_l}(a)e^{\lambda t}; \quad \overline{i_l}(t,a)=\overline{i_l}(a)e^{\lambda t}; \quad \overline{r_l}(t,a)=\overline{r_l}(a)e^{\lambda t}; \\ \overline{s_{v_l}}(t,a)=\overline{s_{v_l}}(a)e^{\lambda t}; \quad \overline{i_{v_l}}(t,a)=\overline{i_{v_l}}(a)e^{\lambda t}; \quad with \quad l=1,2,3 \end{cases}$$

These solutions satisfy the following system of equations:

$$\begin{cases} \frac{d}{da}\overline{s_1}(a) &= \alpha_1\overline{r_1}(a)+\alpha_2\overline{r_2}(a)+\eta_{21}\overline{s_2}(a)+\eta_{31}\overline{s_3}(a)-\left(\eta_{12}+\eta_{13}+b(a)+\lambda\right)\overline{s_1}(a) \\ & \quad -\left[\Gamma_1(a)\overline{\Gamma_1}-\sum_{l=1}^{3}\mu_l(a)\overline{i_l}(a)\right]s_1^0(a) \\ \frac{d}{da}\overline{e_1}(a) &= \Gamma_1(a)\overline{\Gamma_1}s_1^0(a)+\epsilon_{21}\overline{e_2}(a)+\epsilon_{31}\overline{e_3}(a)-\left[\epsilon_{12}+\epsilon_{13}+\beta_1(a)+b(a)+\lambda\right]\overline{e_1}(a) \\ \frac{d}{da}\overline{i_1}(a) &= \beta_1(a)\overline{e_1}(a)+\rho_{21}\overline{i_2}(a)+\rho_{31}\overline{i_3}(a)-\left[\gamma_1(a)+\rho_{12}+\rho_{13}+\mu_1(a)+b(a)+\lambda\right]\overline{i_1}(a) \\ \frac{d}{da}\overline{r_1}(a) &= \gamma_1(a)\overline{i_1}(a)-\left[3\alpha_1+b(a)+\lambda\right]\overline{r_1}(a) \\ \frac{d}{da}\overline{s_2}(a) &= \alpha_1\overline{r_1}(a)+\alpha_2\overline{r_2}(a)+\eta_{12}\overline{s_1}(a)+\eta_{32}\overline{s_3}(a)+\left(\eta_{21}+\eta_{23}+b(a)+\lambda\right)\overline{s_2}(a)+\gamma_2(a)\overline{i_2}(a) \\ & \quad -\left[\Gamma_2(a)\overline{\Gamma_2}-\sum_{l=1}^{3}\mu_l(a)\overline{i_l}(a)\right]s_2^0(a) \\ \frac{d}{da}\overline{e_2}(a) &= \Gamma_2(a)\overline{\Gamma_2}s_2^0(a)+\epsilon_{12}\overline{e_1}(a)+\epsilon_{32}\overline{e_3}(a)-\left[\epsilon_{21}+\epsilon_{23}+\beta_2(a)+b(a)+\lambda\right]\overline{e_2}(a) \\ \frac{d}{da}\overline{i_2}(a) &= \beta_2(a)e_2^0(a)+\rho_{12}\overline{i_1}(a)+\rho_{32}\overline{i_3}(a)-\left[\rho_{21}+\rho_{23}+\mu_2(a)+\gamma_2(a)+b(a)+\lambda\right]\overline{i_2}(a) \\ \frac{d}{da}\overline{s_3}(a) &= \alpha_1\overline{r_1}(a)+\alpha_2\overline{r_2}(a)+\eta_{13}\overline{s_1}(a)+\eta_{23}\overline{s_2}(a)-\left(\eta_{31}+\eta_{32}+b(a)+\lambda\right)\overline{s_3}(a) \\ & \quad -\left[\Gamma_3(a)\overline{\Gamma_3}-\sum_{l=1}^{3}\mu_l(a)\overline{i_l}(a)\right]s_3^0(a) \\ \frac{d}{da}\overline{e_3}(a) &= \Gamma_3(a)\overline{\Gamma_3}s_3^0(a)+\epsilon_{13}\overline{e_1}(a)+\epsilon_{23}\overline{e_2}(a)-\left[\epsilon_{31}+\epsilon_{32}+\beta_3(a)+b(a)+\lambda\right]\overline{e_3}(a) \\ \frac{d}{da}\overline{i_3}(a) &= \beta_3(a)\overline{e_3}(a)+\rho_{13}\overline{i_1}(a)+\rho_{23}\overline{i_2}(a)-\left[\gamma_3(a)+\rho_{31}+\rho_{32}+\mu_3(a)+b(a)+\lambda\right]\overline{i_3}(a) \\ \frac{d}{da}\overline{r_3}(a) &= \gamma_3(a)\overline{i_3}(t,a)-\left[3\alpha_3+b(a)+\lambda\right]\overline{r_3}(t,a) \\ \frac{d}{da}\overline{s_{v_l}}(a) &= b_{v_l}(a)-\left(\mu_v(a)+\lambda\right)\overline{s_{v_l}}(a)-\Gamma_{v_l}(a)\overline{\Gamma}_l s_{v_l}^0(a) \\ \frac{d}{da}\overline{i_{v_l}}(a) &= \Gamma_{v_l}(a)\overline{\Gamma}_l s_{v_l}^0(a)-\left(\mu_v(a)+\lambda\right)\overline{i_{v_l}}(a) \end{cases}$$

where

$$\overline{\Gamma_l}=\int_0^{a_+}\varphi_l(a)\left(\frac{1}{\theta_l}\overline{i_l}(a)+\theta_l\overline{i_{v_l}}(a)\right)da \qquad and \qquad \theta_l=\frac{N_l(t,a)}{N_{v_l}(t,a)}$$

with the following boundary conditions:

$$\overline{s_l}(0)=\overline{e_l}(0)=\overline{i_l}(0)=\overline{r_l}(0)=\overline{s_{v_l}}(0)=\overline{i_{v_l}}(0)=0, \quad with\, l=1,2,3.$$

We have the following expressions for the latent and infectious classes:

$$\begin{cases} \frac{d}{da}\overline{e_l}(a)=\Gamma_l(a)\overline{\Gamma_l}s_l^0(a)+\sum_{j=1}^{3}\epsilon_{jl}\overline{e_j}(a)-\left[\sum_{j=1}^{3}\epsilon_{lj}+\beta_l(a)+b(a)+\lambda\right]\overline{e_l}(a) \\ \frac{d}{da}\overline{i_l}(a)=\beta_l(a)\overline{e_l}(a)+\sum_{j=1}^{3}\rho_{jl}\overline{i_j}(a)-\left[\sum_{j=1}^{3}\rho_{lj}+\gamma_l(a)+\mu_l(a)+b(a)+\lambda\right]\overline{i_l}(a) \end{cases} \tag{9}$$

where l=1,2,3 and $l\neq j$. Consider the constants

$$\overline{\eta}_{lj}, \qquad \overline{\epsilon_{jl}} \quad and \quad \overline{\rho_{jl}}$$

such as

$$\sum_{j=1}^{3}\eta_{jl}\overline{e_j}(a)=\sum_{j=1}^{3}\overline{\eta_{jl}e_l}, \qquad \sum_{j=1}^{3}\epsilon_{jl}\overline{e_j}(a)=\sum_{j=1}^{3}\overline{\epsilon_{jl}e_l}(a) \qquad and \qquad \sum_{j=1}^{3}\rho_{jl}\overline{i_l}(a)=\sum_{j=1}^{3}\overline{\rho_{jl}}\overline{i_l}(a)$$

where l=1,2,3 and $l\neq j$. So we can rewrite the expressions for the latent and infectious classes as follows:

$$\begin{cases} \frac{d}{da}\overline{e_l}(a)=\Gamma_l(a)\overline{\Gamma_l}s_l^0(a)-\left[\sum_{j=1}^{3}\left(\epsilon_{lj}-\overline{\epsilon}_{jl}\right)+\beta_l(a)+b(a)+\lambda\right]\overline{e_l}(a) \\ \frac{d}{da}\overline{i_l}(a)=\beta_l(a)\overline{e_l}(a)-\left[\sum_{j=1}^{3}\left(\rho_{lj}-\overline{\rho}_{jl}\right)+\gamma_l(a)+\mu_l(a)+b(a)+\lambda\right]\overline{i_l}(a) \end{cases} \tag{10}$$

where l=1,2,3 and $l\neq j$.
Following an integration of the system (10), we obtain:

$$\overline{e_l}(a)=\overline{\Gamma_l}\int_0^a e^{-\int_\eta^a\left(\sum_{j=1}^{3}\epsilon_{lj}-\sum_{j=1}^{3}\overline{\epsilon_{jl}}+\beta_l(\tau)+b(\tau)+\lambda\right)d\tau}\Gamma_l(\eta)\,s_l^0(\eta)\,d\eta \tag{11}$$

and

$$\overline{i_l}(a)=\int_0^a e^{-\int_\eta^a\left(\sum_{j=1}^{3}\rho_{lj}-\sum_{j=1}^{3}\overline{\rho_{jl}}+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)+\lambda\right)d\tau}\beta_l(\eta)\,\overline{e_l}(\eta)\,d\eta \tag{12}$$

By using (11) into (12), we obtain:

$$\overline{i_l}(a)=\overline{\Gamma_l}\int_0^a e^{-\int_\eta^a\left(\sum_{j=1}^3\rho_{lj}-\sum_{j=1}^3\overline{\rho_{jl}}+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)+\lambda\right)d\tau}\beta_l(\eta)\int_0^\eta e^{-\int_\zeta^\eta\left(\sum_{j=1}^3\epsilon_{lj}-\sum_{j=1}^3\overline{\epsilon_{jl}}+\beta_l(\tau)+b(\tau)+\lambda\right)d\tau}\Gamma_l(\zeta)\,s_l^0(\zeta)\,d\zeta d\eta \quad (13)$$

To simplify writing the expression for (13), we reverse the order of integration using region equivalence:

$$\{(\eta,\zeta),0\le\eta\le a,0\le\zeta\le\eta\}\Longleftrightarrow\{(\eta,\zeta),\zeta\le\eta\le a,0\le\zeta\le a\}$$

and changing the integration borns, we obtain:

$$\overline{i_l}(a)=\overline{\Gamma_l}\int_0^a e^{-\int_\zeta^a(b(\tau)+\lambda)d\tau}\Gamma_l(\zeta)\,s_l^0(\zeta)\int_\zeta^a e^{-\int_\eta^a\left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)\right)d\tau}e^{-\int_\zeta^\eta\left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)\right)d\tau}\beta_l(\eta)\,d\eta d\zeta$$

By permuting the variables $\eta$ and $\zeta$, we obtain :

$$\overline{i_l}(a)=\overline{\Gamma_l}\int_0^a e^{-\int_\eta^a(b(\tau)+\lambda)d\tau}\Gamma_l(\eta)\,s_l^0(\eta)\int_\eta^a e^{-\int_\zeta^a\left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)\right)d\tau}e^{-\int_\eta^\zeta\left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)\right)d\tau}\beta_l(\zeta)\,d\zeta d\eta \quad (14)$$

We have

$$\overline{i_{v_l}}(a)=\overline{\Gamma_l}\int_0^a e^{-\int_\eta^a(\mu_v(\tau)+\lambda)d\tau}\Gamma_{v_l}(\eta)\,s_{v_l}^0(\eta)\,d\eta \quad (15)$$

By inserting (14) and (15) in the expression of $\overline{\Gamma_l}$, we obtain :

$$\begin{aligned}\overline{\Gamma_l} &= \overline{\Gamma_l}\int_0^{a_+}\varphi_l(a)\Big[\tfrac{1}{\theta_l}\int_0^a e^{-\int_\eta^a(b(\tau)+\lambda)d\tau}\Gamma_l(\eta)\,s_l^0(\eta)\int_\eta^a e^{-\int_\zeta^a\left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)\right)d\tau}\\ &\qquad e^{-\int_\eta^\zeta\left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)\right)d\tau}\beta_l(\zeta)\,d\zeta d\eta\\ &\qquad +\theta_l\int_0^a e^{-\int_\eta^a(\mu_v(\tau)+\lambda)d\tau}\Gamma_{v_l}(\eta)\,s_{v_l}^0(\eta)\,d\eta\Big]\,da\end{aligned} \quad (16)$$

By dividing (16) by $\overline{\Gamma_l}$ (like $\overline{\Gamma_l}\neq 0$), we get the characteristic equation

$$\begin{aligned}1 &= \int_0^{a_+}\varphi_l(a)\Big[\tfrac{1}{\theta_l}\int_0^a e^{-\int_\eta^a(b(\tau)+\lambda)d\tau}\Gamma_l(\eta)\,s_l^0(\eta)\int_\eta^a e^{-\int_\zeta^a\left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)\right)d\tau}\\ &\qquad e^{-\int_\eta^\zeta\left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)\right)d\tau}\beta_l(\zeta)\,d\zeta d\eta\\ &\qquad +\theta_l\int_0^a e^{-\int_\eta^a(\mu_v(\tau)+\lambda)d\tau}\Gamma_{v_l}(\eta)\,s_{v_l}^0(\eta)\,d\eta\Big]\,da\end{aligned} \quad (17)$$

Denote the right term of (16) by $G_l(\lambda)$, i.e,

$$\begin{aligned}G_l(\lambda) &= \int_0^{a_+}\varphi_l(a)\Big[\tfrac{1}{\theta_l}\int_0^a e^{-\int_\eta^a(b(\tau)+\lambda)d\tau}\Gamma_l(\eta)\,s_l^0(\eta)\int_\eta^a e^{-\int_\zeta^a\left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)\right)d\tau}\\ &\qquad e^{-\int_\eta^\zeta\left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)\right)d\tau}\beta_l(\zeta)\,d\zeta d\eta\\ &\qquad +\theta_l\int_0^a e^{-\int_\eta^a(\mu_v(\tau)+\lambda)d\tau}\Gamma_{v_l}(\eta)\,s_{v_l}^0(\eta)\,d\eta\Big]\,da\end{aligned} \quad (18)$$

We define the basic reproduction number by $R_0^l=R^l(0)=G_l(0)$, i.e,

$$\begin{aligned}R_0^l &= \Lambda_l\int_0^{a_+}\varphi_l(a)\Big[\tfrac{1}{\theta_l}\int_0^a e^{-\int_\eta^a(b(\tau))d\tau}\Gamma_l(\eta)\int_\eta^a e^{-\int_\zeta^a\left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)\right)d\tau}\\ &\qquad \times e^{-\int_\eta^\zeta\left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)\right)d\tau}\beta_l(\zeta)\,d\zeta d\eta\\ &\qquad +\theta_l\int_0^a e^{-\int_\eta^a(\mu_v(\tau))d\tau}\Gamma_{v_l}(\eta)\,d\eta\Big]\,da\end{aligned}$$

The reproduction number is influenced only by the migration of infectious and latent as evidenced by its expression. It makes it possible to study the local and global asymptotic stability of the disease-free equilibrium . Let's note that $R_0=\max R_0^l$ where l=1,2,3

### 4.3. Local stability of the disease-free equilibrium

**Theorem 4.1.**
*The disease-free equilibrium point of the (2) is locally asymptotically stable (l.a.s.) if $R_0 < 1$ and unstable if $R_0 > 1$.*

*Proof.* It is obvious to see that $G_l'(\lambda) < 0$, $\lim_{\lambda\to+\infty} G_l(\lambda) = 0$ and $\lim_{\lambda\to-\infty} G_l(\lambda) = +\infty$. We know that the equation ([eq:17]) has a unique negative solution $\lambda^*$ if, and only if, $G_l(0) < 1$, hence, $R_0 < 1$. Also, it admits a unique real positive (or zero) solution if $G_l(0) > 0$ ,hence, $R_0 > 1$ ($G_l(0) = 1$). To show that $\lambda^*$ is the dominant real part of the solutions of $G_l(\lambda)$ we consider $\lambda = x + iy$ an arbitrary complex solution of the equation (18). Note that :

$$1 = G_l(\lambda) = \left|G_l(x+iy)\right| \le G_l(x)$$

which indicates that $R_e(\lambda) \le \lambda^*$. This proves that the disease-free equilibrium point is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$. □

Local stability in one subpopulation while the situation is unstable in the other is not possible because through migration the latent, infectious compartments will not be empty. This situation does not allow stability in one subpopulation if the situation in the other is unstable. In other words if $R_0^l < 1$ does not mean to say that we have stability in the patch l, because if $R_0^l > 1$, through migration we will have the presence of latents, infectious and retired in this subpopulation and vice versa.

### 4.4. Global stability of the disease-free equilibrium

In this subsection we study the asymptotic behavior of the equilibrium point without disease if $R_0 < 1$. Let us first note that as $\mu_l$ and $i_l$ are bounded functions and the borns of the integral are finite, there is a positive constant C which satisfies the relation:

$$0 \le \int_\eta^a \sum_{l=1}^{3} \mu_l(\tau)\, i_l(t-a+\tau,\tau)\, d\tau \le C \tag{19}$$

We have the following result:

**Theorem 4.2.**
*The disease-free equilibrium point of the (2) is globally asymptotically stable if $R_0 < 1$ and $C < \ln\left(\frac{1}{R_0}\right)$.*

*Proof.* The proof of the theorem consists in showing that $i_l(t,a) \to 0$; $e_l(t,a) \to 0$; $r_l(t,a) \to 0$ and $s_l(t,a) \to \Lambda_l$, when $t \to +\infty$. We can summarize the system (2) as follows:

$$Proof.\begin{cases}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) s_l(t,a) &= \pi_l b_l(a) + \alpha_1 r_1(t,a) + \alpha_3 r_3(t,a) + \left[\gamma_2(a)\, i_2(t,a)\ si\ l=2\right] \\
& \quad - \left[\sum_{j=1}^3 \left(\eta_{lj} - \overline{\eta_{jl}}\right) + b(a) + \Gamma_l(a)\,\lambda_l(t) - \sum_{l=1}^3 \mu_l(a)\, i_l(t,a)\right] s_l(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) e_l(t,a) &= \Gamma_l(a)\,\lambda_l(t)\, s_l(t,a) - \left[\sum_{j=1}^3 \left(\epsilon_{lj} - \overline{\epsilon_{jl}}\right) + \beta_l(a) + b(a) - \sum_{l=1}^3 \mu_l(a)\, i_l(t,a)\right] e_l(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) i_l(t,a) &= \beta_l(a)\, e_l(t,a) - \left[\sum_{j=1}^3 \left(\rho_{lj} - \overline{\rho_{jl}}\right) + \gamma_l(a) + \mu_l(a) + b(a) - \sum_{l=1}^3 \mu_l(a)\, i_l(t,a)\right] i_l(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) r_m(t,a) &= \gamma_m(a)\, i_m(t,a) - \left[3\alpha_l + b(a) - \sum_{l=1}^3 \mu_l(a)\, i_l(t,a)\right] r_m(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) s_{v_l}(t,a) &= b_{v_l}(a) - \left[\Gamma_{v_l}(a)\,\lambda_l(t) + \mu_v(a)\right] s_{v_l}(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) i_{v_l}(t,a) &= \Gamma_{v_l}(a)\,\lambda_l(t)\, s_{v_l}(t,a) - \mu_v(a)\, i_{v_l}(t,a) \\
\lambda_l(t) &= \int_0^{a_+} \varphi_l(a) \left(\frac{1}{\theta_l} i_l(t,a) + \theta_l i_{v_l}(t,a)\right)
\end{cases} \tag{20}$$

with l=1,2,3 ($l \neq j$) and m=1,3. By solving the latent evolution equation of the system (20) by the method of characteristics, we obtain the following system of ordinary differential equations:

$$\begin{cases}
\frac{dt}{1} &= \frac{da}{1} \\
\frac{da}{1} &= \frac{de_l(t,a)}{\Gamma_l(a)\lambda_l(t)s_l(t,a) - \left[\sum_{j=1}^3 \left(\epsilon_{lj} - \overline{\epsilon_{jl}}\right) + \beta_l(a) + b(a) - \sum_{l=1}^3 \mu_l(a) i_l(t,a)\right] e_l(t,a)} \\
\frac{dt}{1} &= \frac{de_i(t,a)}{\Gamma_l(a)\lambda_l(t)s_l(t,a) - \left[\sum_{j=1}^3 \left(\epsilon_{lj} - \overline{\epsilon_{jl}}\right) + \beta_l(a) + b(a) - \sum_{l=1}^3 \mu_l(a) i_l(t,a)\right] e_l(t,a)}
\end{cases} \tag{21}$$

The first equation of the system (21) leads to the solution t - a = c which is the characteristic curve, while the second and third equations of (21) give respectively :

$$\frac{de_l(t,a)}{da} = \Gamma_l(a)\lambda_l(t)s_l(t,a) - \left[\sum_{j=1}^{3}\left(\epsilon_{lj}-\overline{\epsilon_{jl}}\right)+\beta_l(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)i_l(t,a)\right]e_l(t,a) \tag{22}$$

and

$$\frac{de_l(t,a)}{dt} = \Gamma_l(a)\lambda_l(t)s_l(t,a) - \left[\sum_{j=1}^{3}\left(\epsilon_{lj}-\overline{\epsilon_{jl}}\right)+\beta_l(a)+b(a)-\sum_{l=1}^{3}\mu_l(a)i_l(t,a)\right]e_l(t,a) \tag{23}$$

For next steps, we use only the solution of the equation (22), we obtain :

$$e_l(t,a) = \int_0^a e^{-\int_\eta^a\left(\sum_{j=1}^{3}(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{l=1}^{3}\mu_l(\tau)i_l(t-a+\tau,\tau)\right)d\tau}\Gamma_l(\eta)\lambda_l(t-a+\eta)s_l(t-a+\eta,\eta)\,d\eta,\ if\ a\le t \tag{24}$$

By proceeding in the same way, on the equations reflecting the evolution of the infectious in the system (20), we obtain:

$$i_l(t,a) = \int_0^a e^{-\int_\zeta^a\left(\sum_{j=1}^{3}(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^{3}\mu_l(\tau)i_l(t-a+\tau,\tau)\right)d\tau}\beta_l(\zeta)e_l(t-a+\zeta,\zeta)\,d\zeta,\ if\ a\le t \tag{25}$$

By using (24) into (25), we obtain:

$$\begin{aligned} i_l(t,a) = {} & \int_0^a e^{-\int_\zeta^a\left(\sum_{j=1}^{3}(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^{3}\mu_l(\tau)i_l(t-a+\tau,\tau)\right)d\tau}\beta_l(\zeta) \\ & \times\int_0^\zeta e^{-\int_\eta^\zeta\left(\sum_{j=1}^{3}(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{l=1}^{3}\mu_l(\tau)i_l(t-a+\tau,\tau)\right)d\tau}\Gamma_l(\eta)\lambda_l(t-a+\eta)s_l(t-a+\eta,\eta)\,d\eta d\zeta,\ if\ a\le t \end{aligned} \tag{26}$$

After reversing the order of integration using the following domain equivalence:

$\{(\eta,\zeta), 0\le\eta\le a, 0\le\zeta\le\eta\} \Longleftrightarrow \{(\eta,\zeta), \zeta\le\eta\le a, 0\le\zeta\le a\}$

and by permuting the integration borns, (26) becomes:

$$\begin{aligned} i_l(t,a) = {} & \int_0^a e^{\int_\eta^a\sum_{l=1}^{3}\mu_l(\tau)i_l(t-a+\tau,\tau)d\tau}e^{-\int_\eta^a\left(\sum_{j=1}^{3}(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)\right)d\tau}\Gamma_l(\eta)\lambda_l(t-a+\eta)s_l(t-a+\eta,\eta) \\ & \times\int_\eta^a e^{-\int_\xi^\eta\left(\sum_{j=1}^{3}(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)\right)d\tau}\beta_l(\zeta)\,d\eta d\zeta,\ if\ a\le t \end{aligned} \tag{27}$$

Similarly we obtain

$$i_{v_l}(t,a) = \int_0^a e^{-\int_\eta^a\mu_v(\tau)d\tau}\Gamma_{v_l}(\eta)\lambda_l(t-a+\eta)s_{v_l}^0(t-a+\eta,\eta)\,d\eta \tag{28}$$

By injecting the equations (27) and (28) into the

$$\lambda_l(t) = \int_0^{a_+}\varphi_l(a)\left(\frac{1}{\theta_l}i_l(t,a)+\theta_l i_{v_l}(t,a)\right),$$

we obtain :

$$\begin{aligned} \lambda_l(t) = {} & \int_0^{a_+}\varphi_l(a)\Bigg[\frac{1}{\theta_l}\int_0^a e^{\int_\eta^a\sum_{l=1}^{3}\mu_l(\tau)i_l(t-a+\tau,\tau)d\tau}e^{-\int_\eta^a\left(\sum_{j=1}^{3}(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)\right)d\tau}\Gamma_l(\eta)\lambda_l(t-a+\eta)s_l(t-a+\eta,\eta) \\ & \times\int_\eta^a e^{-\int_\xi^\eta\left(\sum_{j=1}^{3}(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)\right)d\tau}\beta_l(\zeta)\,d\eta d\zeta \\ & +\theta_l\int_0^a e^{-\int_\eta^a\mu_v(\tau)d\tau}\Gamma_{v_l}(\eta)\lambda_l(t-a+\eta)s_{v_l}^0(t-a+\eta,\eta)\,d\eta\Bigg]da \end{aligned} \tag{29}$$

Using the fact that $s_l(t-a+\eta,\eta)\le\Lambda_l$ and $s_{v_l}(t-a+\eta,\eta)\le\Lambda_l$, we obtain:

$$\begin{aligned} \lambda_l(t)\le\Lambda_l \quad & \int_0^{a_+}\varphi_l(a)\Bigg[\frac{1}{\theta_l}\int_0^a e^{\int_\eta^a\sum_{l=1}^{3}\mu_l(\tau)i_l(t-a+\tau,\tau)d\tau}e^{-\int_\eta^a\left(\sum_{j=1}^{3}(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)\right)d\tau}\Gamma_l(\eta)\lambda_l(t-a+\eta) \\ & \times\int_\eta^a e^{-\int_\xi^\eta\left(\sum_{j=1}^{3}(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)\right)d\tau}\beta_l(\zeta)\,d\eta d\zeta+\theta_l\int_0^a e^{-\int_\eta^a\mu_v(\tau)d\tau}\Gamma_{v_l}(\eta)\lambda_l(t-a+\eta)\,d\eta\Bigg]da \end{aligned} \tag{30}$$

By using the suplimit of (30), using the relation (19) and Fatou's lemma, we obtain:

$$\lim_{t\to+\infty}\lambda_l(t)\le e^C R_l^0\limsup_{t\to+\infty}\lambda_l(t)$$

As $e^C R_l^0<1$, we then have $\limsup\limits_{t\to+\infty}\lambda_l(t)=0$ and, according to (27) and (28), $i_l(t,a)\to0$ and $i_{v_l}(t,a)\to0$ when $t\to+\infty$. Thus, $e_l(t,a)\to0$ when $t\to+\infty$. Using the fact that

$s_l(t,a)+e_l(t,a)+i_l(t,a)+r_l(t,a)=\Lambda_l$ *and* $s_{v_l}(t,a)+i_{v_l}(t,a)=\Lambda_l$,

we obtain $s_l(t,a)\to\Lambda_l$ because $r_l(t,a)\to0$ when $t\to+\infty$. This completes the proof of the theorem. □

Similar to local stability, migration flow prevents us from having the global stability of the disease-free equilibrium in one subpopulation if it is not in the other.

## 5. Existence of the endemic equilibrium

In this section, we will demonstrate the existence of a single endemic equilibrium point.
The existence of this point is given by the following result:

**Theorem 5.1.**
*The unique endemic equilibrium point is of the form*

$E^* = \left(s_1^*(a), e_1^*(a), i_1^*(a), r_2^*(a), s_2^*(a), e_2^*(a), i_2^*(a), s_3^*(a), e_3^*(a), i_3^*(a), r_3^*(a), s_{v_1}^*(a), i_{v_1}^*(a), s_{v_2}^*(a), i_{v_2}^*(a), s_{v_3}^*(a), i_{v_3}^*(a)\right)$

*when* $R_0^l > 1$, *which corresponds to the case where the disease persists in the three subpopulations where l=1,2,3.*

*Proof.*

$E^* = \left(s_1^*(a), e_1^*(a), i_1^*(a), r_2^*(a), s_2^*(a), e_2^*(a), i_2^*(a), s_3^*(a), e_3^*(a), i_3^*(a), r_3^*(a), s_{v_1}^*(a), i_{v_1}^*(a), s_{v_2}^*(a), i_{v_2}^*(a), s_{v_3}^*(a), i_{v_3}^*(a)\right)$

satisfies the following system of differential equations:

$$\begin{cases} \frac{d}{da}s_l^*(a) &=& \pi_l b_l(a) + \alpha_1 r_1^*(a) + \alpha_3 r_3^*(a) - \left[\sum_{j=1}^3 \left(\eta_{lj} - \overline{\eta_{jl}}\right) + b(a) + \Gamma_l(a)\lambda_l^* - \sum_{l=1}^3 \mu_l(a) i_l^*(a)\right] s_l^*(a) \\ && + \left(\gamma_2(a) i_2^*(a) \; if \; l=2\right) \\ \frac{d}{da}e_l^*(a) &=& \Gamma_l(a)\lambda_l^* s_l^*(a) - \left[\sum_{j=1}^3 \left(\epsilon_{lj} - \overline{\epsilon_{jl}}\right) + \beta_l(a) + b(a) - \sum_{l=1}^3 \mu_l(a) i_l^*(a)\right] e_l^*(a) \\ \frac{d}{da}i_l(a) &=& \beta_l(a) e_l^*(a) - \left[\sum_{j=1}^3 \left(\rho_{lj} - \overline{\rho_{jl}}\right) + \gamma_l(a) + \mu_l(a) + b(a) - \sum_{l=1}^3 \mu_l(a) i_l^*(a)\right] i_l^*(a) \\ \frac{d}{da}r_m^*(a) &=& \gamma_m(a) i_m^*(a) - \left[3\alpha_l + b(a) - \sum_{l=1}^3 \mu_l(a) i_l^*(a)\right] r_m^*(a) \\ \frac{d}{da}s_3(a) &=& \pi_3 b_3(a) + \alpha_1 r_1^*(a) + \alpha_3 r_3^*(a) + \gamma_3(a) i_3^* - \left[\sum_{j=1}^3 \left(\eta_{lj} - \overline{\eta_{jl}}\right) + b(a) + \Gamma_3(a)\lambda_3^* - \sum_{l=1}^3 \mu_l(a) i_l^*(a)\right] s_3^*(a) \\ \frac{d}{da}s_{v_l}^*(a) &=& b_{v_l}(a) - \left[\Gamma_{v_l}(a)\lambda_l^* + \mu_v(a)\right] s_{v_l}^*(a) \\ \frac{d}{da}i_{v_l}^*(a) &=& \Gamma_{v_l}(a)\lambda_l^* s_{v_l}^*(a) - \mu_v(a) i_{v_l}^*(a) \\ \lambda_l^* &=& \int_0^{a+} \varphi_l(a)\left(\frac{1}{\theta_l} i_l^*(a) + \theta_l i_{v_l}^*(a)\right) da \end{cases} \tag{31}$$

with l=1,2,3 and $l \neq j$ and m=1,3. The edge conditions satisfy: $s_l^*(0) = \Lambda_l = s_{v_l}^*(0)$, $e_l^*(0) = i_l^*(0) = r_l^*(0) = 0$ where $\Lambda_l$ denotes the proportion of the initial population of patch l.
By integrating the system (31), we obtain:

$$\begin{aligned} s_l^*(a) &= \Lambda_l e^{-\int_0^a \left(\sum_{j=1}^3 (\eta_{lj} - \overline{\eta_{jl}}) + b(\tau) + \Gamma_l(\tau)\lambda_l^* - \sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right) d\tau} + \int_0^a \left[\pi_l b_l(\eta) + \sum_{j=1}^3 \alpha_l r_l^*(\eta) + \left(\gamma_2(\eta) i_2^*(\eta) \; if \; l=2\right)\right] \\ & \quad \times e^{-\int_\eta^a \left(\sum_{j=1}^3 (\eta_{lj} - \overline{\eta_{jl}}) + b(\tau) + \Gamma_l(\tau)\lambda_l^* - \sum_{l=1}^3 \mu_l(\tau) i_l^*\right) d\tau} d\eta \end{aligned} \tag{32}$$

$$e_l^*(a) = \int_0^a \Gamma_l(\zeta)\lambda_l^* s_l^*(\zeta) e^{-\int_\zeta^a \left(\sum_{j=1}^3 (\epsilon_{lj} - \overline{\epsilon_{jl}}) + \beta_l(\tau) + b(\tau) - \sum_{l=1}^3 \mu_l(\tau) i_l^*\right) d\tau} d\zeta, \tag{33}$$

$$i_l^*(a) = \int_0^a \beta_l(\eta) e_l^*(\eta) e^{-\int_\eta^a \left(\sum_{j=1}^3 (\rho_{lj} - \overline{\rho_{jl}}) + \gamma_l(\tau) + \mu_l(\tau) + b(\tau) - \sum_{l=1}^3 \mu_l(\tau) i_l^*\right) d\tau} d\eta, \tag{34}$$

$$r_l^*(a) = \int_0^a \gamma_i(\eta) i_i^*(\eta) e^{-\int_\eta^a \left(3\alpha_i + b(\tau) - \sum_{l=1}^3 \mu_i(\tau) i_i^*(\tau)\right) d\tau} d\eta, \tag{35}$$

$$s_{v_l}^*(a) = \Lambda_l e^{-\int_0^a \left(\Gamma_{v_l}(\tau)\lambda_l^* + \mu_v(\tau)\right) d\tau} + \int_0^a b_{v_l}(\eta) e^{-\int_\eta^a \int_0^a \left(\Gamma_{v_l}(\tau)\lambda_l^* + \mu_v(\tau)\right) d\tau} d\eta \tag{36}$$

and

$$i_{v_l}^*(a) = \int_0^a \Gamma_{v_l}(\eta)\lambda_l^* s_{v_l}^*(\eta) e^{-\int_\eta^a \mu_v(\tau) d\tau} d\eta. \tag{37}$$

By using (33) into (34), we obtain:

$$\begin{aligned} i_l^*(a) &= \int_0^a \beta_l(\eta) \int_0^\eta \Gamma_l(\zeta)\lambda_l^* s_l^*(\zeta) e^{-\int_\zeta^\eta \left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} d\zeta \\ &\quad \times e^{-\int_\eta^a \left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} d\eta \end{aligned}$$

Similarly, to calculate the inner integral, let's reverse the order of integration and use the equivalence of regions:

$$\{(\eta,\zeta), 0 \le \eta \le a, 0 \le \zeta \le \eta\} \Longleftrightarrow \{(\eta,\zeta), \zeta \le \eta \le a, 0 \le \zeta \le a\}$$

A change of integration bounds we then obtain:

$$\begin{aligned} i_l^*(a) &= \int_0^a \Gamma_l(\zeta)\lambda_l^* s_l^*(\zeta) e^{-\int_\zeta^a \left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} \\ &\quad \times \int_\zeta^a \beta_l(\eta) e^{-\int_\eta^a \left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} d\eta d\zeta \end{aligned}$$

and permuting $\eta$ and $\zeta$ variables , we obtain :

$$\begin{aligned} i_l^*(a) &= \int_0^a \Gamma_l(\eta)\lambda_l^* s_l^*(\eta) e^{-\int_\eta^\zeta \left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} \\ &\quad \times \int_\eta^a \beta_l(\zeta) e^{-\int_\zeta^a \left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} d\zeta d\eta \end{aligned} \tag{38}$$

By using (37) and (38) into the expression of $\lambda_l^*$, we obtain :

$$\begin{aligned} \lambda_l^* &= \int_0^{a_+} \varphi_l(a) \Bigg[ \frac{1}{\theta_i} \int_0^a \Gamma_l(\eta)\lambda_l^* s_l^*(\eta) e^{-\int_\eta^\zeta \left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} \\ &\quad \times \int_\eta^a \beta_l(\zeta) e^{-\int_\zeta^a \left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} d\zeta d\eta \\ &\quad + \theta_l \int_0^a \Gamma_{v_l}(\eta)\lambda_l^* s_{v_l}^*(\eta) e^{-\int_\eta^a \mu_v(\tau)d\tau} d\eta \Bigg] da \end{aligned} \tag{39}$$

By dividing (39) by $\lambda_l^*$ (like $\lambda_l^* \neq 0$) we get :

$$\begin{aligned} 1 &= \int_0^{a_+} \varphi_l(a) \Bigg[ \frac{1}{\theta_i} \int_0^a \Gamma_l(\eta) s_l^*(\eta) e^{-\int_\eta^\zeta \left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{j=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} \\ &\quad \times \int_\eta^a \beta_l(\zeta) e^{-\int_\zeta^a \left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} d\zeta d\eta \\ &\quad + \theta_l \int_0^a \Gamma_{v_l}(\eta) s_{v_l}^*(\eta) e^{-\int_\eta^a \mu_v(\tau)d\tau} d\eta \Bigg] da \end{aligned} \tag{40}$$

Let $G_l^*(\lambda_l^*)$ the function given by the right-hand side of the equation (40), we deduce the existence of an endemic equilibrium point when the equation (40) admits a positive solution. For $\lambda_l^* = 0$, we obtain $s_l^*(a) = s_l^0(a)$ and $s_{v_l}^*(a) = s_{v_l}^0(a)$ so the basic reproduction number is given by:

$$\begin{aligned} R_0^l &= \int_0^{a_+} \varphi_l(a) \Bigg[ \frac{1}{\theta_i} \int_0^a \Gamma_l(\eta) s_l^0(\eta) e^{-\int_\eta^\zeta \left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} \\ &\quad \times \int_\eta^a \beta_l(\zeta) e^{-\int_\zeta^a \left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} d\zeta d\eta \\ &\quad + \theta_l \int_0^a \Gamma_{v_l}(\eta) s_{v_l}^0(\eta) e^{-\int_\eta^a \mu_v(\tau)d\tau} d\eta \Bigg] da \end{aligned} \tag{41}$$

To complete the proof of the existence of the endemic equilibrium state, it suffices to show that the equation (40) admits a positive solution. Since $G_l(0) = R_0^l > 1$ and,

$$i_l^*(a) < 1, \; and \; i_{v_l}^*(a) < 1$$

we deduce from (40) that for $\lambda_l^* > 0$ and $\theta_l$ constant:

$$\begin{aligned} \lambda_l^* G_l^*(\lambda_l^*) &= \int_0^{a_+} \varphi_l(a) \lambda_l^* \Bigg[ \frac{1}{\theta_l} \int_0^a \Gamma_l(\eta) s_l^0(\eta) e^{-\int_\eta^\zeta \left(\sum_{j=1}^3(\epsilon_{lj}-\overline{\epsilon_{jl}})+\beta_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} \\ &\quad \times \int_\eta^a \beta_l(\zeta) e^{-\int_\zeta^a \left(\sum_{j=1}^3(\rho_{lj}-\overline{\rho_{jl}})+\gamma_l(\tau)+\mu_l(\tau)+b(\tau)-\sum_{l=1}^3 \mu_l(\tau) i_l^*(\tau)\right)d\tau} d\zeta d\eta \\ &\quad + \theta_l \int_0^a \Gamma_{v_l}(\eta) s_{v_l}^0(\eta) e^{-\int_\eta^a \mu_v(\tau)d\tau} d\eta \Bigg] da < \frac{\theta_l^2+1}{\theta_l} \int_0^{a_+} \varphi_l(a) da \end{aligned} \tag{42}$$

So for $\lambda_l^* \neq 0$, starting from the relation (42), we obtain:

$$G_l^*(\lambda_l^*) < \frac{\theta_l^2+1}{\theta_l} \frac{1}{\lambda_l^*} \int_0^{a_+} \varphi_l(a) da \tag{43}$$

The right hand side of (43) tends to zero when $\lambda_l^*$ tends to infinity. However $G_l^*(\lambda_l^*) = 1$ admits a unique solution in the interval $[0, +\infty[$. Moreover, if $\lambda_l^* > \int_0^{a_+} \varphi_l(a) da$ then has $G_l(\lambda_l^*) < 1$. Hence the equation (40) admits a unique solution in the interval $]0, \int_0^{a_+} \varphi_l(a) da[$ which proves the existence of a single point of endemic equilibrium. □

**Table 1.** parameter values and their dimensions

| Parameters | High transmission | Medium transmission | low transmission | References |
|---|---|---|---|---|
| $\Gamma_l(a)$ | 0.022 | 0.017 | 0.012 | [2, 14, 8] |
| $\Gamma_{v_l}(a)$ | 0.48 | 0.36 | 0.24 | [2, 10, 16] |
| $\varphi_l(a))$ | 0.40 | 0.325 | 0.25 | [2, 10, 16] |
| $b_l(a)$ | 0.00011 | 0.0000825 | 0.000055 | [2, 10] |
| $b_{v_l}(a)$ | 0.13 | 0.13 | 0.13 | [2, 10] |
| $\mu(a)$ | 0.033 | 0.033 | 0.033 | [2, 10] |
| $\mu_v(a)$ | 0.052 | 0.052 | 0.052 | [2, 10, 16] |
| $\mu_l(a)$ | 0.00009 | 0.000054 | (0.000018 | [2, 10] |
| $\beta_l(a)$ | 0.1 | 0.1 | 0.1 | [2, 10] |
| $\gamma_l(a)$ | 0.00351 | 0.00351 | 0.00351 | [10] |
| $\alpha_l(a)$ | 0.00055 | 0.001625 | 0.0027 | [10] |
| $\eta_{lj}$ | $[j=1]3\sum\left(\eta_{lj}-\overline{\eta}_{jl}\right)$=-0.277% | $\sum_{j=1}^{3}\left(\eta_{lj}-\overline{\eta}_{jl}\right)$=0.17% | $\sum_{j=1}^{3}\left(\eta_{lj}-\overline{\eta}_{jl}\right)$=0.33% | Estimation |
| $\epsilon_{lj}$ | $\sum_{j=1}^{3}\left(\epsilon_{lj}-\bar{\epsilon}_{jl}\right)$=-0.135% | $\sum_{j=1}^{3}\left(\epsilon_{lj}-\bar{\epsilon}_{jl}\right)$=0.085% | $\sum_{j=1}^{3}\left(\epsilon_{lj}-\bar{\epsilon}_{jl}\right)$=0.165% | Estimation |
| $\rho_{lj}$ | $\sum_{j=1}^{3}\left(\rho_{lj}-\overline{\rho}_{jl}\right)$=-0.135% | $\sum_{j=1}^{3}\left(\rho_{lj}-\overline{\rho}_{jl}\right)$=0.085% | $\sum_{j=1}^{3}\left(\rho_{lj}-\overline{\rho}_{jl}\right)$=0.165% | Estimation |
| $\Lambda_l$ | 0,033 | 0,033 | 0,033 | [2, 16] |
| $N_l$ | 5 453 983 | 3 235 656 | 2 467 055 | [20] |
| **Malaria prevalence** | [37,4 - 83] | [12,9 - 27,7] | [1,3 - 6,3] | [21] |

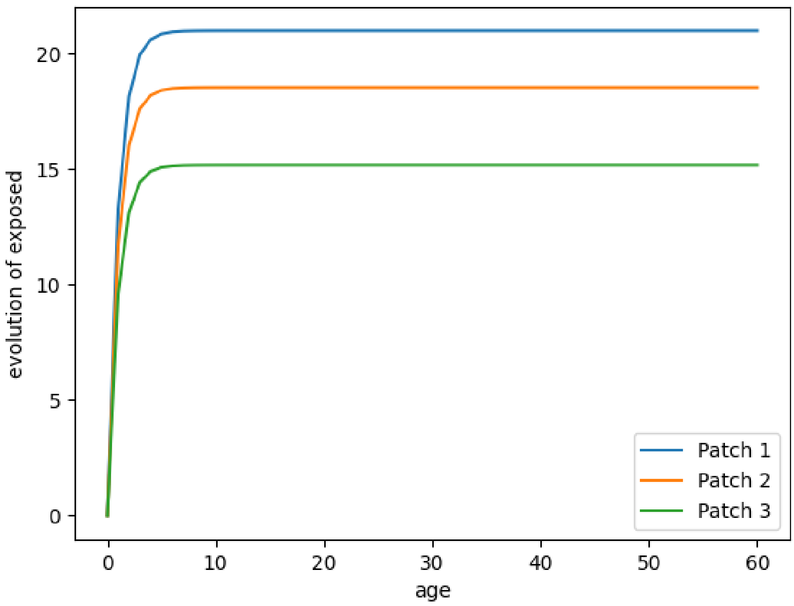


**Fig. 2.** Evolution of the human population of latent when $R_0 = 0.7 < 1$.

## 6. Numerical simulations

In this section, we propose numerical simulations which aim to illustrate the asymptotic behavior of the number of latent and infected individuals as a function of the basic reproduction numbers $R_0^l$ as predicted by the properties from Section 4.

The parameters used for the simulation are given in the following table:

The curves of Figures 2 and 3 are obtained by choosing the values of the parameters such that $R_0 = 0.7 < 1$. For these two figures, we used the same parameters above to follow the evolution of exposed and infectious individuals for each zone.

It appears from these curves that the most exposed population class is that of children under five (5) years of age. This could be explained by the fact that living individuals in this age group have no temporary immunity against the disease.

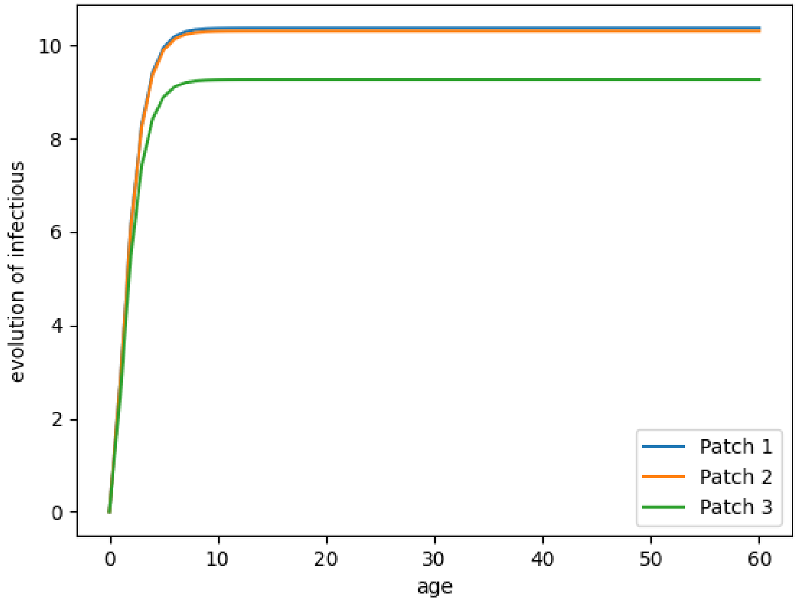


**Fig. 3.** Evolution of the human population of infectious when $R_0 = 0.7 < 1$.

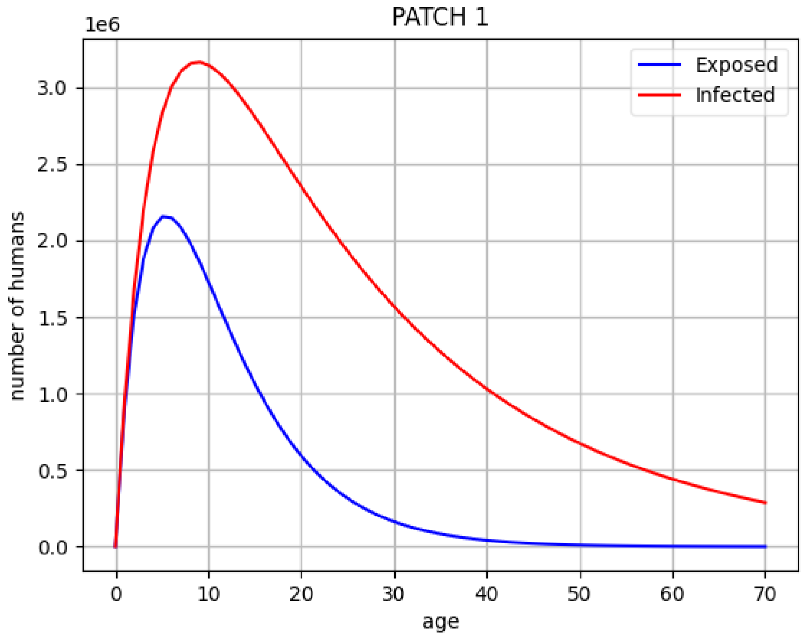


**Fig. 4.** Evolution of the human population of exposed and infectious when $R_0 = 0.7 < 1$.

Figures 4, 5 and 6 show the evolution of these same individuals (exposed and infectious), but each patch separately. These figures were obtained using the high, medium, and low transmission parameters, respectively. The curves in Figures 4 and 5 show that the number of individuals affected by malaria is growing up: the disease is endemic in the age group of children. The curve in Figure 6 corresponds to the situation in which there is extinction of the disease in the long term.

The curves of Figures 7 and 8 show the asymptotic behavior of the variation in the number of malaria cases in the first two patches for $R_0 > 1$. An endemic equilibrium is reached and is stable as stated by Theorem 6.

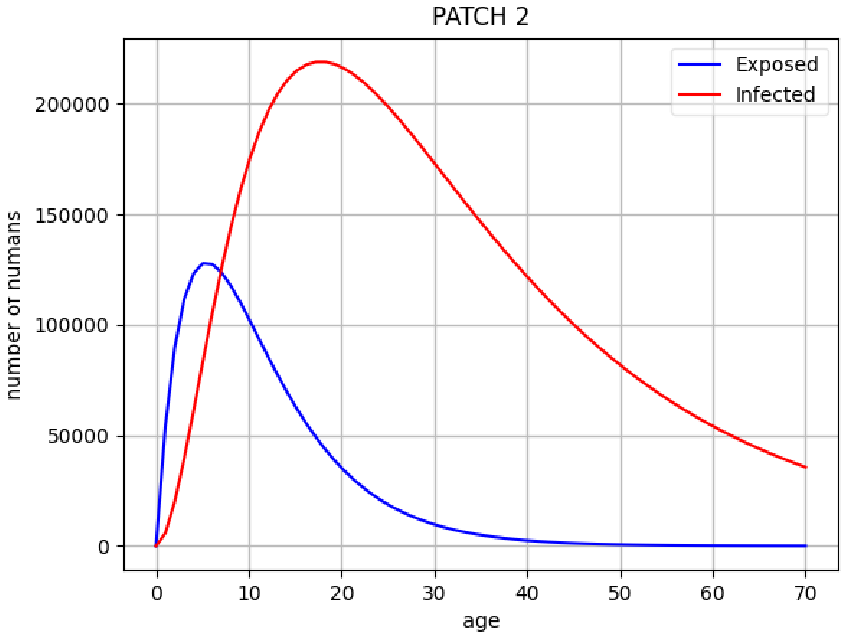


**Fig. 5.** Evolution of the human population of exposed and infectious when $R_0 = 0.605 < 1$.

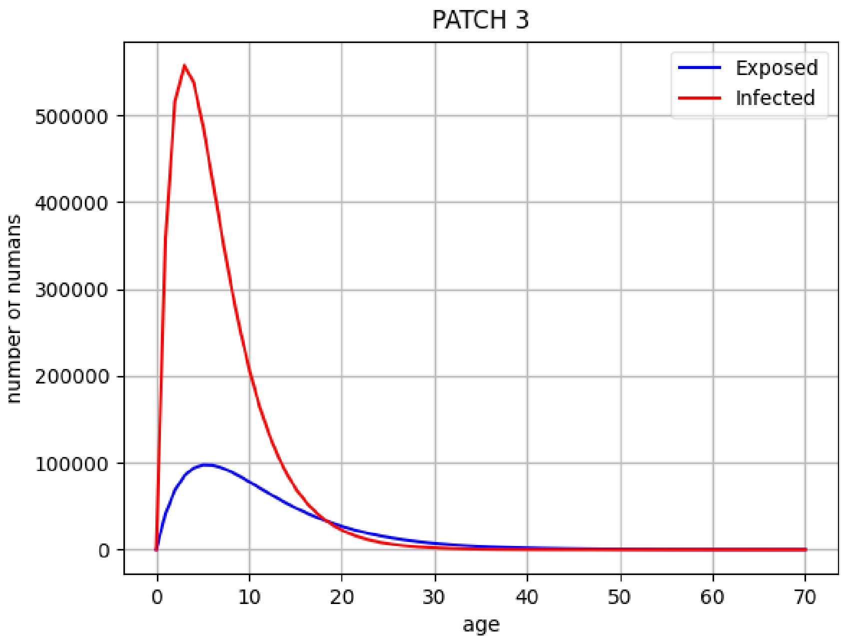


**Fig. 6.** Evolution of the human population of exposed and infectious when $R_0 = 0.58 < 1$.

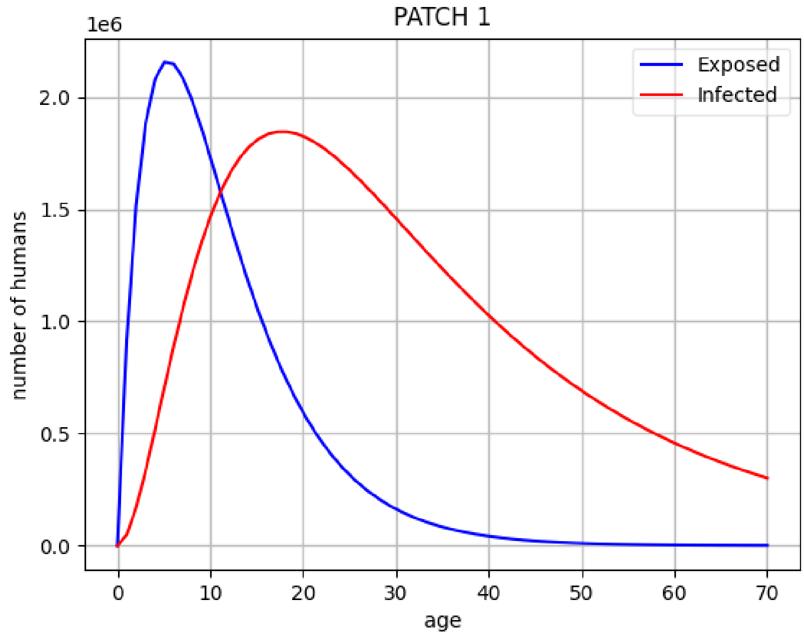


**Fig. 7.** Evolution of exposed and infectious when $R_0 = 13.58 > 1$.

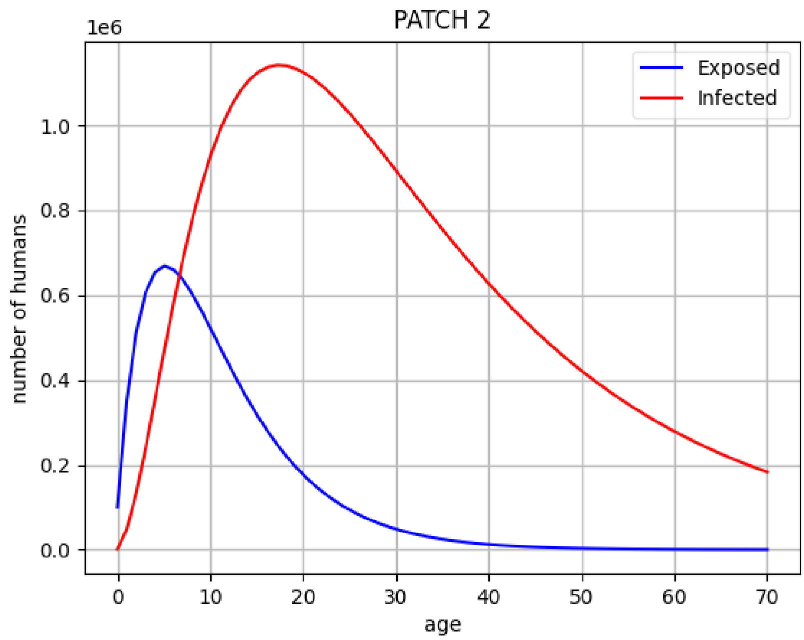


**Fig. 8.** Evolution of exposed and infectious when $R_0 = 13.58 > 1$.

## 7. Conclusion

We studied a three-patch age-structured model applied to the transmission dynamics of the malaria. We have divided the population into three areas according to its epidemiological status, and we have allowed migration in all patches. Numerical simulations show that for each figure, the These curves illustrate that when $R_0 < 1$, the disease is more prevalent in children than in adults. We notice through these figures that beyond 30 years the more the age advances, the disease becomes less important. This is because these individuals have acquired some partial immunity. For $R_0 > 1$, Figures 7 and 8 show that the disease still persists in the regions of high and medium transmission in the patch 1 and 3.

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