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On Sums and Generating Functions of Horadam Polynomials

Research Article

Yüksel Soykan *

Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey

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Abstract: In this paper, we present sum formulas $\sum_{k=0}^{n} kz^k W_{mk+j}$ and generating functions $\sum_{n=0}^{\infty} nW_{mn+j} z^n$ for Horadam (generalized Fiboacci) polynomials and special cases. Moreover, we evaluate the infinite sums of special cases of Horadam polynomials.

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1. Introduction and Preliminaries: Generalized Fibonacci Polynomials

The generalized Fibonacci polynomials (or Horadam polynomials or x-Horadam numbers or generalized $(r(x), s(x))$-polynomials or $(r(x), s(x))$ Horadam polynomials or 2-step Fibonacci polynomials)

$$\{W_n(W_0, W_1; r(x), s(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n(x) = r(x)W_{n-1}(x) + s(x)W_{n-2}(x), \quad W_0(x) = a(x), W_1(x) = b(x), \quad n \geq 2 \tag{1}$$

where $W_0(x), W_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x) \neq 0$, $s(x) \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = -\frac{r(x)}{s(x)} W_{-(n-1)}(x) + \frac{1}{s(x)} W_{-(n-2)}(x)$$

for $n = 1, 2, 3, \ldots$ when $s(x) \neq 0$. Therefore, recurrence (1) holds for all integers n. Note that $W_{-n}(x)$ need not to be a polynomial in the ordinary sense. For more details on generalized Fibonacci (Horadam) polynomials, see [10]. For some references on special cases of Horadam polynomials see [3–5, 9, 17, 18] for papers and [1, 2, 6–8, 11, 16] for books.

Binet's formula of generalized Fibonacci (Horadam) polynomials can be calculated using its characteristic equation (the quadratic equation, polynomial) which is given as

$$y^2 - r(x)y - s(x) = 0. \tag{2}$$

* *E-mail address(es):* yuksel_soykan@hotmail.com (Yüksel Soykan).

The roots of characteristic equation are

$$\alpha(x) := \alpha = \frac{r(x)+\sqrt{r^2(x)+4s(x)}}{2}, \quad \beta(x) := \beta = \frac{r(x)-\sqrt{r^2(x)+4s(x)}}{2}, \tag{3}$$

Now, we define two special cases of the polynomials $W_n(x)$. $(r(x), s(x))$-Fibonacci polynomials $\{G_n(0,1; r(x), s(x))\}_{n\geq 0}$ (or shortly $G_n(x)$) and $(r(x), s(x))$-Lucas polynomials $\{H_n(2, r(x); r(x), s(x))\}_{n\geq 0}$ (or shortly $H_n(x)$) are defined, respectively, by the second-order recurrence relations

$$G_{n+2}(x) = r(x)G_{n+1} + s(x)G_n(x), \quad G_0(x) = 0, G_1(x) = 1, \tag{4}$$
$$H_{n+2}(x) = r(x)H_{n+1} + s(x)H_n(x), \quad H_0(x) = 2, H_1(x) = r(x). \tag{5}$$

The (sequences of polinomials) $\{G_n(x)\}_{n\geq 0}$ and $\{H_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n}(x) = -\frac{r(x)}{s(x)}G_{-(n-1)}(x) + \frac{1}{s(x)}G_{-(n-2)}(x),$$
$$H_{-n}(x) = -\frac{r(x)}{s(x)}H_{-(n-1)}(x) + \frac{1}{s(x)}H_{-(n-2)}(x),$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (4) and (5) hold for all integers n.

NOTE: For the sake of simplicity throughout the rest of the paper, we use

$$W_n, r, s, W_0, W_1, \alpha, \beta, G_n, H_n, G_0, G_1, H_0, H_1$$

instead of

$$W_n(x), r(x), s(x), W_0(x), W_1(x), \alpha(x), \beta(x), G_n(x), H_n(x), G_0(x), G_1(x), H_0(x), H_1(x),$$

respectively, unless otherwise stated. . For example, we write

$$W_n = rW_{n-1} + sW_{n-2}, \quad W_0 = a, W_1 = b, \ n \geq 2$$

for the equation (1).

Using the roots α, β and recurrence relation (1), Binet's formula can be given as follows:

Theorem 1.1 ([10], Theorem 2).
The general term of the generalized Fibonacci (Horadam) polynomials W_n can be presented by the following Binet's formula:

$$W_n = \begin{cases} \frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases} \tag{6}$$
$$= \begin{cases} \frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \frac{r}{2}(n-1)W_0)\left(\frac{r}{2}\right)^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.$$

We can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Fibonacci polynomials (in terms of elements of the sequence of generalized Fibonacci polynomials).

Theorem 1.2 ([15], Theorem 4.1).
For all integers m and j, we have the following sum formulas.

(a) *If $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 \neq 0$ then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4}{z^2\Gamma_1 + z\Gamma_2 + \Gamma_3} = \frac{\Theta_W(z)}{\Gamma_W(z)} \tag{7}$$

where

$\Theta_W(z) = z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4 = z^{n+2}((W_jW_{m+1} - W_{j+1}W_m)W_{mn+m+1} + ((W_{j+1} - rW_j)W_{m+1} - sW_jW_m)W_{mn+m}) + z^{n+1}((W_0W_{j+1} - W_1W_j)W_{mn+m+1} + (-W_1W_{j+1} + (rW_1 + sW_0)W_j)W_{m+mn}) + z((-W_0W_{j+1} + (-W_1 + rW_0)W_j)W_{m+1} + (W_1W_{j+1} + sW_0W_j)W_m) + (W_1^2 - sW_0^2 - rW_0W_1)W_j$

$z^{n+2}\Theta_1 = z^{n+2}((W_jW_{m+1} - W_{j+1}W_m)W_{mn+m+1} + ((W_{j+1} - rW_j)W_{m+1} - sW_jW_m)W_{mn+m})$

$z^{n+1}\Theta_2 = z^{n+1}((W_0W_{j+1} - W_1W_j)W_{mn+m+1} + (-W_1W_{j+1} + (rW_1 + sW_0)W_j)W_{m+mn})$

$z\Theta_3 = z((-W_0W_{j+1} + (-W_1 + rW_0)W_j)W_{m+1} + (W_1W_{j+1} + sW_0W_j)W_m)$

$\Theta_4 = (W_1^2 - sW_0^2 - rW_0W_1)W_j$

i.e.,

$\Theta_W(z) = z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4 = z^{n+2}(-1)(W_1^2 - sW_0^2 - rW_0W_1)((-H_mG_j + G_{m+j})W_{mn+m+1} + (-H_m(G_{j+1} - rG_j) + G_{m+j+1} - rG_{m+j})W_{mn+m}) + z^{n+1}(-1)(W_1^2 - sW_0^2 - rW_0W_1)(G_jW_{mn+m+1} + (G_{j+1} - rG_j)W_{m+mn}) + z(W_1^2 - sW_0^2 - rW_0W_1)(G_mW_{j+1} - G_{m+1}W_j) + (W_1^2 - sW_0^2 - rW_0W_1)W_j$

$z^{n+2}\Theta_1 = z^{n+2}(-1)(W_1^2 - sW_0^2 - rW_0W_1)((-H_mG_j + G_{m+j})W_{mn+m+1} + (-H_m(G_{j+1} - rG_j) + G_{m+j+1} - rG_{m+j})W_{mn+m})$

$z^{n+1}\Theta_2 = z^{n+1}(-1)(W_1^2 - sW_0^2 - rW_0W_1)(G_jW_{mn+m+1} + (G_{j+1} - rG_j)W_{m+mn})$

$z\Theta_3 = z(W_1^2 - sW_0^2 - rW_0W_1)(G_mW_{j+1} - G_{m+1}W_j)$

$\Theta_4 = (W_1^2 - sW_0^2 - rW_0W_1)W_j$

and

$\Gamma_W(z) = z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = z^2(-1)^m s^m(W_1^2 - sW_0^2 - rW_0W_1) + z(-1)H_m(W_1^2 - sW_0^2 - rW_0W_1) + (W_1^2 - sW_0^2 - rW_0W_1)$

$z^2\Gamma_1 = z^2(W_{m+1}^2 - sW_m^2 - rW_mW_{m+1})$

$z\Gamma_2 = z((-2W_1 + rW_0)W_{m+1} + (rW_1 + 2sW_0)W_m)$

$\Gamma_3 = W_1^2 - sW_0^2 - rW_0W_1$

i.e.,

$\Gamma_W(z) = z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = (z^2(-1)^m s^m + z(-1)H_m + 1)(W_1^2 - sW_0^2 - rW_0W_1)$

$z^2\Gamma_1 = z^2(-1)^m s^m(W_1^2 - sW_0^2 - rW_0W_1)$

$z\Gamma_2 = z(-1)H_m(W_1^2 - sW_0^2 - rW_0W_1)$

$\Gamma_3 = W_1^2 - sW_0^2 - rW_0W_1$

(b) *If* $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = u(z-a)(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $z = a$ *or* $z = b$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+2)z^{n+1}\Theta_1 + (n+1)z^n\Theta_2 + \Theta_3}{2z\Gamma_1 + \Gamma_2}.$$

(c) *If* $z^2\Gamma_1 + z\Gamma_2 + \Gamma_3 = u(z-a)^2 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $z = a$*, then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+2)(n+1)z^n\Theta_1 + (n+1)nz^{n-1}\Theta_2}{2\Gamma_1}.$$

2. The Weighted Sum Formula $\sum_{k=0}^{n} kz^k W_{mk+j}$ of Generalized Fibonacci Polynomials

By using Theorem 1.2 (a), we can give the sum formula $\sum_{k=0}^{n} kz^k W_{mk+j}$ of generalized Fibonacci polynomials (in terms of elements of the sequence of generalized Fibonacci polynomials).

Theorem 2.1.
Let z be a non-zero complex (or real) number. For all integers m and j, we have the following sum formulas.

(a) *If* $(z^2(-1)^m s^m + z(-1)H_m + 1)^2(W_1^2 - sW_0^2 - rW_0W_1)^2 \neq 0$ *then*

$$\sum_{k=0}^{n} kz^k W_{mk+j} = \frac{z(\Gamma_W(z)\frac{d}{dz}\Theta_W(z) - \Theta_W(z)\frac{d}{dz}\Gamma_W(z))}{(\Gamma_W(z))^2} = \frac{(W_1^2 - sW_0^2 - rW_0W_1)\Delta_W(z)}{(z^2(-1)^m s^m + z(-1)H_m + 1)^2(W_1^2 - sW_0^2 - rW_0W_1)^2} \quad (8)$$

where

$\Theta_W(z) = z^{n+2}\Theta_1 + z^{n+1}\Theta_2 + z\Theta_3 + \Theta_4 = z^{n+2}((W_jW_{m+1} - W_{j+1}W_m)W_{mn+m+1} + ((W_{j+1} - rW_j)W_{m+1} - sW_jW_m)W_{mn+m}) + z^{n+1}((W_0W_{j+1} - W_1W_j)W_{mn+m+1} + (-W_1W_{j+1} + (rW_1 + sW_0)W_j)W_{m+mn}) + z((-W_0W_{j+1} + (-W_1 + rW_0)W_j)W_{m+1} + (W_1W_{j+1} + sW_0W_j)W_m) + (W_1^2 - sW_0^2 - rW_0W_1)W_j,$

$\frac{d}{dz}\Theta_W(z) = (n+2)z^{n+1}\Theta_1+(n+1)z^n\Theta_2+\Theta_3 = (n+2)z^{n+1}((W_jW_{m+1}-W_{j+1}W_m)W_{mn+m+1}+((W_{j+1}-rW_j)W_{m+1}-sW_jW_m)W_{mn+m})+(n+1)z^n((W_0W_{j+1}-W_1W_j)W_{mn+m+1}+(-W_1W_{j+1}+(rW_1+sW_0)W_j)W_{m+mn})+((-W_0W_{j+1}+(-W_1+rW_0)W_j)W_{m+1}+(W_1W_{j+1}+sW_0W_j)W_m),$

and

$\Gamma_W(z) = z^2\Gamma_1+z\Gamma_2+\Gamma_3 = (z^2(-1)^m s^m+z(-1)H_m+1)(W_1^2-sW_0^2-rW_0W_1),$

$\frac{d}{dz}\Gamma_W(z) = 2z\Gamma_1+\Gamma_2 = (2z(-1)^m s^m+(-1)H_m)(W_1^2-sW_0^2-rW_0W_1),$

$\Delta_W(z) = n(-s)^m z^{n+4}((-W_mW_{j+1}+W_jW_{m+1})W_{m+mn+1}+((W_{j+1}-rW_j)W_{m+1}-sW_jW_m)W_{m+mn})+z^{n+3}((-(n+1)W_{m+1}W_jH_m+(n+1)W_{j+1}W_mH_m+(-s)^m(n-1)(W_0W_{j+1}-W_1W_j))W_{m+mn+1}+((n+1)(rW_j-W_{j+1})W_{m+1}H_m+s(n+1)W_jW_mH_m+(-s)^m(n-1)\left(-W_1W_{j+1}+(rW_1+sW_0)W_j\right))W_{m+mn})+z^{n+2}(((n+2)W_jW_{m+1}-(n+2)W_{j+1}W_m-n(W_0W_{j+1}-W_1W_j)H_m)W_{m+mn+1}+((n+2)(W_{j+1}-rW_j)W_{m+1}-s(n+2)W_jW_m+n((W_{j+1}-rW_j)W_1-sW_0W_j)H_m)W_{m+mn})+(n+1)z^{n+1}((W_0W_{j+1}-W_1W_j)W_{m+mn+1}-(W_1W_{j+1}-(rW_1+sW_0)W_j)W_{m+mn})+z^3(-s)^m((W_1W_j+W_0(W_{j+1}-rW_j))W_{m+1}-W_m(W_1W_{j+1}+sW_0W_j))-2z^2(-s)^m(W_1^2-sW_0^2-rW_0W_1)W_j+z((W_1^2-sW_0^2-rW_0W_1)W_jH_m-(W_1W_j+(W_{j+1}-rW_j)W_0)W_{m+1}+(W_1W_{j+1}+sW_0W_j)W_m),$

$z(\Gamma_W(z)\frac{d}{dz}\Theta_W(z)-\Theta_W(z)\frac{d}{dz}\Gamma_W(z)) = (W_1^2-sW_0^2-rW_0W_1)\Delta_W(z).$

(b) *If* $(z^2(-1)^m s^m+z(-1)H_m+1)^2(W_1^2-sW_0^2-rW_0W_1)^2 = u(z-a)^2(z-b)^2 = 0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then*

$$\sum_{k=0}^{n} kz^kW_{mk+j} = \frac{\frac{d^2}{dz^2}(z(\Gamma_W(z)\frac{d}{dz}\Theta_W(z)-\Theta_W(z)\frac{d}{dz}\Gamma_W(z)))}{\frac{d^2}{dz^2}(\Gamma_W(z))^2} = \frac{\frac{d^2}{dz^2}((W_1^2-sW_0^2-rW_0W_1)\Delta_W(z))}{(12z^2s^{2m}-12z(-1)^m s^mH_m+2(H_m^2+2(-1)^m s^m))(W_1^2-sW_0^2-rW_0W_1)^2}$$

(c) *If* $(z^2(-1)^m s^m+z(-1)H_m+1)^2(W_1^2-sW_0^2-rW_0W_1)^2 = u(z-a)^4 = 0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, then*

$$\sum_{k=0}^{n} kz^kW_{mk+j} = \frac{\frac{d^4}{dz^4}(z(\Gamma_W(z)\frac{d}{dz}\Theta_W(z)-\Theta_W(z)\frac{d}{dz}\Gamma_W(z)))}{\frac{d^4}{dz^4}(\Gamma_W(z))^2} = \frac{\frac{d^4}{dz^4}((W_1^2-sW_0^2-rW_0W_1)\Delta_W(z))}{24s^{2m}(W_1^2-sW_0^2-rW_0W_1)^2}$$

Proof. Note that

$$\begin{aligned}
(\Gamma_W(z))^2 &= (z^2(-1)^m s^m+z(-1)H_m+1)^2(W_1^2-sW_0^2-rW_0W_1)^2\\
&= (z^4s^{2m}-2z^3(-1)^m s^mH_m+z^2(H_m^2+2(-1)^m s^m)-2zH_m+1)(W_1^2-sW_0^2-rW_0W_1)^2,\\
\frac{d}{dz}(\Gamma_W(z))^2 &= (4z^3s^{2m}-6z^2(-1)^m s^mH_m+2z(H_m^2+2(-1)^m s^m)-2H_m)(W_1^2-sW_0^2-rW_0W_1)^2,\\
\frac{d^2}{dz^2}(\Gamma_W(z))^2 &= (12z^2s^{2m}-12z(-1)^m s^mH_m+2(H_m^2+2(-1)^m s^m))(W_1^2-sW_0^2-rW_0W_1)^2,\\
\frac{d^3}{dz^3}(\Gamma_W(z))^2 &= (24zs^{2m}-12(-1)^m s^mH_m)(W_1^2-sW_0^2-rW_0W_1)^2,\\
\frac{d^4}{dz^4}(\Gamma_W(z))^2 &= 24s^{2m}(W_1^2-sW_0^2-rW_0W_1)^2.
\end{aligned}$$

(a) From Theorem 1.2 (a), we know that

$$\sum_{k=0}^{n} z^kW_{mk+j} = \frac{\Theta_W(z)}{\Gamma_W(z)}.$$

By taking the derivative of the both sides of the above formulas with respect to z, we get

$$\sum_{k=0}^{n} kz^{k-1}W_{mk+j} = \frac{\Gamma_W(z)\frac{d}{dz}\Theta_W(z)-\Theta_W(z)\frac{d}{dz}\Gamma_W(z)}{\left((z^2(-1)^m s^m+z(-1)H_m+1)(W_1^2-sW_0^2-rW_0W_1)\right)^2}$$

where $\frac{d}{dz}\Theta_W(z) = \Theta'_W(z)$ and $\frac{d}{dz}\Gamma_W(z) = \Gamma'_W(z)$ denotes the derivatives of $\Theta_W(z)$ and $\Gamma_W(z)$, respectively. Then it follows that

$$\sum_{k=0}^{n} kz^kW_{mk+j} = z\times\frac{\Gamma_W(z)\frac{d}{dz}\Theta_W(z)-\Theta_W(z)\frac{d}{dz}\Gamma_W(z)}{(z^2(-1)^m s^m+z(-1)H_m+1)^2(W_1^2-sW_0^2-rW_0W_1)^2}.$$

(b) We use (8). For $z=a$ and $z=b$, the right hand side of the sum formula (8) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (ii) by using

$$\sum_{k=0}^{n} ka^k W_{mk+j} = \left.\frac{\frac{d^2}{dz^2}(z(\Gamma_W(z)\frac{d}{dz}\Theta_W(z)-\Theta_W(z)\frac{d}{dz}\Gamma_W(z)))}{\frac{d^2}{dz^2}(\Gamma_W(z))^2}\right|_{z=a}$$

and

$$\sum_{k=0}^{n} kb^k W_{mk+j} = \left.\frac{\frac{d^2}{dz^2}(z(\Gamma_W(z)\frac{d}{dz}\Theta_W(z)-\Theta_W(z)\frac{d}{dz}\Gamma_W(z)))}{\frac{d^2}{dz^2}(\Gamma_W(z))^2}\right|_{z=b}$$

(c) For $z=a$, the right hand side of the sum formula (8) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (iii) by using

$$\sum_{k=0}^{n} ka^k W_{mk+j} = \left.\frac{\frac{d^4}{dz^4}(z(\Gamma_W(z)\frac{d}{dz}\Theta_W(z)-\Theta_W(z)\frac{d}{dz}\Gamma_W(z)))}{\frac{d^4}{dz^4}(\Gamma_W(z))^2}\right|_{z=a}. \quad \square$$

Now, we consider special cases of Theorem 1.2.

Theorem 2.2.
Let z be a non-zero complex (or real) number. For all integers m and j, we have the following sum formulas.

(a) $(m=1, j=0)$.

(i) *If $(sz^2+rz-1)^2 \neq 0$, i.e., if $z \neq \frac{1}{2s}(-r-\sqrt{r^2+4s})$, $z \neq \frac{1}{2s}(-r+\sqrt{r^2+4s})$ then*

$$\sum_{k=0}^{n} kz^k W_k = \frac{\Omega_1}{(sz^2+rz-1)^2}$$

where
$\Omega_1 = ns^2z^{n+4}W_n + sz^{n+3}((n-1)W_{n+1}+r(n+1)W_n)+z^{n+2}(nrW_{n+1}-s(n+2)W_n)-(n+1)z^{n+1}W_{n+1}+sz^3(W_1-rW_0)+2sz^2W_0+W_1z$

(ii) *If $(sz^2+rz-1)^2 = 0$ provided that $r^2+4s \neq 0$, i.e., if $z = \frac{1}{2s}(-r-\sqrt{r^2+4s})$ or $z = \frac{1}{2s}(-r+\sqrt{r^2+4s})$ provided that $s \neq -\frac{r^2}{4}$ then*

$$\sum_{k=0}^{n} kz^k W_k = \frac{\Omega_2}{2(6s^2z^2+6rsz+r^2-2s)}$$

where
$\Omega_2 = n(n+4)(n+3)s^2z^{n+2}W_n + s(n+3)(n+2)z^{n+1}((n-1)W_{n+1}+r(n+1)W_n)+(n+2)(n+1)z^n(nrW_{n+1}-s(n+2)W_n)-n(n+1)^2z^{n-1}W_{n+1}+6sz(W_1-rW_0)+4sW_0$

(iii) *If $(sz^2+rz-1)^2 = (z-\frac{2}{r})^4 = 0$ provided that $r^2+4s=0$, i.e., if $z = -\frac{r}{2s} = \frac{2}{r}$, $s = -\frac{r^2}{4}$ then*

$$\sum_{k=0}^{n} kz^k W_k = \frac{(n+1)nz^{n-3}}{24s^2}\Omega_3$$

where
$\Omega_3 = s^2z^3(n+4)(n+3)(n+2)W_n + sz^2(n+3)(n+2)((n-1)W_{n+1}+r(n+1)W_n)+z(n+2)(n-1)(nrW_{n+1}-s(n+2)W_n)-(n+1)(n-1)(n-2)W_{n+1}$

(b) $(m=2, j=0)$.

(i) *If* $(r^2z-s^2z^2+2sz-1)^2\neq 0$, *i.e., if* $z\neq \frac{1}{2s^2}((r^2+2s)+r\sqrt{r^2+4s})$, $z\neq \frac{1}{2s^2}((r^2+2s)-r\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} kz^kW_{2k}=\frac{\Omega_1}{(r^2z-s^2z^2+2sz-1)^2}$$

where

$\Omega_1=ns^4z^{n+4}W_{2n}+s^2z^{n+3}(r(1-n)W_{2n+1}-((r^2+3s)n+r^2+s)W_{2n})+z^{n+2}(nr(2s+r^2)W_{2n+1}+s(nr^2+3ns+2s)W_{2n})-(n+1)z^{n+1}(rW_{2n+1}+sW_{2n})+s^2z^3(-rW_1+sW_0+r^2W_0)+-2s^2z^2W_0+z(rW_1+sW_0)$

(ii) *If* $(r^2z-s^2z^2+2sz-1)^2=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2s^2}((r^2+2s)+r\sqrt{r^2+4s})$ *or* $z=\frac{1}{2s^2}((r^2+2s)-r\sqrt{r^2+4s})$ *provided that* $s\neq -\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{2k}=\frac{\Omega_2}{2z(6s^4z^2-6s^2(r^2+2s)z+r^4+4r^2s+6s^2)}$$

where

$\Omega_2=ns^4(n+4)(n+3)z^{n+3}W_{2n}+s^2(n+3)(n+2)z^{n+2}(-r(n-1)W_{2n+1}-(nr^2+3ns+s+r^2)W_{2n})+(n+2)(n+1)z^{n+1}(nr(2s+r^2)W_{2n+1}+s(nr^2+3ns+2s)W_{2n})-n(n+1)^2z^n(rW_{2n+1}+sW_{2n})+6s^2z^2(-rW_1+sW_0+r^2W_0)-4s^2zW_0$

(iii) *If* $(r^2z-s^2z^2+2sz-1)^2=(z-\frac{4}{r^2})^4=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=\frac{r^2+2s}{2s^2}=\frac{4}{r^2}$, $s=-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{2k}=\frac{(n+1)\,nz^{n-3}}{24s^4}\Omega_3$$

where

$\Omega_3=s^4z^3(n+4)(n+3)(n+2)W_{2n}+s^2z^2(n+3)(n+2)(-r(n-1)W_{2n+1}-(nr^2+3ns+s+r^2)W_{2n})+z(n+2)(n-1)(nr(2s+r^2)W_{2n+1}+s(nr^2+3ns+2s)W_{2n})-(n+1)(n-1)(n-2)(rW_{2n+1}+sW_{2n})$

(c) $(m=2, j=1)$.

(i) *If* $(r^2z-s^2z^2+2sz-1)^2\neq 0$, *i.e., if* $z\neq \frac{1}{2s^2}((r^2+2s)+r\sqrt{r^2+4s})$, $z\neq \frac{1}{2s^2}((r^2+2s)-r\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} kz^kW_{2k+1}=\frac{1}{(r^2z-s^2z^2+2sz-1)^2}\Omega_1$$

where

$\Omega_1=ns^4z^{n+4}W_{2n+1}-s^2z^{n+3}((2nr^2+3ns+s)W_{2n+1}+rs(n-1)W_{2n})+z^{n+2}((nr^4+3nr^2s+3ns^2+2s^2)W_{2n+1}+nrs(2s+r^2)W_{2n})-z^{n+1}(n+1)((s+r^2)W_{2n+1}+rsW_{2n})+s^3z^3(W_1-rW_0)-2s^2z^2W_1+z((s+r^2)W_1+rsW_0)$

(ii) *If* $(r^2z-s^2z^2+2sz-1)^2=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2s^2}((r^2+2s)+r\sqrt{r^2+4s})$ *or* $z=\frac{1}{2s^2}((r^2+2s)-r\sqrt{r^2+4s})$ *provided that* $s\neq -\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{2k+1}=\frac{1}{2z(6s^4z^2-6s^2z(2s+r^2)+r^4+4r^2s+6s^2)}\Omega_2$$

where

$\Omega_2=s^4n(n+4)(n+3)z^{n+3}W_{2n+1}-s^2(n+3)(n+2)z^{n+2}((s+2nr^2+3ns)W_{2n+1}+rs(n-1)W_{2n})+(n+2)(n+1)z^{n+1}(((r^4+3r^2s+3s^2)n+2s^2)W_{2n+1}+nrs(2s+r^2)W_{2n})-n(n+1)^2z^n((s+r^2)W_{2n+1}+rsW_{2n})+6s^3z^2(W_1-rW_0)-4s^2zW_1$

(iii) *If* $(r^2z-s^2z^2+2sz-1)^2=(z-\frac{4}{r^2})^4=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=\frac{r^2+2s}{2s^2}=\frac{4}{r^2}$, $s=-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{2k+1}=\frac{n\,(n+1)\,z^{n-3}}{24s^4}\Omega_3$$

where

$\Omega_3=s^4(n+4)(n+3)(n+2)z^3W_{2n+1}-s^2(n+3)(n+2)z^2((s+2nr^2+3ns)W_{2n+1}+rs(n-1)W_{2n})+(n+2)(n-1)z((nr^4+3nr^2s+3ns^2+2s^2)W_{2n+1}+nrs(2s+r^2)W_{2n})-(n+1)(n-1)(n-2)(s+r^2)W_{2n+1}-rs(n+1)(n-1)(n-2)W_{2n}$

(d) $(m=-1, j=0)$.

(i) *If* $(z^2-rz-s)^2\neq 0$, *i.e., if* $z\neq \frac{1}{2}(r+\sqrt{r^2+4s})$, $z\neq \frac{1}{2}(r-\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} kz^kW_{-k}=\frac{1}{(z^2-rz-s)^2}\Omega_1$$

where

$\Omega_1 = nz^{n+4}W_{-n}-z^{n+3}(-s(n-1)W_{-n-1}+r(n+1)W_{-n})-sz^{n+2}((n+2)W_{-n}+nrW_{-n-1})-s^2(n+1)z^{n+1}W_{-n-1}+z^3W_1+2sz^2W_0+z(sW_1-rsW_0)$

$= nz^{n+4}W_{-n}+z^{n+3}((n-1)W_{-n+1}-2nrW_{-n})+z^{n+2}(-nrW_{-n+1}+(nr^2-ns-2s)W_{-n})-s(n+1)z^{n+1}(W_{-n+1}-rW_{-n})+z^3W_1+2sz^2W_0+z(sW_1-rsW_0)$

(ii) *If* $(z^2-rz-s)^2=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2}(r+\sqrt{r^2+4s})$ *or* $z=\frac{1}{2}(r-\sqrt{r^2+4s})$ *provided that* $s\neq -\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{-k}=\frac{1}{2z(6z^2-6rz+r^2-2s)}\Omega_2$$

where

$\Omega_2 = n(n+4)(n+3)z^{n+3}W_{-n}-(n+3)(n+2)z^{n+2}(r(n+1)W_{-n}-s(n-1)W_{-n-1})-s(n+2)(n+1)z^{n+1}(2W_{-n}+n(rW_{-n-1}+W_{-n}))-ns^2(n+1)^2z^nW_{-n-1}+6z^2W_1+4szW_0$

$= n(n+4)(n+3)z^{n+3}W_{-n}-(n+3)(n+2)z^{n+2}(2nrW_{-n}-(n-1)W_{-n+1})-(n+2)(n+1)z^{n+1}(nrW_{-n+1}+(-nr^2+ns+2s)W_{-n})-ns(n+1)^2z^n(W_{-n+1}-rW_{-n})+6z^2W_1+4szW_0$

(iii) *If* $(z^2-rz-s)^2=(z-\frac{r}{2})^4=0$ *provided that* $r^2+4s=0$, *i.e., if* $z=\frac{r}{2}$, $s=-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{-k}=\frac{n(n+1)z^{n-3}}{24}\Omega_3$$

where

$\Omega_3 = (n+4)(n+3)(n+2)z^3W_{-n}-z^2(n+3)(n+2)(r(n+1)W_{-n}-s(n-1)W_{-n-1})-s(n+2)(n-1)z((n+2)W_{-n}+nrW_{-n-1})-s^2(n+1)(n-1)(n-2)W_{-n-1}$

$= (n+4)(n+3)(n+2)z^3W_{-n}-z^2(n+3)(n+2)(-(n-1)W_{-n+1}+2nrW_{-n})-(n+2)(n-1)z(nrW_{-n+1}+(ns-nr^2+2s)W_{-n})-s(n+1)(n-1)(n-2)(W_{-n+1}-rW_{-n})$

(e) $(m=-2, j=0)$.

(i) *If* $(z^2-(r^2+2s)z+s^2)^2\neq 0$, *i.e., if* $z\neq \frac{1}{2}((r^2+2s)+r\sqrt{r^2+4s})$, $z\neq \frac{1}{2}((r^2+2s)-r\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} kz^kW_{-2k}=\frac{1}{(z^2-(r^2+2s)z+s^2)^2}\Omega_1$$

where

$\Omega_1 = nz^{n+4}W_{-2n}-z^{n+3}((nr^2+3ns+s+r^2)W_{-2n}-rs(n-1)W_{-2n-1})+sz^{n+2}((nr^2+3ns+2s)W_{-2n}-nr(2s+r^2)W_{-2n-1})+s^3(n+1)z^{n+1}(-W_{-2n}+rW_{-2n-1})+z^3(rW_1+sW_0)-2s^2z^2W_0+s^2z(-rW_1+(s+r^2)W_0)$

$= nz^{n+4}W_{-2n}+z^{n+3}(r(n-1)W_{-2n+1}-(2nr^2+3ns+s)W_{-2n})+z^{n+2}(-nr(2s+r^2)W_{-2n+1}+((r^4+3r^2s+3s^2)n+2s^2)W_{-2n})+s^2(n+1)z^{n+1}(rW_{-2n+1}-(r^2+s)W_{-2n})+z^3(rW_1+sW_0)-2s^2z^2W_0+s^2z(-rW_1+(s+r^2)W_0)$

(ii) *If* $(z^2-(r^2+2s)z+s^2)^2=0$ *provided that* $r^2+4s\neq 0$, *i.e., if* $z=\frac{1}{2}((r^2+2s)+r\sqrt{r^2+4s})$ *or* $z=\frac{1}{2}((r^2+2s)-r\sqrt{r^2+4s})$ *provided that* $s\neq -\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{-2k}=\frac{1}{2z(6z^2-6z(r^2+2s)+r^4+4r^2s+6s^2)}\Omega_2$$

where

$\Omega_2 = n(n+4)(n+3)z^{n+3}W_{-2n}-(n+3)(n+2)z^{n+2}((nr^2+3ns+r^2+s)W_{-2n}-rs(n-1)W_{-2n-1})+s(n+2)(n+1)z^{n+1}((nr^2+3ns+2s)W_{-2n}-nr(2s+r^2)W_{-2n-1})+ns^3(n+1)^2z^n(-W_{-2n}+rW_{-2n-1})+6z^2(rW_1+sW_0)-4s^2zW_0$

$= n(n+4)(n+3)z^{n+3}W_{-2n}-(n+3)(n+2)z^{n+2}(r(1-n)W_{-2n+1}+(2nr^2+3ns+s)W_{-2n})+(n+2)(n+1)z^{n+1}(-nr(r^2+2s)W_{-2n+1}+(3nr^2s+nr^4+3ns^2+2s^2)W_{-2n})+ns^3(n+1)^2z^n(-W_{-2n}+r\frac{1}{s}(W_{-2n+1}-rW_{-2n}))+6z^2(rW_1+sW_0)-4s^2zW_0$

(iii) *If* $(z^2-(r^2+2s)z+s^2)^2=(z-\frac{r^2}{4})^4=0$ *provided that* $r^2+4s=0$*, i.e., if* $z=\frac{r^2+2s}{2}=\frac{r^2}{4}$, $s=-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{-2k}=\frac{n(n+1)z^{n-3}}{24}\Omega_3$$

where

$\Omega_3=z^3(n+4)(n+3)(n+2)W_{-2n}-z^2(n+3)(n+2)((nr^2+3ns+r^2+s)W_{-2n}-rs(n-1)W_{-2n-1})+sz(n+2)(n-1)((nr^2+3ns+2s)W_{-2n}-nr(r^2+2s)W_{-2n-1})-s^3(n+1)(n-1)(n-2)W_{-2n}+rs^3(n+1)(n-1)(n-2)W_{-2n-1}$
$=z^3(n+4)(n+3)(n+2)W_{-2n}-z^2(n+3)(n+2)(r(1-n)W_{-2n+1}+(2nr^2+3ns+s)W_{-2n})+z(n+2)(n-1)(-nr(r^2+2s)W_{-2n+1}+(nr^4+3ns^2+3nr^2s+2s^2)W_{-2n})-s^2(n+1)(n-1)(n-2)(r^2+s)W_{-2n}+rs^2(n+1)(n-1)(n-2)W_{-2n+1}$

(f) $(m=-2, j=1)$.

(i) *If* $(z^2-(r^2+2s)z+s^2)^2\neq 0$*, i.e., if* $z\neq\frac{1}{2}((r^2+2s)+r\sqrt{r^2+4s})$, $z\neq\frac{1}{2}((r^2+2s)-r\sqrt{r^2+4s})$ *then*

$$\sum_{k=0}^{n} kz^kW_{-2k+1}=\frac{1}{(z^2-(r^2+2s)z+s^2)^2}\Omega_1$$

where

$\Omega_1=nz^{n+4}(rW_{-2n}+sW_{-2n-1})-z^{n+3}(r(n+1)(r^2+2s)W_{-2n}+s(nr^2+3ns+r^2+s)W_{-2n-1})+s^2z^{n+2}(r(n+2)W_{-2n}+(nr^2+3ns+2s)W_{-2n-1})-s^4(n+1)z^{n+1}W_{-2n-1}+z^3(r^2+s)W_1+sz^2(-2sW_1+rzW_0)+s^3z(W_1-rW_0)$
$=nz^{n+4}W_{-2n+1}-z^{n+3}((nr^2+3ns+r^2+s)W_{-2n+1}-rs(n-1)W_{-2n})+sz^{n+2}((nr^2+3ns+2s)W_{-2n+1}-nr(r^2+2s)W_{-2n})-s^3(n+1)z^{n+1}(W_{-2n+1}-rW_{-2n})+z^3(r^2+s)W_1+sz^2(-2sW_1+rzW_0)+s^3z(W_1-rW_0)$

(ii) *If* $(z^2-(r^2+2s)z+s^2)^2=0$ *provided that* $r^2+4s\neq 0$*, i.e., if* $z=\frac{1}{2}((r^2+2s)+r\sqrt{r^2+4s})$ *or* $z=\frac{1}{2}((r^2+2s)-r\sqrt{r^2+4s})$ *provided that* $s\neq-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{-2k+1}=\frac{1}{2z(6z^2-6z\left((r^2+2s)\right)+r^4+4r^2s+6s^2)}\Omega_2$$

where

$\Omega_2=nz^{n+3}(n+4)(n+3)(rW_{-2n}+sW_{-2n-1})-z^{n+2}(n+3)(n+2)(r(n+1)(r^2+2s)W_{-2n}+s(nr^2+3ns+r^2+s)W_{-2n-1})+s^2z^{n+1}(n+2)(n+1)(r(n+2)W_{-2n}+(nr^2+3ns+2s)W_{-2n-1})-ns^4z^n(n+1)^2W_{-2n-1}+6z^2(sW_1+r^2W_1+rsW_0)-4s^2zW_1$
$=nz^{n+3}(n+4)(n+3)W_{-2n+1}-z^{n+2}(n+3)(n+2)((nr^2+3ns+r^2+s)W_{-2n+1}-rs(n-1)W_{-2n})+sz^{n+1}(n+2)(n+1)((nr^2+3ns+2s)W_{-2n+1}-nr(r^2+2s)W_{-2n})-ns^3z^n(n+1)^2(W_{-2n+1}-rW_{-2n})+6z^2(sW_1+r^2W_1+rsW_0)-4s^2zW_1$

(iii) *If* $(z^2-(r^2+2s)z+s^2)^2=(z-\frac{r^2}{4})^4=0$ *provided that* $r^2+4s=0$*, i.e., if* $z=\frac{r^2+2s}{2}=\frac{r^2}{4}$, $s=-\frac{r^2}{4}$ *then*

$$\sum_{k=0}^{n} kz^kW_{-2k+1}=\frac{n(n+1)z^{n-3}}{24}\Omega_3$$

where

$\Omega_3=z^3(n+4)(n+3)(n+2)(rW_{-2n}+sW_{-2n-1})-z^2(n+3)(n+2)(r(n+1)(r^2+2s)W_{-2n}+s(nr^2+3ns+s+r^2)W_{-2n-1})+s^2z(n+2)(n-1)(r(n+2)W_{-2n}+(nr^2+3ns+2s)W_{-2n-1})-s^4(n+1)(n-1)(n-2)W_{-2n-1}$
$=z^3(n+4)(n+3)(n+2)W_{-2n+1}-z^2(n+3)(n+2)((nr^2+3ns+s+r^2)W_{-2n+1}-rs(n-1)W_{-2n})+sz(n+2)(n-1)((nr^2+3ns+2s)W_{-2n+1}-nr(r^2+2s)W_{-2n})-s^3(n+1)(n-1)(n-2)(W_{-2n+1}-rW_{-2n})$

3. Weighted Generating Function $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ of Generalized Fibonacci Polynomials

In this section, we present weighted generating function of the sequence W_{mn+j} and its special cases.

Next, we give the ordinary weighted generating function $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Fibonacci polynomials and (r,s)-Fibonacci and (r,s)-Fibonacci-Lucas polynomials).

Lemma 3.1.
Assume that $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} nW_{mn+j}z^n$ *is the ordinary weighted generating function of the generalized Fibonacci (sequence of) polynomials* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} nW_{mn+j}z^n = \frac{\Psi(z)}{(z^2(-1)^m s^m + z(-1)H_m + 1)^2(W_1^2 - sW_0^2 - rW_0W_1)}$$

where
$\Psi(z) = z^3(-s)^m((W_1W_j + W_0(W_{j+1} - rW_j))W_{m+1} - W_m(W_1W_{j+1} + sW_0W_j)) - 2z^2(-s)^m(W_1^2 - sW_0^2 - rW_0W_1)W_j + z((W_1^2 - sW_0^2 - rW_0W_1)W_jH_m - (W_1W_j + (W_{j+1} - rW_j)W_0)W_{m+1} + (W_1W_{j+1} + sW_0W_j)W_m)$.

Proof. Use Theorem 2.1 (a) and Theorem 1.1. □
Now, we consider special cases of Lemma 3.1.

Corollary 3.1.
The ordinary weighted generating functions of the sequences $W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1}$ *are given as follows:*

(a) $(m = 1, j = 0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}\})$.

$$\sum_{n=0}^{\infty} nW_nz^n = \frac{sz^3(W_1 - rW_0) + 2sz^2W_0 + W_1z}{(sz^2 + rz - 1)^2}$$

(b) $(m = 2, j = 0, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}\})$.

$$\sum_{n=0}^{\infty} nW_{2n}z^n = \frac{s^2z^3(-rW_1 + sW_0 + r^2W_0) - 2s^2z^2W_0 + z(rW_1 + sW_0)}{(r^2z - s^2z^2 + 2sz - 1)^2}$$

(c) $(m = 2, j = 1, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}\})$.

$$\sum_{n=0}^{\infty} nW_{2n+1}z^n = \frac{s^3z^3(W_1 - rW_0) - 2s^2z^2W_1 + z((s + r^2)W_1 + rsW_0)}{(r^2z - s^2z^2 + 2sz - 1)^2}$$

(d) $(m = -1, j = 0, |z| < \min\{|\alpha|, |\beta|\})$.

$$\sum_{n=0}^{\infty} nW_{-n}z^n = \frac{z^3W_1 + 2sz^2W_0 + z(sW_1 - rsW_0)}{(z^2 - rz - s)^2}$$

(e) $(m = -2, j = 0, |z| < \min\{|\alpha|^2, |\beta|^2\})$.

$$\sum_{n=0}^{\infty} nW_{-2n}z^n = \frac{z^3(rW_1 + sW_0) - 2s^2z^2W_0 + s^2z(-rW_1 + (s + r^2)W_0)}{(z^2 - (r^2 + 2s)z + s^2)^2}$$

(f) $(m = -2, j = 1, |z| < \min\{|\alpha|^2, |\beta|^2\})$.

$$\sum_{n=0}^{\infty} nW_{-2n+1}z^n = \frac{z^3((r^2 + s)W_1 + rsW_0) - 2s^2z^2W_1 + s^3z(W_1 - rW_0)}{(z^2 - (r^2 + 2s)z + s^2)^2}$$

Proof. Use Lemma 3.1 (or Theorem 2.2). □

4. Special Cases of Generating Function of Generalized Fibonacci Polynomials

In this section, we present special cases of the ordinary weighted generating function of generalized Fibonacci polynomials.

4.1. Weighted Generating Function of Generalized Fibonacci Numbers

In this subsection, we consider the case $r = 1, s = 1$. A generalized Fibonacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = W_{n-1} + W_{n-2}, \tag{9}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (9) holds for all integer n. The Binet formula of generalized Fibonacci numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n \tag{10}$$

where α and β are the roots of the quadratic equation $x^2 - x - 1 = 0$. Moreover

$$\alpha = \frac{1+\sqrt{5}}{2}$$
$$\beta = \frac{1-\sqrt{5}}{2}$$

So

$$W_n = \frac{(W_1 - \beta W_0)\left(\frac{1+\sqrt{5}}{2}\right)^n - (W_1 - \alpha W_0)\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Now, we define two special cases of the sequence $\{W_n\}$. Fibonacci sequence $\{F_n\}_{n\geq 0}$ and Lucas sequence $\{L_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1, \tag{11}$$
$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, L_1 = 1, \tag{12}$$

The sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$F_{-n} = F_{-(n-1)} + F_{-(n-2)},$$
$$L_{-n} = L_{-(n-1)} + L_{-(n-2)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (11)-(12) hold for all integer n.For all integers n, Fibonacci and Lucas numbers can be expressed using Binet's formulas as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$L_n = \alpha^n + \beta^n,$$

respectively. Note that here, $G_n = F_n$ and $H_n = L_n$.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_{mn+j}z^n$ of the generalized Fibonacci numbers.

Lemma 4.1.
Assume that $|z| < \min\{\left|\frac{1+\sqrt{5}}{2}\right|^{-m}, \left|\frac{1-\sqrt{5}}{2}\right|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} nW_{mn+j}z^n$ *is the ordinary generating function of the generalized Fibonacci numbers* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} nW_{mn+j}z^n = \frac{\Psi(z)}{(z^2(-1)^m + z(-1)L_m + 1)^2(W_1^2 - W_0^2 - W_0W_1)}$$

where

$\Psi(z) = z^3(-1)^m((W_1W_j + W_0(W_{j+1} - W_j))W_{m+1} - W_m(W_1W_{j+1} + W_0W_j)) - 2z^2(-1)^m(W_1^2 - W_0^2 - W_0W_1)W_j + z((W_1^2 - W_0^2 - W_0W_1)W_jL_m - (W_1W_j + (W_{j+1} - W_j)W_0)W_{m+1} + (W_1W_{j+1} + W_0W_j)W_m)$.

Proof. Set $r = 1, s = 1, G_n = F_n$ and $H_n = L_n$ in Lemma 3.1. □

Now, we consider special cases of the last Lemma.

Corollary 4.1.
The ordinary weighted generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z|<\left|\frac{1+\sqrt{5}}{2}\right|^{-1}\simeq 0.618033)$.

$$\sum_{n=0}^{\infty} nW_n z^n = \frac{z^3(W_1-W_0)+2z^2W_0+W_1z}{(z^2+z-1)^2}.$$

(b) $(m=2, j=0, |z|<\left|\frac{1+\sqrt{5}}{2}\right|^{-2}\simeq 0.381966)$.

$$\sum_{n=0}^{\infty} nW_{2n} z^n = \frac{z^3(-W_1+2W_0)-2z^2W_0+z(W_1+W_0)}{(z^2-3z+1)^2}.$$

(c) $(m=2, j=1, |z|<\left|\frac{1+\sqrt{5}}{2}\right|^{-2}\simeq 0.381966)$.

$$\sum_{n=0}^{\infty} nW_{2n+1} z^n = \frac{z^3(W_1-rW_0)-2z^2W_1+z(2W_1+W_0)}{(z^2-3z+1)^2}.$$

(d) $(m=-1, j=0, |z|<\left|\frac{1-\sqrt{5}}{2}\right|\simeq 0.618033)$.

$$\sum_{n=0}^{\infty} nW_{-n} z^n = \frac{z^3W_1+2z^2W_0+z(W_1-W_0)}{(z^2-z-1)^2}.$$

(e) $(m=-2, j=0, |z|<\left|\frac{1-\sqrt{5}}{2}\right|^{2}\simeq 0.381966)$.

$$\sum_{n=0}^{\infty} nW_{-2n} z^n = \frac{z^3(W_1+W_0)-2z^2W_0+z(-W_1+2W_0)}{(z^2-3z+s^2)^2}.$$

(f) $(m=-2, j=1, |z|<\left|\frac{1-\sqrt{5}}{2}\right|^{2}\simeq 0.381966)$.

$$\sum_{n=0}^{\infty} nW_{-2n+1} z^n = \frac{z^3(2W_1+W_0)-2z^2W_1+z(W_1-W_0)}{(z^2-3z+1)^2}.$$

The last Lemma gives the following results as particular examples (weighted generating functions of Fibonacci and Fibonacci-Lucas numbers).

Corollary 4.2.
Assume that $|z|<\min\{\left|\frac{1+\sqrt{5}}{2}\right|^{-m}, \left|\frac{1-\sqrt{5}}{2}\right|^{-m}\}$. Weighted generating functions of Fibonacci and Fibonacci-Lucas numbers are given, respectively, as follows:

(a)

$$\sum_{n=0}^{\infty} nF_{mn+j} z^n = \frac{z^3(-1)^m(-F_mF_{j+1}+F_jF_{m+1})-2z^2(-1)^mF_j+z(F_mF_{j+1}-F_jF_{m+1}+F_jL_m)}{(z^2(-1)^m+z(-1)L_m+1)^2}.$$

(b)

$$\sum_{n=0}^{\infty} nL_{mn+j} z^n = \frac{\begin{array}{c} z^3(-1)^m((2L_{j+1}-L_j)L_{m+1}-L_m(2L_j+L_{j+1}))+10z^2(-1)^mL_j \\ +z((-2L_{j+1}+L_j)L_{m+1}+(L_{j+1}-3L_j)L_m) \end{array}}{-5(z^2(-1)^m+z(-1)L_m+1)^2}.$$

Now, we consider special cases of the last two corollaries.

Corollary 4.3.
The ordinary weighted generating functions of the sequences F_n, F_{2n}, F_{2n+1}, F_{-n}, F_{-2n}, F_{-2n+1} and L_n, L_{2n}, L_{2n+1}, L_{-n}, L_{-2n}, L_{-2n+1} are given as follows:

(a) $\left(m=1, j=0, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-1} \simeq 0.618033\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nF_n z^n &= \frac{z^3+z}{(z^2+z-1)^2}, \\ \sum_{n=0}^{\infty} nL_n z^n &= \frac{-z^3+4z^2+z}{(z^2+z-1)^2}. \end{aligned}$$

(b) $\left(m=2, j=0, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-2} \simeq 0.381966\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nF_{2n} z^n &= \frac{z-z^3}{(z^2-3z+1)^2}, \\ \sum_{n=0}^{\infty} nL_{2n} z^n &= \frac{3z^3-4z^2+3z}{(z^2-3z+1)^2}. \end{aligned}$$

(c) $\left(m=2, j=1, |z| < \left|\frac{1+\sqrt{5}}{2}\right|^{-2} \simeq 0.381966\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nF_{2n+1} z^n &= \frac{z^3-2z^2+2z}{(z^2-3z+1)^2}, \\ \sum_{n=0}^{\infty} nL_{2n+1} z^n &= \frac{-z^3-2z^2+4z}{(z^2-3z+1)^2}. \end{aligned}$$

(d) $\left(m=-1, j=0, |z| < \left|\frac{1-\sqrt{5}}{2}\right| \simeq 0.618033\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nF_{-n} z^n &= \frac{z^3+z}{(z^2-z-1)^2}, \\ \sum_{n=0}^{\infty} nL_{-n} z^n &= \frac{z^3+4z^2-z}{(z^2-z-1)^2}. \end{aligned}$$

(e) $\left(m=-2, j=0, |z| < \left|\frac{1-\sqrt{5}}{2}\right|^{2} \simeq 0.381966\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nF_{-2n} z^n &= \frac{z^3-z}{(z^2-3z+1)^2}, \\ \sum_{n=0}^{\infty} nL_{-2n} z^n &= \frac{3z^3-4z^2+3z}{(z^2-3z+1)^2}. \end{aligned}$$

(f) $\left(m=-2, j=1, |z| < \left|\frac{1-\sqrt{5}}{2}\right|^{2} \simeq 0.381966\right)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nF_{-2n+1} z^n &= \frac{2z^3-2z^2+z}{(z^2-3z+1)^2}, \\ \sum_{n=0}^{\infty} nL_{-2n+1} z^n &= \frac{4z^3-2z^2-z}{(z^2-3z+1)^2}. \end{aligned}$$

From the last corollary, we obtain the following results for Fibonacci and Fibonacci-Lucas numbers.

Corollary 4.4.
Infinite sums of $F_n, F_{2n}, F_{2n+1}, F_{-n}, F_{-2n}, F_{-2n+1}$ *and* $L_n, L_{2n}, L_{2n+1}, L_{-n}, L_{-2n}, L_{-2n+1}$ *are given as follows:*

(a) $z=\frac{1}{2}$.

$$\begin{aligned} \sum_{n=0}^{\infty} n\frac{F_n}{2^n} &= 10, \\ \sum_{n=0}^{\infty} n\frac{L_n}{2^n} &= 22. \end{aligned}$$

(b) $z=\frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{F_{2n}}{3^n} &= 24,\\ \sum_{n=0}^{\infty} n\frac{L_{2n}}{3^n} &= 54.\end{aligned}$$

(c) $z=\frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{F_{2n+1}}{3^n} &= 39,\\ \sum_{n=0}^{\infty} n\frac{L_{2n+1}}{3^n} &= 87.\end{aligned}$$

(d) $z=\frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{F_{-n}}{2^n} &= \frac{2}{5},\\ \sum_{n=0}^{\infty} n\frac{L_{-n}}{2^n} &= \frac{2}{5}.\end{aligned}$$

(e) $z=\frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{F_{-2n}}{3^n} &= -24,\\ \sum_{n=0}^{\infty} n\frac{L_{-2n}}{3^n} &= 54.\end{aligned}$$

(f) $z=\frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{F_{-2n+1}}{3^n} &= 15,\\ \sum_{n=0}^{\infty} n\frac{L_{-2n+1}}{3^n} &= -33.\end{aligned}$$

4.2. Weighted Generating Function of Generalized Pell Numbers

In this subsection, we consider the case $r=2, s=1$. A generalized Pell sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 2W_{n-1}+W_{n-2}, \tag{13}$$

with the initial values $W_0=c_0, W_1=c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -2W_{-(n-1)}+W_{-(n-2)}$$

for $n=1,2,3,....$ Therefore, recurrence (13) holds for all integer n.

The Binet formula of generalized Pell numbers can be written as

$$W_n = \frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n \tag{14}$$

where α and β are the roots of the quadratic equation $x^2-2x-1=0$. Moreover

$$\begin{aligned}\alpha &= 1+\sqrt{2},\\ \beta &= 1-\sqrt{2}.\end{aligned}$$

So

$$W_n = \frac{(W_1-\beta W_0)(1+\sqrt{2})^n-(W_1-\alpha W_0)(1-\sqrt{2})^n}{2\sqrt{2}}.$$

Now, we define two special cases of the sequence $\{W_n\}$. Pell sequence $\{P_n\}_{n\geq 0}$ and Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 1, P_1 = 0, \tag{15}$$
$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad Q_0 = 2, Q_1 = 2, \tag{16}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -2P_{-(n-1)} + P_{-(n-2)}$$
$$Q_{-n} = -2Q_{-(n-1)} + Q_{-(n-2)}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (15)-(16) hold for all integer n.

For all integers n, Pell and Pell-Lucas numbers can be expressed using Binet's formulas as

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$Q_n = \alpha^n + \beta^n,$$

respectively. Here, $G_n = P_n$ and $H_n = Q_n$.

Next, we give the ordinary weighted generating function $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ of the generalized Pell numbers $\{W_{mn+j}\}$.

Lemma 4.2.
Assume that $|z| < \min\{|1+\sqrt{2}|^{-m}, |1-\sqrt{2}|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} nW_{mn+j}z^n$ *is the ordinary weighted generating function of the generalized Pell numbers* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} nW_{mn+j}z^n = \frac{\Psi(z)}{(z^2(-1)^m + z(-1)Q_m + 1)^2(W_1^2 - W_0^2 - 2W_0W_1)}$$

where

$\Psi(z) = z^3(-1)^m((W_1W_j + W_0(W_{j+1} - 2W_j))W_{m+1} - W_m(W_1W_{j+1} + W_0W_j)) - 2z^2(-1)^m(W_1^2 - W_0^2 - 2W_0W_1)W_j + z((W_1^2 - W_0^2 - 2W_0W_1)W_jQ_m - (W_1W_j + (W_{j+1} - 2W_j)W_0)W_{m+1} + (W_1W_{j+1} + W_0W_j)W_m).$

Proof. Set $r = 2, s = 1, G_n = P_n$ and $H_n = Q_n$ in Lemma 3.1. □

Now, we consider special cases of the last Lemma.

Corollary 4.5.
The ordinary weighted generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < |1+\sqrt{2}|^{-1} \simeq 0.414213)$.

$$\sum_{n=0}^{\infty} nW_nz^n = \frac{z^3(W_1 - 2W_0) + 2z^2W_0 + W_1z}{(z^2 + 2z - 1)^2}.$$

(b) $(m = 2, j = 0, |z| < |1+\sqrt{2}|^{-2} \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} nW_{2n}z^n = \frac{z^3(-2W_1 + W_0 + 4W_0) - 2z^2W_0 + z(2W_1 + W_0)}{(-z^2 + 6z - 1)^2}.$$

(c) $(m = 2, j = 1, |z| < |1+\sqrt{2}|^{-2} \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} nW_{2n+1}z^n = \frac{z^3(W_1 - 2W_0) - 2z^2W_1 + z(5W_1 + 2W_0)}{(-z^2 + 6z - 1)^2}.$$

(d) $(m = -1, j = 0, |z| < |1-\sqrt{2}| \simeq 0.414213)$.

$$\sum_{n=0}^{\infty} nW_{-n}z^n = \frac{z^3W_1 + 2z^2W_0 + z(W_1 - 2W_0)}{(z^2 - 2z - 1)^2}.$$

(e) $(m = -2, j = 0, |z| < |1-\sqrt{2}|^2 \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} nW_{-2n}z^n = \frac{z^3(2W_1 + W_0) - 2z^2W_0 + z(-2W_1 + 5W_0)}{(z^2 - 6z + 1)^2}.$$

(f) $(m=-2, j=1, |z|<\left|1-\sqrt{2}\right|^{2}\simeq 0.171572)$.

$$\sum_{n=0}^{\infty} nW_{-2n+1}z^{n}=\frac{z^{3}(5W_{1}+2W_{0})-2z^{2}W_{1}+z(W_{1}-2W_{0})}{(z^{2}-6z+1)^{2}}$$

The last Lemma gives the following results as particular examples (weighted generating functions of Pell and Pell-Lucas numbers).

Corollary 4.6.
Assume that $|z|<\min\{\left|1+\sqrt{2}\right|^{-m},\left|1-\sqrt{2}\right|^{-m}\}$. *Weighted generating functions of Pell and Pell-Lucas numbers are given, respectively, as follows:*

(a)

$$\sum_{n=0}^{\infty} nP_{mn+j}z^{n}=\frac{z^{3}(-1)^{m}(P_{j}P_{m+1}-P_{m}P_{j+1})-2z^{2}(-1)^{m}P_{j}+z(P_{j}Q_{m}-P_{j}P_{m+1}+P_{j+1}P_{m})}{(z^{2}(-1)^{m}+z(-1)Q_{m}+1)^{2}}.$$

(b)

$$\sum_{n=0}^{\infty} nQ_{mn+j}z^{n}=\frac{\begin{array}{c} z^{3}(-1)^{m}(2(Q_{j+1}-Q_{j})Q_{m+1}-2(Q_{j+1}+Q_{j})Q_{m})+16z^{2}(-1)^{m}Q_{j} \\ +2z((Q_{j}-Q_{j+1})Q_{m+1}+(Q_{j+1}-3Q_{j})Q_{m}) \end{array}}{-8(z^{2}(-1)^{m}+z(-1)Q_{m}+1)^{2}}.$$

Now, we consider special cases of the last two corollaries.

Corollary 4.7.
The ordinary weighted generating functions of the sequences P_{n}, P_{2n}, P_{2n+1}, P_{-n}, P_{-2n}, P_{-2n+1} *and* Q_{n}, Q_{2n}, Q_{2n+1}, Q_{-n}, Q_{-2n}, Q_{-2n+1} *are given as follows:*

(a) $(m=1, j=0, |z|<\left|1+\sqrt{2}\right|^{-1}\simeq 0.414213)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nP_{n}z^{n} &= \frac{z^{3}+z}{(z^{2}+2z-1)^{2}},\\ \sum_{n=0}^{\infty} nQ_{n}z^{n} &= \frac{-2z^{3}+4z^{2}+2z}{(z^{2}+2z-1)^{2}}.\end{aligned}$$

(b) $(m=2, j=0, |z|<\left|1+\sqrt{2}\right|^{-2}\simeq 0.171572)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nP_{2n}z^{n} &= \frac{-2z^{3}+2z}{(-z^{2}+6z-1)^{2}},\\ \sum_{n=0}^{\infty} nQ_{2n}z^{n} &= \frac{6z^{3}-4z^{2}+6z}{(-z^{2}+6z-1)^{2}}.\end{aligned}$$

(c) $(m=2, j=1, |z|<\left|1+\sqrt{2}\right|^{-2}\simeq 0.171572)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nP_{2n+1}z^{n} &= \frac{z^{3}-2z^{2}+5z}{(-z^{2}+6z-1)^{2}},\\ \sum_{n=0}^{\infty} nQ_{2n+1}z^{n} &= \frac{-2z^{3}-4z^{2}+14z}{(-z^{2}+6z-1)^{2}}.\end{aligned}$$

(d) $(m=-1, j=0, |z|<\left|1-\sqrt{2}\right|\simeq 0.414213)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nP_{-n}z^{n} &= \frac{z^{3}+z}{(z^{2}-2z-1)^{2}},\\ \sum_{n=0}^{\infty} nQ_{-n}z^{n} &= \frac{2z^{3}+4z^{2}-2z}{(z^{2}-2z-1)^{2}}.\end{aligned}$$

(e) $(m = -2, j = 0, |z| < \left|1-\sqrt{2}\right|^2 \simeq 0.171572)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nP_{-2n}z^n &= \frac{2z^3-2z}{(z^2-6z+1)^2},\\ \sum_{n=0}^{\infty} nQ_{-2n}z^n &= \frac{6z^3-4z^2+6z}{(z^2-6z+1)^2}.\end{aligned}$$

(f) $(m = -2, j = 1, |z| < \left|1-\sqrt{2}\right|^2 \simeq 0.171572)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nP_{-2n+1}z^n &= \frac{5z^3-2z^2+z}{(z^2-6z+1)^2},\\ \sum_{n=0}^{\infty} nQ_{-2n+1}z^n &= \frac{14z^3-4z^2-2z}{(z^2-6z+1)^2}.\end{aligned}$$

From the last corollary, we obtain the following results for Pell and Pell-Lucas numbers.

Corollary 4.8.
Infinite sums of P_n, P_{2n}, P_{2n+1}, P_{-n}, P_{-2n}, P_{-2n+1} and Q_n, Q_{2n}, Q_{2n+1}, Q_{-n}, Q_{-2n}, Q_{-2n+1} are given as follows:

(a) $z = \frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{P_n}{3^n} &= \frac{15}{2},\\ \sum_{n=0}^{\infty} n\frac{Q_n}{3^n} &= 21.\end{aligned}$$

(b) $z = \frac{1}{6}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{P_{2n}}{6^n} &= 420,\\ \sum_{n=0}^{\infty} n\frac{Q_{2n}}{6^n} &= 1188.\end{aligned}$$

(c) $z = \frac{1}{6}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{P_{2n+1}}{6^n} &= 1014,\\ \sum_{n=0}^{\infty} n\frac{Q_{2n+1}}{6^n} &= 2868.\end{aligned}$$

(d) $z = \frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{P_{-n}}{3^n} &= \frac{15}{98},\\ \sum_{n=0}^{\infty} n\frac{Q_{-n}}{3^n} &= -\frac{3}{49}.\end{aligned}$$

(e) $z = \frac{1}{6}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{P_{-2n}}{6^n} &= -420,\\ \sum_{n=0}^{\infty} n\frac{Q_{-2n}}{6^n} &= 1188.\end{aligned}$$

(f) $z = \frac{1}{6}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{P_{-2n+1}}{6^n} &= 174,\\ \sum_{n=0}^{\infty} n\frac{Q_{-2n+1}}{6^n} &= -492.\end{aligned}$$

4.3. Weighted Generating Function of Generalized Jacobsthal Numbers

In this subsection, we consider the case $r = 1, s = 2$. A generalized Jacobsthal sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = W_{n-1} + 2W_{n-2}, \tag{17}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (17) holds for all integer n.

The Binet formula of generalized Jacobsthal numbers can be written as

$$\begin{aligned} W_n &= \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n \\ &= \frac{p_1\alpha^n - p_2\beta^n}{\alpha - \beta} \end{aligned} \tag{18}$$

where α and β are the roots of the quadratic equation $x^2 - x - 2 = 0$ and

$$\begin{aligned} p_1 &= W_1 - \beta W_0 \\ p_2 &= W_1 - \alpha W_0. \end{aligned}$$

Moreover

$$\begin{aligned} \alpha &= 2, \\ \beta &= -1. \end{aligned}$$

So

$$W_n = \frac{(W_1 - \beta W_0) \times 2^n - (W_1 - \alpha W_0) \times (-1)^n}{3}.$$

Now, we define two special cases of the sequence $\{W_n\}$. Jacobsthal sequence $\{J_n\}_{n\geq 0}$ and Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, J_1 = 1, \tag{19}$$

$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2, j_1 = 1, \tag{20}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)} + \frac{1}{2}J_{-(n-2)} \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)} + \frac{1}{2}j_{-(n-2)} \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (19)-(20) hold for all integer n.

For all integers n, Jacobsthal and Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned} J_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{3}, \\ j_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively. Here, $G_n = J_n$ and $H_n = j_n$.

Next, we give the ordinary weighted generating function $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ of the generalized Jacobsthal numbers $\{W_{mn+j}\}$.

Lemma 4.3.
Assume that $|z| < \min\{2^{-m}, 1\}$. Suppose that $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} nW_{mn+j}z^n$ is the ordinary weighted generating function of the generalized Jacobsthal numbers $\{W_{mn+j}\}$. Then, $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ is given by

$$\sum_{n=0}^{\infty} nW_{mn+j}z^n = \frac{\Psi(z)}{(z^2(-2)^m + z(-1)j_m + 1)^2(W_1^2 - 2W_0^2 - W_0W_1)}$$

where

$\Psi(z) = z^3(-2)^m((W_1W_j + W_0(W_{j+1} - W_j))W_{m+1} - W_m(W_1W_{j+1} + 2W_0W_j)) - 2z^2(-2)^m(W_1^2 - 2W_0^2 - W_0W_1)W_j + z((W_1^2 - 2W_0^2 - W_0W_1)W_jj_m - (W_1W_j + (W_{j+1} - W_j)W_0)W_{m+1} + (W_1W_{j+1} + 2W_0W_j)W_m).$

Proof. Set $r = 1, s = 2, G_n = J_n$ and $H_n = j_n$ in Lemma 3.1. □
Now, we consider special cases of the last Lemma.

Corollary 4.9.
The ordinary weighted generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:

(a) $(m = 1, j = 0, |z| < \frac{1}{2})$.

$$\sum_{n=0}^{\infty} nW_n z^n = \frac{2z^3(W_1 - W_0) + 4z^2 W_0 + W_1 z}{(2z^2 + z - 1)^2}.$$

(b) $(m = 2, j = 0, |z| < \frac{1}{4})$.

$$\sum_{n=0}^{\infty} nW_{2n} z^n = \frac{4z^3(-W_1 + 2W_0 + W_0) - 8z^2 W_0 + z(W_1 + 2W_0)}{(-4z^2 + 5z - 1)^2}.$$

(c) $(m = 2, j = 1, |z| < \frac{1}{4})$.

$$\sum_{n=0}^{\infty} nW_{2n+1} z^n = \frac{8z^3(W_1 - W_0) - 8z^2 W_1 + z(3W_1 + 2W_0)}{(-4z^2 + 5z - 1)^2}.$$

(d) $(m = -1, j = 0, |z| < 1)$.

$$\sum_{n=0}^{\infty} nW_{-n} z^n = \frac{z^3 W_1 + 4z^2 W_0 + 2z(W_1 - W_0)}{(z^2 - z - 2)^2}.$$

(e) $(m = -2, j = 0, |z| < 1)$.

$$\sum_{n=0}^{\infty} nW_{-2n} z^n = \frac{z^3(W_1 + 2W_0) - 8z^2 W_0 + 4z(-W_1 + 3W_0)}{(z^2 - 5z + 4)^2}.$$

(f) $(m = -2, j = 1, |z| < 1)$.

$$\sum_{n=0}^{\infty} nW_{-2n+1} z^n = \frac{z^3(3W_1 + 2W_0) - 8z^2 W_1 + 8z(W_1 - W_0)}{(z^2 - 5z + 4)^2}.$$

The last Lemma gives the following results as particular examples (weighted generating functions of Jacobsthal and Jacobsthal-Lucas numbers).

Corollary 4.10.
Assume that $|z| < \min\{2^{-m}, 1\}$. Weighted Generating functions of Jacobsthal and Jacobsthal-Lucas numbers are given, respectively, as follows:

(a)

$$\sum_{n=0}^{\infty} nJ_{mn+j} z^n = \frac{z^3(-2)^m (J_j J_{m+1} - J_m J_{j+1}) - 2z^2(-2)^m J_j + z(J_j j_m - J_j J_{m+1} + J_{j+1} J_m)}{(z^2(-2)^m + z(-1) j_m + 1)^2}.$$

(b)

$$\sum_{n=0}^{\infty} nj_{mn+j} z^n = \frac{\begin{array}{c} z^3(-2)^m((2j_{j+1} - j_j)j_{m+1} - j_m(4j_j + j_{j+1})) + 18z^2(-2)^m j_j \\ +z((-2j_{j+1} + j_j)j_{m+1} + (j_{j+1} - 5j_j)j_m) \end{array}}{-9(z^2(-2)^m + z(-1)j_m + 1)^2}.$$

Now, we consider special cases of the last two corollaries.

Corollary 4.11.
The ordinary weighted generating functions of the sequences J_n, J_{2n}, J_{2n+1}, J_{-n}, J_{-2n}, J_{-2n+1} and j_n, j_{2n}, j_{2n+1}, j_{-n}, j_{-2n}, j_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z|<\frac{1}{2})$.

$$\begin{aligned}\sum_{n=0}^{\infty} nJ_n z^n &= \frac{2z^3+z}{(2z^2+z-1)^2},\\ \sum_{n=0}^{\infty} nj_n z^n &= \frac{-2z^3+8z^2+z}{(2z^2+z-1)^2}.\end{aligned}$$

(b) $(m=2, j=0, |z|<\frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} nJ_{2n} z^n &= \frac{z-4z^3}{(-4z^2+5z-1)^2},\\ \sum_{n=0}^{\infty} nj_{2n} z^n &= \frac{20z^3-16z^2+5z}{(-4z^2+5z-1)^2}.\end{aligned}$$

(c) $(m=2, j=1, |z|<\frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} nJ_{2n+1} z^n &= \frac{8z^3-8z^2+3z}{(-4z^2+5z-1)^2},\\ \sum_{n=0}^{\infty} nj_{2n+1} z^n &= \frac{-8z^3-8z^2+7z}{(-4z^2+5z-1)^2}.\end{aligned}$$

(d) $(m=-1, j=0, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nJ_{-n} z^n &= \frac{z^3+2z}{(z^2-z-2)^2},\\ \sum_{n=0}^{\infty} nj_{-n} z^n &= \frac{z^3+8z^2-2z}{(z^2-z-2)^2}.\end{aligned}$$

(e) $(m=-2, j=0, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nJ_{-2n} z^n &= \frac{z^3-4z}{(z^2-5z+4)^2},\\ \sum_{n=0}^{\infty} nj_{-2n} z^n &= \frac{5z^3-16z^2+20z}{(z^2-5z+4)^2}.\end{aligned}$$

(f) $(m=-2, j=1, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nJ_{-2n+1} z^n &= \frac{3z^3-8z^2+8z}{(z^2-5z+4)^2},\\ \sum_{n=0}^{\infty} nj_{-2n+1} z^n &= \frac{7z^3-8z^2-8z}{(z^2-5z+4)^2}.\end{aligned}$$

From the last corollary, we obtain the following results for Jacobsthal and Jacobsthal-Lucas numbers.

Corollary 4.12.
Infinite sums of $J_n, J_{2n}, J_{2n+1}, J_{-n}, J_{-2n}, J_{-2n+1}$ and $j_n, j_{2n}, j_{2n+1}, j_{-n}, j_{-2n}, j_{-2n+1}$ are given as follows:

(a) $z=\frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{J_n}{3^n} &= \frac{33}{16},\\ \sum_{n=0}^{\infty} n\frac{j_n}{3^n} &= \frac{93}{16}.\end{aligned}$$

(b) $z = \frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{J_{2n}}{5^n} &= \frac{105}{16},\\ \sum_{n=0}^{\infty} n\frac{j_{2n}}{5^n} &= \frac{325}{16}.\end{aligned}$$

(c) $z = \frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{J_{2n+1}}{5^n} &= \frac{215}{16},\\ \sum_{n=0}^{\infty} n\frac{j_{2n+1}}{5^n} &= \frac{635}{16}.\end{aligned}$$

(d) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{J_{-n}}{2^n} &= \frac{2}{9},\\ \sum_{n=0}^{\infty} n\frac{j_{-n}}{2^n} &= \frac{2}{9}.\end{aligned}$$

(e) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{J_{-2n}}{2^n} &= -\frac{30}{49},\\ \sum_{n=0}^{\infty} n\frac{j_{-2n}}{2^n} &= \frac{106}{49}.\end{aligned}$$

(f) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{J_{-2n+1}}{2^n} &= \frac{38}{49},\\ \sum_{n=0}^{\infty} n\frac{j_{-2n+1}}{2^n} &= -\frac{82}{49}.\end{aligned}$$

4.4. Weighted Generating Function of Generalized Mersenne Numbers

In this subsection, we consider the case $r = 3, s = -2$. A generalized Mersenne sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 3W_{n-1} - 2W_{n-2} \tag{21}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{3}{2}W_{-(n-1)} - \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (21) holds for all integer n. For more information on generalized Mersenne numbers, see Soykan [12].

The Binet formula of generalized Mersenne numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n$$

where α and β are the roots of the quadratic equation $x^2 - 3x + 2 = 0$. Moreover

$$\begin{aligned}\alpha &= 2\\ \beta &= 1\end{aligned}$$

So

$$W_n=(W_1-W_0)2^n-(W_1-2W_0). \tag{22}$$

Now, we define two special cases of the sequence $\{W_n\}$. Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$\begin{aligned} M_n &= 3M_{n-1}-2M_{n-2}, \quad M_0=0, M_1=1, & (23)\\ H_n &= 3H_{n-1}-2H_{n-2}, \quad H_0=2, H_1=3, & (24) \end{aligned}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)}-\frac{1}{2}M_{-(n-2)},\\ H_{-n} &= \frac{3}{2}H_{-(n-1)}-\frac{1}{2}H_{-(n-2)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (23)-(24) hold for all integer n.

For all integers n, Mersenne and Mersenne-Lucas can be expressed using Binet's formulas as

$$\begin{aligned} M_n &= \frac{\alpha^n}{(\alpha-\beta)}+\frac{\beta^n}{(\beta-\alpha)}=2^n-1,\\ H_n &= \alpha^n+\beta^n=2^n+1, \end{aligned}$$

respectively. Here, $G_n=M_n$ and $H_n:=H_n$.

Next, we give the ordinary weighted generating function $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ of the generalized Mersenne numbers $\{W_{mn+j}\}$.

Lemma 4.4.
Assume that $|z|<\min\{2^{-m},1\}$. Suppose that $f_{W_{mn+j}}(z)=\sum_{n=0}^{\infty} nW_{mn+j}z^n$ is the ordinary weighted generating function of the generalized Mersenne numbers $\{W_{mn+j}\}$. Then, $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ is given by

$$\sum_{n=0}^{\infty} nW_{mn+j}z^n=\frac{\Psi(z)}{(z^2 2^m+z(-1)H_m+1)^2(W_1^2+2W_0^2-3W_0W_1)}$$

where
$\Psi(z)=z^3 2^m((W_1W_j+W_0(W_{j+1}-3W_j))W_{m+1}-W_m(W_1W_{j+1}-2W_0W_j))-z^2 2^{m+1}(W_1^2+2W_0^2-3W_0W_1)W_j+z((W_1^2+2W_0^2-3W_0W_1)W_jH_m-(W_1W_j+(W_{j+1}-3W_j)W_0)W_{m+1}+(W_1W_{j+1}-2W_0W_j)W_m)$.

Proof. Set $r=3, s=-2, G_n=M_n$ and $H_n:=H_n$ in Lemma 3.1. □

Now, we consider special cases of the last Lemma.

Corollary 4.13.
The ordinary weighted generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z|<\frac{1}{2})$.

$$\sum_{n=0}^{\infty} nW_nz^n=\frac{-2z^3(W_1-3W_0)-4z^2W_0+W_1z}{(2z^2-3z+1)^2}.$$

(b) $(m=2, j=0, |z|<\frac{1}{4})$.

$$\sum_{n=0}^{\infty} nW_{2n}z^n=\frac{4z^3(-3W_1-2W_0+9W_0)-8z^2W_0+z(3W_1-2W_0)}{(4z^2-5z+1)^2}.$$

(c) $(m=2, j=1, |z|<\frac{1}{4})$.

$$\sum_{n=0}^{\infty} nW_{2n+1}z^n=\frac{-8z^3(W_1-3W_0)-8z^2W_1+z(7W_1-6W_0)}{(4z^2-5z+1)^2}.$$

(d) $(m=-1, j=0, |z|<1)$.

$$\sum_{n=0}^{\infty} nW_{-n}z^n = \frac{z^3W_1 - 4z^2W_0 + 2z(-W_1+3W_0)}{(z^2-3z+2)^2}.$$

(e) $(m=-2, j=0, |z|<1)$.

$$\sum_{n=0}^{\infty} nW_{-2n}z^n = \frac{z^3(3W_1-2W_0) - 8z^2W_0 + 4z(-3W_1+7W_0)}{(z^2-5z+4)^2}.$$

(f) $(m=-2, j=1, |z|<1)$.

$$\sum_{n=0}^{\infty} nW_{-2n+1}z^n = \frac{z^3(7W_1-6W_0) - 8W_1z^2 - 8z(W_1-3W_0)}{(z^2-5z+4)^2}.$$

The last Lemma gives the following results as particular examples (weighted generating functions of Mersenne and Mersenne-Lucas numbers).

Corollary 4.14.
Assume that $|z| < \min\{2^{-m}, 1\}$. *Weighted Generating functions of Mersenne and Mersenne-Lucas numbers are given, respectively, as follows:*

(a)

$$\sum_{n=0}^{\infty} nM_{mn+j}z^n = \frac{z^3 2^m (M_jM_{m+1} - M_mM_{j+1}) - z^2 2^{m+1} M_j + z(H_mM_j - M_jM_{m+1} + M_{j+1}M_m)}{(z^2 2^m + z(-1)H_m + 1)^2}.$$

(b)

$$\sum_{n=0}^{\infty} nH_{mn+j}z^n = \frac{\begin{array}{c} z^3 2^m((2H_{j+1}-3H_j)H_{m+1} - (3H_{j+1}-4H_j)H_m) + z^2 2^{m+1}H_j \\ +z((3H_j-2H_{j+1})H_{m+1} + (3H_{j+1}-5H_j)H_m)) \end{array}}{-(z^2 2^m + z(-1)H_m + 1)^2}.$$

Now, we consider special cases of the last two corollaries.

Corollary 4.15.
The ordinary weighted generating functions of the sequences M_n, M_{2n}, M_{2n+1}, M_{-n}, M_{-2n}, M_{-2n+1} *and* H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} *are given as follows:*

(a) $(m=1, j=0, |z|<\frac{1}{2})$.

$$\begin{aligned} \sum_{n=0}^{\infty} nM_nz^n &= \frac{z-2z^3}{(2z^2-3z+1)^2}, \\ \sum_{n=0}^{\infty} nH_nz^n &= \frac{6z^3-8z^2+3z}{(2z^2-3z+1)^2}. \end{aligned}$$

(b) $(m=2, j=0, |z|<\frac{1}{4})$.

$$\begin{aligned} \sum_{n=0}^{\infty} nM_{2n}z^n &= \frac{3z-12z^3}{(4z^2-5z+1)^2}, \\ \sum_{n=0}^{\infty} nH_{2n}z^n &= \frac{20z^3-16z^2+5z}{(4z^2-5z+1)^2}. \end{aligned}$$

(c) $(m=2, j=1, |z|<\frac{1}{4})$.

$$\begin{aligned} \sum_{n=0}^{\infty} nM_{2n+1}z^n &= \frac{-8z^3-8z^2+7z}{(4z^2-5z+1)^2}, \\ \sum_{n=0}^{\infty} nH_{2n+1}z^n &= \frac{24z^3-24z^2+9z}{(4z^2-5z+1)^2}. \end{aligned}$$

(d) $((m=-1, j=0, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nM_{-n}z^n &= \frac{z^3-2z}{(z^2-3z+2)^2},\\ \sum_{n=0}^{\infty} nH_{-n}z^n &= \frac{3z^3-8z^2+6z}{(z^2-3z+2)^2}.\end{aligned}$$

(e) $(m=-2, j=0, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nM_{-2n}z^n &= \frac{3z^3-12z}{(z^2-5z+4)^2},\\ \sum_{n=0}^{\infty} nH_{-2n}z^n &= \frac{5z^3-16z^2+20z}{(z^2-5z+4)^2}.\end{aligned}$$

(f) $(m=-2, j=1, |z|<1)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nM_{-2n+1}z^n &= \frac{7z^3-8z^2-8z}{(z^2-5z+4)^2},\\ \sum_{n=0}^{\infty} nH_{-2n+1}z^n &= \frac{9z^3-24z^2+24z}{(z^2-5z+4)^2}.\end{aligned}$$

From the last corollary, we obtain the following results for Mersenne and Mersenne-Lucas numbers.

Corollary 4.16.
Infinite sums of M_n, M_{2n}, M_{2n+1}, M_{-n}, M_{-2n}, M_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} are given as follows:

(a) $z=\frac{1}{3}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{M_n}{3^n} &= \frac{21}{4},\\ \sum_{n=0}^{\infty} n\frac{H_n}{3^n} &= \frac{27}{4}.\end{aligned}$$

(b) $z=\frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{M_{2n}}{5^n} &= \frac{315}{16},\\ \sum_{n=0}^{\infty} n\frac{H_{2n}}{5^n} &= \frac{325}{16}.\end{aligned}$$

(c) $z=\frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{M_{2n+1}}{5^n} &= \frac{635}{16},\\ \sum_{n=0}^{\infty} n\frac{H_{2n+1}}{5^n} &= \frac{645}{16}.\end{aligned}$$

(d) $z=\frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{M_{-n}}{2^n} &= -\frac{14}{9},\\ \sum_{n=0}^{\infty} n\frac{H_{-n}}{2^n} &= \frac{22}{9}.\end{aligned}$$

(e) $z = \frac{1}{2}$.

$$\begin{aligned} \sum_{n=0}^{\infty} n\frac{M_{-2n}}{2^n} &= -\frac{90}{49}, \\ \sum_{n=0}^{\infty} n\frac{H_{-2n}}{2^n} &= \frac{106}{49}. \end{aligned}$$

(f) $z = \frac{1}{2}$.

$$\begin{aligned} \sum_{n=0}^{\infty} n\frac{M_{-2n+1}}{2^n} &= -\frac{82}{49}, \\ \sum_{n=0}^{\infty} n\frac{H_{-2n+1}}{2^n} &= \frac{114}{49}. \end{aligned}$$

4.5. Weighted Generating Function of Generalized balancing Numbers

In this subsection, we consider the case $r = 6, s = -1$. A generalized balancing sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 6W_{n-1} - W_{n-2} \tag{25}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 6W_{-(n-1)} - W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (25) holds for all integer n. For more information on generalized balancing numbers, see Soykan [13].

The Binet formula of generalized balancing numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n$$

where α and β are the roots of the quadratic equation $x^2 - 6x + 1 = 0$.Moreover

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2}, \\ \beta &= 3 - 2\sqrt{2}. \end{aligned}$$

So

$$W_n = \frac{W_1 - (3 - 2\sqrt{2})W_0}{4\sqrt{2}}(3 + 2\sqrt{2})^n - \frac{W_1 - (3 + 2\sqrt{2})W_0}{4\sqrt{2}}(3 - 2\sqrt{2})^n. \tag{26}$$

Now, we define three special cases of the sequence $\{W_n\}$. balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$\begin{aligned} B_n &= 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1, && (27) \\ H_n &= 6H_{n-1} - H_{n-2}, \quad H_0 = 2, H_1 = 6, && (28) \\ C_n &= 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3. && (29) \end{aligned}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)} - B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)} - H_{-(n-2)}, \\ C_{-n} &= 6C_{-(n-1)} - C_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (27)-(29) hold for all integer n.

For all integers n, balancing, modified Lucas-balancing and Lucas-balancing numbers can be expressed using Binet's formulas as

$$\begin{aligned} B_n &= \frac{\alpha^n}{(\alpha - \beta)} + \frac{\beta^n}{(\beta - \alpha)}, \\ H_n &= \alpha^n + \beta^n, \\ C_n &= \frac{\alpha^n + \beta^n}{2}, \end{aligned}$$

respectively. Here, $G_n = B_n$ and $H_n := H_n$.

Next, we give the ordinary weighted generating function $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ of the generalized balancing numbers $\{W_{mn+j}\}$.

Lemma 4.5.
Assume that $|z| < \min\{|3+2\sqrt{2}|^{-m}, |3-2\sqrt{2}|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} nW_{mn+j}z^n$ *is the ordinary weighted generating function of the generalized balancing numbers* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} nW_{mn+j}z^n = \frac{\Psi(z)}{(z^2 - zH_m + 1)^2(W_1^2 + W_0^2 - 6W_0W_1)}$$

where
$\Psi(z) = z^3((W_1W_j + W_0(W_{j+1} - 6W_j))W_{m+1} - W_m(W_1W_{j+1} - W_0W_j)) - 2z^2(W_1^2 + W_0^2 - 6W_0W_1)W_j + z((W_1^2 + W_0^2 - 6W_0W_1)W_jH_m - (W_1W_j + (W_{j+1} - 6W_j)W_0)W_{m+1} + (W_1W_{j+1} - W_0W_j)W_m)$.

Proof. Set $r = 6, s = -1, G_n = B_n$ and $H_n := H_n$ in Lemma 3.1. □
Now, we consider special cases of the last Lemma.

Corollary 4.17.
The ordinary weighted generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < |3+2\sqrt{2}|^{-1} \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} nW_nz^n = \frac{-z^3(W_1 - 6W_0) - 2z^2W_0 + W_1z}{(-z^2 + 6z - 1)^2}.$$

(b) $(m = 2, j = 0, |z| < |3+2\sqrt{2}|^{-2} \simeq 0.029437)$.

$$\sum_{n=0}^{\infty} nW_{2n}z^n = \frac{z^3(-6W_1 - W_0 + 36W_0) - 2z^2W_0 + z(6W_1 - W_0)}{(z^2 - 34z + 1)^2}.$$

(c) $(m = 2, j = 1, |z| < |3+2\sqrt{2}|^{-2} \simeq 0.029437)$.

$$\sum_{n=0}^{\infty} nW_{2n+1}z^n = \frac{-z^3(W_1 - 6W_0) - 2z^2W_1 + z(35W_1 - 6W_0)}{(z^2 - 34z + 1)^2}.$$

(d) $(m = -1, j = 0, |z| < |3-2\sqrt{2}| \simeq 0.171572)$.

$$\sum_{n=0}^{\infty} nW_{-n}z^n = \frac{z^3W_1 - 2z^2W_0 + z(-W_1 + 6W_0)}{(z^2 - 6z + 1)^2}.$$

(e) $(m = -2, j = 0, |z| < |3-2\sqrt{2}|^2 \simeq 0.029437)$.

$$\sum_{n=0}^{\infty} nW_{-2n}z^n = \frac{z^3(6W_1 - W_0) - 2z^2W_0 + z(-6W_1 + 35W_0)}{(z^2 - 34z + 1)^2}.$$

(f) $(m = -2, j = 1, |z| < |3-2\sqrt{2}|^2 \simeq 0.029437)$.

$$\sum_{n=0}^{\infty} nW_{-2n+1}z^n = \frac{z^3(35W_1 - 6W_0) - 2z^2W_1 - z(W_1 - 6W_0)}{(z^2 - 34z + 1)^2}.$$

The last Lemma gives the following results as particular examples (weighted generating functions of balancing, modified Lucas balancing, Lucas-balancing numbers).

Corollary 4.18.
Assume that $|z| < \min\{|3+2\sqrt{2}|^{-m}, |3-2\sqrt{2}|^{-m}\}$. *Weighted Generating functions of balancing, modified Lucas balancing, Lucas-balancing numbers are given, respectively, as follows:*

(a)

$$\sum_{n=0}^{\infty} nB_{mn+j}z^n = \frac{z^3(B_jB_{m+1} - B_{j+1}B_m) - 2z^2B_j + z(B_jH_m - B_jB_{m+1} + B_{j+1}B_m)}{(z^2 - zH_m + 1)^2}.$$

(b)

$$\sum_{n=0}^{\infty} nH_{mn+j}z^n = \frac{\begin{array}{c} z^3((2H_{j+1}-6H_j)H_{m+1}-(6H_{j+1}-2H_j)H_m)+64z^2H_j \\ +z(-32H_jH_m-(2H_{j+1}-6H_j)H_{m+1}+2(3H_{j+1}-H_j)H_m) \end{array}}{-32(z^2-zH_m+1)^2}.$$

(c)

$$\sum_{n=0}^{\infty} nC_{mn+j}z^n = \frac{\begin{array}{c} z^3((C_{j+1}-3C_j)C_{m+1}-C_m(C_1C_{j+1}-C_0C_j))+16z^2C_j \\ +z(-8C_jH_m-(C_{j+1}-3C_j)C_{m+1}+(3C_{j+1}-C_j)C_m) \end{array}}{-8(z^2-zH_m+1)^2}.$$

Now, we consider special cases of the last two corollaries.

Corollary 4.19.
The ordinary weighted generating functions of the sequences B_n, B_{2n}, B_{2n+1}, B_{-n}, B_{-2n}, B_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} and C_n, C_{2n}, C_{2n+1}, C_{-n}, C_{-2n}, C_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z| < |3+2\sqrt{2}|^{-1} \simeq 0.171572)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nB_nz^n &= \frac{z-z^3}{(-z^2+6z-1)^2}, \\ \sum_{n=0}^{\infty} nH_nz^n &= \frac{6z^3-4z^2+6z}{(-z^2+6z-1)^2}, \\ \sum_{n=0}^{\infty} nC_nz^n &= \frac{3z^3-2z^2+3z}{(-z^2+6z-1)^2}. \end{aligned}$$

(b) $(m=2, j=0, |z| < |3+2\sqrt{2}|^{-2} \simeq 0.029437)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nB_{2n}z^n &= \frac{6z-6z^3}{(z^2-34z+1)^2}, \\ \sum_{n=0}^{\infty} nH_{2n}z^n &= \frac{34z^3-4z^2+34z}{(z^2-34z+1)^2}, \\ \sum_{n=0}^{\infty} nC_{2n}z^n &= \frac{17z^3-2z^2+17z}{(z^2-34z+1)^2}. \end{aligned}$$

(c) $(m=2, j=1, |z| < |3+2\sqrt{2}|^{-2} \simeq 0.029437)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nB_{2n+1}z^n &= \frac{-z^3-2z^2+35z}{(z^2-34z+1)^2}, \\ \sum_{n=0}^{\infty} nH_{2n+1}z^n &= \frac{6z^3-12z^2+198z}{(z^2-34z+1)^2}, \\ \sum_{n=0}^{\infty} nC_{2n+1}z^n &= \frac{3z^3-6z^2+99z}{(z^2-34z+1)^2}. \end{aligned}$$

(d) $(m=-1, j=0, |z| < |3-2\sqrt{2}| \simeq 0.171572)$.

$$\begin{aligned} \sum_{n=0}^{\infty} nB_{-n}z^n &= \frac{z^3-z}{(z^2-6z+1)^2}, \\ \sum_{n=0}^{\infty} nH_{-n}z^n &= \frac{6z^3-4z^2+6z}{(z^2-6z+1)^2}, \\ \sum_{n=0}^{\infty} nC_{-n}z^n &= \frac{3z^3-2z^2+3z}{(z^2-6z+1)^2}. \end{aligned}$$

(e) $(m=-2, j=0, |z|<\left|3-2\sqrt{2}\right|^2 \simeq 0.029437)$.

$$\sum_{n=0}^{\infty} nB_{-2n}z^n = \frac{6z^3-6z}{(z^2-34z+1)^2},$$
$$\sum_{n=0}^{\infty} nH_{-2n}z^n = \frac{34z^3-4z^2+34z}{(z^2-34z+1)^2},$$
$$\sum_{n=0}^{\infty} nC_{-2n}z^n = \frac{17z^3-2z^2+17z}{(z^2-34z+1)^2}.$$

(f) $(m=-2, j=1, |z|<\left|3-2\sqrt{2}\right|^2 \simeq 0.029437)$.

$$\sum_{n=0}^{\infty} nB_{-2n+1}z^n = \frac{35z^3-2z^2-z}{(z^2-34z+1)^2},$$
$$\sum_{n=0}^{\infty} nH_{-2n+1}z^n = \frac{198z^3-12z^2+6z}{(z^2-34z+1)^2},$$
$$\sum_{n=0}^{\infty} nC_{-2n+1}z^n = \frac{99z^3-6z^2+3z}{(z^2-34z+1)^2}.$$

From the last corollary, we obtain the following results for balancing, modified Lucas balancing, Lucas-balancing numbers.

Corollary 4.20.
Infinite sums of B_n, B_{2n}, B_{2n+1}, B_{-n}, B_{-2n}, B_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} and C_n, C_{2n}, C_{2n+1}, C_{-n}, C_{-2n}, C_{-2n+1} are given as follows:

(a) $z=\frac{1}{6}$.

$$\sum_{n=0}^{\infty} n\frac{B_n}{6^n} = 210,$$
$$\sum_{n=0}^{\infty} n\frac{H_n}{6^n} = 1188,$$
$$\sum_{n=0}^{\infty} n\frac{C_n}{6^n} = 594.$$

(b) $z=\frac{1}{36}$.

$$\sum_{n=0}^{\infty} n\frac{B_{2n}}{36^n} = \frac{279720}{5329},$$
$$\sum_{n=0}^{\infty} n\frac{H_{2n}}{36^n} = \frac{1582344}{5329},$$
$$\sum_{n=0}^{\infty} n\frac{C_{2n}}{36^n} = \frac{791172}{5329}.$$

(c) $z=\frac{1}{36}$.

$$\sum_{n=0}^{\infty} n\frac{B_{2n+1}}{36^n} = \frac{1630332}{5329},$$
$$\sum_{n=0}^{\infty} n\frac{H_{2n+1}}{36^n} = \frac{9222552}{5329},$$
$$\sum_{n=0}^{\infty} n\frac{C_{2n+1}}{36^n} = \frac{4611276}{5329}.$$

(d) $z = \frac{1}{6}$.

$$\begin{aligned} \sum_{n=0}^{\infty} n\frac{B_{-n}}{6^n} &= -210, \\ \sum_{n=0}^{\infty} n\frac{H_{-n}}{6^n} &= 1188, \\ \sum_{n=0}^{\infty} n\frac{C_{-n}}{6^n} &= 594. \end{aligned}$$

(e) $z = \frac{1}{36}$.

$$\begin{aligned} \sum_{n=0}^{\infty} n\frac{B_{-2n}}{36^n} &= -\frac{279720}{5329}, \\ \sum_{n=0}^{\infty} n\frac{H_{-2n}}{36^n} &= \frac{1582344}{5329}, \\ \sum_{n=0}^{\infty} n\frac{C_{-2n}}{36^n} &= \frac{791172}{5329}. \end{aligned}$$

(f) $z = \frac{1}{36}$.

$$\begin{aligned} \sum_{n=0}^{\infty} n\frac{B_{-2n+1}}{36^n} &= -\frac{47988}{5329}, \\ \sum_{n=0}^{\infty} n\frac{H_{-2n+1}}{36^n} &= \frac{271512}{5329}, \\ \sum_{n=0}^{\infty} n\frac{C_{-2n+1}}{36^n} &= \frac{135756}{5329}. \end{aligned}$$

4.6. Weighted Generating Function of Generalized Oresme Numbers

In this subsection, we consider the case $r = 1, s = -\frac{1}{4}$. A generalized Oresme sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} \tag{30}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 4W_{-(n-1)} - 4W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (30) holds for all integer n. For more information on generalized Oresme numbers, see Soykan [14].

Binet formula of generalized Oresme numbers can be given as

$$W_n = (D_1 + D_2 n)\alpha^n \tag{31}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}(W_1 - \alpha W_0). \end{aligned}$$

i.e.,

$$W_n = (W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)\, n)\alpha^n$$

Here, $\alpha = \beta = \frac{1}{2}$ are the roots of the quadratic equation

$$x^2 - x + \frac{1}{4} = 0. \tag{32}$$

i.e. the roots of characteristic equation (32) are equal. Note that

$$W_n = (W_0 + 2\left(W_1 - \frac{1}{2}W_0\right) n) \times \frac{1}{2^n}.$$

Now, we define three special cases of the sequence $\{W_n\}$. Modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = G_{n+1} - \frac{1}{4}G_n, \quad G_0 = 0, G_1 = 1, \tag{33}$$

$$H_{n+2} = H_{n+1} - \frac{1}{4}H_n, \quad H_0 = 2, H_1 = 1, \tag{34}$$

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, \quad O_0 = 0, O_1 = \frac{1}{2}. \tag{35}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= 4G_{-(n-1)} - 4G_{-(n-2)}, \\ H_{-n} &= 4H_{-(n-1)} - 4H_{-(n-2)}, \\ O_{-n} &= 4O_{-(n-1)} - 4O_{-(n-2)}, \end{aligned}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (33)-(35) hold for all integer n.

For all integers n, modified Oresme, Oresme-Lucas and Oresme numbers can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1} = \frac{n}{2^{n-1}}, \\ H_n &= 2\alpha^n = \frac{1}{2^{n-1}}, \\ O_n &= n\alpha^n = \frac{n}{2^n}, \end{aligned}$$

respectively. Here, $G_n := G_n$ and $H_n := H_n$.

Next, we give the ordinary weighted generating function $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ of the generalized Oresme numbers $\{W_{mn+j}\}$.

Lemma 4.6.
Assume that $|z| < 2^m$. Suppose that $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} nW_{mn+j}z^n$ is the ordinary weighted generating function of the generalized Oresme numbers $\{W_{mn+j}\}$. Then, $\sum_{n=0}^{\infty} nW_{mn+j}z^n$ is given by

$$\sum_{n=0}^{\infty} nW_{mn+j}z^n = \frac{\Psi(z)}{(z^2(\frac{1}{4})^m + z(-1)H_m + 1)^2}$$

where

$$\Psi(z) = z^3\left(\tfrac{1}{4}\right)^m\left(-\frac{1}{2^{m+j}}\right)(2(m-j)W_1 + (j-m-1)W_0) - 2z^2\left(\tfrac{1}{4}\right)^m W_j + z(W_jH_m + \frac{1}{2^{m+j}}(2(m-j)W_1 + (j-m-1)W_0))$$

Proof. Set $r = 1, s = -\frac{1}{4}, G_n := G_n$ and $H_n := H_n$ in Lemma 3.1. Note that

$$(W_1^2 + \frac{1}{4}W_0^2 - W_0W_1) = \frac{1}{4}(W_0 - 2W_1)^2.$$

Note also that using the Binet's formula

$$W_n = (W_0 + 2\left(W_1 - \frac{1}{2}W_0\right) n) \times \frac{1}{2^n}$$

we obtain

$$\begin{aligned} &((W_1W_j + W_0(W_{j+1} - W_j))W_{m+1} - W_m(W_1W_{j+1} - \frac{1}{4}W_0W_j)) \\ = &-\frac{1}{2^{m+j}}\frac{1}{4}(W_0 - 2W_1)^2(2(m-j)W_1 + (j-m-1)W_0), \\ &-(W_1W_j + (W_{j+1} - W_j)W_0)W_{m+1} + (W_1W_{j+1} - \frac{1}{4}W_0W_j)W_m \\ = &\frac{1}{2^j2^m}\frac{1}{4}(W_0 - 2W_1)^2(2(m-j)W_1 + (j-m-1)W_0). \end{aligned}$$

□

Now, we consider special cases of the last Lemma.

Corollary 4.21.
The ordinary weighted generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z|<2)$.

$$\sum_{n=0}^{\infty} nW_n z^n = \frac{4z^2(W_0-W_1)-8zW_1}{(z-2)^3}.$$

(b) $(m=2, j=0, |z|<4)$.

$$\sum_{n=0}^{\infty} nW_{2n} z^n = \frac{4z^2(3W_0-4W_1)+16z(W_0-4W_1)}{(z-4)^3}.$$

(c) $(m=2, j=1, |z|<4)$.

$$\sum_{n=0}^{\infty} nW_{2n+1} z^n = \frac{4z^2(W_0-W_1)+16z(W_0-3W_1)}{(z-4)^3}.$$

(d) $(m=-1, j=0, |z|<\frac{1}{2})$.

$$\sum_{n=0}^{\infty} nW_{-n} z^n = \frac{8z^2W_1+4z(W_1-W_0)}{(2z-1)^3}.$$

(e) $(m=-2, j=0, |z|<\frac{1}{4})$.

$$\sum_{n=0}^{\infty} nW_{-2n} z^n = \frac{16z^2(4W_1-W_0)+4z(4W_1-3W_0)}{(4z-1)^3}.$$

(f) $(m=-2, j=1, |z|<\frac{1}{4})$.

$$\sum_{n=0}^{\infty} nW_{-2n+1} z^n = \frac{16z^2(3W_1-W_0)+4z(W_1-W_0)}{(4z-1)^3}.$$

The last Lemma üstteki lemma gives the following results as particular examples (weighted generating functions of modified Oresme, Oresme-Lucas and Oresme numbers).

Corollary 4.22.
Assume that $|z|<2^m$. Weighted Generating functions of modified Oresme, Oresme-Lucas and Oresme numbers are given, respectively, as follows:

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} nG_{mn+j} z^n &= \frac{-z^3 2^{1-3m-j}(m-j)-z^2 2^{1-2m}G_j+z(G_jH_m+2^{1-m-j}(m-j))}{(z^2 2^{-2m}+z(-1)H_m+1)^2}.\\ &= \frac{z^3\left(\frac{1}{4}\right)^m(G_jG_{m+1}-G_{j+1}G_m)-2z^2\left(\frac{1}{4}\right)^m G_j+z(G_jH_m-G_jG_{m+1}+G_{j+1}G_m)}{(z^2(\frac{1}{4})^m+z(-1)H_m+1)^2}.\end{aligned}$$

(b)

$$\sum_{n=0}^{\infty} nH_{mn+j} z^n = \frac{z^3 2^{1-3m-j}-z^2 2^{1-2m}H_j+z(H_jH_m-2^{1-m-j})}{(z^2 2^{-2m}+z(-1)H_m+1)^2}.$$

(c)

$$\begin{aligned}\sum_{n=0}^{\infty} nO_{mn+j} z^n &= \frac{-z^3 2^{-3m-j}(m-j)-z^2 2^{1-2m}O_j+z(O_jH_m+2^{-m-j}(m-j))}{(z^2 2^{-2m}+z(-1)H_m+1)^2}\\ &= \frac{z^3\left(\frac{1}{4}\right)^m\frac{1}{2}(O_jO_{m+1}-O_{j+1}O_m)-2z^2\left(\frac{1}{4}\right)^{m+1}O_j+z(\frac{1}{4}O_jH_m-\frac{1}{2}O_jO_{m+1}+\frac{1}{2}O_{j+1}O_m)}{\frac{1}{4}(z^2(\frac{1}{4})^m+z(-1)H_m+1)^2}.\end{aligned}$$

Now, we consider special cases of the last two corollaries.

Corollary 4.23.
The ordinary weighted generating functions of the sequences G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} and O_n, O_{2n}, O_{2n+1}, O_{-n}, O_{-2n}, O_{-2n+1} are given as follows:

(a) $(m=1, j=0, |z|<2)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_n z^n &= \frac{-4z^2-8z}{(z-2)^3},\\ \sum_{n=0}^{\infty} nH_n z^n &= \frac{4z^2-8z}{(z-2)^3},\\ \sum_{n=0}^{\infty} nO_n z^n &= \frac{-2z^2-4z}{(z-2)^3}.\end{aligned}$$

(b) $(m=2, j=0, |z|<4)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_{2n} z^n &= \frac{-16z^2-64z}{(z-4)^3},\\ \sum_{n=0}^{\infty} nH_{2n} z^n &= \frac{8z^2-32z}{(z-4)^3},\\ \sum_{n=0}^{\infty} nO_{2n} z^n &= \frac{-8z^2-32z}{(z-4)^3}.\end{aligned}$$

(c) $(m=2, j=1, |z|<4)$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_{2n+1} z^n &= \frac{-4z^2-48z}{(z-4)^3},\\ \sum_{n=0}^{\infty} nH_{2n+1} z^n &= \frac{4z^2-16z}{(z-4)^3},\\ \sum_{n=0}^{\infty} nO_{2n+1} z^n &= \frac{-2z^2-24z}{(z-4)^3}.\end{aligned}$$

(d) $(m=-1, j=0, |z|<\frac{1}{2})$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_{-n} z^n &= \frac{8z^2+4z}{(2z-1)^3},\\ \sum_{n=0}^{\infty} nH_{-n} z^n &= \frac{8z^2-4z}{(2z-1)^3},\\ \sum_{n=0}^{\infty} nO_{-n} z^n &= \frac{4z^2+2z}{(2z-1)^3}.\end{aligned}$$

(e) $(m=-2, j=0, |z|<\frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_{-2n} z^n &= \frac{64z^2+16z}{(4z-1)^3},\\ \sum_{n=0}^{\infty} nH_{-2n} z^n &= \frac{32z^2-8z}{(4z-1)^3},\\ \sum_{n=0}^{\infty} nO_{-2n} z^n &= \frac{32z^2+8z}{(4z-1)^3}\end{aligned}$$

(f) $(m=-2, j=1, |z|<\frac{1}{4})$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_{-2n+1} z^n &= \frac{48z^2+4z}{(4z-1)^3},\\ \sum_{n=0}^{\infty} nH_{-2n+1} z^n &= \frac{16z^2-4z}{(4z-1)^3},\\ \sum_{n=0}^{\infty} nO_{-2n+1} z^n &= \frac{24z^2+2z}{(4z-1)^3}.\end{aligned}$$

From the last corollary, we obtain the following results for modified Oresme, Oresme-Lucas and Oresme numbers.

Corollary 4.24.
Infinite sums of G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} and O_n, O_{2n}, O_{2n+1}, O_{-n}, O_{-2n}, O_{-2n+1} are given as follows:

(a) $z=1$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_n &= 12,\\ \sum_{n=0}^{\infty} nH_n &= 4,\\ \sum_{n=0}^{\infty} nO_n &= 6.\end{aligned}$$

(b) $z=1$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_{2n} &= \frac{80}{27},\\ \sum_{n=0}^{\infty} nH_{2n} &= \frac{8}{9},\\ \sum_{n=0}^{\infty} nO_{2n} &= \frac{40}{27}.\end{aligned}$$

(c) $z=1$.

$$\begin{aligned}\sum_{n=0}^{\infty} nG_{2n+1} &= \frac{52}{27},\\ \sum_{n=0}^{\infty} nH_{2n+1} &= \frac{4}{9},\\ \sum_{n=0}^{\infty} nO_{2n+1} &= \frac{26}{27}.\end{aligned}$$

(d) $z=\frac{1}{4}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{G_{-n}}{4^n} &= -12,\\ \sum_{n=0}^{\infty} n\frac{H_{-n}}{4^n} &= 4,\\ \sum_{n=0}^{\infty} n\frac{O_{-n}}{4^n} &= -6.\end{aligned}$$

(e) $z=\frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{G_{-2n}}{5^n} &= -720,\\ \sum_{n=0}^{\infty} n\frac{H_{-2n}}{5^n} &= 40,\\ \sum_{n=0}^{\infty} n\frac{O_{-2n}}{5^n} &= -360.\end{aligned}$$

(f) $z=\frac{1}{5}$.

$$\begin{aligned}\sum_{n=0}^{\infty} n\frac{G_{-2n+1}}{5^n} &= -340,\\ \sum_{n=0}^{\infty} n\frac{H_{-2n+1}}{5^n} &= 20,\\ \sum_{n=0}^{\infty} n\frac{O_{-2n+1}}{5^n} &= -170.\end{aligned}$$

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