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Structured Model in Age with Delay for the Study of some Diseases

Research Article

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Abstract: This article focuses on the age structured model with delay. Our aim is to study both the stability of the equilibrium points and the vaccination impact on the disease evolution. For this, we first formulate the problem using the VSEIRS model, then we proof the existence and uniqueness of the solution, we determine the basic reproduction number to study the stability of the equilibrium points and finally we present numerical simulations.

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1. Introduction

The distribution and the impact of the vectorial transmission are greated increased with the climate change, urbanization, high population mobility. Understanding the dynamics of infectious diseases transmission become very important. This requires an integrated and multidisciplinary approach. Mathematical modeling is a fundamental tool to understand the interaction between environment, animals, climate and infectious diseases. This remains an important help tool for decision making. The mathematical models can be deterministic [7], based on ordinary or partial differential equations [23], or stochastic [23], which are based on probability theory [7], [23]. In this work we use a deterministic model, so we will classify the population into compartments. The first epidemic model about disease transmission was established by Kermack and McKendrick in 1927 [24]. From this model, there have been many other extensions depending on what the researcher wants to take into account. Nowadays, there are some works devoted to study the models such as, SIR, SIRS [22], SEIR [2, 20], SEIRS [24], and so on.
In our work, we considered the SEIRS model with disease related mortality. In addition, we have added a compartment of vaccinated individuals, hence the choice of VSEIRS model. [3, 10, 19, 24].

2. Model formulation

We consider Vaccinated (V), Susceptible (S), Exposed (E), Infective (I) and Recovered (R) of two age stages in age-structured epidemiological model with delay. The susceptible (S) are people who are able to develop the disease. The vaccinated are a group of individuals who is extracted among the susceptible and vaccinated. The exposed are

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those who have been in contact with the infected, but who do not show the clinical signs of the disease. The infected are those who, in contact with a susceptible person, can transmit the disease. Recovered (people with immunity againts disease either naturally or after vaccination). The individuals of each subpopulation are divided into two age group: a first group goes from birth to maturity (from 0 to a_1) and a second group goes from maturity to maximum age A (from a_1 to A). In the first phase, the individual can be born, grow and die before maturity just as he can reach the age a_1 of maturity and begin to procreate [19], [24]. a_1 is considered the minimum age to be mature, it is the age from which individuals enter in the second phase and can begin to procreate. In the second phase, individuals can grow, procreate, die or reach the maximum age A.

Let V(a,t), S(a,t), E(a,t), I(a,t) and R(a,t) be the respective densities of the Vaccinated denoted V, of the susceptible denoted S, of the exposed denoted E, infected denoted I and recovered denoted R of age a and at time t in the domain $\Omega = \{(a,t)/0 \le t \le T; 0 \le a \le A\}$. The number of Vaccinated, Susceptible, Exposed, Infected, and Recovered individuals is respectively defined as follows:

$N_s(t) = \int_0^A S(a,t)da$, $N_e(t) = \int_0^A E(a,t)da$, $N_i(t) = \int_0^A I(a,t)da$, $N_r(t) = \int_0^A R(a,t)da$, $N_v(t) = \int_0^A V(a,t)da$.

The sum $N_s(t) + N_v(t) + N_e(t) + N_i(t) + N_r(t)$ represent the total number of individuals and is denoted N(t). In constructing the mathematical model, it is assumed that infection in the population does not occur directly in the susceptible after exposure, but at some point with a delay. This slight delay or incubation time is denoted τ. Moreover, as the total number of the initial population is assumed to be finite, then the quantities V(a,t), S(a,t), E(a,t); I(a,t) and R(a,t) belong to $L^1(\Omega)$.

The used parameters are:

•$\theta_i(a,t), i \in \{1,2,3,4,5\}$ is the female fertility rate of each subpopulation of age a with

$$\theta_i(a,t) = \begin{cases} 1 & if \quad a \in [a_1, A] \\ 0 & else \end{cases}$$

•$\hat{\alpha}_(a,t)$ is the natural mortality rate of each subpopulation of age a.

•$\hat{\alpha}_{ad}(a,t)$ is the death rate due to the disease.

•$\hat{\delta}(a,t)$ is the rate of vaccination coverage against the disease

• v is the rate of vaccinated people who did not have the expected effect of the vaccine and who became susceptible again at the same time.

•σ is a parameter provided such that it is equal to 1 if the individuals die while producing and it is zero when the individuals continue to survive by producing; see the papers [19], [24], [20]..

•$\gamma_1(a,t)$ is a transmission coefficient describing the variable probability of infection and is related to a large number of social, environmental and epidemiological factors.

•$\beta(a,a',t)$ is the contact rate between an infected population of age a' and a susceptible population of age a.

•$\tau(\tau \le 0)$ is an incubation period, for fixing the infection.

•γ_2 is the transmission coefficient from the exposed population to the infected population of age a.

•γ_3 is the recovery rate of age a.

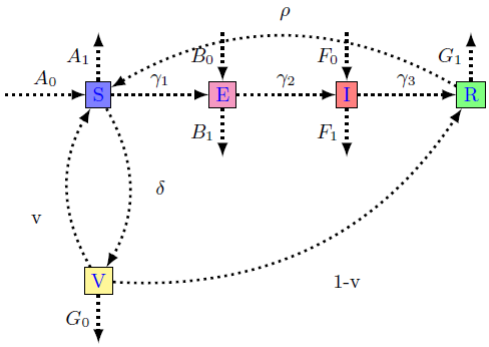
•$\rho(a,t)$ is the conversion rate of the recovered population losing immunity to the susceptible population of age a.

•$\mu_i(i = 1,2,3,4,5)$ is the reproduction rate of each subpopulation at the stage of procreation.

•$p(a,t)$ is a part of the exhibits that give rise to new exposed individuals.

•$q(a,t)$ is a part of infected individuals that give birth to new infected individuals; see the recent review papers [19], [24], [20].

These different processes can be summarized in the following diagram:



Flow diagram of the disease transmission

$A_0 = \mu_1\hat{\theta}_1 S + \mu_2\hat{\theta}_2(1-p_0)E + \mu_3\hat{\theta}_3(1-q_0)I + \mu_4\hat{\theta}_4 R + \mu_5\hat{\theta}_5 V$, $A_1 = (\hat{\alpha} + \sigma\hat{\theta})S$,
$B_0 = \mu_2\hat{\theta}_2 p_0 E, B_1 = (\hat{\alpha} + \sigma\hat{\theta})E, F_0 = \mu_3\hat{\theta}_3 q_0 I, F_1 = (\hat{\alpha} + \alpha_{ad} + \sigma\hat{\theta}_3)$,
$G_0 = (\hat{\alpha} + \sigma\theta_5)V, G_1 = (\hat{\alpha} + \sigma\hat{\theta}_4)R$.

The mathematical model to describe this dynamic is governed by the following partial differential equations:

$$\begin{cases} \frac{\partial S}{\partial t} = -(\hat{\alpha}(a,t) + \sigma\hat{\theta}_1(a,t) - \mu_1\hat{\theta}_1(a,t) + \delta)S(a,t) - \\ \left(\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I(a',t-\tau)da'\right)S(a,t) + \mu_2\hat{\theta}_2(a,t)(1-p)(a,t)E(a,t) + \\ \mu_3\hat{\theta}_3(a,t)(1-q)(a,t)I(a,t) + (\mu_4\hat{\theta}_4(a,t)+\rho)R(a,t) + (\mu_5\hat{\theta}_5(a,t)+\nu)V(a,t), (a,t)\in\Omega, \\ \frac{\partial E}{\partial t} = -\left(\hat{\alpha} + \gamma_2(a,t) + \sigma\theta_2(a,t) - \mu_2\hat{\theta}_2 p(a,t)\right)E(a,t) + \\ \left(\gamma_1(a,t-\tau)\int_0^A \beta(a,a',t-\tau)I(a',t-\tau)da'\right)S(a,t), \\ \frac{\partial I}{\partial t} = -\left(\hat{\alpha}(a,t) + \gamma_3(a,t) + \alpha_{ad}(a,t) + \sigma\theta_3(a,t) - \mu_3\hat{\theta}_3 q(a,t)\right)I(a,t) + \gamma_2(a,t)E(a,t), \\ \frac{\partial R}{\partial t} = -\left(\hat{\alpha}(a,t) + \sigma\hat{\theta}_4(a,t) + \rho)\right)R(a,t) + \gamma_3(a,t)I(a,t) + (1-\nu)V(a,t), \\ \frac{\partial V}{\partial t} = -\left(\hat{\alpha}(a,t) + \sigma\theta_5(a,t) + 1\right)V(a,t) + \delta S(a,t). \end{cases} \tag{1}$$

We assume that at time t=0, we have none recovered and none vaccinated individual. We have the following initials conditions for $a \in [0, A]$ and $t \in [-\tau, 0]$

$$\begin{cases} S(a,0) = S_0(a), \\ E(a,0) = E_0(a), \\ I(a,0) = I_0(a,t), \\ R(a,0) = 0, \\ V(a,0) = 0. \end{cases} \tag{2}$$

3. Existence and uniqueness of solution

By using the characteristic method, we transform the system (1) into a nonlinear integro-differential system with delay [24], we obtain the following system (3):

$$\begin{cases} S_t = -(\hat{\alpha}(u+t,t) + \sigma\hat{\theta}_1(u+t,t) - \mu_1\hat{\theta}_1(u+t,t) + \delta)S(u+t,t) - \\ \left(\gamma_1(u+t,t-\tau)\int_0^A \beta(u+t,a',t-\tau)I(a',t-\tau)da'\right)S(u+t,t) + \mu_2\hat{\theta}_2(1-p)E(u+t,t) + \\ \mu_3\hat{\theta}_3(u+t,t)(1-q)(u+t,t)I(u+t,t) + (\mu_4\hat{\theta}_4(u+t,t)+\rho)R(u+t,t) + (\mu_5\hat{\theta}_5+\nu)V(u+t,t), \\ E_t = -\left(\hat{\alpha} + \gamma_2(u+t,t) + \sigma\theta_2(u+t,t) - \mu_2\hat{\theta}_2 p(u+t,t)\right)E(u+t,t) + \\ \left(\gamma_1(u+t,t-\tau)\int_0^A \beta(u+t,a',t-\tau)I(a',t-\tau)da'\right)S(u+t,t), \\ I_t = -\left(\hat{\alpha} + \alpha_{ad} + \gamma_3(u+t,t) + \alpha_{ad} + \sigma\theta_3(u+t,t) - \mu_3\hat{\theta}_3 q\right)I(u+t,t) + \gamma_2 E(u+t,t), \\ R_t = -\left(\hat{\alpha}(u+t,t) + \sigma\hat{\theta}_4(u+t,t) + \rho\right)R(u+t,t) + \gamma_3 I(u+t,t) + (1-\nu)V(u+t,t), \\ V_t = -\left(\hat{\alpha}_5(u+t,t) + \sigma\theta_5(u+t,t) + 1\right)V(u+t,t) + \delta S(u+t,t). \end{cases} \tag{3}$$

with $(S_t, E_t, I_t, R_t, V_t) = (\frac{\partial S}{\partial t}, \frac{\partial E}{\partial t}, \frac{\partial I}{\partial t}, \frac{\partial R}{\partial t}, \frac{\partial V}{\partial t})$
The initial conditions are:

$$\begin{cases} S(u,0) = S_0(a), a \in [0,A], \\ E(u,0) = E_0(a), a \in [0,A], \\ I(u+t,0) = I_0(u+t,t), t \in [-\tau,0], \\ R(u,0) = 0, \\ V(u,0) = 0. \end{cases} \tag{4}$$

Thus, we show the existence of a unique solution (V, S, E, I, R, S) of the system (3) by using the theorem (3.1). [18]

Theorem 3.1 ([18]).
Assume a continuously differentiable function G and that its partial derivatives are bounded by a constant K. For all i, j =1, ..., d, for all $X \in \mathbb{R}^d$, $\left|\frac{\partial G_i}{\partial X_j}(X)\right| \leq K$.
Then there is a unique solution to the Cauchy problem defined on $[0,\infty[$

It is then sufficient to transform the problem (3) into a Cauchy Lipschitz problem. We pose $X = \begin{pmatrix} S \\ E \\ I \\ R \\ V \end{pmatrix}$; $G = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}$

with

$$\begin{cases} Y_1 = -(\hat{\alpha}(u+t,t) + \sigma\hat{\theta}_1(u+t,t) - \mu_1\hat{\theta}_1(u+t,t) + \delta)S(u+t,t) - \\ \left(\gamma_1(u+t,t-\tau)\int_0^A \beta(u+t,a',t-\tau)I(a',t-\tau)da'\right)S(u+t,t) + \mu_2\hat{\theta}_2(1-p)E(u+t,t) + \\ \mu_3\hat{\theta}_3(u+t,t)(1-q)(u+t,t)I(u+t,t) + (\mu_4\hat{\theta}_4(u+t,t)+\rho)R(u+t,t) + (\mu_5\hat{\theta}_5+\nu)V(u+t,t), \\ Y_2 = -\left(\hat{\alpha} + \gamma_2(u+t,t) + \sigma\theta_2(u+t,t) - \mu_2\hat{\theta}_2 p(u+t,t)\right)E(u+t,t) + \\ \left(\gamma_1(u+t,t-\tau)\int_0^A \beta(u+t,a',t-\tau)I(a',t-\tau)da'\right)S(u+t,t), \\ Y_3 = -\left(\hat{\alpha} + \alpha_{ad} + \gamma_3(u+t,t) + \alpha_{ad} + \sigma\theta_3(u+t,t) - \mu_3\hat{\theta}_3 q\right)I(u+t,t) + \gamma_2 E(u+t,t), \\ Y_4 = -\left(\hat{\alpha}(u+t,t) + \sigma\hat{\theta}_4(u+t,t) + \rho\right)R(u+t,t) + \gamma_3 I(u+t,t) + (1-\nu)V(u+t,t), \\ Y_5 = -\left(\hat{\alpha}(u+t,t) + \sigma\theta_5(u+t,t) + 1\right)V(u+t,t) + \delta S(u+t,t). \end{cases} \tag{5}$$

Which brings us back to the Cauchy problem.(6)

$$\begin{cases} \frac{dX}{dt} = G(t,X) \\ X(t_0) = X_0 \end{cases} \tag{6}$$

To show the existence and uniqueness of solution (V, S, E, I, R, S), we show that the partial derivatives are bounded by using the theorem (3.1). Indeed, there are positive constants $K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13}, K_{14}, K_{15}$ such that:

$$\begin{cases} \left|\frac{\partial Y_1}{\partial S}\right| \leq K_1, \\ \left|\frac{\partial Y_1}{\partial E}\right| \leq K_2, \\ \left|\frac{\partial Y_1}{\partial I}\right| \leq K_3, \\ \left|\frac{\partial Y_1}{\partial R}\right| \leq K_4, \\ \left|\frac{\partial Y_1}{\partial V}\right| \leq K_5, \end{cases} \quad \begin{cases} \left|\frac{\partial Y_2}{\partial S}\right| \leq K_6, \\ \left|\frac{\partial Y_2}{\partial E}\right| \leq K_7 \\ \left|\frac{\partial Y_2}{\partial I}\right| \leq K_8, \end{cases} \quad \begin{cases} \left|\frac{\partial Y_3}{\partial E}\right| \leq K_9, \\ \left|\frac{\partial Y_3}{\partial I}\right| \leq K_10, \end{cases} \quad \begin{cases} \left|\frac{\partial Y_4}{\partial I}\right| \leq K_{11}, \\ \left|\frac{\partial Y_4}{\partial R}\right| \leq K_{12}, \\ \left|\frac{\partial Y_4}{\partial V}\right| \leq K_{13}, \end{cases} \quad \begin{cases} \left|\frac{\partial Y_5}{\partial S}\right| \leq K_{14}, \\ \left|\frac{\partial Y_5}{\partial V}\right| \leq K_{15}. \end{cases}$$

For reasons of simplification of writing, let us omit the writing of (u+t,t) and we obtain:

$$\begin{cases} \left|\frac{\partial Y_1}{\partial S}\right| < |\hat{\alpha}| + \left|\sigma\hat{\theta}1\right| + \left|\mu_1\hat{\theta}_1\right| + \left|\hat{\gamma}\right| \int_0^A \left|\beta_0 I\right| \leq K_1, \\ \left|\frac{\partial Y_1}{\partial E}\right| < \left|\mu_2\theta_2(1-p)\right| < \left|\mu_2\theta_2\right| < K_2, \\ \left|\frac{\partial Y_1}{\partial I}\right| < \left|\gamma_1\right| \frac{\partial}{\partial I}(\int_0^A \beta_0 I da')S|| + \left|\mu_3\theta_3(1-q)\right| < K_3, \\ \left|\frac{\partial Y_1}{\partial R}\right| < \left|\mu_4\theta_4 + \rho\right| < \left|\mu_4\theta_4\right| < \left|\rho\right| < K_4, \\ \left|\frac{\partial Y_1}{\partial V}\right| < \left|\mu_5\theta_5 + v\right| < \left|\mu_5\theta_5\right| < |v| < |v| < K_5. \end{cases}$$

$$\begin{cases} \left|\frac{\partial Y_2}{\partial S}\right| < \left|\gamma_1\right| \left|\int_0^A \beta_0 I\right| da'|S| < \left|\gamma_1\right| \left|\int_0^A \beta_0 b_3\right| da' b_1 < K_6, \\ \left|\frac{\partial Y_2}{\partial E}\right| < \left|-(\alpha - \mu_2\theta_2 p + \gamma_2)\right| < |\alpha_2| + \left|\gamma_2\right| + \left|\mu_2\theta_2\right| < K_7, \\ \left|\frac{\partial Y_2}{\partial I}\right| < \left|\gamma_1 \frac{\partial}{\partial I} \int_0^A \beta_0\right| |I| da'|S| < \left|\gamma_1\right| b_1 c_1 < K_8, \end{cases}$$

$$\begin{cases} \left|\frac{\partial Y_3}{\partial E}\right| < \left|\gamma_1\right| K_9, \\ \left|\frac{\partial Y_3}{\partial I}\right| < |\alpha| + \left|\gamma_3\right| + \left|\mu_3\theta_3 q\right| + |\alpha_{ad}| < K_{10}. \end{cases}$$

$$\begin{cases} \left|\frac{\partial Y_4}{\partial I}\right| < \left|\gamma_3\right| < K_{11}, \\ \left|\frac{\partial Y_4}{\partial R}\right| < \left|\alpha + \rho\right| < |\alpha| + \left|\rho\right| < K_{12}, \\ \left|\frac{\partial Y_4}{\partial V}\right| < |1 - v| < 1 < K_{13}. \end{cases}$$

$$\begin{cases} \left|\frac{\partial Y_5}{\partial S}\right| < |\delta| < 1 < K_{14}, \\ \left|\frac{\partial Y_5}{\partial V}\right| < |\alpha + 1| < |\alpha| + 1 < K_{15}. \end{cases}$$

The partial derivatives are bounded, hence the existence and uniqueness of the solution. Let us now study the stability of the equilibrium points.

4. Stability analysis

Consider the nonlinear autonomous system in equation (3) where the following parameters are constants. To avoid any ambiguity, we will adopt a new notation. Thus, let us note:
$\alpha(a,t) = \hat{\alpha}(a,t) + \sigma\hat{\theta}_{i0}(a,t) = \hat{\alpha}(a,t)$ with $\sigma = 0$ for human being , $\theta_{i0}(i = 1,2,3,4,5)$, $\gamma_i(a,t) = \gamma_i(i = 1,2,3)$, $\beta(a,a',t) = \beta_0$, $p(a,t) = p$, $q(a,t) = q$, $\rho(a,t) = \rho$ where α_{i0}, θ_{i0}, γ_{i0}, p,q, ρ are positive constants. By integrating the equation (1) with respect to the age of a_1 to A and using the real condition S(A,t)=0, E(A,t)=0, I(A,t)=0, R(A,t)=0, the system of ordinary differential equation describing the dynamics of the population is :

$$\begin{cases} N_s' = -(\hat{\alpha} - \mu_1\hat{\theta_1} + \delta)N_s(t) - \gamma_1\beta_0 N_i(t)N_s(t) + \mu_2\hat{\theta_2}(1-p)(a,t)N_e(t) + \\ \mu_3\hat{\theta_3}(1-q)N_i(t) + (\mu_4\hat{\theta_4} + \rho)N_r(t) + (\mu_5\theta_5 + \nu)N_\nu(t), \\ N_e' = -\left(\hat{\alpha} + \gamma_2 - \mu_2\right)pN_e(t) + \gamma_1\beta_0 N_i(t)N_s(t), \\ N_i' = -\left(\hat{\alpha} + \gamma_3 + \gamma_{ad} - \mu_3\right)qN_i(t) + \gamma_2 N_e(t), \\ N_r' = -\left(\hat{\alpha} + p\right)qN_r(t) + \gamma_3 N_i(t) + (1-\nu)N_\nu(t), \\ N_\nu' = -\left(\hat{\alpha} + 1\right)N_\nu(t) + \delta N_s(t). \end{cases} \tag{7}$$

With as initial functions

$$\begin{cases} N_s(0) = \int_0^A S_0(a)da, \\ N_e(0) = \int_0^A E_0(a)da, \\ N_i(0) = \int_0^A I_0(a,t)da, t \in [-\tau, 0], \\ N_r(0) = \int_0^A R_0(a)da, \\ N_\nu(0) = \int_0^A V_0(a)da. \end{cases} \tag{8}$$

Proposition 4.1.
There is a disease when $R_{01} > 1$, $R_{02} > 1$, $R_0 > 1$, *with* $R_{01} = \frac{\gamma_1\beta_0}{\alpha+\gamma_2-\mu_2\theta_2}$, $R_{02} = \frac{\gamma_{20}}{\alpha+\alpha_{ad}+\gamma_{30}-\mu_3\theta_3}$

$R_0 = \frac{\gamma_1\beta_0\gamma_2}{(\alpha+\gamma_2-\mu_2\theta_)(\alpha+\alpha_{ad}+\gamma_{30}-\mu_3\theta_3)}$

Proof. There is an epidemic when the number of new people exposed and the number of new patients increase. That is, if we have the following two cases:
Firstly: there are more new patients exposed than new patients (ie by new patients, those exposed who have just changed status). In other words :
$-\alpha + \mu_2\theta_2 + \gamma_1\beta_0 > \gamma_2, \Longrightarrow \alpha - \mu_2\theta_2 + \gamma_2 < \gamma_1\beta_0 \Longrightarrow \frac{\gamma_1\beta_0}{\alpha+\gamma_2-\mu_2\theta_2} > 1$

Secondly: there are more new patients than new cured (that is to say by newly cured, patients who have just changed status by becoming cured). This is explained by:
$-\alpha - \alpha_{ad} + \gamma_{20} - \mu_3\theta_3 q > \gamma_{30} \Longrightarrow \alpha + \alpha_{ad} + \gamma_{30} - \mu_3\theta_3 q < \gamma_{20}$
$\frac{\gamma_{20}}{\alpha+\alpha_{ad}+\gamma_{30}-\mu_3\theta_3} > 1$

Taking the product of the two inequalities, we get:

$\frac{\gamma_1\beta_0\gamma_2}{(\alpha+\gamma_2-\mu_2\theta_)(\alpha+\alpha_{ad}+\gamma_{30}-\mu_3\theta_3)} > 1$, we notice :

$R_0 = \frac{\gamma_1\beta_0\gamma_2}{(\alpha+\gamma_2-\mu_2\theta_)(\alpha+\alpha_{ad}+\gamma_{30}-\mu_3\theta_3)} > 1$ is the basic reproduction number. □

Using the next generation matrix method, we obtain the same value of R_0 (the principle is the same as [12], [9], [11], [1], [2], [7], [5], [17], [14], [8]) We therefore have an epidemic as soon as $R_0 > 1$. Thus, we have the following theorems :

Theorem 4.1.
For all $\tau \geq 0$ *the equilibrium point without disease noted* $H_0(S_0, 0, 0, 0)$ *is asymptotically locally stable when* $\gamma_1 = 0, R_0 < 1$,
$R_1 < 1, R_2 < 1$ *and* $R_3 < 1$ *with* $R_1 = \frac{\mu_1\theta_1}{\alpha+\gamma_1+\delta}$,

$R_2 = \frac{\mu_2\theta_2 p}{\alpha+\gamma_2}$, $R_3 = \frac{\mu_3\theta_3}{\alpha+\gamma_3+\alpha_{ad}}$.

Proof. When $\gamma_1 = 0$, the system (7) becomes :

$$\begin{cases} N_s' = -(\hat{\alpha} - \mu_1\hat{\theta}_1 + \delta)N_s(t) + \mu_2\hat{\theta}_2(1-p)(a,t)N_e(t) + \mu_3\hat{\theta}_3(1-q)N_i(t) + \\ (\mu_4\hat{\theta}_4 + \rho)N_r(t) + (\mu_5\theta_5 + \nu)N_\nu(t), \\ N_e' = -\left(\hat{\alpha} + \gamma_2 + -\mu_2\right)N_e(t), \\ N_i' = -\left(\hat{\alpha} + \gamma_3 + \gamma_{ad} - \mu_3\right)N_i(t) + \gamma_2 N_e(t), \\ N_r' = -\left(\hat{\alpha} + p\right)N_r(t) + \gamma_3 N_i(t) + (1-\nu)N_\nu(t), \\ N_\nu' = -\left(\hat{\alpha} + 1\right)N_\nu(t) + \delta N_s(t). \end{cases} \tag{9}$$

We will look for the characteristic equation of the system (9). For this, let us set:
$N_j(t) = \psi_j exp(\lambda t)$, j = v, s, e, i, r. By replacing these results in the system (9), we obtain :

$$\begin{cases} (\lambda + \hat{\alpha} - \mu_1\hat{\theta}_1 + \delta)\psi_s(t) - \mu_2\hat{\theta}_2(1-p)(a,t)\psi_e(t) \\ -\mu_3\hat{\theta}_3(1-q)\psi_i(t) - (\mu_4\hat{\theta}_4 + \rho)\psi_r(t) + (\mu_5\theta_5 + \nu)\psi_\nu(t) = 0, \\ \left(\lambda + \hat{\alpha} + \gamma_2 - \mu_2\right)p\psi_e(t) = 0, \\ \left(\lambda + \hat{\alpha} + \gamma_3 + \alpha_{ad} - \mu_3\right)q\psi_i(t) - \gamma_2\psi_e(t) = 0, \\ \left(\lambda + \hat{\alpha} + p\right)\psi_r(t) - \gamma_3 N_i(t) + (1-\nu)\psi_\nu(t) = 0, \\ (\lambda + \hat{\alpha} + 1)\psi_\nu(t) - \delta\psi_s(t) = 0. \end{cases} \tag{10}$$

From system (10), we obtain the following caractéristic equation:

$$(\lambda + \hat{\alpha} - \mu_1\hat{\theta}_1 + \delta)(\lambda + \hat{\alpha} + \gamma_2 - \mu_2)(\lambda + \hat{\alpha} + \gamma_3 + \alpha ad - \mu_3)(\lambda + \hat{\alpha} + p)(\lambda + \hat{\alpha_5} + \sigma\theta_5 + 1) = 0 \tag{11}$$

$\Longrightarrow \lambda = -\hat{\alpha} + \mu_1\hat{\theta}_1 - \delta$, $\lambda = -\hat{\alpha} - \gamma_2 + \mu_2\theta_2 p$, $\lambda = -\hat{\alpha} - \gamma_3 - \alpha_{ad} + \mu_3\theta_3 q$, $\lambda = -\hat{\alpha} - p$, $\lambda = -\hat{\alpha} - 1$
There is stability of the equilibrium point when the roots of the characteristic equation all have negative real parts ($\lambda < 0$), in other words $-\alpha + \mu_1\theta_1 - \delta < 0, R_1 < 1, -\alpha - \gamma_2 + \mu_2\theta_2 p < 0$, where $R_2 < 1, -\alpha_{ad} - \alpha - \gamma_3 + \mu_3\theta_3 q$, hence $R_3 < 1$.
The other two roots ($-\alpha - \rho$ and $-\alpha - 1$) are negative because ρ and α are positive. □

Theorem 4.2.
If $\gamma_{10} > 0$ and system parameters satisfy $R_0 > 1, R_1 > 1, R_2 < 1, R_3 < 1, R_4 < 1$ et $R_5 < 1$ Where R_1, R_2, R_3 are determined in theorem (4.1) and

$$R_4 = \frac{(\hat{\alpha} - \mu_1\hat{\theta}_1 + \delta)(\hat{\alpha_4} + \rho)(\hat{\alpha} + 1)}{\delta(\mu_4\theta_4 + p)(1-\nu)(\mu_5\theta_5 + \nu)(\alpha + \rho)}$$
$$R_5 = \frac{(\alpha + \rho)(\hat{\alpha_3} + \gamma_3 + \alpha ad - \mu_3\theta_3 q)(\hat{\alpha_2} + \gamma_2 - \mu_2\theta_2 p)}{\gamma_2(\mu_4\theta_4 + p) + \gamma_2(\alpha + p)\mu_3\theta_3(1-q)}$$

The system (10) has only one positive endemic equilibrium point noted

$H^* = (V^*, S^*, E^*, I^*, R^*)$ *asymptotically locally stable*

Proof. Our goal here is first to determine the endemic equilibrium point and then study its stability. [1] [24]
The endemic equilibrium point $H^* = (N_s^*, N_e^*, N_i^*, N_r^*, N_\nu^*)$, is the solution of the following system (12) :

$$\begin{cases} 0 = -(\hat{\alpha} - \mu_1\hat{\theta}_1 + \delta)N_s^* - \gamma_1\beta_0 N_i^* N_s^* + \mu_2\hat{\theta}_2(1-p)(a,t)N_e^* + \\ \mu_3\hat{\theta}_3(1-q)N_i^* + (\mu_4\hat{\theta}_4 + \rho)N_r^* + (\mu_5\theta_5 + \nu)N_\nu^*, \\ 0 = -\left(\hat{\alpha} + \gamma_2 - \mu_2\right)N_e^* + \gamma_1\beta_0 N_i(t)N_s^*, \\ 0 = -\left(\hat{\alpha} + \gamma_3 + \alpha ad - \mu_3\right)qN_i(t) + \gamma_2 N_e^*, \\ 0 = -\left(\hat{\alpha} + p\right)N_r^* + \gamma_3 N_i(t) + (1-\nu)N_\nu^*, \\ 0 = -(\hat{\alpha} + 1)N_\nu^* + \delta N_s^*. \end{cases} \tag{12}$$

$H^* = (N_s^*, N_e^*, N_i^*, N_r^*, N_\nu^*)$ are strictly positive constants to be determined.
Let's use substitution to determine $H^* = (N_s^*, N_e^*, N_i^*, N_r^*, N_\nu^*)$

$N_\nu^* = \frac{\delta}{\alpha+1}N_s^* > 0$
$N_e^* = \dfrac{\alpha + \alpha_{ad} + \gamma_{30} - \mu_3\theta_3 q}{\gamma_{20}}N_i^* > 0$
$N_s^* = \dfrac{(\alpha + \alpha_{ad} + \gamma_{30} - \mu_3\theta_3 q)(\alpha + \gamma_{20} - \mu_3\theta_2 p)}{\gamma_2 0\gamma_{10}\beta_0} > 0$
$N_r^* = \dfrac{\gamma_{30}}{\alpha+\rho}N_i^* + \dfrac{(1-\nu)\delta}{(\alpha+\rho)(\alpha+1)}N_s^*$
$C_0 = \dfrac{(\alpha - \mu_1\theta_1 + \delta)(\alpha+\rho)(\alpha+1) - \delta(\mu_4\theta_4 + \rho)(1-\nu) + (\mu_5\theta_5 + \nu)(\alpha+\rho)}{(\alpha+\rho)(\alpha + \alpha_{ad} - \mu_3\theta_3 + \gamma_3)(-\alpha + \mu_2\theta_2 - \gamma_2) + \gamma_2\gamma_3(\mu_4\theta_4 + \rho) + \gamma_2(\alpha+\rho)\mu_3\theta_3(1-q)}$
$N_i^* = \dfrac{\gamma_{20}}{\gamma_{20}+1}C_0 N_s^* > 0$ C_0 is positive according to the conditions on R_4 and R_5

Next, we will study the stability of the endemic equilibrium point H^* by linearizing the autonomous nonlinear system and calculating the characteristic equation. We obtain a polynomial equation, which will allow us to analyze the stability of the system. Thus, after linearization, we obtain the following system:

$$\begin{cases} \psi_s'(t) = -(\hat{\alpha} - \mu_1\hat{\theta}_1 + \delta)\psi_s(t)^* - \gamma_1\beta_0[N_i^*\psi_s(t) + N_s^*\psi_i(t)] + \\ \mu_2\hat{\theta}_2(1-p)(a,t)\psi_e(t) + \mu_3\hat{\theta}_3(1-q)N_i^* + (\mu_4\hat{\theta}_4 + \rho)N_r^* + (\mu_5\theta_5 + \nu)N_\nu^*, \\ \psi_e'(t) = -\left(\hat{\alpha} + \gamma_2 - \mu_2\right)p\psi_e'(t) + \gamma_1\beta_0\psi_i(t)N_s^*, \\ \psi_i'(t) = -\left(\hat{\alpha} + \gamma_3 + \alpha_{ad} - \mu_3\right)q\psi_i(t) + \gamma_2\psi_i(t), \\ \psi_r'(t) = -\left(\hat{\alpha} + p\right)N_r^* + \gamma_3\psi_i(t) + (1-\nu)\psi_\nu(t), \\ \psi_\nu'(t) = -(\hat{\alpha} + 1)\psi_\nu(t) + \delta\psi_s(t). \end{cases} \tag{13}$$

Let's note that $\psi_i(t) = \phi_i exp^{(\lambda t)} \Longrightarrow \psi_i'(t) = \lambda\phi_i exp^{(\lambda t)}$
By replacing $\psi_i(t)$ and $\psi_i'(t)$ by their values in (13), we obtain the following system: (14)

$$\begin{cases} (\lambda + \hat{\alpha} - \mu_1\hat{\theta}_1 + \delta)\phi_s(t)^* + \gamma_1\beta_0[N_i^*\phi_s(t) + N_s^*\phi_i(t)] - \mu_2\hat{\theta}_2(1-p)(a,t)\phi_e(t) - \\ \mu_3\hat{\theta}_3(1-q)N_i^* - (\mu_4\hat{\theta}_4 + \rho)N_r^* - (\mu_5\theta_5 + \nu)N_\nu^* = 0, \\ \left(\lambda + \hat{\alpha} + \gamma_2 - \mu_2\right)p\phi_e(t) - \gamma_1\beta_0\phi_i(t)N_s^* = 0, \\ \left(\lambda + \hat{\alpha} + \gamma_3 + \alpha ad - \mu_3\right)q\phi_i(t) - \gamma_2\phi_i(t) = 0, \\ \left(\lambda + \hat{\alpha} + p\right)N_r^* - \gamma_3\phi_i(t) - (1-\nu)\phi_\nu(t) = 0, \\ (\lambda + \hat{\alpha} + 1)\phi_\nu(t) - \delta\phi_s(t) = 0. \end{cases} \tag{14}$$

Substitute N_s^* and N_i^* by their expressions in (14) while simplifying if necessary and setting: $C_1 = -\hat{\alpha} + \mu_1\hat{\theta}_1 - \delta$, $C_2 = \hat{\alpha} + \gamma_2 - \mu_2\theta_2$, $C_3 = \alpha + \rho$, $C_4 = \hat{\alpha} + \gamma_2 - \mu_2\theta_2 p$, $C_5 = \hat{\alpha} + \alpha_{ad} + \gamma_3 - \mu_3\theta_3 q$, $C_6 = \alpha + 1$, $C_7\mu_5\theta_5 + \nu$, $C_8 = \mu_4\theta_4 + \rho$ with $C_i (i = 1 \quad to \quad 8)$ strictly positive according to the conditions on R_1, R_2, R_3, R_4, R_5) we obtain the following system:

$$\begin{cases} (\lambda - C_1 + \frac{C_0C_4C_5}{C_6})\phi_s(t) + (\frac{C_4C_5}{\gamma_2}e^{-\lambda\tau} - \mu_3\theta_3(1-q))\phi_i(t) - \\ \mu_2\theta_2(1-p)\phi_e(t) - C_8\phi_r(t) - C_7\phi_v(t) = 0, \\ (\lambda + C_4)\phi_e(t) - \frac{C_0C_4C_5}{C_6}\phi_s(t) - \frac{C_4C_5}{\gamma_2 e^{-\lambda\tau}}\phi_i(t) = 0, \\ (\lambda + C_5)\phi_i(t) - \gamma_2\phi_e(t) = 0, \\ (\lambda + C_3)\phi_r(t) - \gamma_3\phi(t) - (1-\nu)\phi_v(t) = 0, \\ (\lambda + C_6)\phi_v(t) - \delta\phi_s(t) = 0. \end{cases} \tag{15}$$

Let's note $M_1 = \Big(-(C_1 + \frac{C_0C_4C_5}{C_6} + C_6 + C_3) + \frac{C_0C_4C_5}{C_6} \Big)$;

$M_2 = C_1(C_6 + C_3) + C_6C_3$;

$M_3 = \delta C_7C_3 + \delta(1-\nu)C_8 + C_1C_6C_3 - C_3C_0C_4C_5 + C_3C_0C_4C_5$; $M_4 = C_4C_5$;

$M_5 = \Big[C_1 + \frac{C_0C_4C_5}{C_6} + C_5 + C_4 + C_6 + C_3 \Big]$;

$M_6 = \Big[-\delta C_7 - \frac{C_0C_4C_5}{C_6}\mu_2\theta_2(1-p) + (C_5 + C_4 + C_6 + C_3)(C_1 + \frac{C_0C_4C_5}{C_6}) + (C_4C_5 + (C_6 + C_3)(C_5 + C_3) + C_6C_3) \Big]$;

$M_7 = \Big[-\delta C_7(C_5 + C_3 + C_4) - \delta(1-\nu)C_8 - \gamma_2\frac{C_0C_4C_5}{C_6}\mu_3\theta_3(1-q) + \Big(C_4C_5 + (C_6 + C_3)(C_5 + C_3) + C_6C_3\Big)(C_1 + \frac{C_0C_4C_5}{C_6}) + C_4C_5(C_6 + C_3) + C_6C_3(C_5 + C_4) - (C_5 + C_3 + C_6)\frac{C_0C_4C_5}{C_6}\mu_2\theta_2(1_p) \Big]$;

$M_8 = \Big[-\delta C_7(C_3C_5 + C_4C_5 + C_3C_4) - \delta(1-\nu)C_8(C_4 + C_5) - \gamma_3C_8\frac{C_0C_4C_5}{C_6}\gamma_2 - (C_6 + C_3)\gamma_2\frac{C_0C_4C_5}{C_6}\mu_3\theta_3(1-q) - (C_3C_5 + C_5C_6 + C_6C_3)\frac{C_0C_4C_5}{C_6}\mu_2\theta_2(1-p) + C_6C_3C_5C_4 + \Big(C_5C_4(C_6 + C_3) + C_6C_3(C_4 + C_5)\Big)(C_1 + \frac{C_0C_4C_5}{C_6}) \Big]$;

$M_9 = +\Big[-\delta C_7C_3C_4C_5 - \delta(1-\nu)C_8C_4C_5 - \gamma_3C_8C_0C_5C_4\gamma_2 - C_3\gamma_2C_0C_4C_5\mu_3\theta_3(1-q) - C_3C_5C_0C_5C_4C_5\mu_2\theta_2(1-p) + C_1C_6C_3C_5C_4 + C_0C_4C_5C_3C_5C_4 \Big]$. □

we obtain this characteristic equation:

$$\lambda^5 - \Big[\lambda^3 + M_1\lambda^2 + (M_2 - \delta C_7)\lambda - M_3\Big]M_4e^{-\lambda\tau} + M_5\lambda^4 + M_6\lambda^3 + M_7\lambda^2M_8\lambda + M_9M_4 = 0 \tag{16}$$

Theorem 4.3.
If $\tau = 0$, for all (i=1,...9), $M_i > 0$ and in addition $M_6 - M_4 > 0$, $M_7 - M_1M_4 > 0$, $M_8 - M_2M_4 + \delta C_7M_4 > 0$, then, we have an asymptotic stability of the endemic equilibrium point.

if $\tau = 0$, we obtain;

$$\lambda^5 + M_5\lambda^4 + (M_6 - M_4)\lambda^3 + (M_7 - M_1M_4)\lambda^2 + (M_8 - M_2M_4 + \delta C_7M_4)\lambda + M_3M_4 + M_9M_4 = 0 \tag{17}$$

It's easy to see that Descartes' rule holds, such that there are as many positive roots as there are sign changes of the coefficients of the polynomial [13], [15], [16]. In this case, all polynomial coefficients are positive, hence all polynomial roots are therefore negative, which demonstrates the endemic asymptotic stability equilibrium point.

Theorem 4.4.
For $\tau > 0$, $M_5 - 2M_6 > 0$, $2(M_8 - M_5M_7) + (M_6+1)(M_6 - M_1) > 0$,

$2(M_5M_9M_4 - M_6M_8 - \delta C_7M_2) + (M_7 - M_1)(M_7 + M_1) > 0$,

$-2(M_1M_3 + M_7M_9M_4) + (M_8 - \delta C_7 + M_2)(M_8 + \delta C_7 - M_2) > 0$, $(M_9M_4 - M_3)(M_9M_4 + M_3) > 0$, *then, we have an asymptotic stability of the endemic equilibrium point.*

Proof. For $\tau > 0$, let $\lambda = iw$,[6] Substituting in the equation (16), we obtain:
$iw^5 - \Big[- iw^3 - M_1w^2 + i(M_2 - \delta C_7)w - M_3\Big]M_4(\cos(w\tau) - i\sin(w\tau)) + M_5w^4 - iM_6w^3 - M_7w^2 + iM_8w + M_9M_4 = 0.$
Separating the real part from the imaginary part, we get:
$\Big[M_5w^4 - M_7w^2 + M_9M_4 + M_4w^3\sin(w\tau) + M_1M_4w^2\cos(w\tau) + M_4(\delta C_7 - M_2)\sin(w\tau)w + M_3M_4\cos(w\tau)\Big] + i\Big[w^5 - M_6w^3 + M_8w + M_4w^3\cos(w\tau) - M_1M_4w^2\sin(w\tau) + M_4(\delta C_7 - M_2)w\cos(w\tau) - M_3M_4\sin(w\tau)\Big] = 0$
We obtain the following system: (18)

$$\begin{cases} M_5w^4 - M_7w^2 + M_9M_4 + M_4w^3\sin(w\tau) + M_1M_4w^2\cos(w\tau) + \\ M_4(\delta C_7 - M_2)\sin(w\tau)w + M_3M_4\cos(w\tau) = 0, \\ w^5 - M_6w^3 + M_8w + M_4w^3\cos(w\tau) - M_1M_4w^2\sin(w\tau) + \\ M_4(\delta C_7 - M_2)w\cos(w\tau) - M_3M_4\sin(w\tau) = 0. \end{cases} \tag{18}$$

By the method of comparison, we get $\sin^2(w\tau)$ and $\cos^2(w\tau)$ and applying $\sin^2(w\tau) + \cos^2(w\tau) = 1$, and let pose $w^2 = x$, we get:
$x^5 + (M_5^2 - 2M_6)x^4 + \Big[2(M_8 - M_5M_7) + (M_6+1)(M_6-1)\Big]x^3 + \Big[2(M_5M_9M_4 - M_6M_8 - \delta C_7 + M_2) + (M_7 - M_1)(M_7 + M_1)\Big]x^2 + \Big[2(-M_1M_3 - M_7M_9M_4) + (M_8 + \delta C_7 - M_2)(M_8 - \delta C_7 + M_2)\Big]x + (M_9M_4 + M_3)(M_9M_4 - M_3) = 0$ (19)
Then, according to the Descartes criterion [24], the roots of the characteristic equations (19) are all negative real parts. Thus, the endemic equilibrium point of the autonomous system in Equation (16) is asymptotically stable. □

5. Simulation

We will now simulate numerically the disease free equilibrium point and the endemic equilibrium point by using python. the simulation time is expressed in days. First let consider the disease free equilibruim $H(S_0, 0, 0, 0$ and the following parametters:
$\alpha = 0.13$ because $\sigma = 0$ for human being cases, $\theta_1 = 0.4$, $\theta_2 = 0.1$, $\theta_3 = 0.1$, $\theta_4 = 0.2$, $\theta_5 = 0.1$, $\gamma_1 = 0.$, $\gamma_2 = 0.15$, $\gamma_3 = 0.1$, $\beta_0 = 0.0023$, $p = 0.1$, $q = 0.15$, $\rho = 0.02$, $\alpha_{ad} = 0.1$, v=0.4, [19],
$(N_s(0), N_e(0), N_i(0), N_r(0), N_v(0))$=(50000, 0, 0, 0, 0)
we observe that without vaccination, the disease free equilibruim H_0 is locally assymptotically stable, there is no one

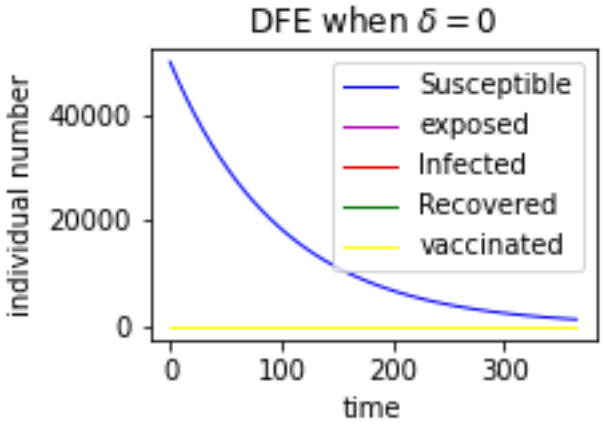


Fig. 1. Local asymptotic stability of the disease-free equilibrium point when δ=0

infected and the susceptibles population disappers, everyone is healthy without being sensitive.
Now we simulate the EE ($\gamma_1 = 0.1$). We will change the value of δ to see the vaccination impact on the disease.
a) First, we consider there is no time delay.
case 1 (Figure (2)); When $\tau = 0$ and $\delta = 0$ we get the following figure:

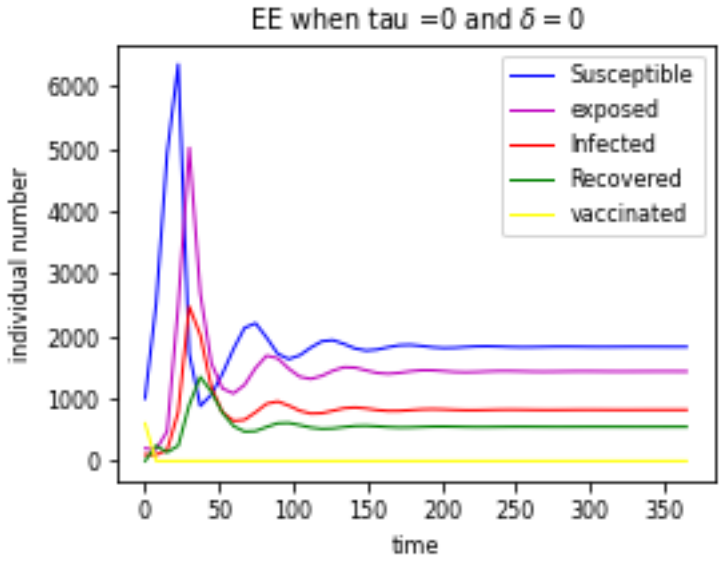


Fig. 2. Local asymptotic stability of the endemic equilibrium point when $\tau = 0$ and δ=0

We can see that there is EE stability after a short oscillation (the curve of susceptible is above all.
case 2 (Figure (3)) ; When $\tau = 0$ and $\delta = 0.1$

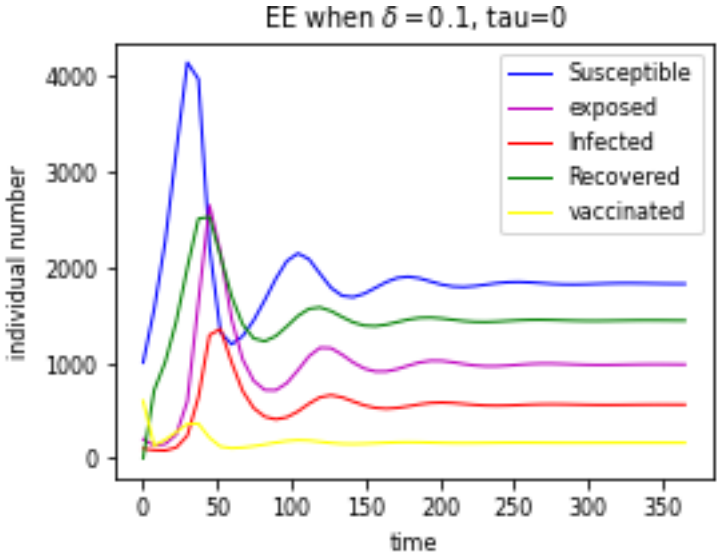


Fig. 3. Local asymptotic stability of the endemic equilibrium point when $\tau = 0$ and δ=0.1

There is EE stability, the curve of the recovered has risen, it comes above the exposed and the infected.
case 3; When tau =0 and $\delta = 0.2$

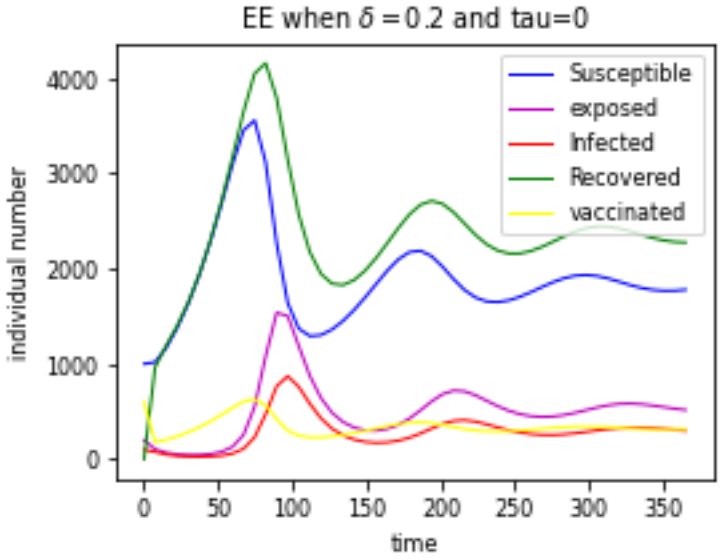


Fig. 4. Local asymptotic stability of the endemic equilibrium point when $\tau = 0$ and δ=0.2

It can be seen that the number of recovered is greater than all. This is due to the fact that the recovered compartment is fed by the large number of vaccinated people who acquire definitive immunity. There is stability of the equilibrium point.
case 4; $\delta = 0.4$:

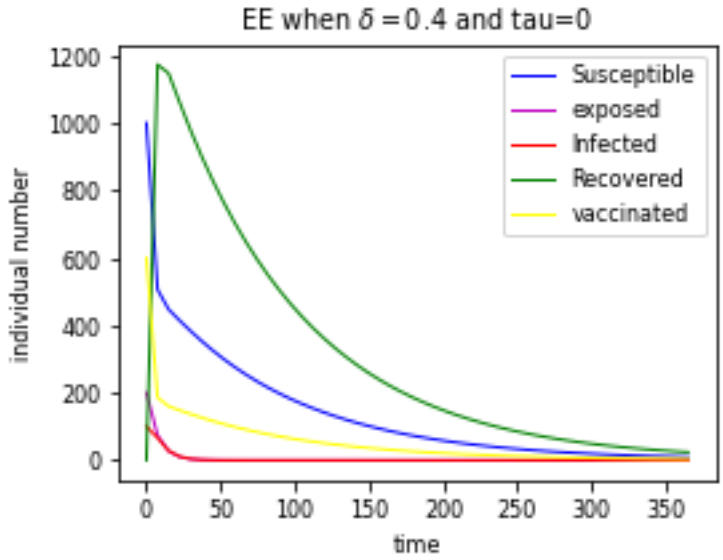


Fig. 5. Local asymptotic stability of the endemic equilibrium point when $\tau = 0$ and δ=0.4

at case 4, $\delta = 0.4$: We noted that the number of infected and exposed disappears in twenty five days. almost everyone has acquired immunity.
case 5; $\delta = 0.5$
if we vaccinate 50 percent of the population, we have the following figure:

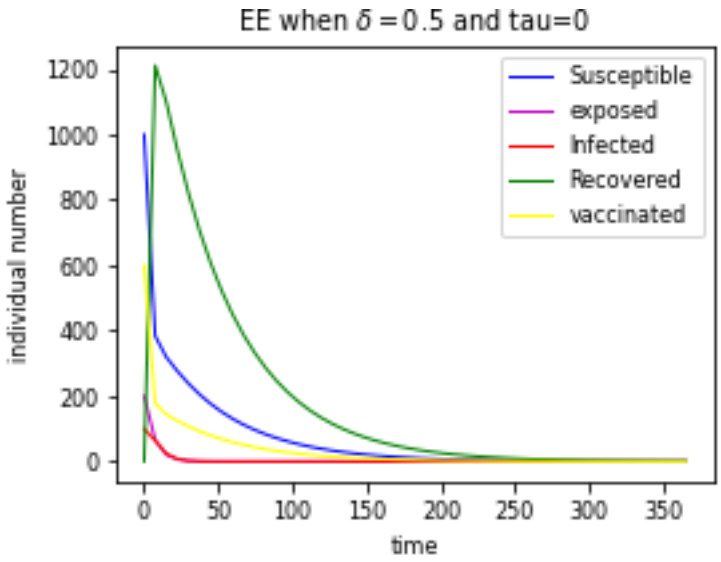


Fig. 6. Local asymptotic stability of the endemic equilibrium point when $\tau > 0$ and δ=0.5

It is found that with higher vaccination coverage, the disease disappears more quickly. the figures (5) and (6) show an endemic equilibrium point stability. The disease completely disappears from the population within a few months.
b) Second We consider that there is time delay:

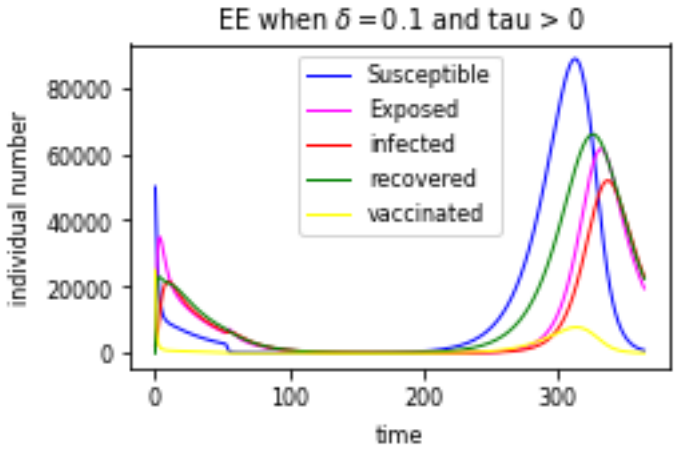


Fig. 7. Instability of the endemic equilibrium point when $\tau > 0$ and δ=0.1

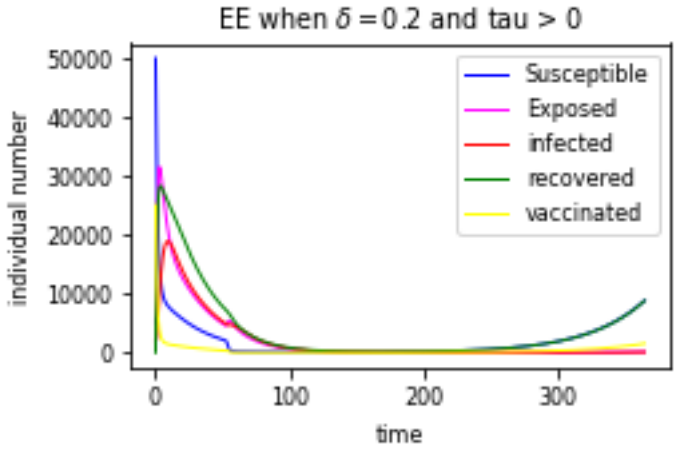


Fig. 8. Local asymptotic stability of the endemic equilibrium point when $\tau > 0$ and δ=0.2

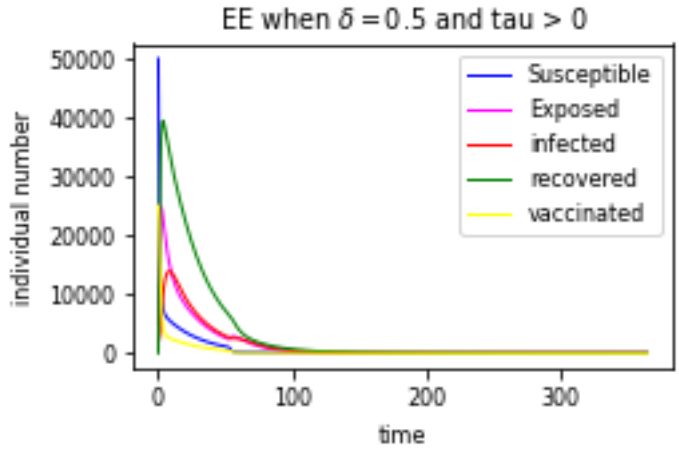


Fig. 9. Local asymptotic stability of the endemic equilibrium point when $\tau > 0$ and δ=0.5

Figures 7, 8 and 9 show a gradual stabilization of the equilibrium point as the vaccination rate increases.

6. Conclusion

In this article we studied an age structured model with delay. We have taken the case where we can vaccinate part of the population, to do so, we choose the model VSEIRS. Tuberculose and COVID 19 are examples of these diseases [22]. We have seen that if there is no time delay, the increase in vaccination coverage accelerates the disappearance of the disease. if there is time delay, vaccination is more than necessary to be able to control the disease. This fact support the reccommendations of health professionals. in the next research we will study an age structured models taking into account the space. For example the case of a latent individual who leaves one area for another area. In this context, how to prevent and provide appropriate solutions ?

References

[1] A. M. Oumarou, S. Bisso, *Modelling and simulating a transmission of Covid-19 disease: Niger Republic case* University Abdou Moumouni, P.O. box 10 662, Niamey, Niger.

[2] A. M Oumarou, S Bisso et B Mampassi, *stability Analysis and simulation of an Age-structured Hepatite B Model without vertical transmission*, Int.J.Diff E.Appl, vol. 14, No.1, (2015), 13-41.

[3] A. Chekroun, M. Nor Frioui , T. Kuniya , and T. Mohammed, *Global stability of an age-structured epidemic model with general Lyapunov functional*, Mathematical Biosciences and Engineering, 16(3): xxx–xxx.

[4] Alpha Omar Diallo, *modélisation et optimisation du controle de l'ancéphalite Japonaise au Cambodge*, Université de Mont Pellier, 27 Novembre 2018
[5] A. Otto , M. Amidou, *A mathematical model for a transmissible disease with two variants*, Int. J. Adv. Appl. Math. and Mech. 10(2) (2022) 1 – 14
[6] Carlo Bianca, Massimiliano Ferrara, and Luca Guerrini, *Qualitative Analysis of a Retarded Mathematical Framework with Applications to Living Systems*
[7] Derdei BICHARA, *Étude de modèles épidémiologiques : Stabilité, observation et estimation de paramètres*, Institut Élie Cartan de Loraine - Site de Metz, ISGMP, Bât A, Ile du Saulcy, 57045, Metz.
[8] Debashis Biswas, Samares Pal,*Stability analysis of a non-linear HIV/AIDS epidemic model with vaccination and antiretroviral therapy*, Int. J. Adv. Appl. Math. and Mech. 5(2) (2017) 41 – 50
[9] O. Diekmann, J. A. P. Heesterbeek , and M. G. Roberts, *The construction of next-generation matrices for compartmental epidemic models*, Department of Mathematics, Utrecht University, Budapestlaan 6, 3584 CD, Utrecht, The Netherlands
[10] Dongmei Li, Chunyu Gui, and Xuefeng Luo, *Impulsive Vaccination SEIR Model with Nonlinear Incidence Rate and Time Delay*, Department of Applied Mathematics, Harbin University of Science and Technology, Harbin 150080, China.
[11] Eduardo Pozo Valdiviezo, *Une brève étude du nombre de reproduction en épidémiologie et leurs applications*, Institute Polytechnique de Paris - Site Ecole Polytechnique, Memoire de master 2
[12] Gabriel Obed Fosu, Emmanuel Akweittey and Albert Adu-Sackey, *Next-generation matrices and basic reproductive numbers for all phases of the Coronavirus disease*, Department of Mathematics, Presbyterian University College, Ghana.
[13] Harouna OUEDRAOGO, *modélisation mathématique de la dengue : Stratégies de controle*, UNIVERSITE POLYTECHNIQUE ET DE RECHERCHE DE BOBO-DIOLASSO (U-P-B), 17 janvier 2017
[14] Luvaha Joel Lutumbi , Akanga Jotham, Chepkwony Isaac, Wali Augustus, *A novel model for female population on the effects of African Stalk Borer on Saccharum officinarum L. under the sterile Insect Technology Interventions*,Int. J. Adv. Appl. Math. and Mech. 10(3) (2023) 52 – 65
[15] Mamadou Lamine Diouf. *Analyse de modèles épidémiologiques à plusieurs classes d'infectés : stabilité et observabilité*. Mathématiques [math]. Université Gaston BERGER de Saint-Louis du Sénégal, 2016. français. tel-01425907
[16] Messaoud Benidir et Michel Barret, *Stabilité des filtres et des systèmes linéaires*, Dunod, 1999, 256 p. (ISBN978-2-10-004432-0).
[17] Saul C. Mpeshe, Nkuba Nyerere, *A Human-Animal Model of Giardiasis Infection in Contaminated Environment*, Int.J. Adv. Appl. Math. and Mech. 8(4) (2021) 37 – 47
[18] Soufiane Elhadi, *Etude d'un modèle d'epidémie SIR avec retard*, Université Mohamed Khider, Biskra, memoire de master 2
[19] Vitalli Akimenko, *Nonlinear age-structured models of polycyclic population dynamics with death rate as power function with exponent n*, faculty of cybernetics and computer science.T.Shevchenko National University of Kyiv, Volodymyrska 64, 01030 kyiv, Ukraine.
[20] Vitalli Akimenko, *An age structured SIR Model with fixed incubation period of infection*, computer and mathematics with Application (2017),htt,// :dx. doi.org/101016/j.camwa 2017.01.022
[21] Wahid, B.K.A., Moustapha, D., Rabi, H.G. and Bisso, S. (2020) *Contribution to the Mathematical Modeling of COVID-19 in Niger*. Applied Mathematics , 11, 427-435.
[22] X-Zhi Li and B. Fang, *Stability of an Age-structured SEIR Epidemic Model with Infectivity in Latent Period*, Xinyang Normal University Henan 464000, P. R. China.
[23] YOUNSI Fatima Zohra, *Mise en place d'un Système d'Information Décisionnel pour le Suivi et la Prévention des Epidémies*, Université d'Oran 1 (thèse 2015/2016).
[24] Z. Yin, Y. Yu and Z. Lu, *stability Analysis of an Age-structrured SEIRS With time delay*, J.M, (2020), 8, 455.