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On the Solution of Partial Differential Equations using the Sumudu and Elzaki Transforms

Research Article

Christian Kasumo*, **Edwin Hapunda**

Department of Mathematics and Statistics, School of Natural and Applied Sciences, Mulungushi University, P O Box 80415, Kabwe, Zambia

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Abstract: We use the Sumudu and Elzaki transforms to obtain solutions to linear partial differential equations, in particular, the heat, wave, Poisson and telegraph equations. Numerical examples are given and the results show a strong relationship between the two transforms and lead to the conclusion that both transforms can be used to obtain solutions for certain types of differential equations as shown in the literature. Our comparative analysis aligns with the literature and demonstrates a plausible identical solutions outcome when these transforms are applied to linear partial differential equations.

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1. Introduction

Linear and nonlinear partial differential equations are used in practically all fields of science and engineering. For example, Schrödinger's equation in quantum mechanics, Maxwell's equations in electrodynamics, reaction-diffusion equations in chemistry and mathematical biology, models for the spatial spread of populations and heat conduction problems, and the Black-Scholes formula for the financial market. In the last few decades, the theory of ordinary differential equations has become one of the most important branches of analysis, with the theory of partial differential equations also receiving much attention in the literature [1].

An equation involving an unknown function of several independent variables and/or its partial derivatives is called a partial differential equation. Examples of partial differential equations are the heat conduction or diffusion equation and wave equation which are useful in many physical processes. Partial differential equations are used in various applications in the study of gravitation, heat transfer, perfect fluids and quantum mechanics. In the literature, partial differential equations have been solved using several methods that include integral transforms such as Fourier and Laplace transforms [2].

This paper focuses on the application of the Sumudu and Elzaki transforms for solving linear PDEs. The Sumudu transform is an integral transform similar to the Laplace transform and was proposed in the early 1990s by Watugala [3, 4] to find solutions to ordinary differential equations and control engineering problems. In fact, the Sumudu

* Corresponding author.

E-mail address(es): ckasumo@gmail.com (Christian Kasumo), edwinhapunda@gmail.com (Edwin Hapunda).

transform is the theoretical dual of the Laplace transform, while the Elzaki transform is a modified Sumudu transform introduced by Elzaki [5–8] to solve initial value problems in control engineering [9]. The Elzaki transform has higher qualities as it is capable of solving differential equations with variable coefficients which could not be solved by the Sumudu transform. Other modifications of the Sumudu transform include the fractional Sumudu decomposition method (FSDM) or the fractional Sumudu variational iteration method (FSVIM) which was applied for obtaining approximate but very accurate solutions of time-fractional Burgers and coupled time-fractional Burgers equations after only a few iterates [10]. The prime objective of this research is to understand both Sumudu and Elzaki transform methods through comparative analysis.

The rest of the paper is structured as follows: Section 2 gives the problem formulation, while Section 3 gives a description of the proposed methods of solution. Section 4 presents the results of numerical experiments based on a selection of test problems and a comparison is made between results from the Sumudu transform and those from the Elzaki transform, and Section 5 presents some conclusions.

2. Problem Formulation

In this section, the classical partial differential equations, specific examples of which will be solved, are identified.

2.1. The Heat Equation

We shall consider the one-dimensional heat equation with lateral heat loss given by

$$u_t = \bar{k}u_{xx} - cu + g(x,t), \quad 0 \le x \le 1, \quad 0 \le t \le 1 \tag{1}$$

and inhomogeneous Dirichlet initial and boundary conditions

$$u(x,0) = f(x),\ u(0,t) = h_1(t),\ u(1,t) = h_2(t), \tag{2}$$

where $f(x)$ and $h_i(t)$ are twice continuously differentiable functions on $[0,1]$ and $u(x,t) \in L^2(\mathbb{R})$, u represents the temperature of the rod at the position x at time t and $\bar{k}$ is the thermal diffusivity of the material and measures the rod's ability to conduct thermal energy relative to its ability to store that energy and $g(x,t)$ is the heat source. The heat equation with lateral heat loss was also considered in [11] and takes the form of the Newell-Whitehead-Segel equation [12].

2.2. The Wave Equation

If $u = \phi(x-ct)$ describes a wave, the question that arises is, 'What type of equation does it satisfy?' We can write $u = \phi(\alpha) = \phi(x-ct)$, i.e., $\alpha = x - ct$. We note that $\frac{\partial u}{\partial x} = \frac{d\phi}{d\alpha}\cdot\frac{\partial \alpha}{\partial x} = \frac{d\phi}{d\alpha}$ (since $\frac{\partial \alpha}{\partial x} = 1$). So

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{d\alpha^2} \tag{3}$$

Similarly, $\frac{\partial u}{\partial t} = \frac{d\phi}{d\alpha}\cdot\frac{\partial \alpha}{\partial t} = -c\frac{d\phi}{d\alpha}$ (since $\frac{\partial \alpha}{\partial t} = -c$). Thus, $\frac{\partial^2 u}{\partial t^2} = -c\frac{\partial}{\partial t}\left(\frac{d\phi}{d\alpha}\right) = -c\frac{d^2\phi}{d\alpha^2}\frac{\partial \alpha}{\partial t} = c^2\frac{d^2\phi}{d\alpha^2}$ and so

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{d^2\phi}{d\alpha^2} \tag{4}$$

From (3) and (4), we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{5}$$

or, in compact form, $u_{xx} = c^{-2}u_{tt}$, which is the linear second-order homogeneous wave equation that describes the propagation of waves with respect to space and time. Here, $c > 0$ is a constant representing the wave velocity which is determined by the physical properties of the material through which the wave propagates. The initial conditions associated with this wave equation are $u(0,t) = f(t),\ u_x(0,t) = g(t)$. Equation (5) admits solutions of the form $u(x,t) = \phi(x-ct)+\psi(x+ct)$ where ϕ and ψ are arbitrary functions. No matter the shape of the function ϕ, the wave $u = \phi(x-ct)$ satisfies the wave equation, since $\frac{\partial \phi}{\partial x} = \phi',\ \frac{\partial^2 \phi}{\partial x^2} = \phi'',\ \frac{\partial \phi}{\partial t} = -c\phi',\ \frac{\partial^2 \phi}{\partial t^2} = (-c)^2\phi'' = c^2\phi''$. Thus,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \phi'' - \frac{1}{c^2}(c^2\phi'') = 0$$

Similarly, any function of the form $u = \phi(x+ct)$ also satisfies the wave equation, i.e., $\frac{\partial \phi}{\partial x} = \phi',\ \frac{\partial^2 \phi}{\partial x^2} = \phi'',\ \frac{\partial \phi}{\partial t} = c\phi',\ \frac{\partial^2 \phi}{\partial t^2} = c^2\phi''$, giving, again,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \phi'' - \frac{1}{c^2}(c^2\phi'') = 0$$

It follows that any function of the form $u = \phi(x-ct)+\psi(x+ct)$ is a solution of the wave equation. We can write the wave equation (5) as

$$u_{tt} = c^2u_{xx}, \quad 0 \le x \le \pi, \quad t > 0 \tag{6}$$

2.3. The Poisson Equation

The Poisson equation describes the physics of several situations (e.g., stress in a metal bar, ideal or irrotational fluid flow, groundwater flow or seepage, electrostatic potential). The Poisson equation is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=G(x,y) \text{ or } \nabla^2 u=G(x,y) \tag{7}$$

where $u(x,y)$ is the dependent variable and $G(x,y)$ is the source term. Equation (7) must be considered subject to the conditions

$$\begin{aligned} &u(x,0)=f_1(x),\ u_x(0,y)=f_2(y)\\ &u(0,y)=g_1(y),\ u_x(x,0)=g_2(t) \end{aligned} \tag{8}$$

2.4. The Telegraph Equation

Also known as a damped wave equation, the telegraph equation finds application in many fields of applied science such as planar random motion of a particle in fluid flow, transmission of electrical impulses in the axons of nerve and muscle cells and propagation of electromagnetic waves in superconducting media. Other applications include propagation of pressure waves occurring in pulsating blood flow in arteries, random motion of a bug along a hedge, digital image processing and propagation of electrical signals along a telegraph or cable transmission line [13]. In this paper, among others, we are concerned with finding solutions to linear telegraph equations using the Sumudu and Elzaki transforms. The general telegraph equation takes the form

$$u_{tt}+(\alpha+\beta)u_t+\alpha\beta F(u)=c^2u_{xx}+f(x,t) \tag{9}$$

in the region $\Omega=\{(x,t)|a<x<b,\ 0<t<T\}$ subject to the initial and boundary conditions

$$\begin{aligned} &u(x,0)=g_1(x),\ u_t(x,0)=g_2(x),\ a\le x\le b\\ &u(0,t)=h_1(t),\ u(\ell,t)=h_2(t), 0\le t\le T. \end{aligned} \tag{10}$$

where ℓ is the length of the cable and $u=u(x,t)$ is the unknown function representing the voltage flowing through the wire at any position x and any time t. Also, $F(u)=a_1u^3+a_2u^2+a_3u$, so that if $a_1=a_2=0$, as in this paper, then (9) is linear, otherwise it is nonlinear. The quantities $\alpha,\ \beta,\ a_1,\ a_2,\ a_3$ are known real constants and $f,\ g_1,\ g_2,\ h_1,\ h_2$ are known continuous real-valued functions. In equation (9), we have

$$\alpha=\frac{G}{C},\ \beta=\frac{R}{L},\ c^2=\frac{1}{LC},$$

where R and G represent the resistance and conductance, respectively, of the resistor per unit length of cable, while C and L are, respectively, the capacitance and conductance of the capacitor per unit length of cable.

3. Description of the Methods

3.1. The Sumudu Transform

The Sumudu transform is one of the methods that can be applied to solution of initial value problems. This transform is defined for functions of exponential order by $\mathbb{S}$

$$G(u)=\mathbb{S}[f(t)]=\begin{cases} \int_0^\infty f(ut)e^{-t}\,dt, & 0\le u<\tau_2\\ \int_0^\infty f(ut)e^{-t}\,dt, & \tau_1<u\le 0 \end{cases}$$

over the set of functions

$$A=\left\{f(t)|\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{\frac{|t|}{\tau_j}}, if\, t\in(-1)^j\times[0,\infty)\right\}.$$

With scale and unit-preserving properties, the Sumudu transform may solve problems without resorting to a new frequency domain. Belgacem et al. [14] applied the Sumudu transform to integral production equations and the idea of duality was analyzed which showed a duality relationship between it and the Laplace. This relationship was then known as the Laplace-Sumudu Duality (LSD)[14].

Belgacem and Karaballi [15] introduced more general shift theorems that seemed to have combinatorial connections

Table 1. Basic properties of the Sumudu transform

Formula	Comment
$G(u)=\mathbb{S}[f(t)]=\int_0^\infty f(ut)e^{-t}\,dt,\ -\tau_1<u<\tau_2$	Definition of Sumudu Transform for $f\in A$
$G(u)=\frac{F(\frac{1}{u})}{u},\ F(s)=\frac{G(\frac{1}{s})}{s}$	Duality with Laplace Transform
$\mathbb{S}[af(t)+bg(t)]=a\mathbb{S}[f(t)]+b\mathbb{S}[g(t)]$	Linearity property
$G_1(u)=\mathbb{S}[f'(t)]=\frac{G(u)-f(0)}{u}=\frac{G(u)}{u}-\frac{f(0)}{u}$	
$G_2(u)=\mathbb{S}[f''(t)]=\frac{G(u)}{u^2}-\frac{f(0)}{u^2}-\frac{f'(0)}{u}$	Sumudu transforms of function derivatives
$G_n(u)=\mathbb{S}[f^n(t)]=\frac{G(u)}{u^n}-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}}.$	
$\mathbb{S}\left[\int_0^t f(\tau)\,d\tau\right]=uG(u)$	Sumudu transform of an integral of a function
$\mathbb{S}[f(at)]=G(au)$	First scale preserving theorem
$\mathbb{S}\left(t\frac{df(t)}{dt}\right)=u\frac{dG(u)}{du}$	Second scale preserving theorem
$\mathbb{S}[\mathrm{e}^{at}f(t)]=\frac{1}{1-au}G\left(\frac{u}{1-au}\right)$	First shifting theorem
$\mathbb{S}[f(t-a)H(t-a)]=\mathrm{e}^{\frac{-a}{u}}G(u)$	Second shifting theorem
$\mathbb{S}\left[\frac{1}{t}\int_0^t f(\tau)\,d\tau\right]=\frac{1}{u}\int_0^t f(u)\,du$	Average preserving theorem
$\lim_{u\to 0}G(u)=\lim_{t\to 0}f(t)$	Initial value theorem
$\lim_{u\to\pm\infty}G(u)=\lim_{t\to\pm\infty}f(t)$	Final value theorem
$\mathbb{S}(f(t))=\frac{\int_0^{\frac{\tau}{u}}f(ut)e^{-t}\,dt}{1-\mathrm{e}^{\frac{-\tau}{u}}}$	Sumudu transform of a T-periodic function
$\mathbb{S}(f*g)=u\mathbb{S}(f(t))\mathbb{S}(g(t))$	Sumudu convolution theorem
$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$	

to generalized Stirling numbers. They also established more general Sumudu differentiation, integration, and convolution theorems and used the Laplace-Sumudu Duality to invoke a Bromwich contour integration formula for the complex inverse Sumudu transform. Kiliçman et al. [16] proved the Sumudu transform of convolution for the matrices and used it to solve the regular system of differential equations. Elzaki [7] applied the Elzaki transform to ordinary differential equations with non-constant coefficients and showed a distinction between the Sumudu and Elzaki transforms. Albayrak et al. [17] gave q-Sumudu transforms of certain q-functions and their special cases.

The Sumudu transform has the following properties [18, 19]:

(i) The transform of a Heaviside unit step function is also a Heaviside unit step function in the transformed domain.

(ii) The transform of a Heaviside unit ramp function is also a Heaviside unit ramp function in the transformed domain.

(iii) The transform of a monomial t^n is the scaled monomial $\mathbb{S}(t^n)=n!u^n$.

(iv) If $f(t)$ is a monotonically increasing function, so is $G(u)$, and the converse is true for decreasing functions.

(v) The Sumudu transform can be defined for functions which are discontinuous at the origin, in which case the two branches of the function should be transformed separately. If $f(t)$ is C^n continuous at the origin, then so is the transformation $G(u)$.

(vi) The limit of $f(t)$ as t tends to infinity is equal to the limit of $G(u)$ as u tends to infinity provided both limits exist.

(vii) Scaling of the function by a factor $a>0$ to form the function $f(at)$ gives the transform $G(au)$ which is the result of scaling by the same factor.

Table 1 gives a summary of the properites of the Sumudu transform.

Theorem 3.1.
Let $G(x,u)$ be the Sumudu transform of $f(x,t)$, then:

1. $G_1(x,u)=\mathbb{S}[f_t(x,t)]=\frac{\bar{G}(x,u)-f(x,0)}{u}$

2. $G_2(x,u)=\mathbb{S}[f_{tt}(x,t)]=\frac{\bar{G}(x,u)}{u^2}-\frac{f(x,0)}{u^2}-\frac{f_t(x,0)}{u}$

Proof. 1. Using integration by parts, we get:

$$\mathbb{S}[f_t(x,t)] = \int_0^\infty f_t(x,ut)e^{-t}\,dt = \frac{\bar{G}(x,u) - f(x,0)}{u} \tag{11}$$

2.

$$\mathbb{S}_t[f_{tt}(x,t)] = \int_0^\infty \frac{\partial^2 f}{\partial t^2}(x,ut)e^{-t}\,dt$$

Let $ut = w$, then $dt = \frac{dw}{u}$, and so:

$$\mathbb{S}_t[f_{tt}(x,t)] = \frac{1}{u}\int_0^\infty \frac{\partial^2 f}{\partial w^2}(x,w)e^{-\frac{w}{u}}\,dw$$

Using integration by parts, we get:

$$\mathbb{S}_t[f_{tt}(x,t)] = \frac{1}{u}\left(-\frac{\partial f}{\partial w}(x,0) + u\int_0^\infty \frac{\partial f}{\partial w}(x,w)e^{-\frac{w}{u}}\,dw\right)$$

$$= \frac{1}{u}\left(-f_t(x,0) + \frac{\bar{G}(x,u)}{u} - \frac{f(x,0)}{u}\right)$$

$$= \frac{\bar{G}(x,u)}{u^2} - \frac{f(x,0)}{u^2} - \frac{f_t(x,0)}{u} \tag{12}$$

□

3.2. The Elzaki Transform

The Elzaki transform, on the other hand, is a modified version of the Sumudu transform introduced by Elzaki and Elzaki [7] for solving ordinary and partial differential equations, as well as integral equations and systems of ordinary and partial differential equations. The Elzaki transform of the function $f(t)$, denoted by $E[f(t)] = T(v)$, is defined by

$$E[f(t)] = T(v) = v\int_0^\infty f(t)e^{\frac{-t}{v}}\,dt, \quad v \in (k_1, k_2), \tag{13}$$

or equivalently

$$E[f(t)] = T(v) = v^2\int_0^\infty f(vt)e^{-t}\,dt, \quad k_1, k_2 > 0 \tag{14}$$

of the functions belonging to a class B, where

$$B = \left\{ f(t) | \exists M, k_1, k_2 > 0 \ni |f(t)| < Me^{\frac{|t|}{k_j}}, if\, t \in (-1)^j \times [0,\infty) \right\}.$$

The ability to transform integral equations and systems of differential equations into algebraic equations is an element that makes the Elzaki transform so appealing [20]. Existing literature on the Elzaki transform method highlights its applications toward finding solutions to linear PDEs. Elzaki and Hilal [5] have argued that some integral transform methods such as the Laplace, Fourier and Sumudu are used to solve linear partial differential equations and their usefulness lies in their ability to transform differential equations into algebraic equations which allows simple and systematic solution procedures. The Elzaki transform also behaves in a similar way as it can be used to solve linear PDEs [5].

A combination of the Homotopy perturbation and Elzaki transform was used to investigate some nonlinear partial differential equations by Elzaki and Hilal [5] who also applied the Double Elzaki transform, a modified version of the Double Sumudu transform, to solve the general linear telegraph equation. Kim [9] proved the time-shifting theorem and the convolution theorem for the Elzaki transform and used this transform to propose the solution of differential equations with variable coefficients.

Elzaki and Alamri [8] presented the combined Elzaki transform and projected differential transform method to solve a nonlinear system of partial differential equations, while Khalid et al. [20] applied the Elzaki transform method to some nonhomogeneous fractional ordinary differential equations. Furthermore, Elzaki and Alkhateeb [6] used the Adomian decomposition method to compute Elzaki transforms of some functions. Verma [21] applied the Elzaki transform in solving differential equations including Leguerre polynomial and proved its applicability in the solution of simultaneous differential equations.

In recent years, several researchers have devoted themselves to finding solutions of nonlinear differential equations using various methods. The use of integral transforms to find solutions to nonlinear differential equations presents great difficulty, for which reason methods such as the Adomian decomposition method, the Tanh method, the homotopy perturbation method (HPM), the differential transform method and the variational iteration method have been explored.

To sum up, the literature considered in this section provides insight into the limitations of the methods, when they are applicable as well as their relationship to other methods. The goal remains to compare the two and conclude which method best suits our interests or objectives in this paper. As mentioned before, since our focus is on observing these methods by applying them to PDEs, this paper will only apply the proposed methods to linear PDEs.

Theorem 3.2.
Let $E(x, v)$ be the Elzaki transform of $f(x, t)$, then:

1. $T_1(x, v) = E[f_t(x, t)] = \frac{T(x,v)}{v} - v f(x, 0)$
2. $T_2(x, v) = E[f_{tt}(x, t)] = \frac{T(x,v)}{v^2} - f(x, 0) - v f'(x, 0)$

Proof.

1. Using integration by parts, we get:

$$E[f_t(x, t)] = v \int_0^\infty f(x, t) e^{\frac{-t}{v}}\, dt = v \int_0^\infty \frac{\partial f}{\partial t}(x, t) e^{\frac{-t}{v}}\, dt = \frac{T(x, v)}{v} - v f(x, 0)$$

2.

$$E[f_{tt}(x, t)] = v \int_0^\infty f_{tt}(x, t) e^{-\frac{t}{v}}\, dt = v \int_0^\infty \frac{\partial^2 f}{\partial t^2}(x, t) e^{-\frac{t}{v}}\, dt$$

Using integration by parts, we get:

$$\begin{aligned} E[f_{tt}(x, t)] &= v\left(-\frac{\partial f}{\partial t}(x, 0) + \frac{1}{v}\int_0^\infty \frac{\partial f}{\partial t}(x, t) e^{-\frac{t}{v}}\, dt\right) \\ &= v\left(-f_t(x, 0) + \frac{1}{v}\left[\frac{\bar{T}(x, v)}{v^2} - f(x, 0)\right]\right) \\ &= \frac{\bar{T}(x, v)}{v^2} - f(x, 0) - v f_t(x, 0) \end{aligned} \tag{15}$$

□

Listed in Table 2 are some basic properties of the Elzaki transform that can be used to obtain solutions to many types of differential equations:

Table 3 gives the Sumudu and Elzaki transforms of some basic functions that satisfy the properties of the transforms:

4. Applications and Discussion

In this section, we illustrate the application of the Sumudu and Elzaki transforms using numerical examples based on the heat, wave, Poisson and telegraph equations. All the computations associated with these examples were performed using a Lenovo PC with an Intel Celeron CPU 1037U at 1.80GHz with 4.0GB internal memory and 64-bit operating system (Windows 10). The figures were constructed using R Version 4.2.1 (2022-06-23 ucrt) and R Studio 9.2.191144 Cherry Blossom May 2023 Release. The results are presented in tables and figures accompanying the discussion.

Example 4.1.
Consider the heat equation (1) with $\bar{k} = 1$, lateral heat loss coefficient $c = 2$ and heat source $g(x, t) = 0$, i.e., we have the homogeneous heat equation

$$u_t = u_{xx} - 2u \tag{16}$$

Table 2. Basic properties of the Elzaki transform

Formula	Comment
$E[f(t)] = T(v) = v\int_0^\infty f(t)e^{\frac{-t}{v}}\,dt, v \in (k_1, k_2)$	Definition of Elzaki Transform for $f \in B$
$E[af(t)+bg(t)] = aE[f(t)] + bE[g(t)]$	Linearity property
$T(v) = vF(\frac{1}{v})$	Duality with Laplace Transform
$E[f'(t)] = T_1(v) = \frac{T(v)}{v} - vf(0)$	
$E[f''(t)] = T_2(v) = \frac{T(v)}{v^2} - f(0) - vf'(0)$	Elzaki transforms of function derivatives
$E[f^n(t)] = T_n(v) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0)$	
$E(e^{at}f(t)) = \frac{1}{1-av}T(\frac{v}{1-av})$	First Shifting Property
$E[f(t-a)v*(t-a)] = e^{\frac{-a}{v}}T(v)$	Second Shifting property
$E(f*g) = \frac{1}{v}E(f)E(g)$	Elzaki Convolution Theorem
$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	

Table 3. Sumudu and Elzaki Transforms of some Functions

$f(t)$	$\mathbb{S}[f(t)] = G(u)$	$E[f(t)] = T(v)$
1	1	v^2
t	u	v^3
$\frac{t^{n-1}}{(n-1)!}, n = 1,2,3,\ldots$	u^{n-1}	v^{n+1}
$\frac{t^{a-1}}{\Gamma(a)}, a > 0$	u^{a-1}	v^{a+1}
e^{at}	$\frac{1}{1-au}$	$\frac{v^2}{1-av}$
te^{at}	$\frac{u}{(1-au)^2}$	$\frac{v^3}{(1-av)^2}$
$\frac{t^{n-1}e^{at}}{(n-1)!}, n = 1,2,3,\ldots$	$\frac{u^{n-1}}{(1-au)^n}$	$\frac{v^{n+1}}{(1-av)^n}$
$\frac{t^{k-1}e^{at}}{\Gamma(k)}, k > 0$	$\frac{u^{k-1}}{(1-au)^k}$	$\frac{v^{k+1}}{(1-av)^k}$
$\sin(at)$	$\frac{au}{1+a^2u^2}$	$\frac{av^3}{1+a^2v^2}$
$\cos(at)$	$\frac{1}{1+a^2u^2}$	$\frac{v^2}{1+a^2v^2}$
$\sinh(at)$	$\frac{au}{1-a^2u^2}$	$\frac{av^3}{1-a^2v^2}$
$\cosh(at)$	$\frac{1}{1-a^2u^2}$	$\frac{av^2}{1-a^2v^2}$
$e^{at}\sin(bt)$	$\frac{bu}{(1-au)^2+b^2u^2}$	$\frac{bv^3}{(1-av)^2+b^2v^2}$
$e^{at}\cos(bt)$	$\frac{(1-au)}{(1-au)^2+b^2u^2}$	$\frac{(1-av)v^2}{(1-av)^2+b^2v^2}$
$t\sin(at)$	$\frac{2au^2}{(1+a^2u^2)^2}$	$\frac{2av^4}{1+a^2v^2}$
$t\cos(at)$	$\frac{u(1-a^2u^2)}{(1+a^2u^2)^2}$	$\frac{v^3(1-a^2v^2)}{(1+a^2v^2)^2}$

with initial and boundary conditions given by

$$\begin{aligned} u(x,0) &= \sinh(x), && 0 \le x \le 1,\\ u(0,t) &= 0, && 0 \le t \le 1,\\ u(1,t) &= \sinh(1)e^{-t}, && 0 \le t \le 1.\end{aligned}$$

The exact solution to this problem is $u(x,t) = e^{-t}\sinh(x)$.

Solution using Sumudu Transform. Let $\mathbb{S}_t[u(x,t)] = \bar{U}(x,w)$ so that $\mathbb{S}_t^{-1}[\bar{U}(x,w)] = u(x,t)$. Taking the Sumudu transform of both sides of the equation (16) gives

$$\mathbb{S}_t[u_t] = \mathbb{S}_t[u_{xx} - 2u],$$

$$\frac{\bar{U}(x,w) - u(x,0)}{w} = \bar{U}_{xx}(x,w) - 2\bar{U}(x,w).$$

Using the initial condition $u(x,0) = \sinh(x)$ gives

$$\bar{U}_{xx}(x,w) - 2\bar{U}(x,w) - \frac{\bar{U}(x,w)}{w} = -\frac{\sinh(x)}{w}.$$

Let $\bar{U}(x,w) = y$, then

$$y'' - \left(2 + \frac{1}{w}\right) y = -\frac{\sinh(x)}{w}.$$

This equation is a linear ordinary differential equation with constant coefficients. Hence, the complementary function of the equation is

$$\mathrm{CF} = c_1 \mathrm{e}^{\sqrt{2+\frac{1}{w}}\,x} + c_2 \mathrm{e}^{-\sqrt{2+\frac{1}{w}}\,x}$$

and the particular integral is

$$\mathrm{PI} = -k\frac{\sinh(x)}{w}.$$

Since $k = -\frac{1}{w+1}$, we have

$$\mathrm{PI} = \frac{\sinh(x)}{w+1}.$$

The complete solution is

$$\bar{U}(x,w) = \mathrm{CF} + \mathrm{PI} = c_1 \mathrm{e}^{\sqrt{2+\frac{1}{w}}\,x} + c_2 \mathrm{e}^{-\sqrt{2+\frac{1}{w}}\,x} + \frac{\sinh(x)}{w+1}.$$

To obtain c_1 and c_2, we employ the boundary conditions

$$u(0,t) = 0,$$
$$u(1,t) = \sinh(1)\mathrm{e}^{-t}.$$

Taking the Sumudu transform for each condition gives, respectively,

$$\mathbb{S}_t[u(0,t)] = \bar{U}(0,w) = 0$$
$$\mathbb{S}_t[u(1,t)] = \bar{U}(1,w) = \frac{\sinh(1)}{w+1}.$$

Using the transform of the second condition on the complete solution gives $c_1 = -c_2$. Hence,

$$\bar{U}(x,w) = c_1 \mathrm{e}^{\sqrt{2+\frac{1}{w}}\,x} - c_1 \mathrm{e}^{-\sqrt{2+\frac{1}{w}}\,x} + \frac{\sinh(x)}{w+1}.$$

Using the transform of the third condition on the above solution gives $c_1 = 0 \Rightarrow c_2 = 0$, hence

$$\bar{U}(x,w) = \frac{\sinh(x)}{w+1}.$$

Taking the inverse Sumudu transform of the above equation

$$\mathbb{S}_t^{-1}[\bar{U}(x,w)] = \mathbb{S}_t^{-1}\left[\frac{\sinh(x)}{w+1}\right].$$

We therefore obtain

$$u(x,t) = \mathrm{e}^{-t}\sinh(x) \tag{17}$$

as required.

Solution using Elzaki Transform. Let $\mathbb{E}_t[u(x,t)] = \bar{U}(x,v)$ so that $\mathbb{E}_t^{-1}[\bar{U}(x,v)] = u(x,t)$. Taking the Elzaki transform of both sides of the equation (16) gives

$$\mathbb{E}_t[u_t] = \mathbb{E}_t[u_{xx} - 2u],$$

$$\frac{\bar{U}(x,v)}{v} - vu(x,0) = \bar{U}_{xx}(x,v) - 2\bar{U}(x,v).$$

Using the initial condition $u(x,0) = \sinh(x)$ gives

$$\bar{U}_{xx}(x,v) - 2\bar{U}(x,v) - \frac{\bar{U}(x,v)}{v} = -v\sinh(x).$$

Let $\bar{U}(x,v) = y$, then

$$y'' - \left(2 + \frac{1}{v}\right)y = -v\sinh(x).$$

This equation is a linear ordinary differential equation with constant coefficients. Hence, the complementary function of the equation is

$$\mathrm{CF} = c_1 e^{\sqrt{2+\frac{1}{v}}x} + c_2 e^{-\sqrt{2+\frac{1}{v}}x}.$$

Now, the particular integral is

$$\mathrm{PI} = -kv\sinh(x).$$

Since $k = -\frac{v}{v+1}$, then

$$\mathrm{PI} = \frac{v^2}{v+1}\sinh(x).$$

The complete solution is therefore

$$\bar{U}(x,v) = \mathrm{CF} + \mathrm{PI},$$

$$\bar{U}(x,v) = c_1 \mathrm{e}^{\sqrt{2+\frac{1}{v}}x} + c_2 \mathrm{e}^{-\sqrt{2+\frac{1}{v}}x} + \frac{v^2}{v+1}\sinh(x).$$

To obtain c_1 and c_2, we employ the boundary conditions

$$u(0,t) = 0,$$

$$u(1,t) = \sinh(1)\mathrm{e}^{-t}.$$

Taking the Elzaki transform for each boundary condition gives

$$\mathbb{E}_t[u(0,t)] = \bar{U}(0,v) = 0,$$

$$\mathbb{E}_t[u(1,t)] = \bar{U}(1,v) = \frac{v^2}{v+1}\sinh(1).$$

Using the transform of the first boundary condition on the complete solution gives $c_1 = -c_2$. Hence,

$$\bar{U}(x,v) = c_1 \mathrm{e}^{\sqrt{2+\frac{1}{v}}x} - c_1 \mathrm{e}^{-\sqrt{2+\frac{1}{v}}x} + \frac{v^2}{v+1}\sinh(x).$$

Using the transform of the second boundary condition on the above solution gives $c_1 = 0 \Rightarrow c_2 = 0$, hence

$$\bar{U}(x,v) = \frac{v^2}{v+1}\sinh(x)$$

Taking the inverse Elzaki transform of the above equation

$$\mathbb{E}_t^{-1}[\bar{U}(x,v)] = \mathbb{E}_t^{-1}\left[\frac{v^2}{v+1}\sinh(x)\right]$$

leads to the exact solution

$$u(x,t) = \mathrm{e}^{-t}\sinh(x). \tag{18}$$

Figure 1 shows an inversely proportional relationship between x and t in the case of fixing one variable and choosing values of the other freely. It should be observed that if t is fixed, then u becomes an increasing function and when x is fixed, u becomes a decreasing function. Figure 2 describes the solution u for different values of t.

Example 4.2.

The following homogeneous wave equation is considered

$$u_{tt} = u_{xx} - 3u, \ 0 \le x \le \pi, \ t > 0, \tag{19}$$

given initial conditions

$$u(x,0) = 0, \ u_t(x,0) = 2\cos(x),$$

and boundary conditions

$$u(0,t) = \sin(2t), \ u(\pi,t) = -\sin(2t).$$

The exact solution to this problem is $u(x,t) = \cos(x)\sin(2t)$.

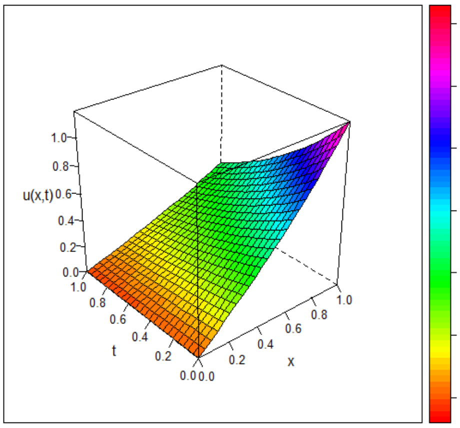

Fig. 1. Surface plot of $u(x,t) = e^{-t}\sinh(x) \forall x, y \in [0,1]$

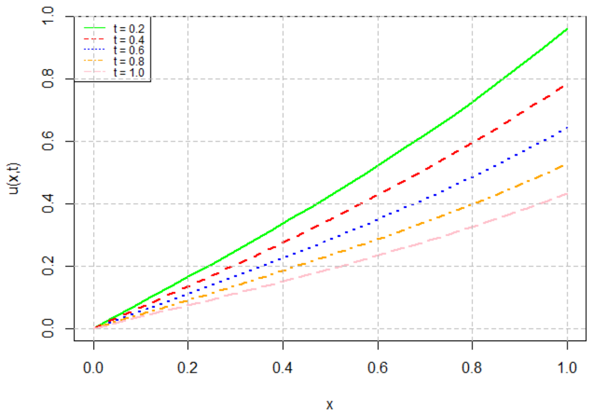

Fig. 2. Exact solution for different values of t

Solution using Sumudu Transform. Let $\mathbb{S}_t[u(x,t)] = \bar{U}(x,w)$ so that $\mathbb{S}_t^{-1}[\bar{U}(x,w)] = u(x,t)$. Taking the Sumudu transform of both sides of the equation (19) gives

$$\mathbb{S}_t[u_{tt}] = \mathbb{S}_t[u_{xx} - 3u],$$

$$\frac{\bar{U}(x,w) - u(x,0)}{w^2} - \frac{u_t(x,0)}{w} = \bar{U}_{xx}(x,w) - 3\bar{U}(x,w).$$

Using $u(x,0) = 0$ and $u_t(x,0) = 2\cos(x)$ gives

$$\bar{U}_{xx}(x,w) - 3\bar{U}(x,w) - \frac{\bar{U}(x,w)}{w^2} = -\frac{2\cos(x)}{w}.$$

Let $\bar{U}(x,w) = y$, then

$$y'' - \left(3 + \frac{1}{w^2}\right) y = -\frac{2\cos(x)}{w}.$$

This is a linear ordinary differential equation with constant coefficients, so its complementary function is

$$\mathrm{CF} = c_1 e^{\sqrt{3+\frac{1}{w^2}}\,x} + c_2 e^{-\sqrt{3+\frac{1}{w^2}}\,x}.$$

Now, the particular integral is

$$\mathrm{PI} = -k\frac{2\cos(x)}{w}.$$

Since $k = -\frac{w^2}{4w^2+1}$, then

$$\mathrm{PI} = \frac{w}{4w^2+1}\cos(x).$$

The complete solution is

$$\bar{U}(x,w) = \mathrm{CF} + \mathrm{PI}$$
$$\bar{U}(x,w) = c_1 e^{\sqrt{3+\frac{1}{w^2}}\,x} + c_2 e^{-\sqrt{3+\frac{1}{w^2}}\,x} + \frac{w}{4w^2+1}\cos(x).$$

To obtain c_1 and c_2, we employ, the boundary conditions

$$u(0,t) = \sin(2t),$$
$$u(\pi,t) = -\sin(2t).$$

Taking the Sumudu transform for each condition gives

$$\mathbb{S}_t[u(0,t)] = \bar{U}(0,w) = \frac{w}{4w^2+1},$$
$$\mathbb{S}_t[u(\pi,t)] = \bar{U}(\pi,w) = -\frac{w}{4w^2+1}.$$

Using the transform of the second condition on the complete solution gives $c_1 = -c_2$, hence

$$\bar{U}(x,w) = c_1 e^{\sqrt{3+\frac{1}{w^2}}\,x} - c_1 e^{-\sqrt{3+\frac{1}{w^2}}\,x} + \frac{w}{4w^2+1}\cos(x).$$

Using the transform of the third condition on the above solution gives $c_1 = 0 \Rightarrow c_2 = 0$, hence

$$\bar{U}(x,w) = \frac{w}{4w^2+1}\cos(x).$$

Taking the inverse Sumudu transform of the above equation

$$\mathbb{S}_t^{-1}[\bar{U}(x,w)] = \mathbb{S}_t^{-1}\left[\frac{w}{4w^2+1}\cos(x)\right]$$

leads to the exact solution

$$u(x,t) = \cos(x)\sin(2t). \tag{20}$$

Solution using Elzaki Transform. Let $\mathbb{E}_t[u(x,t)] = \bar{U}(x,v)$ so that $\mathbb{E}_t^{-1}[\bar{U}(x,v)] = u(x,t)$. Taking the Elzaki transform of both sides of the equation (19) gives

$$\mathbb{E}_t[u_{tt}] = \mathbb{E}_t[u_{xx} - 3u],$$

$$\frac{\bar{U}(x,v)}{v^2} - u(x,0) - v u_t(x,0) = \bar{U}_{xx}(x,v) - 3\bar{U}(x,v).$$

Using $u(x,0) = 0$ and $u_t(x,0) = 2\cos(x)$ gives

$$\bar{U}_{xx}(x,v) - 3\bar{U}(x,v) - \frac{\bar{U}(x,v)}{v^2} = -2v\cos(x).$$

Let $\bar{U}(x,v) = y$, then

$$y'' - \left(3 + \frac{1}{v^2}\right) y = -2v\cos(x).$$

The equation above represents a linear differential equation with constant coefficients. Hence, the complementary function of the equation is

$$\mathrm{CF} = c_1 e^{\sqrt{3+\frac{1}{v^2}}\,x} + c_2 e^{-\sqrt{3+\frac{1}{v^2}}\,x}.$$

Now, the particular integral is

$$\mathrm{PI} = -2kv\cos(x).$$

Since $k = -\frac{v^2}{4v^2+1}$, then

$$\mathrm{PI} = \frac{2v^3}{4v^2+1}\cos(x).$$

The complete solution is

$$\bar{U}(x,v) = \mathrm{CF} + \mathrm{PI},$$

$$\bar{U}(x,v) = c_1 e^{\sqrt{3+\frac{1}{v^2}}\,x} + c_2 e^{-\sqrt{3+\frac{1}{v^2}}\,x} + \frac{2v^3}{4v^2+1}\cos(x).$$

To obtain c_1 and c_2, we employ the boundary conditions

$$u(0,t) = \sin(2t),$$
$$u(\pi,t) = -\sin(2t).$$

Taking the Elzaki transform for each boundary condition gives

$$\mathbb{E}_t[u(0,t)] = \bar{U}(0,v) = \frac{2v^3}{4v^2+1},$$
$$\mathbb{E}_t[u(\pi,t)] = \bar{U}(\pi,v) = -\frac{2v^3}{4v^2+1}.$$

Using the transform of the first boundary condition on the complete solution gives $c_1 = -c_2$, hence

$$\bar{U}(x,v) = c_1 e^{\sqrt{3+\frac{1}{v^2}}\,x} - c_1 e^{-\sqrt{3+\frac{1}{v^2}}\,x} + \frac{2v^3}{4v^2+1}\cos(x).$$

Using the transform of the second boundary condition on the above solution gives $c_1 = 0 \Rightarrow c_2 = 0$, hence

$$\bar{U}(x,v) = \frac{2v^3}{4v^2+1}\cos(x).$$

We take the inverse Elzaki transform of the above equation

$$\mathbb{E}_t^{-1}[\bar{U}(x,v)] = \mathbb{E}_t^{-1}\left[\frac{2v^3}{4v^2+1}\cos(x)\right]$$

and obtain

$$u(x,t) = \cos(x)\sin(2t) \tag{21}$$

as required (see equation (20)). The Sumudu and Elzaki transforms both give the same solution for the wave equation and the results are shown in Figures 3 and 4. These results show that the solution u of the wave equation is periodic and bounded below by -1 and above by 1.

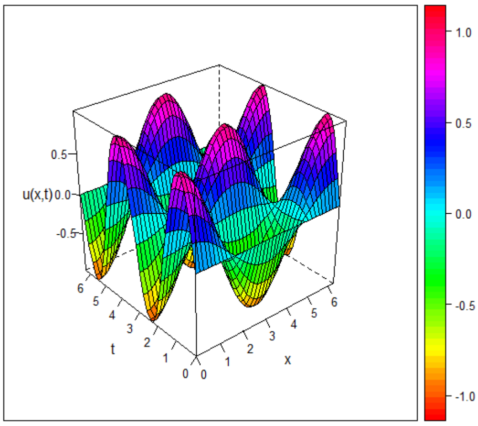

Fig. 3. Surface plot of $u(x,t) = \cos(x)\sin(2t)\,\forall x, t \in [0, 2\pi]$

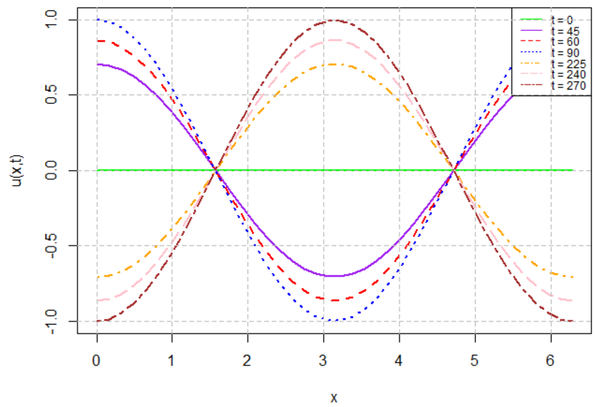

Fig. 4. Exact solution for different values of t

Example 4.3.
Consider the following Poisson equation

$$u_{xx} + u_{yy} = -x\cos(y) \tag{22}$$

(i.e., equation (7) with $G(x,y) = -x\cos(y)$), subject to the initial conditions

$$u(x,0) = x,\ u_x(0,y) = \cos(y),$$
$$u(0,y) = 0,\ u_y(x,0) = 0.$$

The exact solution to this problem is $u(x,y) = x\cos(y)$.

Solution using Sumudu Transform. Let $\mathbb{S}_x[u(x,y)] = \bar{U}(w,y)$ so that $\mathbb{S}_x^{-1}[\bar{U}(w,y)] = u(x,y)$. Taking the Sumudu transform of both sides of the equation (22) gives

$$\mathbb{S}_x[u_{xx} + u_{yy}] = \mathbb{S}_x[-x\cos(y)],$$

$$\frac{\bar{U}(w,y) - u(0,y)}{w^2} - \frac{u_x(0,y)}{w} + \bar{U}_{yy}(w,y) = -w\cos(y).$$

Using the conditions $u(0,y)=0$ and $u_x(0,y)=\cos(y)$ gives

$$\bar{U}_{yy}(w,y)+\frac{\bar{U}(w,y)}{w^2}-\frac{\cos(y)}{w}=-w\cos(y).$$

Let $\bar{U}(w,y)=q$, then we obtain

$$q''-\frac{q}{w^2}=\cos(y)\left(\frac{1-w^2}{w}\right)$$

which is a linear ordinary differential equation with constant coefficients. Hence, the complementary function is

$$\mathrm{CF}=c_1\mathrm{e}^{\frac{i}{w}y}+c_2\mathrm{e}^{-\frac{i}{w}y}.$$

Now, the particular integral is

$$\mathrm{PI}=k\cos(y)\left(\frac{1-w^2}{w}\right).$$

Since $k=\frac{w^2}{1-w^2}$, then

$$\mathrm{PI}=\left(\frac{w^2}{1-w^2}\right)\left(\frac{1-w^2}{w}\right)\cos(y)=w\cos(y).$$

The complete solution is

$$\bar{U}(w,y)=\mathrm{CF}+\mathrm{PI},$$
$$\bar{U}(w,y)=c_1\mathrm{e}^{\frac{i}{w}y}+c_2\mathrm{e}^{-\frac{i}{w}y}+w\cos(y).$$

To obtain c_1 and c_2, we employ the conditions

$$u(x,0)=x,$$
$$u_y(x,0)=0.$$

Taking the Sumudu transform for each condition gives

$$\mathbb{S}_x[u(x,0)]=\bar{U}(w,0)=w,$$
$$\mathbb{S}_x[u_y(x,0)]=\bar{U}(w,0)=0.$$

Using the transform of the second condition on the complete solution gives $c_1=-c_2$, hence

$$\bar{U}(w,y)=c_1\mathrm{e}^{\frac{i}{w}y}-c_1\mathrm{e}^{-\frac{i}{w}y}+w\cos(y).$$

Using the transform of the third condition on the above solution gives $c_1=0\Rightarrow c_2=0$, hence

$$\bar{U}(w,y)=w\cos(y).$$

Taking the inverse Sumudu transform of the above equation

$$\mathbb{S}_x^{-1}[\bar{U}(w,y)]=\mathbb{S}_x^{-1}[w]\cos(y)$$

we get

$$u(x,y)=x\cos(y) \tag{23}$$

as required.

Solution using Elzaki Transform. Let $\mathbb{E}_x[u(x,y)] = \bar{U}(v,y)$ so that $\mathbb{E}_x^{-1}[\bar{U}(v,y)] = u(x,y)$. Taking the Elzaki transform of both sides of the equation (22) gives

$$\mathbb{E}_x[u_{xx}+u_{yy}] = \mathbb{E}_x[-x\cos(y)],$$

$$\frac{\bar{U}(v,y)}{v^2} - u(0,y) - v u_x(0,y) + \bar{U}_{yy}(v,y) = -v^3\cos(y).$$

Applying the conditions $u(0,y)=0$ and $u_x(0,y)=\cos(y)$ gives

$$\bar{U}_{yy}(v,y) + \frac{\bar{U}(v,y)}{v^2} - v\cos(y) = -v^3\cos(y).$$

Let $\bar{U}(v,y) = q$, then

$$q'' - \frac{q}{v^2} = \cos(y)(v - v^3).$$

This is a linear differential equation with constant coefficients and has complementary function

$$\mathrm{CF} = c_1 e^{\frac{i}{v}y} + c_2 e^{-\frac{i}{v}y}.$$

Now, the particular integral is

$$\mathrm{PI} = k\cos(y)(v - v^3).$$

Since $k = \frac{v^2}{1-v^2}$, then

$$\mathrm{PI} = \left(\frac{v^2}{1-v^2}\right)(v - v^3)\cos(y) = v^3\cos(y).$$

The complete solution is

$$\bar{U}(v,y) = \mathrm{CF} + \mathrm{PI},$$

$$\bar{U}(v,y) = c_1 e^{\frac{i}{v}y} + c_2 e^{-\frac{i}{v}y} + v^3\cos(y).$$

To obtain c_1 and c_2, we employ the boundary conditions

$$u(x,0) = x,$$

$$u_y(x,0) = 0.$$

Taking the Elzaki transform for each condition gives

$$\mathbb{E}_x[u(x,0)] = \bar{U}(v,0) = v^3,$$

$$\mathbb{E}_x[u_y(x,0)] = \bar{U}(v,0) = 0.$$

Using the transform of the first boundary condition on the complete solution gives $c_1 = -c_2$, hence

$$\bar{U}(v,y) = c_1 e^{\frac{i}{w}y} - c_1 e^{-\frac{i}{v}y} + v^3\cos(y).$$

Using the transform of the second condition on the above solution gives $c_1 = 0 \Rightarrow c_2 = 0$, hence

$$\bar{U}(v,y) = v^3\cos(y).$$

Taking the inverse Elzaki transform of the above equation

$$\mathbb{E}_x^{-1}[\bar{U}(v,y)] = \mathbb{E}_x^{-1}[v^3]\cos(y)$$

we obtain

$$u(x,y) = x\cos(y) \tag{24}$$

which is the exact solution obtained using the Sumudu transform (equation (23)). The same solution to the Poisson equation is obtained using both the Sumudu and Elzaki transforms, giving an increasing function (not necessarily monotonic) (see Figures 5 and 6). As can be seen from Figure 6, there is periodicity in y.

Example 4.4.

Here, we consider the linear nonhomogeneous telegraph equation be given by

$$u_{xx} = u_{tt} + 3u_t + u - 6e^t\sin(x), \tag{25}$$

with the initial values

$$u(x,0) = \sin(x),\ u_t(x,0) = \sin(x)$$

and the limit values

$$u(0,t) = 0,\ u_x(0,t) = e^t.$$

The exact solution to this problem is $u(x,t) = e^t\sin(x)$.

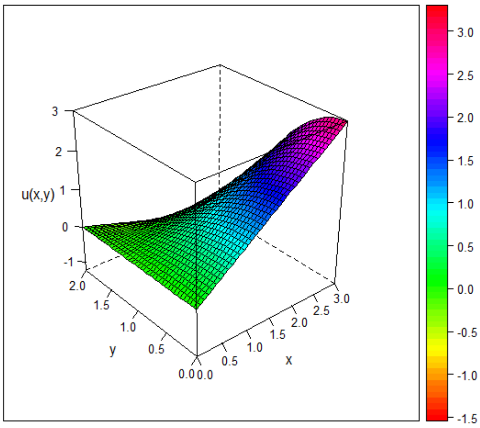

Fig. 5. Surface plot of $u(x,y) = x\cos(y)$ for $x \in [0,3]$ and $y \in [0,2]$

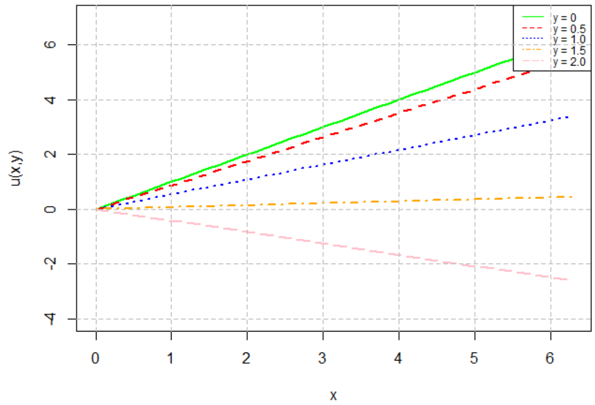

Fig. 6. Exact solution for different values of y.

Solution using Sumudu Transform. Let $\mathbb{S}_x[u(x,t)] = \bar{U}(w,t)$ so that $\mathbb{S}_x^{-1}[\bar{U}(w,t)] = u(x,t)$. Taking the Sumudu transform of both sides of the equation (25) gives

$$\mathbb{S}_x[u_{xx}] = \mathbb{S}_x[u_{tt} + 3u_t + u - 6\mathrm{e}^t \sin(x)],$$

$$\frac{\bar{U}(w,t) - u(0,t)}{w^2} - \frac{u_x(0,t)}{w} = \bar{U}_{tt}(w,t) + 3\bar{U}_t(w,t) + \bar{U}(w,t) - 6\mathrm{e}^t\left(\frac{w}{1+w^2}\right).$$

Using the conditions $u(0,t) = 0$ and $u_x(0,t) = \mathrm{e}^t$ gives

$$\frac{\bar{U}(w,t)}{w^2} - \frac{e^t}{w} = \bar{U}_{tt}(w,t) + 3\bar{U}_t(w,t) + \bar{U}(w,t) - 6\mathrm{e}^t\left(\frac{w}{1+w^2}\right).$$

Let $\bar{U}(w,t) = y$, then we obtain a linear differential equation with constant coefficients

$$y'' + 3y' + y - \frac{y}{w^2} = 6\mathrm{e}^t\left(\frac{u}{1+u^2}\right) - \frac{\mathrm{e}^t}{w}.$$

Hence, the complementary function of this equation is

$$\mathrm{CF} = c_1 \mathrm{e}^{\frac{-3+\sqrt{5-\frac{4}{w^2}}}{2}t} + c_2 \mathrm{e}^{\frac{-3-\sqrt{5-\frac{4}{w^2}}}{2}t}.$$

Now, the particular integral is

$$\mathrm{PI} = k\mathrm{e}^t \frac{5w^2 - 1}{w(1+w^2)}.$$

Since $k = \frac{w^2}{5w^2-1}$, then

$$\mathrm{PI} = \frac{w}{1+w^2}\mathrm{e}^t.$$

The complete solution is

$$\bar{U}(x,w) = \mathrm{CF} + \mathrm{PI},$$

$$\bar{U}(x,w) = c_1 \mathrm{e}^{\frac{-3+\sqrt{5-\frac{4}{w^2}}}{2}t} + c_2 \mathrm{e}^{\frac{-3-\sqrt{5-\frac{4}{w^2}}}{2}t} + \frac{w}{1+w^2}\mathrm{e}^t.$$

To obtain c_1 and c_2, we employ the initial conditions

$$u(x,0) = \sin(x),$$
$$u_t(x,0) = \sin(x).$$

Taking the Sumudu transform for each condition gives

$$\mathbb{S}_x[u(x,0)] = \bar{U}(w,0) = \frac{w}{1+w^2},$$
$$\mathbb{S}_x[u_t(x,0)] = \bar{U}_t(w,0) = \frac{w}{1+w^2}.$$

Using the transform of the first condition on the complete solution gives $c_1 = -c_2$, hence

$$\bar{U}(x,w) = c_1 e^{\frac{-3+\sqrt{5-\frac{4}{w^2}}}{2}t} - c_1 e^{\frac{-3-\sqrt{5-\frac{4}{w^2}}}{2}t} + \frac{w}{1+w^2}e^t.$$

Using the transform of the second condition on the above solution gives $c_1 = 0 \Rightarrow c_2 = 0$, hence

$$\bar{U}(x,w) = \frac{w}{1+w^2}e^t.$$

Taking the inverse Sumudu transform of the above equation

$$\mathbb{S}_x^{-1}[\bar{U}(w,t)] = \mathbb{S}_x^{-1}\left[\frac{w}{1+w^2}\right]e^t$$

results in

$$u(x,t) = e^t \sin(x) \tag{26}$$

as required.

Solution using Elzaki Transform. Let $\mathbb{E}_x[u(x,t)] = \bar{U}(v,t)$ so that $\mathbb{E}_x^{-1}[\bar{U}(v,t)] = u(x,t)$. Taking the Elzaki transform of both sides of the equation (25) gives

$$\mathbb{E}_x[u_{xx}] = \mathbb{E}_x[u_{tt} + 3u_t + u - 6e^t \sin(x)],$$

$$\frac{\bar{U}(v,t)}{v^2} - u(0,t) - vu_x(0,t) = \bar{U}_{tt}(v,t) + 3\bar{U}_t(v,t) + \bar{U}(v,t) - 6e^t\left(\frac{v^3}{1+v^2}\right).$$

Using the initial conditions $u(0,t) = 0$ and $u_x(0,t) = e^t$ gives

$$\frac{\bar{U}(v,t)}{v^2} - ve^t = \bar{U}_{tt}(v,t) + 3\bar{U}_t(v,t) + \bar{U}(v,t) - 6e^t\left(\frac{v^3}{1+v^2}\right).$$

Let $\bar{U}(v,t) = y$, then we have the linear differential equation with constant coefficients

$$y'' + 3y' + y - \frac{y}{v^2} = 6e^t\left(\frac{v^3}{1+v^2}\right) - ve^t.$$

whose complementary function is

$$\text{CF} = c_1 e^{\frac{-3+\sqrt{5-\frac{4}{v^2}}}{2}t} + c_2 e^{\frac{-3-\sqrt{5-\frac{4}{v^2}}}{2}t}.$$

Now, the particular integral is

$$\text{PI} = ke^t \frac{5v^3 - v}{(1+v^2)}.$$

and since $k = \frac{v^2}{5v^2-1}$, we have

$$\text{PI} = \frac{v^3}{1+v^2}e^t.$$

The complete solution is

$$\bar{U}(v,t) = \text{CF} + \text{PI},$$
$$\bar{U}(v,t) = c_1 e^{\frac{-3+\sqrt{5-\frac{4}{v^2}}}{2}t} + c_2 e^{\frac{-3-\sqrt{5-\frac{4}{v^2}}}{2}t} + \frac{v^3}{1+v^2}e^t.$$

To obtain c_1 and c_2, we employ the initial conditions

$$u(x,0) = \sin(x),$$
$$u_t(x,0) = \sin(x).$$

Taking the Elzaki transform for each condition gives

$$\mathbb{E}_x[u(x,0)] = \bar{U}(v,0) = \frac{v^3}{1+v^2},$$
$$\mathbb{E}_x[u_t(x,0)] = \bar{U}_t(v,0) = \frac{v^3}{1+v^2}.$$

Using the transform of the first initial condition on the complete solution gives $c_1 = -c_2$, hence

$$\bar{U}(v,t) = c_1 \mathrm{e}^{\frac{-3+\sqrt{5-\frac{4}{v^2}}}{2}t} - c_1 \mathrm{e}^{\frac{-3-\sqrt{5-\frac{4}{v^2}}}{2}t} + \frac{v^3}{1+v^2}\mathrm{e}^t.$$

Using the transform of the second initial condition on the above solution gives $c_1 = 0 \Rightarrow c_2 = 0$, hence

$$\bar{U}(v,t) = \frac{v^3}{1+v^2}\mathrm{e}^t.$$

Taking the inverse Elzaki transform of the above equation

$$\mathbb{E}_x^{-1}[\bar{U}(v,t)] = \mathbb{E}_x^{-1}\left[\frac{v^3}{1+v^2}\right]\mathrm{e}^t$$

we therefore get

$$u(x,t) = \mathrm{e}^t \sin(x) \tag{27}$$

as required (see equation (26)). As with the previous examples, we obtained the same results for the Telegraph equation from the Sumudu and Elzaki transforms. Also, from the surface graph in Figure 7, it is evident that the further away t goes from 0, the more defined the graph becomes and hence its sinusoidal nature.

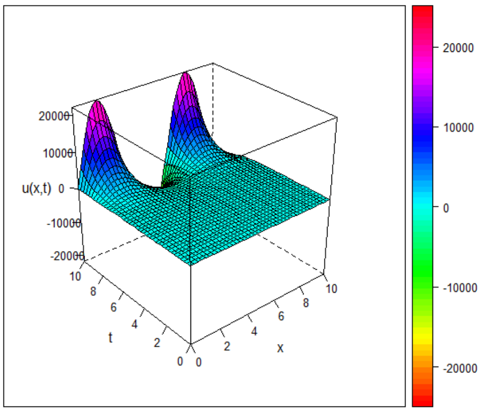

Fig. 7. Surface plot of $u(x,t) = \mathrm{e}^t \sin(x) \forall x, t \in [0,10]$

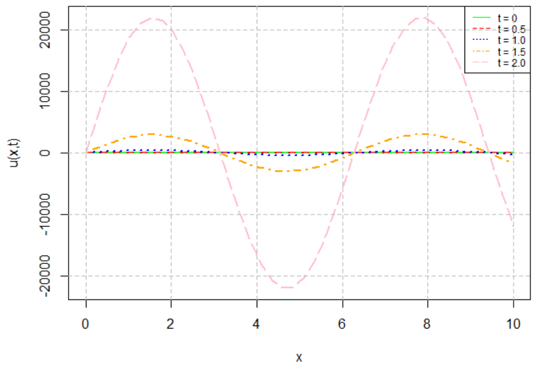

Fig. 8. Exact solution for different values of t

5. Conclusion

This paper has applied the Sumudu and Elzaki transforms and solutions for the heat, wave, Poisson, and telegraph equations have been obtained for each method. The results support the theory that the two transforms can be used to obtain solutions to linear PDEs. However, this does not mean that all linear PDEs can be solved using either or both of these methods. The analysis showed a strong relationship between the Sumudu and Elzaki transforms, since they both yielded exact solutions of the partial differential equations used for numerical experiments.

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