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Comparing iterative methods via basins of attraction

Research Article

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Abstract: Solving nonlinear equations is a key problem across engineering, science, economics, and other fields. Iterative approaches are frequently the only option available. Therefore, studying and comparing different iterative methods is critical, which includes a graphical analysis of the so-called basins of attraction. The analysis of the basins of attraction of iterative techniques to find roots of nonlinear equations constitutes an effective graphical tool for evaluating and contrasting different methods. In order to understand the efficiency and reliability of those root-finding methods, it is essential to have a solid comprehension of their basins of attraction. Intertwined and poorly defined basins of attraction pose additional difficulties in determining the roots of a nonlinear equation. In some instances, the basins of attraction exhibit complex fractal structures, indicating sensitivity to initial conditions. This has significant implications for the implementation of these methods, as it emphasizes the need for careful selection of initial points in order to guarantee convergence to the intended root. The current study makes advances in the knowledge of the basins of attraction of some iterative techniques. In particular, a family of third-order algorithms with a single parameter is addressed and studied. Assessment is given to two transcendental equations.

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Keywords: Nonlinear equations • Transcendental equations • Iterative root-finding methods • Order of convergence • Basins of attraction

1. Introduction

Many mathematical models that represent physical, biological, or economic phenomena are expressed in terms of nonlinear equations. In several branches of applied mathematics, finding their roots is a critical problem. Typically, iterative procedures are required to solve these problems. Therefore, the development of effective iterative processes has been a subject of extensive research and development over time by a large number of studies [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12].

An iterative approach for finding the roots of a nonlinear equation or, equivalently, the zeros of a function, comprises approximating their values by a succession of iterations. In other words, the iterative technique begins with a starting value and applies an equation to generate a new value at each iteration, which is then used for the subsequent iteration, and so on.

Iterative procedures for locating the roots of nonlinear equations are recognized by a number of distinguishing characteristics such as convergence of the process to an equation's root given an initial starting point. Ideally it should

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converge to the nearest root. Besides, the iterative process can be more or less stable in the sense that small variations in the initial approximation or subsequent iterations do not lead to significant differences in convergence to the desired root. The way the iterative process handle complex or multiple roots is also an important characteristic. Concerning the use of computational resources, the iterative processes differ in terms of the number of iterations required to approximate a given root with a particular accuracy level, the amount of functions evaluated in each iteration, or the amount of CPU time required [2], [4], [13], [14],[15].

The behavior of an iterative method applied to a nonlinear equation in relation to its starting points can be shown in a number of different ways. One of these ways is to color code different regions in $\mathbb{C}$ based on the root to which it converges, the so-called basins of attraction. Provided a starting point $z_0 \in \mathbb{C}$, an iterative process T_f generates consecutive points, $z_1 = T_f(z_0),..., z_n = T_f(z_{n-1}) = T_f^2(z_{n-2}) = T_f^n(z_0)$, hoping that it will eventually converge to a zero α of the function f. In order for this to happen, the zero α needs to be an attractive fixed point for T_f, that is, $\lim_{n\to\infty} T_f^n(z_0) = \alpha$. The set of all starting points that converges to a particular solution when the iterative method is repeatedly applied, constitutes a basin of attraction of the method. The basin of attraction is formally defined in Definition 1.1 [16], [5].

Definition 1.1.
The basin of attraction of a zero α of the function f is the set of all points in $\mathbb{C}$ that converge asymptotically to α, that is,
$A(\alpha) = \left\{ z \in \mathbb{C} : T_f^n(z) \to \alpha, n \to \infty \right\}$

Nevertheless, the rate of convergence of the iterative approach cannot be determined from its basins of attraction. One way to visualize the rate of convergence as a function of the various initial values, involves assigning a distinct color to the number of iterations needed to reach a root of the nonlinear equation [17], [10]. It is also possible to adjust the shade of each basin of attraction color to correspond with the proper number of iterations [18], or independently, allow the representation of the basins of attraction and the speed of convergence in different figures [6].

Due to their extensive interwoven relationships, basins of attraction frequently exhibit a fractal-like shape from a geometric viewpoint. In general, basins of attraction are composed of a principal set and a large number of smaller subsets, which are frequently indefinitely distributed on the complex plane [19].

By analyzing the basins of attraction of different iterative procedures, it is possible to figure out which methods are most appropriate for solving a certain problem. An iterative process with small, ill-defined basins of attraction and no clear boundaries is a poorly suited method for identifying roots of nonlinear equations. In contrast, it is better for the iterative technique to have large, well-defined basins of attraction, and with simple and precise boundaries.

The analysis of basins of attraction can also provide information concerning the numerical stability of the iterative processe. If a basin of attraction is extremely sensitive to slight perturbations of the initial conditions or to small numerical errors, this could mean that the method can generate complex basins of attraction or even diverge.

In conclusion, basins of attraction are a useful tool for analyzing and observing graphically the behavior of iterative methods in the complex plane under a variety of starting points. Basins of attraction offer an alternative way to assess the efficacy of various iterative approaches [20], [21], [22], [23], [24], [9], [25], [26], [17], [5], [6], [27], [11]. Many authors, including [22], [23], and [24], have presented and studied the basins of attraction for polynomials with simple and multiple roots as well as real and complex coefficients. The basins of attraction were also investigated for the case of transcendental equations in [5], [6].

This study explores a family of iterative third-order techniques with a single parameter based on its basins of attraction. Two transcendental equations were studied, and a graphical representation of the changes observed in the basins of attraction was obtained by varying the parameter value of the family. In [28], a similar investigation employing the same family applied to a polynomial of high degree was conducted. BSC, Halley's [29] and Euler-Chebyshev's [30] methods are particular cases of this family and are well-described in the literature. The most recent of these, the BSC root-finding technique, was constructed based on Adomian's decomposition technique [31], [32], [33], [34] and on the works of Abbasbandy [2], and Babolian and Biazar [35].

The following outline constitutes the current paper's structure:

First, the one-parameter family and the two transcendental equations are described. The basins of attraction are then illustrated and discussed in terms of complexity, basin boundaries, and convergence. In the conclusion section, the research findings are addressed.

2. The one-parameter family of iterative methods

In this study, the one-parameter family of root-finding methods is represented by equation (1), with the letter A representing the parameter [36], [5].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\left[f(x_n)\right]^2 f''(x_n)}{2\left[f'(x_n)\right]^3 - A\cdot f(x_n) f'(x_n) f''(x_n)} \tag{1}$$

The family (1) of iterative approaches, includes the BSC method for $A = 2$, Halleys's method for $A = 1$, and the Euler–Chebyshev's method for $A = 0$ [4], [30], [29], [5], [36]. These three methods have already been compared using basins of attraction in previous studies [5], [6].

Searching for the zeros of nonlinear functions is equivalent to looking for fixed points of the iterative process. There are, unfortunately, a variety of iterative methods with fixed points that are not zeros of the function. These points are called extraneous fixed points. In the event that these points are attractive, they may be mistaken as roots of the equation. If they are repulsive, they should not pose an issue. However, in this case, the iterative technique may converge on a distant root from the starting point. Halley's method does not have extraneous fixed points [36]. The extraneous fixed points for BSC and Euler–Chebyshev's methods are repulsive [36]. Several investigations indicate that Halley's method is one of the best available iterative third-order root-finding technique [23], [36], [5], [6].

Equation (1) describes a family of third-order convergence iterative procedures. Definition 2.1 outlines the order of convergence of an iterative method [4], [17].

Definition 2.1.
Consider x^* a root of the equation $f(x) = 0$ and $e_n = x^* - x_n$ the truncation error for the nth iterate. The order of convergence for an iterative method is given by a real number $p \geq 1$, so that the following limit holds true for a constant $c \neq 0$:

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = c \tag{2}$$

The constant c is referred to as the asymptotic error constant.

Theorem 2.1 establishes that for every zero of a sufficiently differentiable function, there exists a region in which the iterative methods of the given family (1) converge to it and where the order of convergence is three [36], [4], [5].

Theorem 2.1.
Consider the function $f \in C^5$. Consider the equation $f(x) = 0$. Consider x^*, a root of the equation $f(x) = 0$. If $f'(x^*) \neq 0$, then there exists an interval I containing x^* such that for $x_0 \in I$, the iterative methods belonging to the family (1) converges to the only solution of $f(x) = 0$ belonging to I. Furthermore, the order of convergence is at least 3.

Proof. Consider the iteration function g (3):

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{\left[f(x)\right]^2 f''(x)}{2\left[f'(x)\right]^3 - A\cdot f(x) f'(x) f''(x)} \tag{3}$$

Since $f(x^*) = 0$, $f'(x^*) \neq 0$ and $f \in C^5$, $g(x^*) = x^*$ and $g \in C^3$. The derivative of the function (3) at the point $x = x^*$ is zero, that is, $\left|g'(x^*)\right| = 0 < 1$. Therefore, according to the fixed point theorem, the iterative process $x_{n+1} = g(x_n)$ converges to the unique solution within the interval I.

By taking the second derivative of the function (3) at point $x = x^*$, the value zero is obtained, $g''(x^*) = 0$. By taking the third derivative, one obtains the result (4).

$$g'''\left(x^*\right) = \frac{-2f'''(x^*) f'(x^*) - 3f''(x^*)^2 (A-2)}{2f'(x^*)^2} \tag{4}$$

Computing the Taylor expansion of $g(x_n)$ around x^*, for $\min(x_n, x^*) < \xi_n < \max(x_n, x^*)$, equation (5) is obtained.

$$x_{n+1} - x^* = g(x_n) - g\left(x^*\right) = \frac{g'''(\xi_n)}{6}\left(x_n - x^*\right)^3 \tag{5}$$

For $x_n \neq x^*$:

$$\frac{x_{n+1} - x^*}{(x_n - x^*)^3} = \frac{g'''(\xi_n)}{6} \tag{6}$$

Finally, applying limits to equation (6), since $g \in C^3$,

$$\lim_{n\to\infty} \frac{x_{n+1} - x^*}{(x_n - x^*)^3} = \frac{g'''\left(\lim\limits_{n\to\infty} \xi_n\right)}{6} = \frac{g'''(x^*)}{6} \neq 0 \tag{7}$$

The asymptotic error constant is given by formula (8).

$$\frac{-2f'''(x^*)\,f'(x^*) - 3f''(x^*)^2\,(A-2)}{12f'(x^*)^2} \tag{8}$$

□

In this study, the basins of attraction for the zeros of two transcendental functions, function (9) and function (10) are analyzed.

$$f_1(z) = z^3\exp(-z^2) + \sin^2(2z) - 4\cos(z) \tag{9}$$

$$f_2(z) = z^3\log(z-5) + \frac{3\cos(z)}{z^4 - 3z + 1} - \exp(-z)z^{12} \tag{10}$$

3. Results

The basins of attraction obtained for different values of parameter A are depicted in Figures 1 through **??**. A starting point is deemed non-convergent if, after 200 iterations, it does not approach a root less than 10^{-5}. The regions whose points are not recognized to converge are depicted in white.

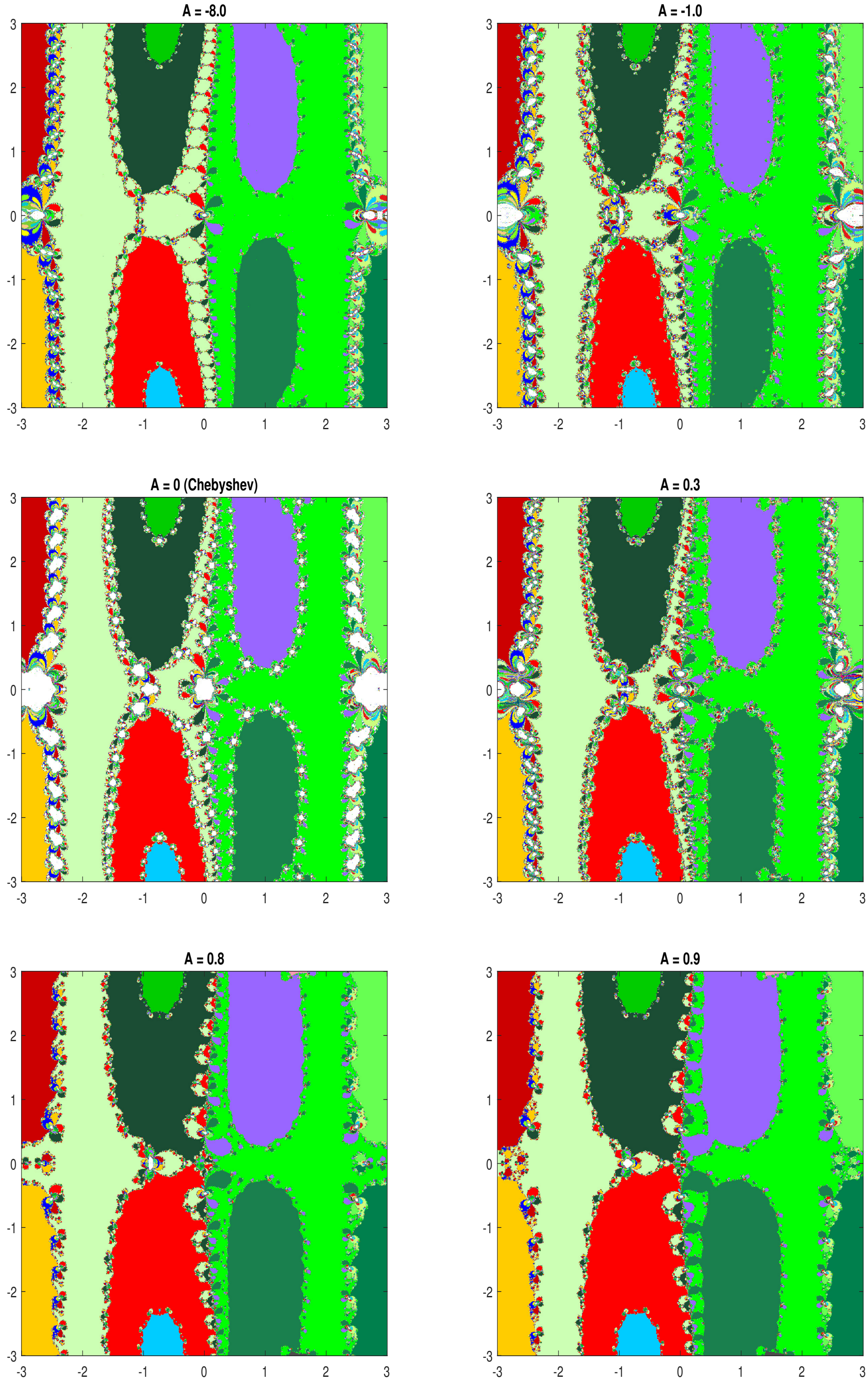

Fig. 1. Function (9). Varying the value of the parameter A from -8.0 to 0.9, with $A = 0$ corresponding to the Euler-Chebyshev's technique.

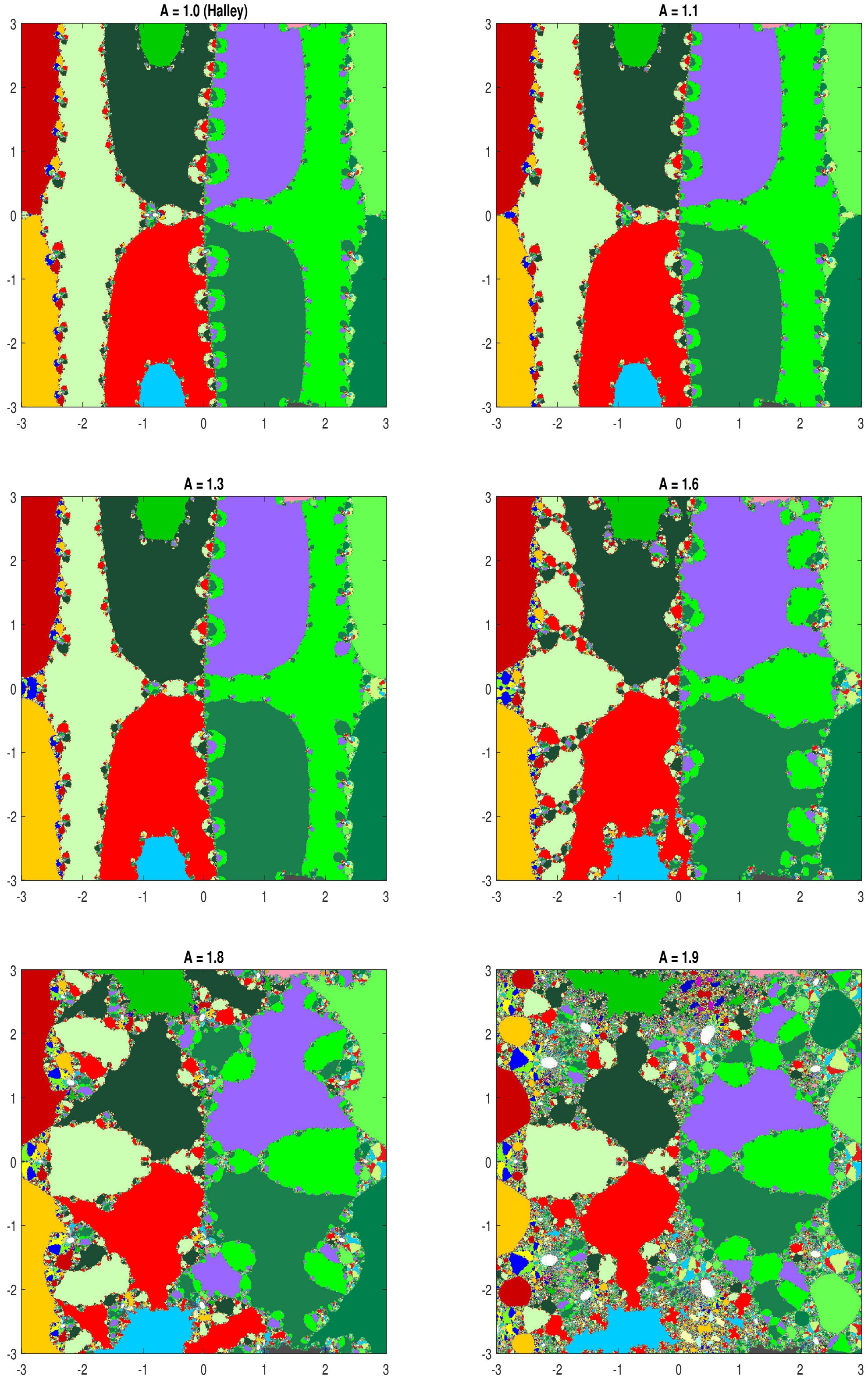

Fig. 2. Function (9). Varying the value of the parameter A from 1 to 1.9, with $A = 1$ corresponding to the Halley's method.

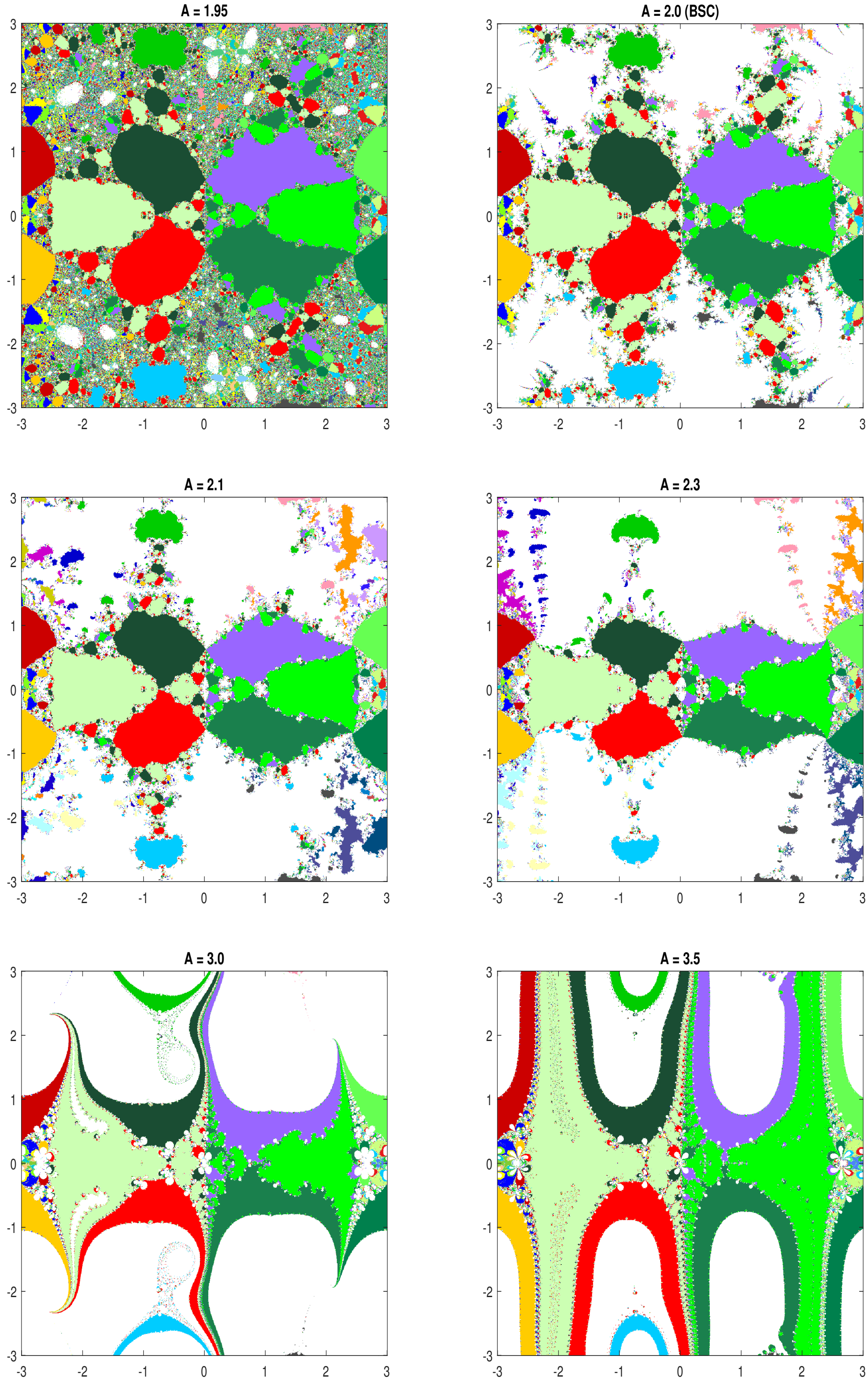

Fig. 3. Function (9). Varying the value of the parameter A from 1.95 to 3.5, with $A = 2$ corresponding to the BSC method.

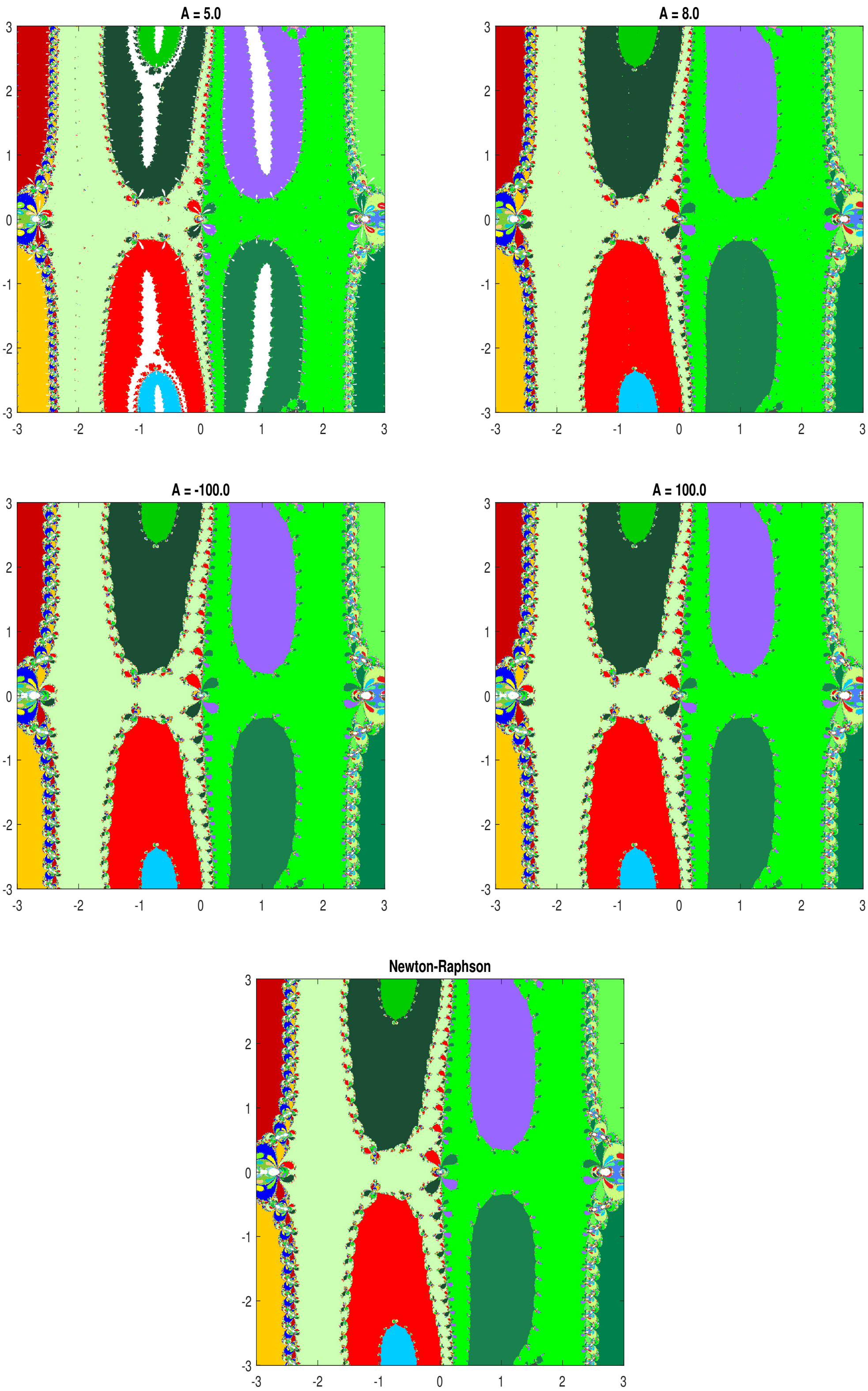

Fig. 4. Function (9). Varying the value of the parameter A from 5.0 to 100. Comparing basins of attraction with high absolute value of the parameter A to basins of attraction of the Newton-Raphson method.

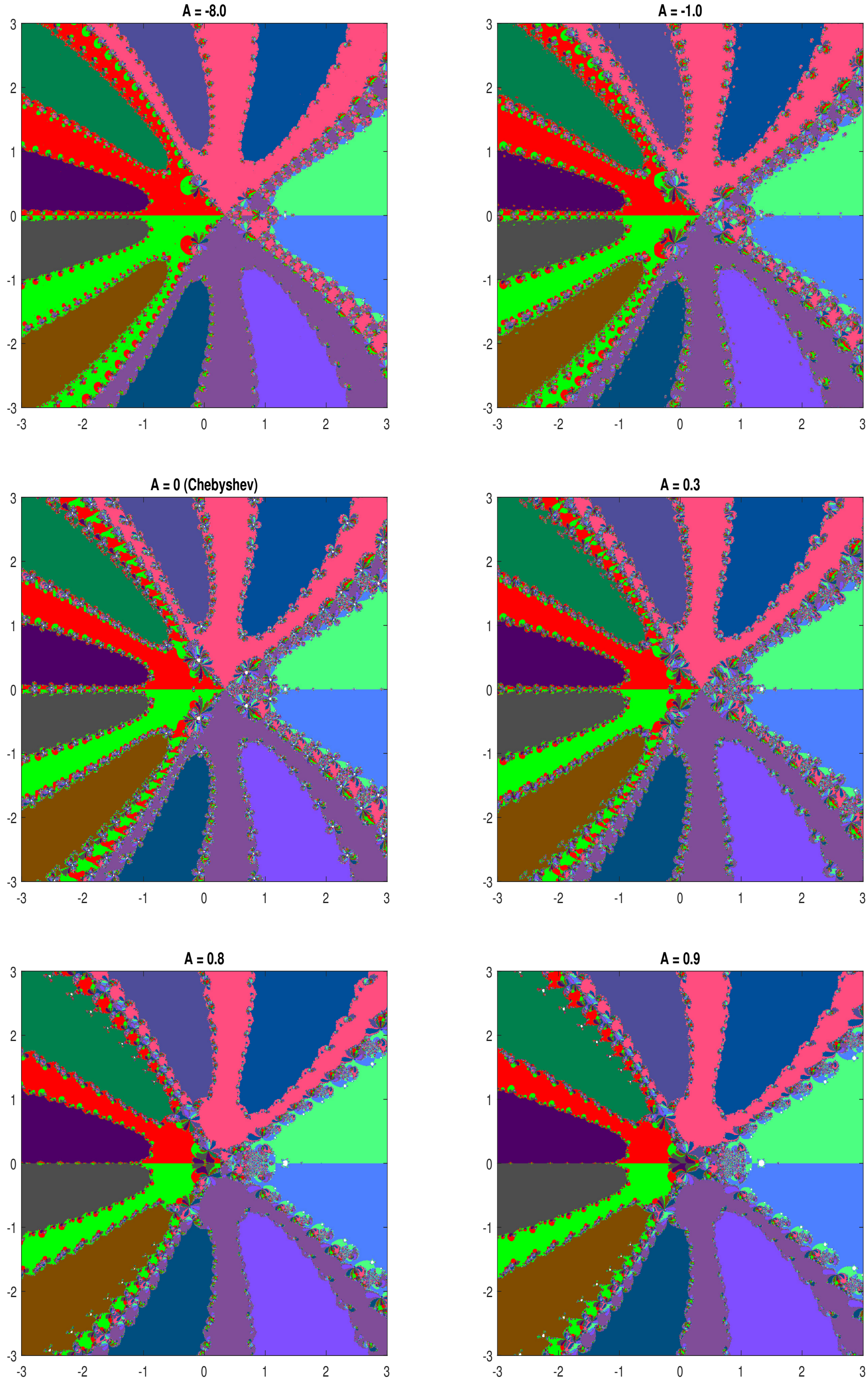

Fig. 5. Function (10). Varying the value of the parameter A from -8.0 to 0.9, with $A = 0$ corresponding to the Euler-Chebyshev's technique.

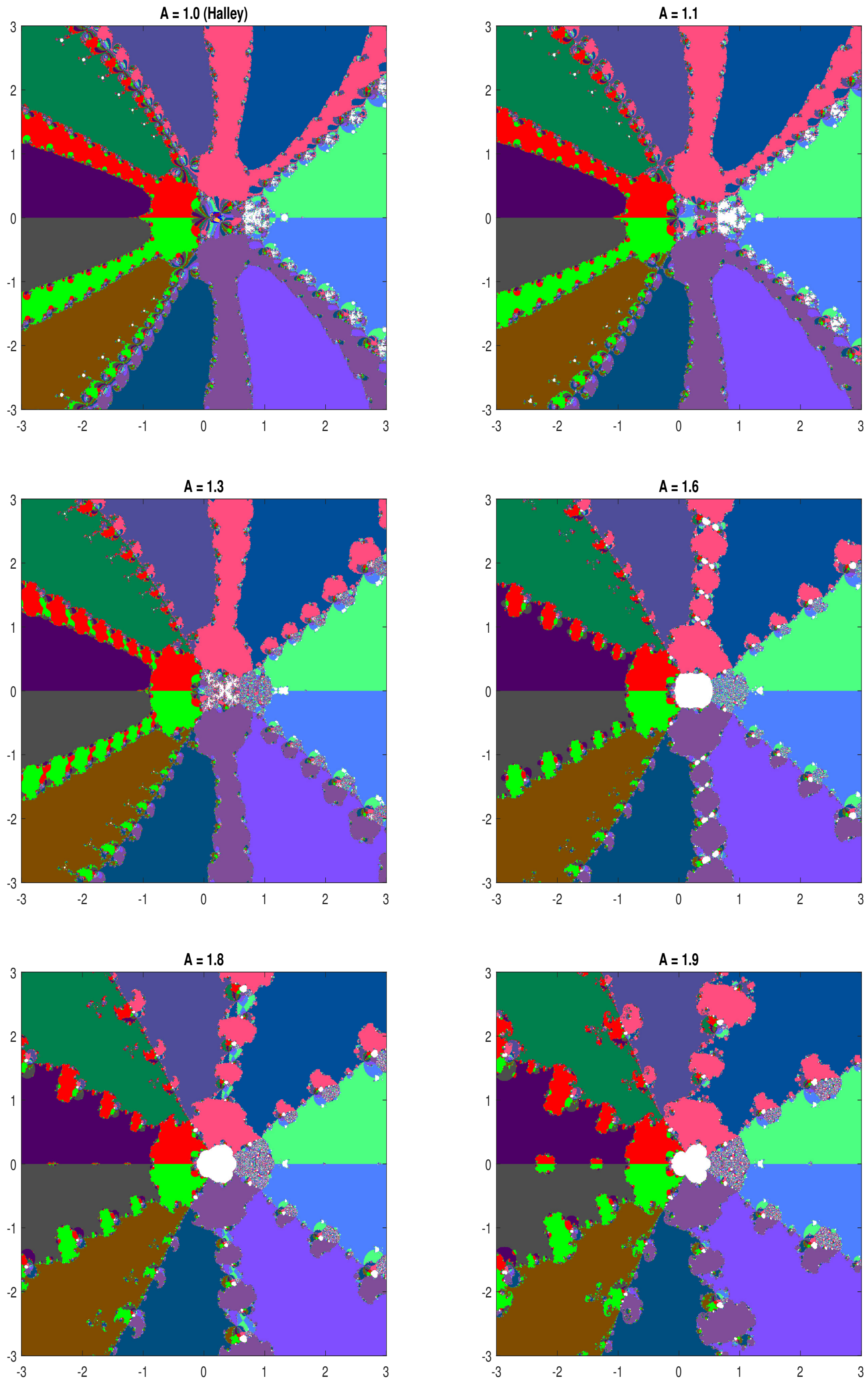

Fig. 6. Function (10). Varying the value of the parameter A from 1 to 1.9, with $A = 1$ corresponding to the Halley's method.

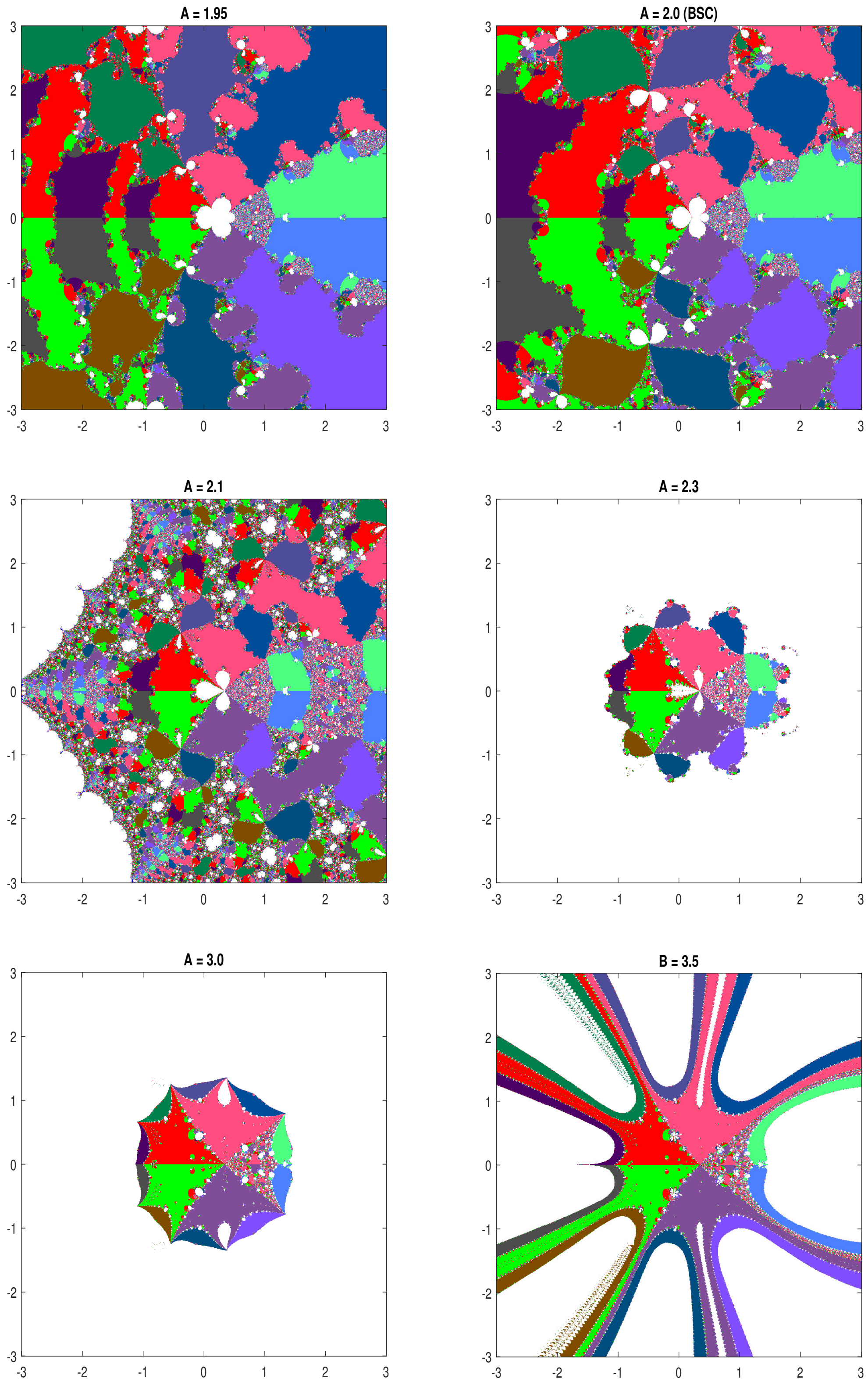

Fig. 7. Function (10). Varying the value of the parameter A from 1.95 to 3.5, with $A = 2$ corresponding to the BSC method.

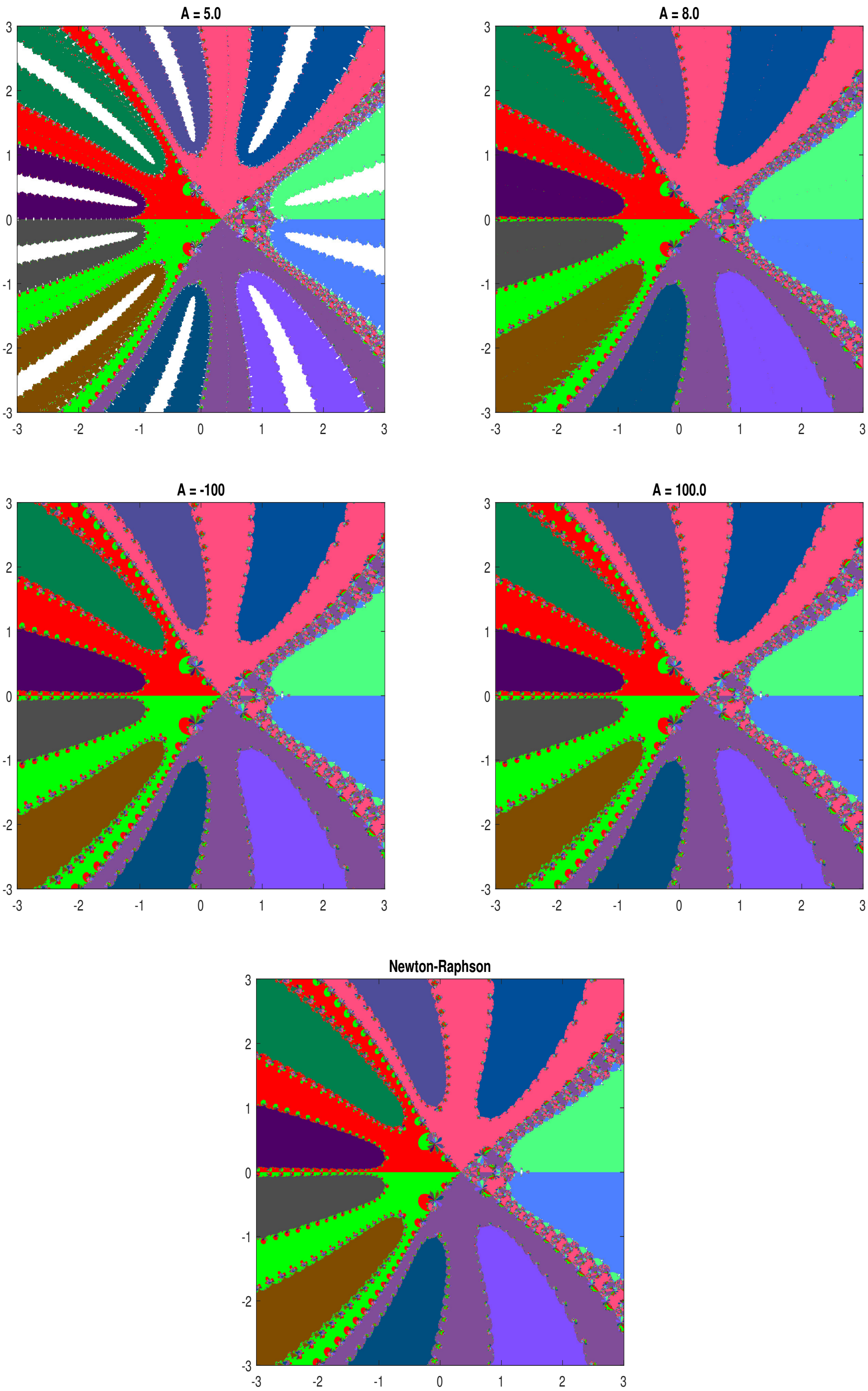

Fig. 8. Function (10). Varying the value of the parameter A from 5.0 to 100. Comparing basins of attraction with high absolute value of the parameter A to basins of attraction of the Newton-Raphson method.

When the parameter A of the family of iterative methods (1) has the value zero, the Euler-Chebyshev technique is reached. In this instance, the simplest mathematical expression for all family members is obtained. By assigning a negative value to A, the possibility for the basins of attraction to increase in complexity is modest. In fact, as A takes on more negative values, the basins of attraction become slightly more simple, with less intricate boundaries. These results are confirmed for both functions (9) and (10) used in this study (Figures 1 and 5). In addition, for function (9), the Chebyshev method's small regions of non-convergence become smaller. These regions of non-convergence are virtually absent in the basins of attraction of function (10).

When the values of A are positive and approach the value 1 that corresponds to Halley's method, a slight improvement is observed in the delineation and boundaries of the basins. Nonetheless, for function (10), minor zones of non-convergence emerge for values of A very close to 1, including $A = 1$ (Figures 5 and 6). This does not happen for function (9) (Figures 1 and 2).

Increasing further the values of A and before achieving the value of $A = 2$ corresponding to the BSC approach, small regions of non-convergence occur when the value of A is increased closer and closer to 2. In addition, the approximation of A to the value corresponding to the BSC approach results in the formation of smaller basins of attraction spread throughout the complex plane (Figures 2, 3, 6 and 7). Once the value $A = 2$ is attained, non-convergent regions become more substantial, notably for function (9) (Figure 3). These regions appear clearly at values slightly above $A = 2$ for function (10) (Figure 7).

Both functions are unstable for values of A above and near $A = 2$. These values of A are the ones where the non-convergent regions are the widest. The basins of attraction begin to stabilize for higher values of A, and for values around and above 8, their basins begin to resemble those of the Newton-Raphson quadratic approach (Figures 4 and 8). Figures 4 and 8 show, for large values of A in both the negative and positive directions, the remarkable similarity with the basins of attraction of the Newton-Raphson method.

It is worthy of emphasis that for values of A in the neighborhood of the BSC method ($A = 2$), the basins of attraction have a very different structure from the basins of the other family members, with smaller basins present in higher numbers and more dispersed through the complex plane. In addition, non-convergent regions emerge. For large negative and positive values of A (approximately for values of A below $A = -8$ or above $A = 8$), the basins of attraction exhibit a behavior similar to that of the quadratic Newton-Raphson method.

Values of A close to $A = 1$, and somewhat large values of A, positive or negative, appear to be best. At least, one main reason appears to be behind the beneficial behavior of the family (1) of iterative methods studied with respect to the basins of attraction, when the value of A increases in both the positive and negative directions. Since parameter A comes in the denominator of the last part of equation (1), its value and weight tend to decrease as parameter A increases, pushing equation (1) closer to that of the Newton-Raphson approach. However, the asymptotic error constant (8) tends to increase and may slows down the rate of convergence.

4. Conclusion

In terms of complexity, size, dispersion, and boundaries of its basins of attraction, it can be concluded that the best iterative approach of the investigated family is likely to be observed when A is close to 1, i.e., Halley's method. For parameter values close to 2, i.e., the BSC method, the basins of attraction of the family of iterative methods exhibit a distinct structure compared to those of the remaining family members. Specifically, there is a greater number of smaller basins that are more widely distributed throughout the complex plane. Furthermore, regions of non-convergence show up. In addition, when the absolute value of the family parameter is high, better defined and simpler basins of attraction are obtained, very similar to those produced by the quadratic Newton-Raphson method.

Halley's technique appears to be among the best, which confirm earlier conclusions made by other studies. However, earlier studies on this family of iterative methods only compared BSC, Euler-Chebyshev's, and Halley's techniques. Values of the parameter A different from 0, 1 or 2 were not included. When selecting a technique from this family of iterative methods, it is advisable to recommend Halley's method.

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