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An investment management with taxes and transaction costs under Vasicek and Constant Elasticity of Variance (CEV) models

Research Article

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Abstract: This paper investigates the investment management problem for a CRRA investor who faces taxes, transaction costs and stochastic environments for power utility functions. Note that the transaction costs and tax rates are charged on risky investments. The study considers an agent who invests in the financial market with one risk-free security (e.g. a money market account or bond) and one risky security (e.g. a stock or stock index). Let the stochastic interest rate follow the Vasicek Model and the risky asset price evolve as the Constant Elasticity of Variance (CEV) model. The objective of the agent is to choose an optimal investment strategy that maximizes the expected discounted utilities derived from terminal wealth over an uncertain lifetime horizon. This stochastic control problem is looked at directly by developing the Hamilton-Jacobi-Bellman Partial Differential Equation (HJB PDE) for CRRA investors via the Dynamic Programming Principle (DPP). We use power utility for our analysis to obtain the value function and optimal policy. Finally, the effects of market parameters on the policy are discussed.

Keywords: Investment management • Taxes and Transaction costs • Vasicek Model • Constant Elasticity of Variance (CEV) price model • Value function • Optimal policy

1. Introduction

This paper is inspired by the scholarly Merton's work [1, 2]. Investment management problems for an agent who faces taxes, transaction costs and a stochastic environment is considered specifically. This study considers stochastic interest rate following the Vasicek Model and the stock price following the Constant Elasticity of Variance (CEV) price model. The introduction of taxes, transaction costs and stochastic environment to any financial model is key as it makes the model realistic, more practical and relevant to agents of all sizes. The Vasicek model assumes that interest rates follow a mean-reverting process, where deviations from the long-term mean are gradually corrected over time. It's one of the popular models for short-term interest rate modeling and is mainly used in various applications in finance, including interest rate derivatives pricing, risk management, and portfolio optimization. The constant elasticity of variance (CEV) model is a natural extension of the GBM. CEV model has advantages over the GBM in that, there is a correlation between the volatility rate and risky asset price. Moreover, CEV can explain volatility smile. The CEV model was introduced by Cox and Ross [3] as an alternative diffusion process for European option pricing. This study investigates investment problems and optimally allocates the wealth between one risk-free asset account and one risky asset account. Such stochastic problems are a major concern to individual and institutional investors

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who seek to allocate wealth among various assets over uncertain time horizons. So far, different researchers have explicitly solved stochastic optimal control problems via methods, such as the dynamic programming principle, the maximum principle (see [4]), and the convex duality martingale method. Merton [1, 2] pioneered in the study of continuous-time stochastic models of financial markets in the absence of taxes and transaction costs. Their results showed that an optimal investment problem can be formulated as a Hamilton-Jacobi-Bellman HJB PDE that allows an explicit solution for a Constant Relative Risk Aversion (CRRA) investor and that the optimal policy (without taxes and transaction costs) is to keep a constant fraction of total wealth in a risk-free asset account and risky asset account but consumption should depend on wealth earned. the goal of this paper is to maximize the discounted expected utility of terminal wealth over an uncertain lifetime horizon when an investor faces taxes and transaction costs. By applying the Dynamic Programming Principle (DPP), we obtain the HJB PDE for the value function. The introduction of taxes and transaction costs to the optimization problem results in a complicated HJB PDE.

2. Links to the Literature

The problem of optimal investment has attracted several extensions. Magill and Constantinides [5] introduced proportional transaction costs to Merton's investment problem. They concluded that there is a no-trading region in the presence of transaction costs. Shreve and Soner [6] applied the viscosity solution approach and showed that the optimal investment and consumption policies are characterized by the transaction regions (selling, buying and no-trade). Janecek and Shreve [7] applied the viscosity solution method to obtain explicit results in solving the infinite horizon investment and consumption problem by expanding the value function into a power series in powers of $\lambda^{\frac{1}{3}}$. Cvitanic and Karatzas [8] used a martingale method to prove the existence of an optimal solution to the portfolio optimization problem for an agent facing transaction costs. Goodman and Ostrov [9] considered optimal trading problems with small proportional transaction costs and showed how the first term in the asymptotical expansion of the value function led to a free boundary problem that minimizes a cost function. Dai et al. [10] considered optimal investment and the consumption problem of a constant relative risk aversion (CRRA) investor who faces proportional transaction costs and a finite horizon. They applied an analytical approach and concluded that the value function satisfies a parabolic variation inequality with gradient constraints characterized by two free boundaries (the optimal buying and selling strategies). Bichuch [11] is considered an agent who invests in a stock and a bond asset account. Explicit results were obtained in solving the infinite horizon investment and consumption problem, in the presence of proportional transaction costs by expanding the value function into a power series in powers of $\lambda^{\frac{1}{3}}$. Davis and Norman [12] considered the optimal investment and consumption problem for an agent who faces transaction costs. The agent has two assets a money account and a risky asset account (stock) whose price follows geometric Brownian motion. They formulated the problem as a free boundary problem where the free boundaries correspond to the optimal buying and selling policies. Øksendal and Sulem [13] investigated optimal consumption and portfolio with both mixed costs fixed and proportional transaction costs (mixed costs) by applying the viscosity solution method and proved that the value function is a viscosity solution of the associated Hamilton-Jacobi-Bellman inequality. Liu and Loewenstein [14] investigated optimal portfolio selection with transaction costs and finite horizons and showed that in the presence of proportional costs, there exists a no-transaction and a transaction region (i.e. free boundaries). We extend Merton's work [1, 2] in a unique way by studying the stochastic control problem for an agent who faces taxes, transaction costs and a stochastic environment. Most research work done in line with our study approached the stochastic control problem differently. They determined optimal policies by defining transaction regions (selling, buying and no-trade) depending on the position of the investor. We have demonstrated that it is possible to find the value function and optimal policies using the Dynamic Programming Principle (DPP) and change of variable method. We show that explicit solutions via the HJB PDE exist. We are not aware of any previous work in our direction.

The outline of this paper is as follows. Section 1 introduction. In section 2, we state links to the literature. Section 3 describes the financial market model. In section 4, we determine the wealth model using models identified in section 3. Section 5 describes the optimization criterion. Section 6 we derive the Hamilton-Jacobi-Bellman HJB PDE and obtain the value function and optimal policies. In section 7, we present simulations and discuss the effects of market parameters on the policies. Section 8 concludes and suggests possible future research work

3. The Financial Market Model

Let $(\Omega, \mathbb{F}, \mathscr{F}, \mathbb{P})$ be a filtered complete probability space with filtration $(\mathscr{F}_t)_{0\le t\le T}$ satisfying the usual conditions such as $(\mathscr{F}_t)_{0\le t\le T}$ being right continuous complete filtration and $\mathbb{P}$-complete. Consider a stochastic control problem of a single investor with a portfolio consisting of one risky-free security (e.g. a money market account or bond) $\mathscr{B}(t)$ and one risky security (e.g. a stock or stock index) $\mathscr{S}(t)$.

Let the price dynamics of the risk-free security $\mathscr{B}(t)$ evolve as follows:

$$d\mathscr{B}(t) = r\mathscr{B}(t)dt, \ \ \mathscr{B}(0) = 1, \tag{1}$$

with stochastic interest rate $r(t)$ following a the Vasicek Model given by:

$$dr(t) = \alpha_0 dt + k_0 dW_0(t), \tag{2}$$

where $r(t)$ is the short-term interest rate at time, α_0 is the long-term mean of the interest rate, k_0 is the instantaneous volatility of the interest rate. $dW_0(t)$ is a stochastic term representing random shocks to the interest rate called one-dimensional Brownian motion.

The risky security $S(t)$ at time t evolves as the CEV process as follows:

$$dS(t) = S(t)[\alpha_1 dt + k_1 S^{\gamma}(t) dW_1(t)], \;\; S(0) = s. \tag{3}$$

Here, $\alpha > r(t)$ and $k_1 > 0$ are constant parameters such that α_1 is the expected instantaneous return rate of the stock. γ is the elasticity parameter and satisfies the general condition $\gamma \leq 0$. $k_1 S^{\gamma}(t)$ is defined as the instantaneous volatility of the stock. $W_1(t)$ is a one-dimensional Brownian motion. The CEV model reduces to a geometric Brownian motion (GBM) when $\gamma = 0$.

4. The Wealth Model

Let θ be the transaction cost rate (fees and duty) and β be the tax rate. Consider an agent whose transaction costs and tax rates are charged on risky investments. Let $\pi(t)$ and $\mathscr{X}(t) - \pi(t)$ represent the amounts of money invested in the risky asset account and the risk-free asset account at time t, respectively.

The net wealth in the presence of taxes and transaction costs evolves as follows:

$$d\mathscr{X}(t) = \Big[(\mathscr{X}(t) - \pi(t))\frac{d\mathscr{B}(t)}{\mathscr{B}(t)}\Big] + \pi(t)\frac{dS(t)}{S(t)} - \theta\pi dt - \beta\pi dt \tag{4}$$

Substituting (1) and (3) into (4) and simplifying gives:

$$d\mathscr{X}(t) = \big[r\mathscr{X}(t) + \pi(t)(\alpha_1 - \theta - \beta - r)\big]dt + \pi(t)k_1 S^{\gamma}(t) dW_1(t), \;\; \mathscr{X}(0) > 0 \tag{5}$$

5. The Optimization Criterion

The trading policy is the choice denoted by $A = (\pi(t), t \in [0, T])$ and it is said to be admissible in A for any $t \in [0, T]$, if the following conditions are satisfied.

(i) $\pi(t)$ is progressively $\mathscr{F}_t - measurable$.

(ii) $\pi(t) \geq 0$.

Note that the set of all admissible strategies is denoted by A. The agent determines the pair $\pi(t) \in A$ that maximizes the discounted expected utility of terminal wealth. That is, a policy which attains the supremum such that

$$V(t, x, r, s) = \sup_{\pi(t)\in A} \mathbb{E}\Big[\int_0^T e^{-\delta t} U(\mathscr{X}(t)) dt \mid S(t) = s, \mathscr{X}(t) = x\Big] \tag{6}$$

subject to (5), (2) and (3).

6. Hamilton-Jacobi-Bellman Partial Differential Equation (HJB PDE) and optimal policy

The concept of dynamic programming technique breaks down the optimization problem into smaller (infinitesimal) sub-problems and uses the results of the sub-problems to get the overall optimum. Note that the infinitesimal generator of an Itô diffusion process turns out to be a second-order differential operator in the case of diffusions driven by the Brownian process. The key component of the theory of stochastic optimal control is the infinitesimal generator of an Itô diffusion. This can be thought of as an extension of the derivative from the deterministic calculus. Here the limit definition has a similar form with an extension to include the stochastic nature of an Itô diffusion. In many applications of the theory of stochastic optimal control, it is key to relate partial differential operator **A** to an Itô diffusion process $\mathscr{X}(t)$. That is, **A** explains the movements of the diffusion process $\mathscr{X}(t)$ in infinitesimal time intervals. In summary, a generator is used to derive ODEs or PDEs relevant to the diffusion process. That is, generators are useful as they link diffusion processes and differential equations. Suppose $\mathscr{X}(t)$ is a degenerate stochastic process given by an ODE, then the ODE for $V(t, \mathscr{X}(t))$ is the generator.

In general, let $\mathscr{X}(t)$ be an Itô diffusion process in $\mathbb{R}^n$ satisfying:

$$d\mathscr{X}(t) = \alpha(t, \mathscr{X}(t))dt + k_2(t, \mathscr{X}(t))dW(t), \;\; \mathscr{X}(t)(0) > 0. \tag{7}$$

. Applying Itô formula for Itô diffusion process to function $V(t, \mathscr{X}(t))$, we obtain the generator as

$$\mathbf{L}^{\star} V(t,x) = \alpha V_x + \frac{1}{2} k_2^2 V_{xx}, \tag{8}$$

resulting in the partial differential equation

$$V_t + \sup_{u(t)\in A} \Big[\alpha V_x + \frac{1}{2} k_2^2 V_{xx}\Big]. \tag{9}$$

In this study, the main goal is to find the optimal control process $\pi^{\star}$ (which gives the optimum of the objective functional) and the value function $V(t,x,r,s)$ (the optimal objective functional). Admissible controls denoted by A are a family of optimal controls from which we select an optimal control. To succeed we will consider different initial times and states along a given trajectory of the controlled diffusion process. Note that π is $\mathbb{F}_t$- measurable implying the control in question has full information about the system up to time t. Also note that under dynamic programming procedure, we vary the initial time and states of the system and select the best control from a set of admissible controls.

The value function is defined as follows:

$$V(t,x,r,s) = \sup_{\pi(t)\in A} \mathbb{E}\Big[\int_0^T e^{-\delta t} U(\mathscr{X}(t)) dt \mid S(t) = s, \mathscr{X}(t)(t) = x\Big] \tag{10}$$

with boundary condition given by $V(T,x,r,s) = e^{-\delta T} U(\mathscr{X}(T))$.

subject to the budget constraint

$$d\mathscr{X}(t) = \big[r\mathscr{X}(t) + \pi(t)(\alpha_1 - \theta - \beta - r)\big]dt + \pi(t) k_1 S^{\gamma}(t) dW_1(t), \ \ \mathscr{X}(0) > 0 \tag{11}$$

$$dr(t) = \alpha_0 dt + k_0 dW_0(t), \tag{12}$$

$$dS(t) = S(t)[\alpha_1 dt + k_1 S^{\gamma}(t) dW_1(t)]. \tag{13}$$

By applying the dynamic programming principle, we obtain the HJB equation given as follows

$$\begin{aligned} V_t + \sup_{\pi(t)\in A} \Big\{ &\Big[rx + \pi(\alpha_1 - \theta - \beta - r)\Big] V_x + \frac{1}{2}\pi^2 k_1^2 s^{2\gamma} V_{xx} + \alpha_0 V_r + \frac{1}{2} k_0^2 V_{rr} \\ &+ \alpha_1 s V_s + \frac{1}{2} k_1^2 s^{2\gamma+2} V_{ss} + \pi k_1^2 s^{2\gamma+1} V_{xs}\Big\} = 0. \end{aligned} \tag{14}$$

where V_t , V_x, V_{xx}, V_r, V_{rr}, V_s, V_{ss} and V_{xs} denote partial derivatives of first-order and second-order for the variables t, x, r and s.

Suppose the HJB equation (14) has a classical solution V. By applying the first-order maximizing conditions for the optimal investment strategy to the problem above, we get the following optimizer

$$V_{\pi} = \big(\alpha_1 - \theta - \beta - r\big) V_x + \pi k_1^2 s^{2\gamma} V_{xx} + k_1^2 s^{2\gamma+1} V_{xs} \tag{15}$$

Set $V_{\pi} = 0$, implying

$$\pi^* = -\frac{(\alpha_1 - \theta - \beta - r) V_x}{k_1^2 s^{2\gamma} V_{xx}} - \frac{k_1^2 s^{2\gamma+1} V_{xs}}{k_1^2 s^{2\gamma} V_{xx}} \tag{16}$$

Substituting the optimizer (16) into the HJB equation (14) we obtain the following partial differential equation

$$\begin{aligned} &V_t + \Big[rx + \Big(-\frac{(\alpha_1 - \theta - \beta - r) V_x}{k_1^2 s^{2\gamma} V_{xx}} - \frac{k_1^2 s^{2\gamma+1} V_{xs}}{k_1^2 s^{2\gamma} V_{xx}}\Big)\big(\alpha_1 - r - \theta - \beta\big)\Big] V_x \\ &+ \frac{1}{2}\Big[-\frac{(\alpha_1 - \theta - \beta - r) V_x}{k_1^2 s^{2\gamma} V_{xx}} - \frac{k_1^2 s^{2\gamma+1} V_{xs}}{k_1^2 s^{2\gamma} V_{xx}}\Big]^2 k_1^2 s^{2\gamma} V_{xx} + \alpha_0 V_r + \frac{1}{2} k_0^2 V_{rr} \\ &+ \alpha_1 s V_s + \frac{1}{2} k_1^2 s^{2\gamma+2} V_{ss} + \Big[-\frac{(\alpha_1 - \theta - \beta - r) V_x}{k_1^2 s^{2\gamma} V_{xx}} - \frac{k_1^2 s^{2\gamma+1} V_{xs}}{k_1^2 s^{2\gamma} V_{xx}}\Big] k_1^2 s^{2\gamma+1} V_{xs} \\ &= 0. \end{aligned} \tag{17}$$

Expanding and simplifying (17) gives the second-order nonlinear PDE as follows:

$$\begin{aligned} &V_t + xr V_x - \frac{(\alpha_1 - r - \theta - \beta)^2 V_x^2}{2 k_1^2 s^{2\gamma} V_{xx}} - \frac{k_1^2 s^{2\gamma+2} V_{xs}^2}{2 V_{xx}} + \alpha_0 V_r \\ &+ \frac{1}{2} k_0 V_{rr} + \alpha_1 s V_s + \frac{1}{2} k_1^2 s^{2\gamma+2} V_{ss} - \frac{[(\alpha_1 - r - \theta - \beta)s] V_x V_{xs}}{V_{xx}} \\ &= 0, \end{aligned} \tag{18}$$

Next, we choose the power utility function for our analysis and apply the variable change method to solve for the value function and optimal policy in (18).

6.1. Power utility function

By choosing a power utility function and making appropriate transformations, we solve the nonlinear PDE equation (18) for V and replace it in (16) to obtain an optimal policy. Suppose the solution to (18) is of the form

$$V(t,r,s,x) = e^{-\lambda t}\frac{x^{\delta}}{\delta}g(t,r,s), \;\; g(T,r,s)=1. \tag{19}$$

Our goal is to determine the function $g(t,r,s)$. Note that the partial derivatives for $V(t,r,s,x)$ are as follows:

$$\begin{aligned} &V_t = -\lambda e^{-\lambda t}\frac{x^{\delta}}{\delta}g + e^{-\lambda t}\frac{x^{\delta}}{\delta}g_t, \;\; V_x = e^{-\lambda t}x^{\delta-1}g, \;\; V_{xx} = e^{-\lambda t}(\delta-1)x^{\delta-2}g,\\ &V_r = e^{-\lambda t}\frac{x^{\delta}}{\delta}g_r, \;\; V_{rr} = e^{-\lambda t}\frac{x^{\delta}}{\delta}g_{rr}, \;\; V_s = e^{-\lambda t}\frac{x^{\delta}}{\delta}g_s, \;\; V_{ss} = e^{-\lambda t}\frac{x^{\delta}}{\delta}g_{ss},\\ &V_{xs} = e^{-\lambda t}x^{\delta-1}g_s. \end{aligned} \tag{20}$$

Substituting (20) into (18) leads to another second order PDE for g of the form

$$\begin{aligned} &g_t + \Big[-\lambda + r\delta - \frac{(\alpha_1 - r-\theta-\beta)^2\delta}{2k_1^2 s^{2\gamma}\delta - 1}\Big]g - \frac{k_1^2 s^{2\gamma+2}\delta g_s^2}{2(\delta-1)g} + \alpha_0 g_r + \frac{1}{2}k_0 g_{rr}\\ &+\alpha_1 s g_s + \frac{1}{2}k_1^2 s^{2\gamma+2}g_{ss} - \Big[\frac{\delta s(\alpha_1 - r-\theta-\beta)}{\delta-1}\Big]g_s = 0 \end{aligned} \tag{21}$$

Note that equation (21) is still a non-linear second-order PDE. Suppose we again transform (21) to be of the form

$$g(t,r,s) = f(t,r,s)^{1-\delta}, \;\; f(T,r,s)=1. \tag{22}$$

Our next goal is to determine the function $f(t,r,s)$ which will subsequently be used to determine optimal policy. The partial derivatives for 22 are as follows:

$$\begin{aligned} &g_t = (1-\delta)f^{-\delta}f_t, \;\; g_r = (1-\delta)f^{-\delta}f_r, \;\; g_{rr} = (1-\delta)(-\delta)f^{-\delta-1}f_r^2 + (1-\delta)f^{-\delta}f_{rr}\\ &g_s = (1-\delta)f^{-\delta}f_s, \;\; g_{ss} = (1-\delta)(-\delta)f^{-\delta-1}f_s^2 + (1-\delta)f^{-\delta}f_{ss} \end{aligned} \tag{23}$$

Substituting (23) into (21), we obtain

$$\begin{aligned} &f_t + \Big[\frac{r\delta-\lambda}{1-\delta} + \frac{(\alpha_1 - r-\theta-\beta)^2\delta}{2(\delta-1)^2 k_1^2 s^{2\gamma}}\Big]f + \alpha_0 f_r + \alpha_1 s f_s + \frac{1}{2}k_0 f_{rr}\\ &-\frac{1}{2}k_0\delta\frac{f_r^2}{f} + \frac{1}{2}k_1^2 s^{2\gamma+2}f_{ss} + \Big[\frac{(\alpha_1 - r-\theta-\beta)s\delta}{\delta-1}\Big]f_s = 0 \end{aligned} \tag{24}$$

At this stage, we cannot solve the PDE. Inspired by the approach of Gao [16]. Let

$$f(t,r,s) = h(t,r,y), \;\; y = s^{-2\gamma}, \;\; h(T,y)=1. \tag{25}$$

Note that the partial derivatives are as follows:

$$\begin{aligned} &f_t = h_t, \;\; f_r = h_r, \;\; f_{rr} = h_{rr}, \;\; f_s = (-2\gamma)s^{-2\gamma-1}h_y,\\ &f_{ss} = (4\gamma^2)s^{-4\gamma-2}h_{yy} + 2\gamma(2\gamma+1)s^{-2\gamma-2}h_y. \end{aligned} \tag{26}$$

Substituting (26) into (24) gives

$$\begin{aligned} &h_t + \Big[\frac{r\delta-\lambda}{1-\delta} + \frac{(\alpha_1 - r-\theta-\beta)^2\delta y}{2k_1^2(\delta-1)^2}\Big]h + \alpha_0 h_r - 2\alpha_1\gamma y h_y + \frac{1}{2}k_0 h_{rr}\\ &-\frac{1}{2}k_0\delta\frac{h_r^2}{h} + 2k_1^2\gamma^2 y h_{yy} + (2\gamma+1)\gamma k_1^2 h_y - \Big[\frac{(\alpha_1 - r-\theta-\beta)2\gamma\delta}{\delta-1}\Big]y h_y = 0 \end{aligned} \tag{27}$$

Note that PDE (18) has been converted to PDE (27) with well defined solutions. Next, we assume the solution to (27) is given in the form

$$h(t,r,y) = \exp\{\Phi(t)y + \Theta(t)r + \Psi(t)\} \tag{28}$$

with the boundary condition $\Phi(T) = \Theta(T) = \Psi(T) = 0$.

From equation (28), we can obtain the following partial derivatives

$$h_t = (\Phi'(t)y + \Theta'(t)r + \Psi'(t))h(t,r,y), \;\; h_r = \Theta(t)h(t,r,y), \;\; h_{rr} = \Theta^2(t)h(t,r,y)$$
$$h_y = \Phi(t)h(t,r,y), \;\; h_{yy} = \Phi^2(t)h(t,r,y). \tag{29}$$

Substituting (29) into (27), we obtain

$$\begin{aligned}&(\Phi'(t)y + \Theta'(t)r + \Psi'(t))h(t,r,y) + \Big[\frac{r\delta - \lambda}{1-\delta} + \frac{(\alpha_1 - r - \theta - \beta)^2\delta y}{2k_1^2(\delta-1)^2}\Big]h(t,r,y)\\ &+ \alpha_0\Theta(t)h(t,r,y) - 2\alpha_1\gamma y\Phi(t)h(t,r,y) + \frac{1}{2}k_0\Theta^2(t)h(t,r,y)\\ &- \frac{1}{2}k_0\delta\Theta^2(t)h(t,r,y) + 2k_1^2\gamma^2 y\Phi^2(t)h(t,r,y) + (2\gamma+1)\gamma k_1^2\Phi(t)h(t,r,y)\\ &- \Big[\frac{2\gamma\delta(\alpha_1 - r - \theta - \beta)}{\delta - 1}\Big]y\Phi(t)h(t,r,y) = 0\end{aligned} \tag{30}$$

Canceling the term $h(t,r,s)$ on both sides of (30) gives

$$\begin{aligned}&(\Phi'(t)y + \Theta'(t)r + \Psi'(t)) + \frac{r\delta - \lambda}{1-\delta}\\ &+ \Big[\frac{(\alpha_1 - r - \theta - \beta)^2\delta y}{2k_1^2(\delta-1)^2}\Big]\\ &+ \alpha_0\Theta(t) - 2\alpha_1\gamma y\Phi(t) + \frac{1}{2}k_0\Theta^2(t) - \frac{1}{2}k_0\delta\Theta^2(t) + 2k_1^2\gamma^2 y\Phi^2(t) + (2\gamma+1)\gamma k_1^2\Phi(t)\\ &- \Big[\frac{2\gamma\delta(\alpha_1 - r - \theta - \beta)}{\delta - 1}\Big]y\Phi(t) = 0\end{aligned} \tag{31}$$

Let $m = \alpha_1 - r - \theta - \beta$ then (31) can be written as:

$$\begin{aligned}&(\Phi'(t)y + \Theta'(t)r + \Psi'(t)) + \frac{r\delta - \lambda}{1-\delta}\\ &+ \Big[\frac{m^2\delta y}{2k_1^2(\delta-1)^2}\Big]\\ &+ \alpha_0\Theta(t) - 2\alpha_1\gamma y\Phi(t) + \frac{1}{2}k_0\Theta^2(t) - \frac{1}{2}k_0\delta\Theta^2(t) + 2k_1^2\gamma^2 y\Phi^2(t) + (2\gamma+1)\gamma k_1^2\Phi(t)\\ &- \Big[\frac{2\gamma\delta m}{\delta - 1}\Big]y\Phi(t) = 0\end{aligned} \tag{32}$$

Collecting like terms in y and r in (32) gives:

$$\begin{aligned}&y\Big[\Phi'(t) + 2k_1^2\gamma^2\Phi^2(t) + 2\gamma\Big(\alpha_1 + \frac{\delta m}{\delta-1}\Big)\Phi^2(t) + \frac{m^2\delta}{2k_1^2(\delta-1)^2}\Big]\\ &+ r\Big(\Theta'(t) + \frac{\delta}{1-\delta}\Big)\\ &+ \Big(\Psi'(t) - \frac{1}{2}k_0(1-\delta)\Theta^2(t) + \alpha_0\Theta(t) + (2\gamma+1)\gamma k_1^2\Phi(t) - \frac{\lambda}{1-\delta}\Big) = 0\end{aligned} \tag{33}$$

Eliminating y and r, we split equation (33) into three ODE's as follows:

$$\Phi'(t) + 2k_1^2\gamma^2\Phi^2(t) + 2\gamma\Big(\alpha_1 + \frac{\delta m}{\delta-1}\Big)\Phi^2(t) + \frac{m^2\delta}{2k_1^2(\delta-1)^2} = 0, \;\; \Phi(T) = 0 \tag{34}$$

$$\Theta'(t) + \frac{\delta}{1-\delta} = 0, \;\; \Theta(T) = 0 \tag{35}$$

$$\Psi'(t) - \frac{1}{2}k_0(1-\delta)\Theta^2(t) + \alpha_0\Theta(t) + (2\gamma+1)\gamma k_1^2\Phi(t) - \frac{\lambda}{1-\delta} = 0, \;\; \Psi(T) = 0 \tag{36}$$

The solution to the equation (34) can equally be obtained directly as follows:

$$\Phi(t) = \Big(-2k_1^2\gamma^2 - 2\gamma\Big(\alpha_1 + \frac{\delta m}{\delta-1}\Big)\Big)\int_t^T \Phi^2(s)ds - \frac{m^2\delta}{2k_1^2(\delta-1)^2}(T-t), \tag{37}$$

The solutions to the equations (35) and (36) can also be obtained directly as follows:

$$\Theta(t) = \frac{\delta}{(1-\delta)a}\left[1 - e^{-a(T-t)}\right] \tag{38}$$

$$\Psi(t) = \frac{1}{2}k_0(1-\delta)\int_t^T \Theta^2(s)ds - \alpha_0\int_t^T \Theta(s)ds - (2\gamma+1)\gamma k_1^2\int_t^T \Phi(s)ds + \frac{\lambda}{1-\delta}(T-t). \tag{39}$$

Note that we characterize the value function as a classical solution of a simpler semi-linear PDE equation. Therefore, following the transformations and computations of $\Phi(t)$, $\Theta(t)$ and $\Psi(t)$ above it is expected that the candidate value function V is represented as follows:

$$\begin{aligned} V(t,r,s,x) &= e^{-\lambda t}\frac{x^\delta}{\delta} f(t,r,y)^{1-\delta} \\ &= e^{-\lambda t}\frac{x^\delta}{\delta}\left[\phi^{\frac{1}{1-\delta}}\int_t^T \exp\{\Phi(u)s + \Theta(u)r + \Psi(u)\}du \right.\\ &\left. + (1-\phi)^{\frac{1}{1-\delta}}\exp\{\Phi(t)s + \Theta(t)r + \Psi(t)\}\right]^{1-\delta}. \end{aligned} \tag{40}$$

where $\Phi(t)$, $\Theta(t)$ and $\Psi(t)$ are given in (37), (38) and (39) respectively.

Note that according to (20) and (23) and the transformation (22),

$$\frac{V_x}{V_{xx}} = \frac{1}{\delta-1}X \tag{41}$$

and

$$\frac{V_{x\eta}}{V_{xx}} = \frac{Xf_s}{f} \tag{42}$$

Considering (16) and the transformation (19) we can finally determine the candidate optimal feedback portfolio functions as

$$\begin{aligned} \pi^* &= -\frac{(\alpha_1-\theta-\beta-r)}{k_1^2 s^{2\gamma}}\cdot\frac{V_x}{V_{xx}} - \frac{k_1^2 s^{2\gamma+1}}{k_1^2 s^{2\gamma}}\cdot\frac{V_{xs}}{V_{xx}} \\ &= -\frac{(\alpha_1-\theta-\beta-r)}{k_1^2 s^{2\gamma}}\cdot\frac{1}{1-\delta}X(t) + \frac{k_1^2 s^{2\gamma+1}}{k_1^2 s^{2\gamma}}\cdot\frac{f_s}{f}X(t). \end{aligned} \tag{43}$$

Where

$$\begin{aligned} f(t,r,s) &= \phi^{\frac{1}{1-\delta}}\int_t^T \exp\{\Phi(u)s + \Theta(u)r + \Psi(u)\}du \\ &+ (1-\phi)^{\frac{1}{1-\delta}}\exp\{\Phi(t)s + \Theta(t)r + \Psi(t)\}. \end{aligned} \tag{44}$$

with $\Phi(t)$, $\Theta(t)$ and $\Psi(t)$ as determined in (37), (38) and (39).

7. Effects of market parameters on the policy

In this section, we study the effect of some sensitive parameters on the optimal investment. we mainly consider the effect of wealth, interest rate, taxes and transaction costs on the optimal strategies. Throughout the sensitive parameter analysis, unless otherwise stated, the basic parameters are given by:

$k_1 = 0.6, \alpha_1 = 0.03, s(0) = 60, \gamma = 0.14, \delta = -1.$

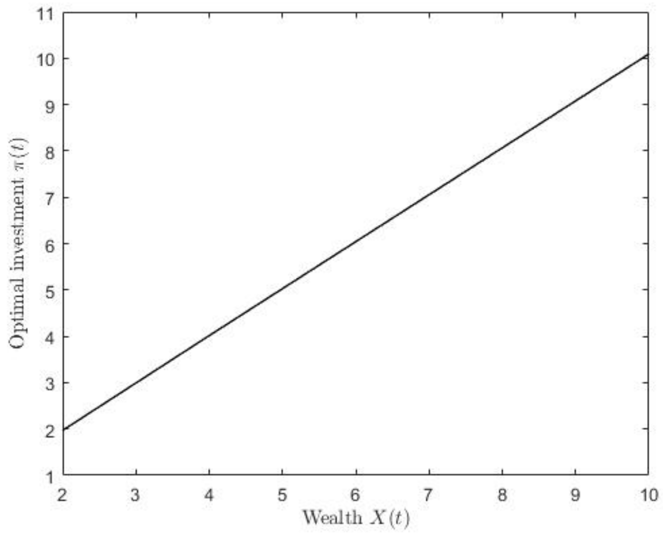

Fig. 1. The effects of wealth $\mathscr{X}(t)$ on investment investment $\pi^*(t)$

In Figure 1 shows the effect of the wealth $\mathscr{X}(t)$ on the optimal investment $\pi^*(t)$. The curve results analysis indicates that wealth affects optimal investment in a positive way. In summary, optimal investment $\pi^*(t)$ increases with the accumulation of the wealth $\mathscr{X}(t)$. This agrees with practical investments and our intuition.

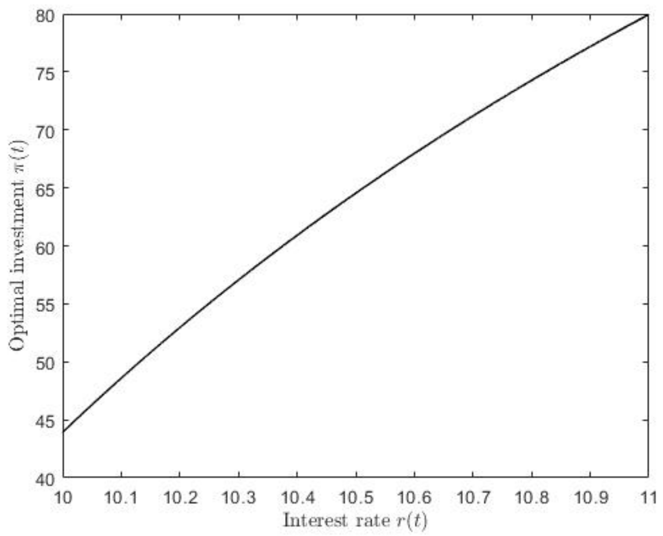

Fig. 2. The effects of the interest rate $r(t)$ on optimal investment $\pi^*(t)$

In Figure 2, the optimal investment strategy increases with an increase in the interest rate for a risk-free asset account. When the interest rate increases, the risky asset account is more attractive. Therefore, an agent will invest more in a risky asset account for more wealth. This agrees with practical investments and our intuition.

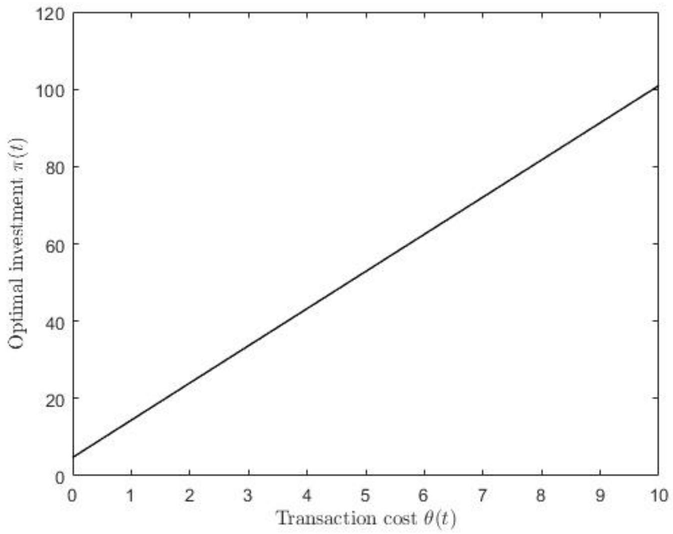

Fig. 3. The effects of the Transaction rate $\theta(t)$ on optimal investment $\pi^*(t)$

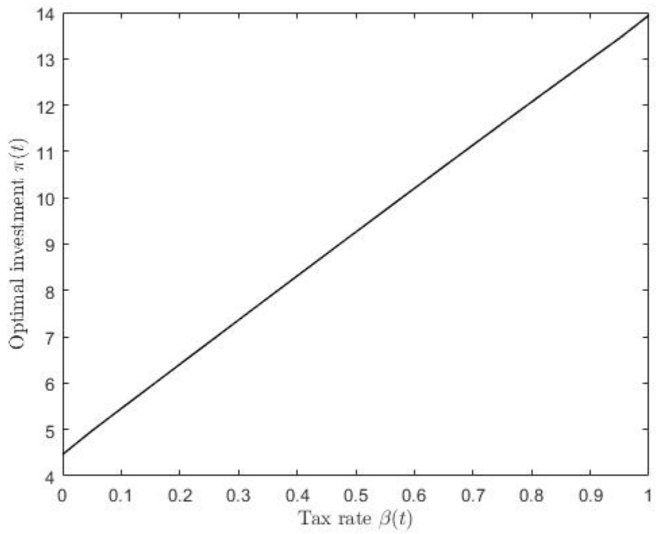

Fig. 4. The effects of the Tax rate $\beta(t)$ on optimal investment $\pi^*(t)$

In Figure 3 and 4, the optimal investment strategy increases with an increase in the Transaction rate and Tax rate. When the transaction rate and tax rate increase, the agent becomes investment aggressive and invests more in both risky and risk-free assets for more wealth.

8. Conclusion

An investment management problem for the Constant Relative Risk Aversion (CRRA) investor who faces taxes, transaction costs and stochastic environment is investigated. We considers stochastic interest rate following the Vasicek Model and the stock price following the Constant Elasticity of Variance (CEV) price model. The introduction of taxes, transaction costs and stochastic environment to any financial model is key as it makes the model realistic, more practical and relevant to agents of all sizes. By applying the Dynamic Programming Principle (DPP) and variable change technique, we determine the value function and optimal investment policy in the power utility case. Most research work done in line with this study approached the problem differently via determining optimal policies by defining transaction regions (selling, buying and no-trade) which depend on the position of the investor. We have demonstrated that it is possible to find the value function and optimal policy using the Dynamic Programming Principle (DPP) and change of variable method. Results show that optimal investment $\pi^*(t)$ increases with the accumulation of the wealth $\mathscr{X}(t)$. In addition, When the interest rate increases in the risk-free asset account, the risky asset account becomes more attractive. Therefore, an agent will invest more in a risky asset account for more wealth. Furthermore, when the transaction rate and tax rate increase, the agent becomes more investment aggressive and invests more in both risky and risk-free assets (Diversification) for more wealth. Our study can be extended in so many directions. For instance, we can introduce complex stock price models that lead to more sophisticated nonlinear second-order partial differential equations. The latest two publications of Mukonda et.al [17] and Juma Oroni et.al [18] this year have motivated us to publish this paper in this journal.

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