Int. J. Adv. Appl. Math. and Mech. 11(4) (2024) 32 – 50 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

New convergence definitions for sequences in quaternion valued generalized metric spaces

Research Article

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Received 05 July 2024; accepted (in revised version) 27 July 2024

Abstract: In this paper, we introduce the concepts of quasi-invariant convergence and quasi-invariant statistical convergence for sequences in quaternion-valued generalized metric spaces. We provide a characterization for a bounded sequence to be quasi-invariant convergent. Additionally, we define quasi-almost convergence, quasi-almost statistical convergence, quasi-strongly almost convergence, and quasi s-strongly almost convergence, examining their interrelationships. We also introduce quasi-lacunary invariant convergence, quasi-strongly lacunary invariant convergence, and quasi-strongly s-lacunary invariant convergence, along with quasi-lacunary invariant statistical convergence. The study explores the relationships among these new types of convergence and existing convergence concepts for sequences.

MSC: 40A05 • 40C05 • 40D25

Keywords: Quaternion valued g-metric space • Statistical convergence • Lacunary sequence • Quasi-invariant convergence • Invariant • Quasi convergence

1. Introduction and Background

Fast [8] explored the concept of statistical convergence, which had a profound impact across scientific disciplines. For further reference, see ([4, 15, 18, 20, 31, 32]). Connor [6] described the connections between statistical convergence and strong p-Cesàro convergence of sequences. Lorentz [11] introduced the concept of almost convergence. Maddox [19] and, independently, Freedman [12] developed the idea of strong almost convergence. Similar concepts are discussed in [10, 17]. Hajdukovic [9] introduced the notion of quasi-almost convergence in a normed space. Later, Nuray [25] explored the ideas of quasi-invariant convergence and quasi-invariant statistical convergence in a normed space.

Fridy and Orhan [13] expanded on these ideas by introducing lacunary statistical convergence using lacunary sequences. Many authors have studied on the concepts of invariant mean and invariant convergence [21–23, 26, 27, 30]. The space of lacunary strong σ-convergent sequences L_∞ was introduced by Savaş [28]. Savaş and Nuray [29] proposed σ-statistical convergence and its lacunary variant, establishing correlation theorems.

In mathematical analysis, the notion of distance is formalized using a distance function or metric, which can be generalized in diverse ways (refer to, for example, [16]). An important extension is the G-metric space, extensively

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explored by Mustafa and Sims [24]. Metrics in this context quantify the separation between three locations. Choi et al. [5] introduced the g-metric as an expansion of traditional distance functions. It generalizes the ordinary distance between two points and the G-metric among three points into the $n+1$ point distance, termed the g-metric of degree n. Abazari further advanced this metric framework with the introduction of statistical g-convergence [1].

Quaternions constitute a number system extending beyond complex numbers, originally formulated by Irish mathematician Hamilton in 1843 to describe mechanics in three dimensions. Quaternions are characterized by noncommutative multiplication, distinguishing them from other systems. Extensive discussions on quaternion analysis are found in [3] and related literature. This paper introduces a new type of convergence in quaternion-valued g-metric spaces, building upon the g-metric spaces by Choi et al. [5], quaternion-valued g-metric spaces by Jan and Jalal [14], and various forms of statistical convergence documented in existing literature. This proposal is motivated by practical applications in quaternions and fixed point theorems.

The study of convergence in quaternion-valued generalized metric spaces is a rapidly evolving field, offering new insights into the behavior of sequences in these complex spaces. This paper introduces several novel concepts, including quasi-invariant convergence, quasi-invariant statistical convergence, quasi-almost convergence, quasi-almost statistical convergence, quasi-strongly almost convergence, and quasi s-strongly almost convergence. Additionally, we define quasi-lacunary invariant convergence, quasi-strongly lacunary invariant convergence, and quasi-strongly s-lacunary invariant convergence, along with quasi-lacunary invariant statistical convergence. These new types of convergence are not merely theoretical constructs; they provide valuable insights into the stability and variability of sequences, which are crucial for advancements in fields ranging from quantum mechanics to signal processing. By examining the intricate relationships between these new concepts and existing convergence types, we offer a comprehensive framework that enhances the robustness of analytical methods in higher-dimensional spaces. This paper bridges gaps in current mathematical literature and sets the stage for future research, potentially leading to significant breakthroughs in various scientific and engineering disciplines.

Let us now establish several definitions and notations that will be utilized in this paper. First, we will introduce some fundamental notations for quaternionic spaces. The four-dimensional real algebra with unity is known as the space of quaternions, denoted by $\mathbf{Q}$. The null element of $\mathbf{Q}$ is denoted by $\mathbf{0_Q}$, and the multiplicative identity of $\mathbf{Q}$ is denoted by $\mathbf{1_Q}$. Within $\mathbf{Q}$, there exist three imaginary units represented by the symbols i, j, k. By definition, these units satisfy:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i \text{ and } ki = -ik = j.$$

For each $\rho = y_0 + y_1 i + y_2 j + y_3 k$; where y_0, y_1, y_2 and y_3 belong to $\mathbb{R}$, the elements $1, i, j, k$ are assumed to constitute a real vector basis of $\mathbf{Q}$. Given $\rho = y_0 + y_1 i + y_2 j + y_3 k \in \mathbf{Q}$, we recall that:
(i) $\bar{\rho} = y_0 - y_1 i - y_2 j - y_3 k$ is the conjugate quaternion of ρ,
(ii) $|\rho| = \sqrt{\rho\bar{\rho}} = \sqrt{y_0^2 + y_1^2 + y_2^2 + y_3^2} \in \mathbb{R}$
(iii) $\mathrm{Re}(\rho) = \frac{1}{2}(\rho + \bar{\rho}) = y_0 \in \mathbb{R}$
(iv) $\mathrm{Im}(\rho) = \frac{1}{2}(\rho - \bar{\rho}) = y_1 i + y_2 j + y_3 k$.

When $\rho = \mathrm{Re}(\rho)$, the element $\rho \in \mathbf{Q}$ is said to be real. It is obvious that ρ is real if and only if $\rho = \bar{\rho}$. If $\bar{\rho} = -\rho$ or $\rho = \mathrm{Im}(\rho)$, ρ is said to be imaginary.

The concept of a complex metric space was introduced by Azam et al. [3] as follows:

Definition 1.1 (Azam et al. [3]).
Let X be a nonempty set, and suppose the mapping $d_{\mathbf{C}} : X \times X \to \mathbf{C}$ satisfies the following conditions:
(i) $0 \prec d_{\mathbf{C}}(\tau_1, \tau_2)$, for all $\tau_1, \tau_2 \in X$ and $d_{\mathbf{C}}(\tau_1, \tau_2) = 0$ if and only if $\tau_1 = \tau_2$,
(ii) $d_{\mathbf{C}}(\tau_1, \tau_2) = d_{\mathbf{C}}(\tau_2, \tau_1)$ for all $\tau_1, \tau_2 \in X$,
(iii) $d_{\mathbf{C}}(\tau_1, \tau_2) \preceq d_{\mathbf{C}}(\tau_1, \tau_3) + d_{\mathbf{C}}(\tau_3, \tau_2)$ for all $\tau_1, \tau_2, \tau_3 \in X$.

Then $(X, d_{\mathbf{C}})$ is called a complex metric space.

Ahmed et al. [7] extended the above definition to Clifford analysis as follows:

Definition 1.2 (Ahmed et al. [7]).
Let X be a nonempty set and suppose that the mapping $d_{\mathbf{Q}} : X \times X \to \mathbf{Q}$ satisfies the following.
(i) $0 \prec d_{\mathbf{Q}}(\tau_1, \tau_2)$, for all $\tau_1, \tau_2 \in X$ and $d_{\mathbf{Q}}(\tau_1, \tau_2) = 0$ if and only if $\tau_1 = \tau_2$,
(ii) $d_{\mathbf{Q}}(\tau_1, \tau_2) = d_{\mathbf{Q}}(\tau_2, \tau_1)$ for all $\tau_1, \tau_2 \in X$,
(iii) $d_{\mathbf{Q}}(\tau_1, \tau_2) \preceq d_{\mathbf{Q}}(\tau_1, \tau_3) + d_{\mathbf{Q}}(\tau_3, \tau_2)$ for all $\tau_1, \tau_2, \tau_3 \in X$.

Then $(X, d_{\mathbf{Q}})$ is called a quaternion valued metric metric space.

Ahmed et al. [7] introduced a partial order $\preceq$ on $\mathbf{Q}$ (space of all quaternions).

Definition 1.3 (Ahmed et al. [7]).
Let $\rho_1, \rho_2 \in \mathbf{Q}$, then $\rho_1 \preceq \rho_2$ if and only if $\operatorname{Re}(\rho_1) \leq \operatorname{Re}(\rho_2)$ and $\operatorname{Im}_{\boldsymbol{s}}(\rho_1) \leq \operatorname{Im}_s(\rho_2), \rho_1, \rho_2 \in \mathbf{Q}, s = i, j, k$ where $\operatorname{Im} m_i = b, \operatorname{Im} m_j = c, \operatorname{Im} m_k = d$. It was observed that $\rho_1 \preceq \rho_2$, if one of the following conditions are satisfied:
(i) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2), \operatorname{Im}_{s_1}(\rho_1) = \operatorname{Im}_{s_1}(\rho_2)$ where $s_1 = j, k, \operatorname{Im}_i(\rho_1) < \operatorname{Im}_i(\rho_2)$;
(ii) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2), \operatorname{Im}_{s_2}(\rho_1) = \operatorname{Im}_{s_2}(\rho_2)$ where $s_2 = i, k, \operatorname{Im}_j(\rho_1) < \operatorname{Im}_j(\rho_2)$;
(iii) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2), \operatorname{Im}_{s_3}(\rho_1) = \operatorname{Im}_{s_3}(\rho_2)$ where $s_3 = i, j, \operatorname{Im}_k(\rho_1) < \operatorname{Im}_k(\rho_2)$;
(iv) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2), \operatorname{Im}_{s_1}(\rho_1) = \operatorname{Im}_{s_1}(\rho_2), \operatorname{Im} m_i(\rho_1) = \operatorname{Im}_i(\rho_2)$;
(v) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2), \operatorname{Im}_{s_2}(\rho_1) = \operatorname{Im}_{s_2}(\rho_2), \operatorname{Im} m_j(\rho_1) = \operatorname{Im} m_j(\rho_2)$;
(vii) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2), \operatorname{Im}_{s_3}(\rho_1) < \operatorname{Im}_{s_3}(\rho_2), \operatorname{Im} m_k(\rho_1) = \operatorname{Im} m_k(\rho_2)$;
(viii) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2), \operatorname{Im}_s(\rho_1) = \operatorname{Im}_s(\rho_2)$;
(ix) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2), \operatorname{Im}_{s_1}(\rho_1) = \operatorname{Im}_{s_1}(\rho_2), \operatorname{Im}_i(\rho_1) < \operatorname{Im}_i(\rho_2)$;
(x) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2), \operatorname{Im}_{s_2}(\rho_1) = \operatorname{Im}_{s_2}(\rho_2), \operatorname{Im} m_j(\rho_1) < \operatorname{Im} m_j(\rho_2)$;
(xi) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2), \operatorname{Im}_{s_l}(\rho_1) = \operatorname{Im}_{s_l}(\rho_2), \operatorname{Im}_k(\rho_1) < \operatorname{Im} m_k(\rho_2)$;
(xii) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2), \operatorname{Im}_{s_1}(\rho_1) < \operatorname{Im}_{s_1}(\rho_2), \operatorname{Im}_i(\rho_1) = \operatorname{Im} m_i(\rho_2)$;
(xiii) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2), \operatorname{Im}_{s_2}(\rho_1) < \operatorname{Im}_{s_2}(\rho_2), \operatorname{Im} m_j(\rho_1) = \operatorname{Im} m_j(\rho_2)$;
(xiv) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2), \operatorname{Im}_{s_3}(\rho_1) = \operatorname{Im}_{s_3}(\rho_2), \operatorname{Im}_k(\rho_1) = \operatorname{Im}(\rho_k)$;
(xv) $\operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_2), \operatorname{Im}_s(\rho_1) < \operatorname{Im}_s(\rho_2)$;
(xvi) $\operatorname{Re}(\rho_1) = \operatorname{Re}(\rho_2), \operatorname{Im}_s(\rho_1) = \operatorname{Im}(m_s)$.

Specifically, we denote $\rho_1 \precnsim \rho_2$ if $\rho_1 \neq \rho_2$ and one from *(i)* to *(xvi)* is satisfied and we will write $\rho_1 \prec \rho_2$ if only (xv) is satisfied.

Remark 1.1.
It should be noted that $\rho_1 \preceq \rho_2 \Rightarrow |\rho_1| \leq |\rho_2|$.

Motivated by Ahmed et al.'s [7] work, Adewale et al. [2] provided the following definition.

Definition 1.4 (Adewale et al. [2]).
Let X be a nonempty set, $\mathbf{Q}$ a set of quaternions and $G^{\mathbf{Q}} : X \times X \times X \to \mathbf{Q}$ be a function satisfying the following properties:
(i) $G^{\mathbf{Q}}(\tau_1, \tau_2, \tau_3) = \mathbf{0_Q}$ if and only if $\tau_1 = \tau_2 = \tau_3$,
(ii) $\mathbf{0_Q} \prec G^{\mathbf{Q}}(\tau_1, \tau_1, \tau_2), \forall \tau_1, \tau_2 \in X$ with $\tau_1 \neq \tau_2$,
(iii) $G^{\mathbf{Q}}(\tau_1, \tau_1, \tau_2) \preceq G^{\mathbf{Q}}(\tau_1, \tau_2, \tau_3), \forall \tau_1, \tau_2, \tau_3 \in X$ with $\tau_3 \neq \tau_2$,
(iv) $G^{\mathbf{Q}}(\tau_1, \tau_2, \tau_3) = G^{\mathbf{Q}}(\tau_2, \tau_3, \tau_1) = G^{\mathbf{Q}}(\tau_2, \tau_3, \tau_1) = \ldots$ (symmetry),
(v) There exists a real number $s \geq 1$ such that

$$G^{\mathbf{Q}}(\tau_1, \tau_2, \tau_3) \preceq s\left[G^{\mathbf{Q}}(\tau_1, a, a) + G^{\mathbf{Q}}(a, \tau_2, \tau_3)\right],$$

$\forall a, \tau_1, \tau_2, \tau_3 \in X$ (rectangle inequality).

Then, the function $G^{\mathbf{Q}}$ is called a quaternion valued G-metric and $(X, G^{\mathbf{Q}})$ is referred to as the quaternion valued $G^{\mathbf{Q}}$-metric space. A $G^{\mathbf{Q}}$-metric space is considered complete if every Cauchy sequence in it is $G^{\mathbf{Q}}$-convergent.

The following is an extension of G-metric space with degree l.

Definition 1.5 (Adewale et al. [2]).
Let X be a non-empty set. A function $g : X^{l+1} \to \mathbb{R}^+$ is called a g-metric space with order l on X if it satisfies the following conditions:
(i) $g(\tau_0, \tau_1, \tau_2, \ldots, \tau_l) = 0$ if and only if $\tau_0 = \tau_1 = \ldots = \tau_l$,
(ii) $g(\tau_0, \tau_1, \tau_2, \ldots, \tau_l) = g(\tau_{\sigma(0)}, \tau_{\sigma(1)}, \tau_{\sigma(2)}, \ldots, \tau_{\sigma(l)})$ for permutation σ on $\{0, 1, 2, \ldots l\}$,
(iii) $g(\tau_0, \tau_1, \tau_2, \ldots, \tau_l) \leq g(y_0, y_1, y_2, \ldots, y_l)$ for all $(\tau_0, \tau_1, \tau_2, \ldots, \tau_l), (y_0, y_1, y_2, \ldots, y_l) \in X^{l+1}$ with $\{\tau_i : i = 0, 1, \ldots, l\} \subseteq \{y_i : i = 0, 1, \ldots, l\}$,
(iv) For all $\tau_0, \tau_1, \ldots, \tau_s, y_0, y_1, \ldots, y_t, w \in X$ with $s + t + 1 = l$,

$$\begin{aligned} &g(\tau_0, \tau_1, \tau_2, \ldots, \tau_s, y_0, y_1, y_2, \ldots, y_t) \leq \\ &g(\tau_0, \tau_1, \tau_2, \ldots, \tau_s, w, w, \ldots, w) + g(y_0, y_1, y_2, \ldots, y_t, w, w, \ldots, w). \end{aligned}$$

The pair (X, g) is called g-metric space with degree l. For $l = 1, 2$ respectively, it is respectively equivalent to metric and G-metric space.

The statistical convergence of real-numbers sequencesis based on the concept of natural density of subsets of $\mathbb{N}$, the set of all positive integers, which is defined as follows: Let (X,d) be a metric space. A real number sequence $\left(\tau_j\right)$ is said to be statistically convergent to the number τ if for every $\varepsilon > 0$,

$$\lim_n n^{-1}\left|\left\{j \le n : d\left(\tau_j,\tau\right) \ge \varepsilon\right\}\right| = 0,$$

where the number of elements in the contained set is indicated by the vertical bars. indicate enclosed set.

The following definitions were given by R.Abazari.

Definition 1.6 (Abazari [1], Definition 2.3).
Let $p \in \mathbb{N}$ and $K \in \mathbb{N}^P$ and

$$K(n) = \left\{\left(i_1, i_2, \ldots, i_p\right) \le n (n \in \mathbb{N}) : \left(i_1, i_2, \ldots, i_p\right) \in K\right\},$$

then

$$\delta_{(p)}(K) = \lim_{n\to\infty} \frac{p!}{n^p}|K(n)|,$$

is called p-dimensional asymptotic (or natural density) of the set K.

Definition 1.7 ([1], Definition 2.4).
Let (x_n) be a sequence in a g-metric space $\left(Y, g\right)$.

(i) (x_n) is statistically convergent to x , provided for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{j!}{n^j}\left|\left\{i_1, i_2, ..., i_j \le n : g\left(x, x_{i_1}, x_{i_2}, ..., x_{i_j}\right) \ge \varepsilon\right\}\right| = 0,$$

and is indicated by $gS\text{-}\lim_{n\to\infty} x_n = x$.

(ii) (x_n) is called to be statistical g-Cauchy, provided for all $\varepsilon > 0$, there exists $i_\varepsilon \in \mathbb{N}$ so that

$$\lim_{n\to\infty} \frac{j!}{n^j}\left|\left\{i_1, i_2, ..., i_j \le n : g\left(x_{i_\varepsilon}, x_{i_1}, x_{i_2}, ..., x_{i_j}\right)\right\}\right| = 0.$$

The following definition was given by Jan and Jalal [14].

Definition 1.8 (Jan and Jalal [14]).
Let X be a non-empty set. A function $g_{\mathbf{Q}} : X^{p+1} \to \mathbf{Q}$ (where $\mathbf{Q}$ is the space of quaternions) is called quaternion valued g-metric space with order p on X if it satisfies the following conditions:
(i) $g_{\mathbf{Q}}\left(\tau_0, \tau_1, \tau_2, \ldots, \tau_p\right) = 0$ if and only if $\tau_0 = \tau_1 = \ldots = \tau_p$,
(ii) $g_{\mathbf{Q}}\left(\tau_0, \tau_1, \tau_2, \ldots, \tau_p\right) = g_{\mathbf{Q}}\left(\tau_{\sigma(0)}, \tau_{\sigma(1)}, \tau_{\sigma(2)}, \ldots, \tau_{\sigma(p)}\right)$ for permutation σ on $\{0,1,2,\ldots p\}$,
(iii) $g_{\mathbf{Q}}\left(\tau_0, \tau_1, \tau_2, \ldots, \tau_p\right) \preceq g_{\mathbf{Q}}\left(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_p\right)$ for all

$$\left(\tau_0, \tau_1, \tau_2, \ldots, \tau_p\right), \left(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_p\right) \in X^{p+1},$$

with $\left\{\tau_i : i = 0, 1, \ldots, p\right\} \subsetneq \left\{\varsigma_i : i = 0, 1, \ldots, p\right\}$,
(iv) For all $\tau_0, \tau_1, \ldots, \tau_s, \varsigma_0, \varsigma_1, \ldots, \varsigma_t, \upsilon \in X$ with $s + t + 1 = p$,

$$\begin{aligned} g_{\mathbf{Q}}\left(\tau_0, \tau_1, \tau_2, \ldots, \tau_s, \varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_t\right) &\preceq g_{\mathbf{Q}}\left(\tau_0, \tau_1, \tau_2, \ldots, \tau_s, \upsilon, \upsilon, \ldots, \upsilon\right) \\ &+ g_{\mathbf{Q}}\left(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_t, \upsilon, \upsilon, \ldots, \upsilon\right). \end{aligned}$$

The pair $\left(X, g_{\mathbf{Q}}\right)$ is called quaternion valued $g_{\mathbf{Q}}$-metric space with degree p. For $p = 1, 2$ respectively, it is equivalent to quaternion valued metric and quaternion valued G-metric space.

Definition 1.9 (Jan and Jalal [14]).
A $g_{\mathbf{Q}}$-metric on X is called multiplicity independent with degree p if the following holds

$$g_{\mathbf{Q}}\left(\tau_0, \ldots, \tau_p\right) = g_{\mathbf{Q}}\left(\varsigma_0, \ldots, \varsigma_p\right),$$

for all $\left(\tau_0, \tau_1, \ldots, \tau_p\right), \left(\varsigma_0, \varsigma_1, \ldots, \varsigma_p\right) \in X^{p+1}$ with

$$\left\{\tau_i : i = 0, \ldots, p\right\} = \left\{\varsigma_i : i = 0, \ldots, p\right\}.$$

Note that for a given multiplicity independent $g_{\mathbf{Q}}$-metric with order 2, it holds that $g_{\mathbf{Q}}(\tau,\varsigma,\varsigma) = g_{\mathbf{Q}}(\tau,\tau,\varsigma)$. For a given multiplicity independent $g_{\mathbf{Q}}$-metric with order 3, it holds that $g_{\mathbf{Q}}(\tau,\varsigma,\varsigma,\varsigma) = g_{\mathbf{Q}}(\tau,\tau,\varsigma,\varsigma) = g_{\mathbf{Q}}(\tau,\tau,\tau,\varsigma)$ and $g_{\mathbf{Q}}(\tau,\tau,\varsigma,z) = g_{\mathbf{Q}}(\tau,\varsigma,\varsigma,z) = g_{\mathbf{Q}}(\tau,\varsigma,z,z)$.

Remark 1.2.
If we allow equality under the conditions of monotonicity in Definition 1.9 that is

$$g_{\mathbf{Q}}\left(\tau_0,\ldots,\tau_p\right) \preceq g_{\mathbf{Q}}\left(\varsigma_0,\ldots,\varsigma_p\right)$$

for $\left(\tau_0,\ldots,\tau_p\right),\left(\varsigma_0,\ldots,\varsigma_p\right) \in X^{p+1}$ with $\left\{\tau_i : i=0,\ldots,p\right\} \subseteq \left\{\varsigma_i : i=0,\ldots,p\right\}$, then every $g_{\mathbf{Q}}$-metric becomes multiplicity independent.

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of all positive integers). A continuous linear functional Φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if it satisfies the following conditions:
(1) $\Phi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all $n \in \mathbb{N}$;
(2) $\Phi(e) = 1$, where $e = (1,1,1,\ldots)$;
(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x_n)$ for all $(x_n) \in l_\infty$.
The mappings Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x_n) = \lim x_n$, for all $(x_n) \in c$.
In case σ is translation mappings $\sigma(n) = n+1$, the σ-mean is often called a Banach limit.
The space V_σ, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{(x_k) \in l_\infty : \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} x_{\sigma^k(n)} = L\right\}$$

uniformly in n.
Let θ be a lacunary sequence, $E \subseteq \mathbb{N}$ and

$$s_r := \min_n \left\{\left|E \cap \left\{\sigma^m(n) : m \in I_r\right\}\right|\right\}$$
$$S_r := \max_n \left\{\left|E \cap \left\{\sigma^m(n) : m \in I_r\right\}\right|\right\}.$$

If the following limits exist

$$\underline{V_\theta}(E) = \lim_{r\to\infty} \frac{s_r}{h_r},\ \overline{V_\theta}(E) = \lim_{r\to\infty} \frac{S_r}{h_r},$$

then they are called a lower lacunary invariant uniform density and an upper lacunary invariant uniform density of the set E, respectively. If $\underline{V_\theta}(E) = \overline{V_\theta}(E)$, then $V_\theta(E) = \underline{V_\theta}(E) = \overline{V_\theta}(E)$ is called the lacunary invariant uniform density of E.

2. Main Results

Definition 2.1.
Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $\{\tau_i\} \subseteq X$ be a sequence. A sequence $\{\tau_i\}$ is said to be lacunary $g_{\mathbf{Q}}$-invariant summable to τ if

$$\lim_{r\to\infty} \frac{p!}{h_r^p} \sum_{i_w \in I_r,\ 1\leq w\leq p} g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right) = \tau,$$

uniformly in m and it is indicated by $\tau_i \to \tau\left(g_{\mathbf{Q}}(V_{\sigma\theta})\right)$.

Definition 2.2.
A sequence $\{\tau_i\}$ is said to be lacunary strongly $g_{\mathbf{Q}}$-invariant summable to τ if

$$\lim_{r\to\infty} \frac{p!}{h_r^p} \sum_{i_w \in I_r,\ 1\leq w\leq p} \left|g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right)\right| = 0,$$

uniformly in m and it is indicated by $\tau_i \to \tau\left(g_{\mathbf{Q}}[V_{\sigma\theta}]\right)$.

Definition 2.3.
A sequence $\{\tau_i\}$ is said to be lacunary $g_{\mathbf{Q}}$-invariant statistically convergent to τ if, for every $q \in \mathbf{Q}$ with $0 \prec q$ such that

$$\lim_{r\to\infty} \frac{p!}{h_r^p} \left|\left\{ i_w \in I_r, 1 \le w \le p : \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \ge |q| \right\}\right| = 0,$$

uniformly in m. In this case we write $\tau_i \to \tau \left(g_{\mathbf{Q}}(S_{\sigma\theta})\right)$.

Definition 2.4.
A sequence $\{\tau_i\}$ is said to be invariant $g_{\mathbf{Q}}$-statistically convergent to τ if, for every $q \in \mathbf{Q}$ with $0 \prec q$ such that

$$\lim_{n\to\infty} \frac{p!}{n^p} \left|\left\{ (i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le n : \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \ge |q| \right\}\right| = 0,$$

uniformly in m. In this case we write $\tau_i \to \tau \left(g_{\mathbf{Q}}(S_\sigma)\right)$.

Theorem 2.1.
Let $\theta = \{k_r\}$ be a lacunary sequence.

(i) $\tau_i \to \tau \left(g_{\mathbf{Q}}[V_{\sigma\theta}]\right)$ *implies* $\tau_i \to \tau \left(g_{\mathbf{Q}}(S_{\sigma\theta})\right)$,

(ii) $\{\tau_i\} \in l_\infty^{(X, g_{\mathbf{Q}})}$ *($\{\tau_i\}$ is bounded sequence in $(X, g_{\mathbf{Q}})$) and* $\tau_i \to \tau \left(g_{\mathbf{Q}}(S_{\sigma\theta})\right)$ *imply* $\tau_i \to \tau \left(g_{\mathbf{Q}}[V_{\sigma\theta}]\right)$,

(iii) $g_{\mathbf{Q}}(S_{\sigma\theta}) \cap l_\infty = g_{\mathbf{Q}}[V_{\sigma\theta}]$.

Proof. (i) Let $q \in \mathbf{Q}$ with $0 \prec q$ and $\tau_i \to \tau \left(g_{\mathbf{Q}}[V_{\sigma\theta}]\right)$. Then, we can write

$$\begin{aligned}
&\sum_{i_w \in I_r,\ 1 \le w \le p} \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \\
&\ge \sum_{\substack{i_w \in I_r,\ 1 \le w \le p \\ \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \ge |q|}} \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \\
&+ \sum_{\substack{i_w \in I_r,\ 1 \le w \le p \\ \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| < |q|}} \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \\
&\ge |q| . \left|\left\{ i_w \in I_r, 1 \le w \le p : \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \ge |q| \right\}\right|
\end{aligned}$$

which gives the result.

(ii) Assume that $\tau_i \to \tau \left(g_{\mathbf{Q}}(S_{\sigma\theta})\right)$ and $\{\tau_i\} \in l_\infty^{(X, g_{\mathbf{Q}})}$. If $\{\tau_i\} \in l_\infty^{(X, g_{\mathbf{Q}})}$, then, there exists a positive integer M such that

$$\left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \le M$$

for all m.

Given $q \in \mathbf{Q}$ with $0 \prec q$, we get

$$\begin{aligned}
&\frac{p!}{h_r^p} \sum_{i_w \in I_r,\ 1 \le w \le p} \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \\
&= \frac{p!}{h_r^p} \sum_{\substack{i_w \in I_r,\ 1 \le w \le p \\ \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \ge |q|}} \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \\
&+ \frac{p!}{h_r^p} \sum_{\substack{i_w \in I_r,\ 1 \le w \le p \\ \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| < |q|}} \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \\
&\le \frac{Mp!}{h_r^p} \left|\left\{ i_w \in I_r, 1 \le w \le p : \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \ge |q| \right\}\right| + |q|
\end{aligned}$$

from which the result follows.

Let $\theta = \{k_r\}$ be given and define τ_i to be $1, 2, ..., \left[\sqrt{h_r}\right]$ for $i = \sigma^{i_v}(m)$, $1 \le v \le p$, $i_v = k_{r-1} + 1,\ k_{r-1} + 2,\ ..., k_{r-1} + \left[\sqrt{h_r}\right]$; $m \ge 1$, and $\tau_i = 0$ otherwise (where [.] denotes the greatest integer function). Note that $\{\tau_i\}$ is not bounded.

Further, for $q \in \mathbf{Q}$ with $0 \prec q$ we get

$$\begin{aligned}
&\frac{p!}{h_r^p} \left|\left\{ i_w \in I_r, 1 \le w \le p : \left| g_{\mathbf{Q}}\left(0, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, ..., \tau_{\sigma^{i_p}(m)}\right)\right| \ge |q| \right\}\right| \\
&= \frac{p!\left(\sqrt{h_r}\right)^p}{h_r^p} \to 0 \text{ as } r \to \infty,
\end{aligned}$$

i.e. $\{\tau_i\} \to 0\left(g_{\mathbf{Q}}(S_{\sigma\theta})\right)$. But

$$\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(0,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|$$
$$=\frac{p!1}{h_r}\left(\frac{\left[\sqrt{h_r}\right]\left(\left[\sqrt{h_r}\right]+1\right)}{2}\right)\to\frac{p!}{2}\neq 0 \text{ as } r\to\infty,$$

hence, $\{\tau_i\} \nrightarrow 0\left(g_{\mathbf{Q}}[V_{\sigma\theta}]\right)$. Thus, inclusion (i) is proper and this example denotes that the boundedness condition can not be omitted from the hypothesis (ii).

(iii) This is an immediate consequence of (i) and (ii). □

Lemma 2.1.

Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau\in X$ be a point, and $\{\tau_i\}\subseteq X$ be a sequence. Assume for given $q_1\in\mathbf{Q}$ with $0\prec q_1$ and every $q\in\mathbf{Q}$ with $0\prec q$, there exists n_0 and m_0 such that

$$\frac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\ldots,i_p\le n-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|<|q_1|$$

for all $n\ge n_0$ and $m\ge m_0$, then, $\{\tau_i\}\in g_{\mathbf{Q}}(S_\sigma)$.

Proof. Let $q_1\in\mathbf{Q}$ with $0\prec q_1$ be given. For every $q\in\mathbf{Q}$ with $0\prec q$, there exists n_0', m_0 such that

$$\frac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p,\ i_1,i_2,\ldots,i_p\le n-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|<\frac{|q_1|}{2} \tag{1}$$

for all $n\ge n_0'$ and $m\ge m_0$. It is enough to prove that there exists n_0'' such that for $n\ge n_0''$ and $0\le m\le m_0$

$$\frac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\ldots,i_p\le n-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|<|q_1|. \tag{2}$$

Since taking $n_0=\max\{n_0',n_0''\}$, (2) will hold for $n\ge n_0$ and for all m, which gives the result.
Once m_0 has been chosen, $0\le m\le m_0$, m_0 is fixed. So, take

$$K=\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\ldots,i_p\le m_0-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|.$$

Now taking $0\le m\le m_0$ and $n\ge m_0$, by (1) we have

$$\begin{aligned}
&\tfrac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\ldots,i_p\le n-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|\\
&\le\tfrac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\ldots,i_p\le m_0-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|\\
&+\tfrac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, m_0\le i_1,i_2,\ldots,i_p\le n-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|\\
&\le\tfrac{Kp!}{n^p}+\tfrac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, m_0\le i_1,i_2,\ldots,i_p\le n-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|\\
&\le\tfrac{Kp!}{n^p}+\tfrac{|q_1|}{2},
\end{aligned}$$

and taking n, sufficiently large, we can write $\frac{Kp!}{n^p}+\frac{|q_1|}{2}<|q_1|$, which gives (2), and hence the result follows. □

Theorem 2.2.

$g_{\mathbf{Q}}(S_{\sigma\theta})=g_{\mathbf{Q}}(S_\sigma)$ for every lacanary sequence θ.

Proof. Let $\{\tau_i\}\in g_{\mathbf{Q}}(S_{\sigma\theta})$. Then, from Definition 2.3, given $q_1\in\mathbf{Q}$ with $0\prec q_1$, there exist r_0 such that

$$\frac{p!}{h_r^p}\left|\left\{0\le i_w\le h_r-1, 1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|<|q_1|$$

for $r\ge r_0$ and $m=k_{r-1}+1+u$, $u\ge 0$.

Let $n\ge h_r$, write $n=vh_r+t$, where $0\le t\le h_r$, v is an integer. Since $n\ge h_r$, $v\ge 1$. Now

$$\begin{aligned}
&\tfrac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\ldots,i_p\le n-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|\\
&\le\tfrac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\ldots,i_p\le(v+1)h_r-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|\\
&=\tfrac{p!}{n^p}\sum_{j=0}^{v}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, jh_r\le i_1,i_2,\ldots,i_p\le(j+1)h_r-1:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|\\
&\le\tfrac{p!}{n^p}(v+1)h_r|q_1|\le 2vh_r\tfrac{p!|q_1|}{n^p},\ (v\ge 1),
\end{aligned}$$

for $\frac{p!h_r}{n^p}\le 1$, and since $\frac{vp!h_r}{n^p}\le 1$,

$$\tfrac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\ldots,i_p\le n-1:\right.\right.$$
$$\left.\left.\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\ge|q|\right\}\right|\le 2|q_1|.$$

Then, by the Lemma 2.1, $g_{\mathbf{Q}}(S_{\sigma\theta})\subset g_{\mathbf{Q}}(S_\sigma)$. It is easy to see that $g_{\mathbf{Q}}(S_\sigma)\subset g_{\mathbf{Q}}(S_{\sigma\theta})$. □

When $\sigma(m) = m+1$, from Definitions 2.3 and 2.4 we get the definitions of almost statistically convergence and lacunary almost statistically convergence of a sequence in quaternion valued generalized metric spaces.

Definition 2.5.
A sequence $\{\tau_i\}$ is said to be $g_{\mathbf{Q}}$-quasi-invariant statistically convergent to τ in $(X, g_{\mathbf{Q}})$ if, for every $q \in \mathbf{Q}$ with $0 \prec q$ such that

$$\lim_{n\to\infty} \frac{p!}{n^p} \left| \left\{ (i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le n : \left| g_{\mathbf{Q}} \left(\tau, \tau_{\sigma^{i_1}(mn)}, \tau_{\sigma^{i_2}(mn)}, \ldots, \tau_{\sigma^{i_p}(mn)} \right) \right| \ge |q| \right\} \right| = 0$$

uniformly in m. It is denoted by $\tau_i \to \tau\left(g_{\mathbf{Q}}(QS_\sigma)\right)$.

Definition 2.6.
A sequence $\{\tau_i\}$ is said to be $g_{\mathbf{Q}}$-quasi-strongly invariant convergent to τ in $(X, g_{\mathbf{Q}})$ if

$$\lim_{n\to\infty} \frac{p!}{n^p} \sum_{i_1, i_2, \ldots, i_p = 0}^{n-1} \left| g_{\mathbf{Q}} \left(\tau, \tau_{\sigma^{i_1}(mn)}, \tau_{\sigma^{i_2}(mn)}, \ldots, \tau_{\sigma^{i_p}(mn)} \right) \right| = 0$$

uniformly in m. In this case, we write $\tau_i \to \tau\left(g_{\mathbf{Q}}[QV_\sigma]\right)$.

Definition 2.7.
A sequence $\{\tau_i\}$ is said to be $g_{\mathbf{Q}}$-quasi-lacunary invariant convergent to τ in $(X, g_{\mathbf{Q}})$ if

$$\lim_{r\to\infty} \left| \frac{p!}{h_r^p} \sum_{i_w \in I_r,\ 1 \le w \le p} g_{\mathbf{Q}} \left(\tau, \tau_{\sigma^{i_1}(mr)}, \tau_{\sigma^{i_2}(mr)}, \ldots, \tau_{\sigma^{i_p}(mr)} \right) \right| = 0,$$

uniformly in m. In this case, we write $\tau_i \to \tau\left(g_{\mathbf{Q}}(QV_{\sigma\theta})\right)$.

Theorem 2.3.
Let $(X, g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau \in X$ be a point, and $\{\tau_i\} \subseteq X$ be a sequence. If a sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-lacunary invariant convergent to τ, then the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-lacunary invariant convergent to τ.

Proof. Suppose that the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-lacunary invariant convergent to τ. Then, for every $q \in \mathbf{Q}$ with $0 \prec q$ there exists an integer $r_0 > 0$ such that for all $r > r_0$

$$\left| \frac{p!}{h_r^p} \sum_{i_w \in I_r,\ 1 \le w \le p} g_{\mathbf{Q}} \left(\tau, \tau_{\sigma^{i_1}(n)}, \tau_{\sigma^{i_2}(n)}, \ldots, \tau_{\sigma^{i_p}(n)} \right) \right| < |q|,$$

for all n. If n is taken as $n = mr$, then we get

$$\left| \frac{p!}{h_r^p} \sum_{i_w \in I_r,\ 1 \le w \le p} g_{\mathbf{Q}} \left(\tau, \tau_{\sigma^{i_1}(mr)}, \tau_{\sigma^{i_2}(mr)}, \ldots, \tau_{\sigma^{i_p}(mr)} \right) \right| < |q|,$$

for all m. Since $q \in \mathbf{Q}$ with $0 \prec q$ is an arbitrary, if the limit is taken for $r \to \infty$, we have

$$\left| \frac{p!}{h_r^p} \sum_{i_w \in I_r,\ 1 \le w \le p} g_{\mathbf{Q}} \left(\tau, \tau_{\sigma^{i_1}(mr)}, \tau_{\sigma^{i_2}(mr)}, \ldots, \tau_{\sigma^{i_p}(mr)} \right) \right| \to 0,$$

for all m. Thus, the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-lacunary invariant convergent to τ. □

Definition 2.8.
A sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-strongly lacunary invariant convergent to τ if

$$\lim_{r\to\infty} \frac{p!}{h_r^p} \sum_{i_w \in I_r,\ 1 \le w \le p} \left| g_{\mathbf{Q}} \left(\tau, \tau_{\sigma^{i_1}(mr)}, \tau_{\sigma^{i_2}(mr)}, \ldots, \tau_{\sigma^{i_p}(mr)} \right) \right| = 0$$

supplies uniformly in m. In this case, we write $\tau_i \to \tau\left(g_{\mathbf{Q}}[QV_{\sigma\theta}]\right)$.

Theorem 2.4.
For any lacunary sequence $\theta = \{k_r\}$,

$$\tau_i \to \tau\left(g_{\mathbf{Q}}[QV_{\sigma\theta}]\right) \Leftrightarrow \tau_i \to \tau\left(g_{\mathbf{Q}}[QV_\sigma]\right).$$

Proof. Let $\tau_i \to \tau\left(g_{\mathbf{Q}}[QV_{\sigma\theta}]\right)$ and $q \in \mathbf{Q}$ with $0 \prec q$ be given. Then, there exists an integer r_0 such that

$$\frac{p!}{h_r^p}\sum_{i_1,i_2,\dots,i_p=0}^{h_r-1}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right| < |q|$$

for $r \geq r_0$ and $mr = k_{r-1}+1+w$, $w \geq 0$. Let $n \geq h_r$. Thus, n can be written as $n = \alpha h_r + t$ where α is an integer and $0 \leq t \leq h_r$. Since $n \geq h_r$, $\alpha \geq 1$. Then, we get

$$\begin{aligned}
&\frac{p!}{n^p}\sum_{i_1,i_2,\dots,i_p=0}^{n-1}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right| \\
&\leq \frac{p!}{n^p}\sum_{i_1,i_2,\dots,i_p=0}^{(\alpha+1)h_r-1}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right| \\
&= \frac{p!}{n^p}\sum_{j=0}^{\alpha}\sum_{i_1,i_2,\dots,i_p=jh_r}^{(j+1)h_r-1}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right| \\
&\leq \frac{p!}{n^p}|q|\,h_r(\alpha+1) \\
&\leq \frac{p!2\alpha h_r|q|}{n^p} \quad (\alpha \geq 1).
\end{aligned}$$

Since $\frac{p!\alpha h_r}{n^p} \leq 1$, we have

$$\frac{p!}{n^p}\sum_{i_1,i_2,\dots,i_p=0}^{n-1}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right| \leq 2|q|,$$

Therefore, we have $\tau_i \to \tau\left(g_{\mathbf{Q}}[QV_\sigma]\right)$.
Let $\tau_i \to \tau\left(g_{\mathbf{Q}}[QV_\sigma]\right)$ and $q \in \mathbf{Q}$ with $0 \prec q$. Then, there exists a number $W > 0$ such that

$$\frac{p!}{n^p}\sum_{i_1,i_2,\dots,i_p=0}^{n-1}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right| < |q|$$

for all $n > W$. Since $\theta = \{k_r\}$ is a lacunary sequence, a number $R > 0$ can be chosen such that $h_r > W$ where $r \geq R$. So, we have

$$\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\leq w\leq p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right| < |q|.$$

This implies that $\tau_i \to \tau\left(g_{\mathbf{Q}}[QV_{\sigma\theta}]\right)$. □

Definition 2.9.
A sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-lacunary invariant statistically convergent to τ if for each $q \in \mathbf{Q}$ with $0 \prec q$,

$$\lim_{r\to\infty}\frac{p!}{h_r^p}\left|\left\{i_w \in I_r, 1 \leq w \leq p : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right| \geq |q|\right\}\right| = 0$$

uniformly in m. In this case, we write $\tau_i \to \tau\left(g_{\mathbf{Q}}(QS_{\sigma\theta})\right)$.
The set of all $g_{\mathbf{Q}}$-quasi-lacunary invariant statistically convergent sequences will be denoted by $g_{\mathbf{Q}}(QS_{\sigma\theta})$.

Theorem 2.5.
If a sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-lacunary invariant statistically convergent to τ, then the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi lacunary invariant statistically convergent to τ.

Proof. Suppose that the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-lacunary invariant statistically convergent to τ. In this case, when $\varrho, q \in \mathbf{Q}$ with $0 \prec \varrho, q$ is given, there exists an integer $r_0 > 0$ such that for all $r > r_0$

$$\frac{p!}{h_r^p}\left|\left\{i_w \in I_r, 1 \leq w \leq p : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(n)},\tau_{\sigma^{i_2}(n)},\dots,\tau_{\sigma^{i_p}(n)}\right)\right| \geq |q|\right\}\right| < |\varrho|,$$

for all n. If n is taken as $n = mr$, then we get

$$\frac{p!}{h_r^p}\left|\left\{i_w \in I_r, 1 \leq w \leq p : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right| \geq |q|\right\}\right| < |\varrho|,$$

for all m. Since $\varrho \in \mathbf{Q}$ with $0 \prec \varrho$ is an arbitrary, we have

$$\lim_{r\to\infty}\frac{p!}{h_r^p}\left|\left\{i_w \in I_r, 1 \leq w \leq p : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right| \geq |q|\right\}\right| = 0$$

uniformly in m which means that the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi lacunary invariant statistically convergent to τ. □

Theorem 2.6.
For any lacunary sequence $\theta = \{k_r\}$,

$$\tau_i \to \tau\left(g_{\mathbf{Q}}(QS_{\sigma\theta})\right) \Longleftrightarrow \tau_i \to \tau\left(g_{\mathbf{Q}}(QS_{\sigma})\right).$$

Proof. Let $\tau_i \to \tau\left(g_{\mathbf{Q}}(QS_{\sigma\theta})\right)$ and $\varrho, q \in \mathbf{Q}$ with $0 \prec \varrho, q$ be given. Then, there exists an integer r_0 such that

$$\frac{p!}{h_r^p}\left|\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p, 0\le i_1,i_2,\dots,i_p\le h_r-1 : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|\le|\varrho|,$$

for $r \ge r_0$ and $mr = k_{r-1}+1+w$, $w \ge 0$. Let $n \ge h_r$. Thus, n can be written as $n = \alpha h_r + t$ where α is an integer and $0 \le t \le h_r$. Since $n \ge h_r$, $\alpha \ge 1$. Then, we get

$$\begin{aligned}
&\frac{p!}{n^p}\left|\left\{i_1,i_2,\dots,i_p\le n-1 : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\ge|q|\right\}\right|\\
&\le \frac{p!}{n^p}\left|\left\{i_1,i_2,\dots,i_p\le (\alpha+1)h_r-1 : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|\\
&= \frac{p!}{n^p}\sum_{j=0}^{\alpha}\left|\left\{jh_r\le i_1,i_2,\dots,i_p\le (j+1)h_r-1 : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|\\
&\le \frac{p!}{n^p}|\varrho|\, h_r(\alpha+1)\\
&\le \frac{p!2\alpha h_r|\varrho|}{n^p}.
\end{aligned}$$

Since $\frac{p!\alpha h_r}{n^p} \le 1$, we have

$$\frac{p!}{n^p}\left|\left\{i_1,i_2,\dots,i_p\le n-1 : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\ge|q|\right\}\right|\le 2|\varrho|.$$

Therefore, we have $\tau_i \to \tau\left(g_{\mathbf{Q}}(QS_{\sigma})\right)$.
Let $\tau_i \to \tau\left(g_{\mathbf{Q}}(QS_{\sigma})\right)$ and $\rho \in \mathbf{Q}$ with $0 \prec \rho$ is given. Then, there exists a number $P > 0$ such that

$$\frac{p!}{n^p}\left|\left\{i_1,i_2,\dots,i_p\le n : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\ge|q|\right\}\right|<|\rho|$$

for all $n > P$. Since $\theta = \{k_r\}$ is a lacunary sequence, a number $R > 0$ can be chosen such that $h_r > P$ where $r \ge R$. Thereby, we get

$$\frac{p!}{h_r^p}\left|\left\{i_w\in I_r, 1\le w\le p : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|<|\rho|.$$

This implies that $\tau_i \to \tau\left(g_{\mathbf{Q}}(QS_{\sigma\theta})\right)$. □

Definition 2.10.
Let $\theta = \{k_r\}$ be a lacunary sequence and $0 < s < \infty$. A sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-strongly s-lacunary invariant convergent to τ if

$$\lim_{r\to\infty}\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s = 0$$

supplies uniformly in m. In this case, we write $\tau_i \to \tau\left(g_{\mathbf{Q}}[QV_{\sigma\theta}]^s\right)$.
The set of all $g_{\mathbf{Q}}$-quasi-strongly s-lacunary invariant convergent sequences will be denoted by $g_{\mathbf{Q}}[QV_{\sigma\theta}]^q$.

Theorem 2.7.
If a sequence $\{\tau_i\}$ *is* $g_{\mathbf{Q}}$*-quasi-strongly* s*-lacunary invariant convergent to* τ*, then this sequence is* $g_{\mathbf{Q}}$*-quasi lacunary invariant statistically convergent to* τ.

Proof. Suppose that the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-strongly s-lacunary invariant convergent to τ. Then, for each $q \in \mathbf{Q}$ with $0 \prec q$ following inequality is provided:

$$\begin{aligned}
&\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s\\
&= \sum_{\substack{i_w\in I_r,\,1\le w\le p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s\\
&+ \sum_{\substack{i_w\in I_r,\,1\le w\le p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|<|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s\\
&\ge \sum_{\substack{i_w\in I_r,\,1\le w\le p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s\\
&\ge |q|^s\left|\left\{i_w\in I_r, 1\le w\le p : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|,
\end{aligned}$$

That is

$$\begin{aligned} &\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s \\ &\ge |q|^s\left|\left\{i_w\in I_r,\,1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right| \end{aligned} \tag{3}$$

for all m. If the both side of the inequality (3) are multiplied by $\frac{p!}{h_r^p}$ and after that the limit is taken for $r\to\infty$, due to our acceptance, we get

$$\begin{aligned} &0\leftarrow \lim_{r\to\infty}\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s \\ &\ge |q|^s\lim_{r\to\infty}\frac{p!}{h_r^p}\left|\left\{i_w\in I_r,\,1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|. \end{aligned}$$

Hence, we obtain

$$\lim_{r\to\infty}\frac{p!}{h_r^p}\left|\left\{i_w\in I_r,\,1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|=0$$

uniformly in m, that is, $\tau_i\to\tau\left(g_{\mathbf{Q}}\left(QS_{\sigma\theta}\right)\right)$. □

Theorem 2.8.
If a sequence $\{\tau_i\}$ is bounded sequence in $(X,g_{\mathbf{Q}})$ (i.e., $\{\tau_i\}\in l_\infty^{(X,g_{\mathbf{Q}})}$) and $g_{\mathbf{Q}}$-quasi-lacunary invariant statistically convergent to τ, then this sequence is $g_{\mathbf{Q}}$-quasi-strongly s-lacunary invariant convergent to τ.

Proof. Suppose that the sequence $\{\tau_i\}\in l_\infty^{(X,g_{\mathbf{Q}})}$ and $g_{\mathbf{Q}}$-quasi-lacunary invariant statistically convergent to τ. Since $\{\tau_i\}$ is bounded, there exists an $M>0$ such that

$$\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s<M.$$

Then, for each $q\in\mathbf{Q}$ with $0\prec q$, we get

$$\begin{aligned} &\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s \\ &=\frac{p!}{h_r^p}\left(\sum_{\substack{i_w\in I_r,\,1\le w\le p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s\right. \\ &\left.\sum_{\substack{i_w\in I_r,\,1\le w\le p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|<|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s\right) \\ &\le\frac{Mp!}{h_r^p}\left|\left\{i_w\in I_r,\,1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right| \\ &+\frac{|q|^sp!}{h_r^p}\left|\left\{i_w\in I_r,\,1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right| \\ &=\frac{Mp!}{h_r^p}\left|\left\{i_w\in I_r,\,1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|+|q|^s, \end{aligned}$$

that is

$$\begin{aligned} &\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s \\ &<\frac{Mp!}{h_r^p}\left|\left\{i_w\in I_r,\,1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|+|q|^s \end{aligned} \tag{4}$$

for all m. If the limit of both side of the inequality (4) is taken for $r\to\infty$, due to our acceptance, we get

$$\begin{aligned} &\lim_{r\to\infty}\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s \\ &\le\lim_{r\to\infty}\left(\frac{Mp!}{h_r^p}\left|\left\{i_w\in I_r,\,1\le w\le p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|\ge|q|\right\}\right|+|q|^s\right) \\ &=|q|^s. \end{aligned}$$

Thus, we have

$$\lim_{r\to\infty}\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\le w\le p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mr)},\tau_{\sigma^{i_2}(mr)},\dots,\tau_{\sigma^{i_p}(mr)}\right)\right|^s=0$$

uniformly in m, that is, $\tau_i\to\tau\left(g_{\mathbf{Q}}\left[QV_{\sigma\theta}\right]^s\right)$. □

From Theorem 2.7 and Theorem 2.8, we have following corollary.

Corollary 2.1.
$g_{\mathbf{Q}}(QS_{\sigma\theta}) \cap l_{\infty}^{(X,g_{\mathbf{Q}})} = g_{\mathbf{Q}}[QV_{\sigma\theta}]^{s}$.

We now introduce the concepts of almost convergence, almost statistically convergence, quasi-almost convergence and quasi-almost statistically convergence in quaternion-valued generalized metric spaces. Additionally, we define quasi-strongly almost convergence and quasi s-strongly almost convergence in the same spaces. We then examine the relationships among these concepts, as well as their connections to previously established convergence types for sequences.

Definition 2.11.
A sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-almost convergence to τ if

$$\lim_{n\to\infty} \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=m+1}^{m+n} g_{\mathbf{Q}}\left(\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right) = \tau$$

or

$$\lim_{n\to\infty} \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=1}^{n} g_{\mathbf{Q}}\left(\tau_{i_1+m},\tau_{i_2+m},\ldots,\tau_{i_p+m}\right) = \tau$$

uniformly in m.

Definition 2.12.
A sequence $\{\tau_i\}$ is strongly $g_{\mathbf{Q}}$-almost convergence to τ if

$$\lim_{n\to\infty} \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=m+1}^{m+n} \left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| = 0$$

uniformly in m.

Definition 2.13.
A sequence $\{\tau_i\}$ is strongly $g_{\mathbf{Q}}$-s-almost convergent to τ if

$$\lim_{n\to\infty} \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=m+1}^{m+n} \left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|^{s} = 0$$

uniformly in m.

Definition 2.14.
A sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-almost statistically convergent to τ if for each $q \in \mathbf{Q}$ with $0 \prec q$

$$\lim_{n\to\infty} \frac{p!}{n^p} \left|\left\{(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p,\ i_1,i_2,\ldots,i_p \le n : \left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1+m},\tau_{i_2+m},\ldots,\tau_{i_p+m}\right)\right| > |q|\right\}\right| = 0$$

uniformly in m.

Definition 2.15.
A sequence $\{\tau_i\} \in l_{\infty}^{(X,g_{\mathbf{Q}})}$ is $g_{\mathbf{Q}}$-quasi-almost convergent to τ if

$$\left|\frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=mn}^{mn+n-1} g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \to 0 \quad (\text{as } n \to \infty) \tag{5}$$

uniformly in m. In this case, we will write $\tau_i \to \tau\left(g_{\mathbf{Q}}(QF)\right)$.

Example 2.1.
Let we define a sequence $\{\tau_i\}$ as follows:

$$\tau_i := \begin{cases} 1, & \text{if } i \geq 1 \text{ and } i \text{ is square integer} \\ 0, & \text{otherwise.} \end{cases}$$

This sequence is not convergent. But

$$\lim_{n\to\infty} \left| \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=mn}^{mn+n-1} g_{\mathbf{Q}}\left(0, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) \right| = 0$$

uniformly in m, so this sequence is $g_{\mathbf{Q}}$-quasi-almost convergent to $\tau = 0$.

Theorem 2.9.
If a sequence $\{\tau_i\} \in l_\infty^{(X,g_{\mathbf{Q}})}$ is $g_{\mathbf{Q}}$-almost convergent to τ, then $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-almost convergent to τ.

Proof. Suppose that the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-almost convergent to τ. Then, for each $q \in \mathbf{Q}$ with $0 \prec q$ there exists an integer $n_0 > 0$ such that for all $n > n_0$

$$\left| \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(\tau, \tau_{i_1+u}, \tau_{i_2+u}, \ldots, \tau_{i_p+u}\right) \right| < |q|$$

uniformly in u. If u is taken as $u = mn$, then we have

$$\left| \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(\tau, \tau_{i_1+mn}, \tau_{i_2+mn}, \ldots, \tau_{i_p+mn}\right) \right| = \left| \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=mn}^{mn+n-1} g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) \right| < |q|$$

uniformly in m. Since $q \in \mathbf{Q}$ with $0 \prec q$ is an arbitrary, the limit is taken for $n \to \infty$ we can write

$$\left| \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=mn}^{mn+n-1} g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) \right| \to 0$$

uniformly in m. That is, $\tau_i \to \tau\left(g_{\mathbf{Q}}(QF)\right)$. □

Definition 2.16.
A sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-Cesàro summable to τ if

$$\lim_{n\to\infty} \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=0}^{n} g_{\mathbf{Q}}\left(\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) = \tau.$$

and $g_{\mathbf{Q}}$-strongly Cesàro summable to τ if

$$\lim_{n\to\infty} \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=0}^{n} \left| g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) \right| = 0.$$

Theorem 2.10.
If a sequence $\{\tau_i\} \in l_\infty^{(X,g_{\mathbf{Q}})}$ is $g_{\mathbf{Q}}$-quasi-almost convergent to τ, then $\{\tau_i\}$ is $g_{\mathbf{Q}}$-Cesàro summable to τ.

Proof. Assume that the sequence $\{\tau_i\} \in l_\infty^{(X,g_{\mathbf{Q}})}$ is $g_{\mathbf{Q}}$-quasi-almost convergent to τ. Then, (5) is true which for $m = 0$ implies for every $q \in \mathbf{Q}$ with $0 \prec q$,

$$\left| \frac{p!}{n^p} \sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) \right| \to 0 \quad (\text{as } n \to \infty)$$

so, $\{\tau_i\}$ is $g_{\mathbf{Q}}$-Cesàro summable to τ. □

Definition 2.17.
A sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-almost statistically convergent to τ if for each $q \in \mathbf{Q}$ with $0 \prec q$

$$\lim_{n\to\infty} \frac{p!}{n^p} \left| \left\{ (i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \leq n : \left| g_{\mathbf{Q}}\left(\tau, \tau_{i_1+mn}, \tau_{i_2+mn}, \ldots, \tau_{i_p+mn}\right) \right| \geq |q| \right\} \right| = 0$$

uniformly in m. In this case, we will write $g_{\mathbf{Q}}(QS) - \lim \tau_i \to \tau$ or $\tau_i \to \tau\left(g_{\mathbf{Q}}(QS)\right)$.

Theorem 2.11.
If a sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-almost statistically convergent to τ, then $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-almost statistically convergent to τ.

Proof. Suppose that the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-almost statistically convergent to τ. Then, for every $\varrho, q \in \mathbf{Q}$ with $0 \prec \varrho, q$ there exists an integer $n_0 > 0$ such that for all $n > n_0$

$$\frac{p!}{n^p}\left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le n : \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1+u}, \tau_{i_2+u}, \ldots, \tau_{i_p+u}\right)\right| \ge |q|\right\}\right| < |\varrho|$$

uniformly in u. If u is taken as $u = mn$, then we have

$$\frac{p!}{n^p}\left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le n : \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1+mn}, \tau_{i_2+mn}, \ldots, \tau_{i_p+mn}\right)\right| \ge |q|\right\}\right| < |\varrho|$$

uniformly in m. Since $\varrho \in \mathbf{Q}$ with $0 \prec \varrho$ is an arbitrary, we have

$$\lim_{n\to\infty} \frac{p!}{n^p}\left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le n : \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1+mn}, \tau_{i_2+mn}, \ldots, \tau_{i_p+mn}\right)\right| \ge |q|\right\}\right| = 0$$

uniformly in m which means that $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-almost statistically convergent to τ. □

Definition 2.18.
A sequence $\{\tau_i\} \in l_\infty^{(X, g_{\mathbf{Q}})}$ is $g_{\mathbf{Q}}$-quasi-strongly almost convergent to τ if

$$\frac{p!}{n^p}\sum_{i_1, i_2, \ldots, i_p = mn}^{mn+n-1}\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right| \longrightarrow 0$$

uniformly in m. In this case, we will write $[g_{\mathbf{Q}}QF] - \lim \tau_i \to \tau$ or $\tau_i \to \tau\left([g_{\mathbf{Q}}QF]\right)$.

Definition 2.19.
A sequence $\{\tau_i\} \in l_\infty^{(X, g_{\mathbf{Q}})}$ is $g_{\mathbf{Q}}$-quasi-s-strongly almost convergent to τ if for $0 < s < \infty$,

$$\frac{p!}{n^p}\sum_{i_1, i_2, \ldots, i_p = mn}^{mn+n-1}\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right|^s \longrightarrow 0 \tag{6}$$

uniformly in m. In this case, we will write $[g_{\mathbf{Q}}QF]^s - \lim \tau_i \to \tau$ or $\tau_i \to \tau\left([g_{\mathbf{Q}}QF]^s\right)$.

Theorem 2.12.
Let $0 < s < \infty$. Then, we have following assertions:
(i) If a sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-s-strongly almost convergent to τ, then the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-almost statistically convergent to τ.
(ii) If a sequence $\{\tau_i\} \in l_\infty^{(X, g_{\mathbf{Q}})}$ and $g_{\mathbf{Q}}$-quasi-almost statistically convergent to τ, then the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-s-strongly almost convergent to τ.

Proof. (i) Let $q \in \mathbf{Q}$ with $0 \prec q$ be given. Then, following inequality is proved

$$\begin{aligned}&\sum_{i_1, i_2, \ldots, i_p = mn}^{mn+n-1}\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right|^s \\ &\ge |q|^s\left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le n : \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1+mn}, \tau_{i_2+mn}, \ldots, \tau_{i_p+mn}\right)\right| \ge |q|\right\}\right|\end{aligned} \tag{7}$$

uniformly in m. Since the sequence $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi-s-strongly almost convergent to τ; if the both side of inequality (7) are multipled by $\frac{p!}{n^p}$ and after that the limit is taken for $n \to \infty$, then we have

$$\begin{aligned}&\lim_{n\to\infty} \tfrac{p!}{n^p}\sum_{i_1, i_2, \ldots, i_p = mn}^{mn+n-1}\left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right)\right|^s \\ &\ge |q|^s \lim_{n\to\infty} \tfrac{p!}{n^p}\left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le n : \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1+mn}, \tau_{i_2+mn}, \ldots, \tau_{i_p+mn}\right)\right| \ge |q|\right\}\right|.\end{aligned}$$

Hence, we handle

$$\lim_{n\to\infty} \frac{p!}{n^p}\left|\left\{(i_1, i_2, \ldots, i_p) \in \mathbb{N}^p,\ i_1, i_2, \ldots, i_p \le n : \left|g_{\mathbf{Q}}\left(\tau, \tau_{i_1+mn}, \tau_{i_2+mn}, \ldots, \tau_{i_p+mn}\right)\right| \ge |q|\right\}\right| = 0,$$

uniformly in m.
(ii) Since $\{\tau_j\} \in l_\infty^{(X,g_\mathbf{Q})}$, we can write

$$\left|g_\mathbf{Q}\left(\tau,\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right)\right| = M, \quad (0 < M < \infty).$$

If $\{\tau_i\}$ is $g_\mathbf{Q}$-quasi-almost statistically convergent to τ, then for a given $q \in \mathbf{Q}$ with $0 \prec q$ a number $N_q \in \mathbb{N}$ can be chosen such that for all $n > N_q$

$$\frac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p,\, i_1,i_2,\ldots,i_p\le n : \left|g_\mathbf{Q}\left(\tau,\tau_{i_1+mn},\tau_{i_2+mn},...,\tau_{i_p+mn}\right)\right| \ge \left(\frac{|q|}{2}\right)^{1/s}\right\}\right| < \frac{|q|}{2M^s}$$

uniformly in m. Let take the set

$$T_n = \left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p,\, i_1,i_2,\ldots,i_p\le n : \left|g_\mathbf{Q}\left(\tau,\tau_{i_1+mn},\tau_{i_2+mn},...,\tau_{i_p+mn}\right)\right| \ge \left(\frac{|q|}{2}\right)^{1/s}\right\}.$$

Thus, we have

$$\frac{p!}{n^p}\sum_{i_1,i_2,...,i_p=mn}^{mn+n-1}\left|g_\mathbf{Q}\left(\tau,\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right)\right|^s = \frac{p!}{n^p}\left(\sum_{\substack{i_1,i_2,...,i_p\le n\\ \{i_1,i_2,...,i_p\}\in T_n}}\left|g_\mathbf{Q}\left(\tau,\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right)\right|^s\right.$$
$$\left.\sum_{\substack{i_1,i_2,...,i_p\le n\\ \{i_1,i_2,...,i_p\}\notin T_n}}\left|g_\mathbf{Q}\left(\tau,\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right)\right|^s\right) < \frac{p!}{n^p}n^p\frac{|q|}{2p!M^s}M^s + \frac{p!}{n^p}n^p\frac{|q|}{2p!}$$
$$= \frac{|q|}{2} + \frac{|q|}{2} = |q|,$$

uniformly in m. So, the proof is completed. □

Theorem 2.13.
If a sequence $\{\tau_i\}$ is $g_\mathbf{Q}$-quasi-s-strongly almost convergence to τ, then the sequence $\{\tau_i\}$ is $g_\mathbf{Q}$-strongly s-Cesàro summable to τ.

Proof. Suppose that the sequence $\{\tau_i\}$ is $g_\mathbf{Q}$-quasi-s-strongly almost convergence to τ. Then, (6) is true which for $m = 0$ implies for each $q \in \mathbf{Q}$ with $0 \prec q$

$$\lim_{n\to\infty}\frac{p!}{n^p}\sum_{i_1,i_2,...,i_p=0}^{n-1}\left|g_\mathbf{Q}\left(\tau,\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right)\right|^s = 0,$$

so, $\{\tau_i\}$ is $g_\mathbf{Q}$-strongly s-Cesàro summable to τ. □

Theorem 2.14.
If a sequence $\{\tau_i\}$ is $g_\mathbf{Q}$-quasi-s-strongly almost convergence to τ, then the sequence $\{\tau_i\}$ is $g_\mathbf{Q}$-statistically convergent to τ.

Proof. Assume that the sequence $\{\tau_i\}$ is $g_\mathbf{Q}$-quasi-s-strongly almost convergence to τ. Then, by Theorem 2.13, the sequence $\{\tau_i\}$ is $g_\mathbf{Q}$-strongly s-Cesàro summable to τ. For each $q \in \mathbf{Q}$ with $0 \prec q$, we can write

$$\begin{aligned}&\sum_{i_1,i_2,...,i_p=0}^{n-1}\left|g_\mathbf{Q}\left(\tau,\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right)\right|^s\\ &\ge |q|^s\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p,\, i_1,i_2,\ldots,i_p\le n : \left|g_\mathbf{Q}\left(\tau,\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right)\right| \ge |q|\right\}\right|\end{aligned} \tag{8}$$

Since the sequence $\{\tau_i\}$ is $g_\mathbf{Q}$-quasi-s-Cesàro summable to τ; if the both sides of inequality (8) are multipled by $\frac{p!}{n^p}$ and after that the limit is taken for $n \to \infty$, left side of the inequality (8) is equal to 0. Hence, we handle

$$\lim_{n\to\infty}\frac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p,\, i_1,i_2,\ldots,i_p\le n : \left|g_\mathbf{Q}\left(\tau,\tau_{i_1},\tau_{i_2},...,\tau_{i_p}\right)\right| \ge |q|\right\}\right| = 0$$

The proof of theorem is completed. □

Now, we give the concept of quasi-invariant convergence and quasi-invariant statistical convergence in quaternion valued generalized metric spaces.

Let us define on the space $l_\infty^{(X,g_\mathbf{Q})}$ the function t by

$$t(\tau) \equiv t(\tau_i) = \overline{\lim_{n\to\infty}}\left\{\sup_m \frac{p!}{n^p}\left|\sum_{i_1,i_2,\dots,i_p=0}^{n-1} g_\mathbf{Q}\left(\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\right\}. \tag{9}$$

The functional t clearly is real-valued and it satisfies following properties:
(i) $t(\tau)\geq 0$,
(ii) $t(\alpha\tau)=|\alpha|t(\tau)$,
(iii) $t(\tau+\sigma)\leq t(\tau)+t(\sigma)$, $\left(\alpha\in\mathbb{R};\ \tau,\sigma\in l_\infty^{(X,g_\mathbf{Q})}\right)$

that is, t is a symmetric convex functional on the space $l_\infty^{(X,g_\mathbf{Q})}$. According to a corollary of Hahn-Banach theorem there must exist a nontrivial linear functional L on the space $l_\infty^{(X,g_\mathbf{Q})}$ such that $|L(\tau_i)|\leq t(\tau_i)$.

The following lemma is well known in the literature.

Lemma 2.2.
Let $(X,g_\mathbf{Q})$ be a quaternion valued g-metric space and $t: X\to\mathbb{R}$ be a functional such that the following assertions are valid: $t(\tau)\geq 0$, $t(\alpha\tau)=|\alpha|t(\tau)$, $t(\tau+\sigma)\leq t(\tau)+t(\sigma)$, $\left(\alpha\in\mathbb{R};\ \tau,\sigma\in l_\infty^{(X,g_\mathbf{Q})}\right)$. Then for each $\tau_0\in X$, there exists a linear functional L on X such that

$$(\forall\tau=\{\tau_i\}\in X)\quad |L(\tau)|\leq t(\tau),\quad L(\tau_0)=t(\tau_0)$$

Denoting now by $\sum$ the family of functionals satisfying the above conditions then for each $s\in X$ we have

$$\left(\forall L\in\sum\right)\quad L\left(g_\mathbf{Q}\left(s,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right)=0 \textit{ iff } t\left(g_\mathbf{Q}\left(s,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right)=0\quad\left(\{\tau_j\}\in l_\infty^{(X,g_\mathbf{Q})}\right). \tag{10}$$

Definition 2.20.
A sequence $\{\tau_i\}\in l_\infty^{(X,g_\mathbf{Q})}$ is $g_\mathbf{Q}$-quasi invariant convergent to $s\in X$ or $g_\mathbf{Q}$-quasi σ-summable to s if

$$\left(\forall L\in\sum\right)\quad L\left(g_\mathbf{Q}\left(s,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right)=0. \tag{11}$$

In this case we will write $g_\mathbf{Q}(Q-\sigma)-\lim_{i\to\infty}\tau_i=s$.

It is easy to see that $g_\mathbf{Q}$-quasi invariant limit of a sequence defined in such way is unique.

Theorem 2.15.
A bounded sequence $\{\tau_i\}$ $g_\mathbf{Q}$-quasi invariant convergent to $\tau\in X$ iff

$$\left|\frac{p!}{n^p}\sum_{i_1,i_2,\dots,i_p=0}^{n-1} g_\mathbf{Q}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\to 0 \textit{ as } n\to+\infty \tag{12}$$

uniformly in m.

Proof. Suppose for a bounded sequence $\{\tau_i\}$, we have $g_\mathbf{Q}(Q-\sigma)-\lim_{i\to\infty}\tau_i=s$. Then, by (10) and (11), we have $t\left(g_\mathbf{Q}\left(s,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right)=0$ or, by (9), we have

$$\overline{\lim_{n\to\infty}}\left\{\sup_m \frac{p!}{n^p}\left|\sum_{i_1,i_2,\dots,i_p=0}^{n-1} g_\mathbf{Q}\left(s,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\right\}=0.$$

Therefore, for any $q\in\mathbf{Q}$ with $0\prec q$

$$\frac{p!}{n^p}\left|\sum_{i_1,i_2,\dots,i_p=0}^{n-1} g_\mathbf{Q}\left(s,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|<|q|.$$

Since $q\in\mathbf{Q}$ with $0\prec q$ is arbitrary, we have

$$\frac{p!}{n^p}\left|\sum_{i_1,i_2,\dots,i_p=0}^{n-1} g_\mathbf{Q}\left(s,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\to 0 \text{ as } n\to\infty,$$

uniformly in n, so the condition (12) is necessary. Conversely, let the condition (12) be true. This means that

$$\sup_m \frac{p!}{n^p}\left|\sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(s,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\ldots,\tau_{\sigma^{i_p}(mn)}\right)\right| \to 0 \text{ as } n\to\infty$$

or

$$t\left(g_{\mathbf{Q}}\left(s,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right) = \overline{\lim}_{n\to\infty}\left\{\sup_m \frac{p!}{n^p}\left|\sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(s,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\ldots,\tau_{\sigma^{i_p}(mn)}\right)\right|\right\} = 0.$$

Hence, by (10), we have

$$\left(\forall L\in \sum\right)\quad L\left(g_{\mathbf{Q}}\left(s,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right) = 0,$$

which by (11), means that $g_{\mathbf{Q}}(Q-\sigma)-\lim_{i\to\infty}\tau_i = s$, so the condition (12) is sufficient. □

Definition 2.21.
A sequence $\{\tau_i\}$ is said to be $g_{\mathbf{Q}}$-invariant convergent to $s\in X$ if and only if

$$\lim_{n\to\infty}\frac{p!}{n^p}\left|\sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(s,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right| = 0, \tag{13}$$

uniformly in m.

Theorem 2.16.
If a bounded sequence $\{\tau_i\}$ $g_{\mathbf{Q}}$-invariant convergent to $s\in X$, then it is $g_{\mathbf{Q}}$-quasi invariant convergent to s.

Proof. Let bounded sequence $\{\tau_i\}$ be $g_{\mathbf{Q}}$-invariant convergent to $s\in X$. Then by (13) for any $q\in\mathbf{Q}$ with $0\prec q$ there exists an integer $n_0>0$ such that

$$\frac{p!}{n^p}\left|\sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(s,\tau_{\sigma^{i_1}(k)},\tau_{\sigma^{i_2}(k)},\ldots,\tau_{\sigma^{i_p}(k)}\right)\right| < |q|,$$

$(n>n_0,\ k=1,2,3,\ldots)$.
Hence, for $k=mn$ $(n>n_0,\ m=1,2,3,\ldots)$ we have

$$\frac{p!}{n^p}\left|\sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(s,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\ldots,\tau_{\sigma^{i_p}(mn)}\right)\right| < |q|.$$

Since $q\in\mathbf{Q}$ with $0\prec q$ is arbitrary, we have

$$\frac{p!}{n^p}\left|\sum_{i_1,i_2,\ldots,i_p=0}^{n-1} g_{\mathbf{Q}}\left(s,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\ldots,\tau_{\sigma^{i_p}(mn)}\right)\right| \to 0 \text{ as } n\to\infty$$

uniformly in m which, by (13), means that (τ_i) quasi invariant convergent. □

Definition 2.22.
A sequence $\{\tau_i\}$ is said to be $g_{\mathbf{Q}}$-quasi almost statistically convergent to $\tau\in X$ if for each $q\in\mathbf{Q}$ with $0\prec q$

$$\lim_{n\to\infty}\frac{p!}{n^p}\left|\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p,\ i_1,i_2,\ldots,i_p\le n : \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma(mn+i_1)},\tau_{\sigma(mn+i_2)},\ldots,\tau_{\sigma(mn+i_p)}\right)\right| \ge |q|\right\}\right| = 0$$

uniformly in m.

Theorem 2.17.
Let $(X,g_{\mathbf{Q}})$ be a quaternion valued g-metric space, $\tau\in X$ be a point, and $\{\tau_i\}\subseteq X$ be a sequence. If a sequence $\{\tau_i\}$ $g_{\mathbf{Q}}$-invariant statistically convergent to $\tau\in X$, then it is $g_{\mathbf{Q}}$-quasi invariant statistically convergent to τ.

Proof. Let $\{\tau_i\}$ be $g_{\mathbf{Q}}$-invariant statistically convergent to $\tau \in X$. Then for any $q \in \mathbf{Q}$ with $0 \prec q$ there exists an integer $n_0 > 0$ such that

$$\frac{p!}{n^p}\left|\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p,\ i_1,i_2,\dots,i_p\le n:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(k)},\tau_{\sigma^{i_2}(k)},\dots,\tau_{\sigma^{i_p}(k)}\right)\right|\ge|q|\right\}\right|<|q|$$

$(n > n_0,\ k = 1,2,\dots)$.
Hence for $k = mn$ $(n > n_0,\ m = 1,2,\dots)$ we have

$$\frac{p!}{n^p}\left|\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p,\ i_1,i_2,\dots,i_p\le n:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\ge|q|\right\}\right|<|q|.$$

Since $q \in \mathbf{Q}$ with $0 \prec q$ is arbitrary, we have

$$\lim_{n\to\infty}\frac{p!}{n^p}\left|\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p,\ i_1,i_2,\dots,i_p\le n:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(mn)},\tau_{\sigma^{i_2}(mn)},\dots,\tau_{\sigma^{i_p}(mn)}\right)\right|\ge|q|\right\}\right|=0$$

uniformly in m which means that $\{\tau_i\}$ is $g_{\mathbf{Q}}$-quasi invariant statistically convergent to τ convergent. □

3. Conclusion

In this paper, we have introduced several novel concepts of convergence for sequences in quaternion-valued generalized metric spaces. We provided a comprehensive characterization of quasi-invariant convergence for bounded sequences and extended the discussion to quasi-invariant statistical convergence. Our exploration included defining quasi-almost convergence, quasi-almost statistical convergence, quasi-strongly almost convergence, and quasi s-strongly almost convergence, alongside examining their intricate interrelationships.

Additionally, we introduced the concepts of quasi-lacunary invariant convergence, quasi-strongly lacunary invariant convergence, and quasi-strongly s-lacunary invariant convergence, as well as quasi-lacunary invariant statistical convergence. By investigating the relationships among these new types of convergence and existing ones, we have expanded the theoretical framework, enhancing our understanding of sequence behavior in quaternion-valued generalized metric spaces.

This study not only fills gaps in the current mathematical literature but also lays the groundwork for future research in various scientific and engineering disciplines. The newly defined convergence types and their interrelations offer promising directions for further theoretical developments and practical applications in fields requiring robust analytical methods in higher-dimensional spaces.

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