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On rough $\mathscr{I}_2$-lacunary statistical convergence of double sequences in cone metric spaces

Research Article

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Abstract: This study aims to explore recent advancements in the convergence of double sequences within cone metric spaces (CMS) and their practical applications. We introduce the novel concept of rough $\mathscr{I}_2$-lacunary statistical convergence in CMS, extending the notion of rough convergence. Additionally, we delve into the concept of rough $\mathscr{I}_2$-lacunary statistical limit sets for double sequences in CMS. The investigation is challenged by the ordering structure inherent in CMS, yet we uncover key properties of sequences related to rough $\mathscr{I}_2$-lacunary statistical convergence in this context.

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1. Introduction

Fast [10] pioneered the statistical convergence of sequences within the real number domain, as discussed in their work. Pringsheim [33], on the other hand, explored the convergence of real double sequences. The extension of this idea to statistical convergence of real double sequences was carried out by Mursaleen and Edely [27]. Various studies, identified by different references such as [4, 12, 18, 25, 26, 35, 36], delved into this concept under different names. Das et al. [6] employed ideals in $\mathbb{N}^2$ to generalize the statistical convergence of double sequences to $\mathscr{I}$-convergence of double sequences. Further insights into this topic can be found in [7, 13, 14, 38, 40], among others.

Fridy and Orhan's exploration of lacunary statistical convergence, as documented in [11], has sparked significant research into the summability studies of classical theories. Patterson and Savaş [31] reconsidered the lacunary statistical convergence of double sequences, unveiling various properties associated with this form of convergence. A wealth of investigations on lacunary statistical convergence can be explored in [5, 20, 39], among others.

Phu initiated the exploration of rough convergence, as detailed in [32]. Malik et al. [22] delved into rough convergence for double sequences within normed linear spaces. Building upon this, Malik et al. [23] extended rough convergence of double sequences to rough statistical convergence. Dündar et al. [9] then generalized rough statistical convergence of double sequences to rough $\mathscr{I}$-convergence of double sequences. Malik and Ghosh [24] introduced the concepts of rough $\mathscr{I}$-statistical convergence convergence of double sequences in normed linear spaces.

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The introduction of CMS, credited to Huang and Xian [15], involved replacing the distance between two points in a real Banach space with elements of that space. CMS serves as a clear generalization of the ordinary metric space concept. In the exploration by Banerjee and Mondal [2], the rough convergence of sequences in a CMS was thoroughly investigated. CMS, a concept defined by various authors over the years, have been discussed under different names in the literature, as evidenced by references like [1, 3, 8, 15, 30, 34, 37, 41].

Section 2 of this article will acquaint readers with fundamental concepts of $\mathscr{I}$-statistical convergence for both single and double sequences. It will delve into the implications of this convergence, as well as provide definitions and properties of CMS. Additionally, the article will explore the concept of rough convergence and rough $\mathscr{I}$-convergence of sequences within a CMS. Moving on to Section 3, the focus will shift to introducing rough $\mathscr{I}_2$-lacunary statistical convergence and rough $\mathscr{I}_2^*$-lacunary statistical convergence for double sequences within CMS.

2. Preliminaries

In this section, we will gather essential results and techniques that form the foundation for achieving our main objectives. To start, we'll delve into crucial terminology and concepts.

Definition 2.1 (Kostyrko et al. [19]).
Assuming $Y \neq \emptyset$, let $\mathscr{I} \subset 2^Y$ be designated as an ideal on Y under the following conditions:
1. For each $U, V \in \mathscr{I}$, $U \cup V \in \mathscr{I}$;
2. For each $U \in \mathscr{I}$ and $V \subset P$, $V \in \mathscr{I}$.

Definition 2.2 (Kostyrko et al. [19]).
Assuming $Y \neq \emptyset$, let $\mathscr{F} \subset 2^Y$ be designated as a filter on Y under the following conditions:
1. For all $U, V \in \mathscr{F}$, $U \cap V \in \mathscr{F}$;
2. For all $U \in \mathscr{F}$ and $V \supset P$, $V \in \mathscr{F}$.

An ideal $\mathscr{I}$ is considered non-trivial if $Y \notin \mathscr{I}$ and $\mathscr{I} \neq \emptyset$. A non-trivial ideal $\mathscr{I} \subset P(Y)$ is termed an admissible ideal in Y iff $\mathscr{I} \supset \{\{w\} : w \in Y\}$. Subsequently, the filter $F = F(\mathscr{I}) = \{Y - S : S \in \mathscr{I}\}$ is referred to as the filter associated with the ideal.

By employing the concept of ideals, Kostyrko et al. [19] established the notions of $\mathscr{I}$ and $\mathscr{I}^*$-convergence. Additionally, Kostyrko et al. [19] provided the definition of the (AP) condition for an admissible ideal and investigated the relationship between $\mathscr{I}$ and $\mathscr{I}^*$-convergence under the (AP) condition.

Refer to the citations in [28, 29] for further details on $\mathscr{I}$-convergence.

Now, we will introduce the concept of $\mathscr{I}_2$-asymptotic density of $\mathbb{N}^2$.

$K \subset \mathbb{N}^2$ is a subset with $\mathscr{I}_2$-asymptotic density $d_{\mathscr{I}_2}(K)$ when

$$d_{\mathscr{I}_2}(K) = \mathscr{I}_2 - \lim_{m,n\to\infty} \frac{|K(m,n)|}{m.n},$$

where

$$K(m,n) = \left\{(s,t) \in \mathbb{N} \times \mathbb{N} : s \leq m, t \leq n; \left(j,k\right) \in K\right\},$$

and $|K(m,n)|$ demonstrates number of elements of the set $K(m,n)$.

A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N}^2$ is termed strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathscr{I}_2$ for each $i \in \mathbb{N}$.

Throughout the study, we consider $\mathscr{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.3 (Das et al. [6]).
Suppose that (Y, ρ) be a metric space. A double sequence $y = (y_{st})$ is called to be $\mathscr{I}_2$-convergent to y^*, if for any $\sigma > 0$ we have

$$A(\sigma) := \{(u,v) \in \mathbb{N} \times \mathbb{N} : \rho(y_{st}, y^*) \geq \sigma\} \in \mathscr{I}_2.$$

We write

$$\mathscr{I}_2 - \lim_{s,t\to\infty} y_{st} = y^*.$$

Definition 2.4 (Das et al. [6]).
We state that an admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}^2}$ satisfies condition $(AP2)$ if, for every countable family of mutually disjoint sets $\{U_1, U_2, ...\} \in \mathscr{I}_2$, there exists a countable family of sets $\{V_1, V_2, ...\} \in \mathscr{I}_2$ such that $U_j \Delta V_j \in \mathscr{I}_0$ i.e., $U_j \Delta V_j$ is included in the finite union of rows and columns in $\mathbb{N}^2$ for each $j \in \mathbb{N}$ and $V = \bigcup_{j=1}^{\infty} V_j \in \mathscr{I}_2$ (so $V_j \in \mathscr{I}_2$ for all $j \in \mathbb{N}$).

A double sequence $y = (y_{st})$ in a normed linear space $(Y, \|.\|)$ is considered rough convergent (r-convergent) to y^* with the roughness degree r, denoted by $y_{st} \xrightarrow{r} y^*$ provided that

$$\forall \sigma > 0 \ \exists k_\sigma \in \mathbb{N} : s, t \geq k_\sigma \Rightarrow \| y_{st} - y^* \| < r + \sigma,$$

or identically, if

$$\limsup \| y_{st} - y^* \| \leq r.$$

A double sequence $y = (y_{st})$ is called to be r-$\mathscr{I}_2$-convergent to y^* with the roughness degree r, denoted by $y_{st} \xrightarrow{r-\mathscr{I}_2} y^*$ provided that

$$\{(s,t) \in \mathbb{N} \times \mathbb{N} : \| y_{st} - y^* \| \geq r + \sigma\} \in \mathscr{I}_2,$$

for all $\sigma > 0$.

A double sequence $\overline{\theta} = \theta_{mn} = \{(k_m, l_n)\}$ is termed a double lacunary sequence when there exist two increasing sequences of integers (k_m) and (l_n) such that

$$k_0 = 0, \ h_m = k_m - k_{m-1} \to \infty \ \text{ and } \ l_0 = 0, \ \overline{h}_n = l_n - l_{n-1} \to \infty, \quad m, n \to \infty.$$

We will use the following notations $k_{mn} := k_m l_n$, $h_{mn} := h_m \overline{h}_n$ and θ_{mn} is defined by

$$I_{mn} := \{(k,l) : k_{m-1} < k \leq k_m \text{ and } l_{n-1} < l \leq l_n\},$$

$$q_m := \frac{k_m}{k_{m-1}}, \ \overline{q}_n := \frac{l_n}{l_{n-1}} \text{ and } q_{mn} := q_m \overline{q}_n.$$

Throughout the paper, by $\theta_2 = \theta_{mn} = \{(k_m, l_n)\}$ we will denote a double lacunary sequence.

A double sequence $y = (y_{st})$ is said to be $\mathscr{I}_{\theta_2}$-statistical convergent or $S_{\theta_2}(\mathscr{I}_2)$-convergent to y^*, provided that for all $\sigma, \delta > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : \left| y_{st} - y^* \right| \geq \sigma\right\}\right| \geq \delta\right\} \in \mathscr{I}_2.$$

Symbolically, we denote $y_{st} \to y^* \left(S_{\theta_2}(\mathscr{I}_2)\right)$ or $S_{\theta_2}(\mathscr{I}_2)$-$\lim_{s,t\to\infty} y_{st} = y^*$.

A double sequence $y = (y_{st})$ is said to be rough $\mathscr{I}_2$-lacunary statistical convergent to y^* or r-$\mathscr{I}_{\theta_2}$-statistical convergent to y^* if for any $\sigma, \delta > 0$

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : \left\| y_{st} - y^* \right\| \geq r + \sigma\right\}\right| \geq \delta\right\} \in \mathscr{I}_2.$$

Symbolically, we indicate $y_{st} \xrightarrow{r-\mathscr{I}_{\theta_2}-st} y^*$.

We now revisit the fundamental concepts from [15, 16], which are essential for the rest of the article.

Definition 2.5.
Let E be a Hausdorff topological vector space (tvs) with the zero vector 0. A subset P of E is called a (convex) cone if it satisfies the following conditions:
(i) $P \neq \{0\}, P \neq \emptyset$ and P is closed;
(ii) $\lambda P \subset P$ for $\forall \lambda \geq 0$ and $P + P \subset P$;
(iii) $\{0\} = P \cap (-P)$.

If we have a cone $P \subset E$, we can establish a partial ordering $\preceq$ with respect to P by defining $x \preceq y \iff y - x \in P$. We will use $x \prec y$ to signify that $x \preceq y$ but $x \neq y$, and $x \ll y$ to denote $y - x \in \operatorname{int} P$. The sets of interior points of P are denoted by $\operatorname{int} P$. Order-intervals are sets of the form $[x, y]$ and are defined as follows:

$$[x, y] = \{z \in E : x \preceq z \preceq y\}.$$

The order-intervals are found to be convex. If $[x, y] \subset A$ while $x, y \in A$ and $x \preceq y$, then $A \subset E$ is *order-convex*.

If an ordered tvs (E, P) possesses neighborhoods with a base of 0 consisting of *order-convex* sets, it is considered order-convex. Consequently, the cone P is termed a normal cone. In the context of a normed space, this condition implies that the unit ball is *order-convex*, which is equivalent to the existence of a constant k such that for any $x, y \in E$ and $0 \preceq x \preceq y$, it holds that $\|x\| \leq k \|y\|$. The normal constant of P is defined as the smallest constant k [16].

If every increasing sequence bounded by P is convergent, we can characterize P as a regular cone. In other words, if there exists a sequence $\{x_n\}$ such that

$$x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq \ldots \preceq y,$$

for some $y \in E$, then there exists $y \in E$ so that $\lim_{n\to\infty} \|x_n - x\| = 0$. Similarly, the cone P is regular when all decreasing sequences that are bounded from below converge. If P is a regular cone, it is known to be a normal cone.

Let E be a tvs, $V \subset E$ be an absolutely convex and absorbent subset, and

$$x \mapsto f_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}$$

be the corresponding Minkowski functional $f_V : E \to \mathbb{R}$. On E, it is a semi-norm. Let V be an absolutely convex neighborhood of $0 \in E$. Then f_V is continuous and

$$\{x \in E : f_V(x) < 1\} = \operatorname{int} V \subset V \subset \overline{V} = \{x \in E : f_V(x) \leq 1\}.$$

Let $e \in \operatorname{int} P$ and (E, P) be an ordered tvs. After that

$$[-e, e] = (P - e) \cap (e - P) = \{z \in E : -e \preceq z \preceq e\}$$

is an absolutely convex neighborhood of 0. The corresponding Minkowski functional $f_{[-e,e]}$ is denoted by f_e. It is possible to prove that $\operatorname{int}[-e, e] = (\operatorname{int} P - e) \cap (e - \operatorname{int} P)$. The Minkowski functional f_e is the norm on E when P is normal and solid. It is also an increasing function on P. In fact, for $0 \preceq x_1 \preceq x_2$ the set $\{\lambda : x_1 \in \lambda[-e, e]\}$ is the subset of $\{\lambda : x_2 \in \lambda[-e, e]\}$ and it follows that $f_e(x_1) \leq f_e(x_2)$.

Definition 2.6 (Huang and Zhang [15]).
Take $Y \neq \emptyset$. Suppose that $\rho : Y \times Y \to W$ supplies
(d_1) $\rho(u, v) = 0$ iff $u = v$ and $0 \preceq \rho(u, v)$ for $\forall u, v \in Y$;
(d_2) $\rho(v, u) = \rho(u, v)$ for $\forall u, v \in Y$;
(d_3) $\rho(u, v) \preceq \rho(u, w) + \rho(w, v)$ for $\forall u, v, w \in Y$.
Then, ρ is referred to as a cone metric on Y. (Y, ρ) is termed a CMS. Obviously, the concept of CMS generalizes the notion of metric spaces.

Definition 2.7 (Huang and Zhang [15]).
Let (Y, ρ) be an CMS. $\{y_s\}_{s\in\mathbb{N}}$ be a sequence in CMS Y and assume $y^* \in Y$. If for $\forall c \in Y$ with $0 \ll c$ there is $N \in \mathbb{N}$ so that for all $s > N$, $\rho(y_s, y^*) \ll c$, then $\{y_s\}_{s\in\mathbb{N}}$ is called to be convergent to y^* and it is named the limit of the sequence $\{y_s\}_{s\in\mathbb{N}}$.

Definition 2.8 (Huang and Zhang [15]).
A sequence $\{y_s\}_{s\in\mathbb{N}}$ in Y is called to be $\mathscr{I}^*$-convergent to $y^* \in Y$ iff there is a set $M \in \mathscr{F}(\mathscr{I})$, $M = \{m_1 < m_2 < \cdots < m_j < \ldots\}$ so that $\lim_{j\to\infty} y_{m_j} = y^*$, that is for $\forall c \in Y$ with $c \ll 0$, there is $p \in \mathbb{N}$ such that $c - \rho\left(y_{m_j}, y^*\right) \in \operatorname{int} P$, for $\forall j \geq p$.

Lemma 2.1 (Khani and Pourmahdian [17]).
Assume (Y, W) be an CMS with $x \in P$ and $y \in intP$. Then, one can find $n \in \mathbb{N}$ such that $x << ny$.

Theorem 2.1 (Banerjee and Monda [2]).
Assume that W be a real Banach space and let P be a cone in W.
(i) Let $x_0 \in intP$ and $\alpha (> 0) \in \mathbb{R}$. Then, we have $\alpha x_0 \in intP$.
(ii) Let $x_0 \in P$ and $y_0 \in intP$. Then, we have $x_0 + y_0 \in intP$.
(iii) Let $x_0, y_0 \in intP$. Then, we have $x_0 + y_0 \in intP$.
(iv) We have $0 \notin intP$.

Definition 2.9 (Banerjee and Monda [2]).
Let $\{y_s\}_{s\in\mathbb{N}}$ be a sequence in CMS (Y, ρ). A point $c \in Y$ is said to be a cluster point of $\{y_s\}$ provided that for any $(0 <<) \sigma$ in W and for any $p \in \mathbb{N}$, there is a $p_1 \in \mathbb{N}$ so that $p_1 > p$ with $\rho\left(y_{p_1}, c\right) << \sigma$.

Definition 2.10 (Pal et al. [30]).
[Pal et al. [30]] A sequence $\{y_s\}$ in Y is called to be $\mathscr{I}$-convergent to $y^* \in Y$ if for any $c \in W$ with $(0 <<) c$ the set

$$\{s \in \mathbb{N} : c - \rho\left(y_s, y^*\right) \notin intP\} \in \mathscr{I}.$$

Definition 2.11 (Pal et al. [30]).
A sequence $\{y_s\}$ in Y is named to be $\mathscr{I}^*$-convergent to $y^* \in Y$ iff there is a set $M \in \mathscr{F}(\mathscr{I})$, $M = \{m_1 < m_2 < \cdots < m_j < \ldots\}$ so that $\{y_s\}_{s\in M}$ is convergent to y^* i.e., for any $c \in W$ with $(0 <<) c$ there exists $p \in \mathbb{N}$ such that

$$c - \rho\left(y_{m_k}, y^*\right) \notin intP$$

for all $k \geq p$.

As it is well known [41], every cone metric space is a first countable Hausdorff topological space with the topology imposed by the open balls naturally specified for each element z in X and for each element c in int P. As demonstrated in [21], $\mathscr{I}^*$-convergence always entails $\mathscr{I}$-convergence, but the converse is not true. The two concepts are equivalent if and only if the ideal $\mathscr{I}$ has condition (AP).

Definition 2.12 (Banerjee and Monda [2]).
Let (Y,ρ) be an CMS. A sequence $\{y_s\}$ in Y is said to be rough convergent of roughness degree r to $y^* \in Y$ for some $(0 << r) \in W$ or $r = 0$ if for any $\sigma > 0$ with $(0 <<) \sigma$ there exists a $m \in \mathbb{N}$ so that $\rho\left(y_s, y^*\right) << r + \sigma$ for all $s \geq m$.

Definition 2.13 (Banerjee and Paul [3]).
A sequence $\{y_s\}$ in Y is called to be rough $\mathscr{I}$-convergent of roughness degree r to $y^* \in Y$ for some $(0 << r) \in W$ or $r = 0$ if for any $\sigma > 0$ with $(0 <<) \sigma$ the set

$$A(\sigma) = \left\{s \in \mathbb{N} : \left(r + \sigma - \rho\left(y_s, y^*\right)\right) \notin intP\right\} \in \mathscr{I}.$$

3. Main Results

Throughout our work (Y,ρ) stands for an CMS where $\rho : Y \times Y \to W$ is the cone metric, W being a real Banach space and $\mathscr{I}_2$ stands for a strongly admissible ideal in $\mathbb{N}^2$.

Definition 3.1.
Suppose (Y,ρ) be an CMS. A sequence $\{y_{st}\}$ is said to be lacunary statistically convergent to $y^* \in Y$ for any $\sigma > 0$ with $(0 <<) \sigma$ there exists a $(s,t) \in I_{mn}$ such that

$$\lim_{m,n\to\infty} \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : \sigma - \rho\left(y_{st}, y^*\right) \notin intP\right\}\right| = 0.$$

Symbolically, we denote $S_{\theta_2} - \lim y_{st} = y^*$.

Definition 3.2.
A sequence $y = \{y_{st}\}$ in Y is called to be $\mathscr{I}_2$-lacunary statistically convergent to $y^* \in Y$ provided that for any $(0 <<) \sigma \in W$ and for all $\kappa > 0$, the set

$$T(\sigma) := \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : \sigma - \rho\left(y_{st}, y^*\right) \notin intP\right\}\right| \geq \kappa\right\} \in \mathscr{I}_2.$$

Symbolically, we indicate $y_{st} \overset{\mathscr{I}_{\theta_2}-st}{\to} y^*$.

Definition 3.3.
A sequence $\{y_{st}\}$ in Y is said to be rough lacunary statistically convergent of roughness degree r to $y^* \in Y$ for some $(0 << r) \in W$ or $r = 0$ i.e., for any $\sigma > 0$ with $(0 <<) \sigma$ there exists a $(s,t) \in I_{mn}$ such that

$$\lim_{m,n\to\infty} \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : \left(r + \sigma - \rho\left(y_{st}, y^*\right)\right) \notin intP\right\}\right| = 0.$$

Symbolically, we denote $r - S_{\theta_2} - \lim y_{st} = y^*$.

Definition 3.4.
A sequence $\{y_{st}\}$ is called to be rough $\mathscr{I}_2$-convergent of roughness degree r to $y^* \in Y$ for some $r \in W$ with $0 << r$ or $r = 0$ provided that for any $(0 <<) \sigma \in W$, the set

$$T(\sigma) := \left\{(s,t) \in \mathbb{N}^2 : \left(r + \sigma - \rho\left(y_{st}, y^*\right)\right) \notin intP\right\} \in \mathscr{I}_2.$$

Symbolically, we demonstrate $y_{st} \overset{r-\mathscr{I}_2}{\to} y^*$ or $r - \mathscr{I}_2 - \lim y_{st} = y^*$.

Definition 3.5.
A sequence $y=\{y_{st}\}$ in Y is named to be rough $\mathscr{I}_2$-lacunary statistically convergent of roughness degree r to $y^*\in Y$ for some $r\in W$ with $0<<r$ or $r=0$ provided that for any $(0<<)\sigma\in W$ and for all $\kappa>0$, the set

$$T(\sigma):=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\left(r+\sigma-\rho\left(y_{st},y^*\right)\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2.$$

Symbolically, we indicate $y_{st}\overset{r-\mathscr{I}_{\theta_2}-st}{\rightarrow}y^*$.

For $r=0$ the description of rough $\mathscr{I}_{\theta_2}$-statistically convergence reduces to the description of $\mathscr{I}_{\theta_2}$-statistically convergence of sequence in an CMS. When a sequence $y=\{y_{st}\}$ is rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r to $y^*\in Y$ then y^* is named the $r-\mathscr{I}_{\theta_2}-st$-limit of $y=\{y_{st}\}$. Generally, the $r-\mathscr{I}_{\theta_2}-st$-limit of a sequence $y=\{y_{st}\}$ is not unique which can be examined from the following example. As a result, the set of all $r-\mathscr{I}_{\theta_2}-st$-limits of a sequence $y=\{y_{st}\}$ indicated by $\mathscr{I}_{\theta_2}$-st-LIM$^r y$ is known as the $r-\mathscr{I}_{\theta_2}-st$-limit set of a sequence $y=\{y_{st}\}$ i.e.,

$$\mathscr{I}_{\theta_2}\text{-}st\text{-LIM}^r y:=\left\{y^*\in Y:y_{st}\overset{r-\mathscr{I}_{\theta_2}-st}{\rightarrow}y^*\right\}.$$

Hence, a sequence $y=\{y_{st}\}$ is called to be rough $\mathscr{I}_{\theta_2}$-statistically convergent in an CMS when $\mathscr{I}_{\theta_2}$-st-LIM$^r y\neq\emptyset$.

Example 3.1.
Presume $Y=\mathbb{R}$, $W=\mathbb{R}^2$, $P=\{(u,v)\in W:u,v\geq 0\}\subset W$ and $\rho:Y\times Y\to W$ be a metric. At that time, (Y,ρ) is an CMS. Let us examine the ideal in $\mathbb{N}^2$ which consists of sets whose natural density are zero i.e., $\mathscr{I}_2=\mathscr{I}_2^d$. Also, let us contemplate the sequence $y=\{y_{st}\}$ in Y identified by

$$y_{st}=\begin{cases} st, & \text{if } u-\left[\sqrt{h_m}\right]+1\leq s<u,\ v-\left[\sqrt{h_n}\right]+1\leq t<v\\ 0, & \text{if not.}\end{cases}$$

Now, we can get that for any $r=(r_1,r_2)\in W$ with $0<<r$, when $\min(r_1,r_2)=r^*$ and $r^*\geq 1$ then

$$\mathscr{I}_{\theta_2}-st-LIM^r y=\left[-\left(r^*-1\right),\left(r^*-1\right)\right],$$

as for any $y^*\in[-(r^*-1),(r^*-1)]$ with $r^*\geq 1$ we get

$$\lim_{m,n\to\infty}\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\left(r+\sigma-\rho\left(y_{st},y^*\right)\right)\notin intP\right\}\right|=0,$$

and

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\ \left(r+\sigma-\rho\left(y_{st},y^*\right)\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2.$$

When $r^*<1$ or $r=0$ then $\mathscr{I}_{\theta_2}$-st-LIM$^r y=\emptyset$.

From the above example, we can observe that, in general $\mathscr{I}_{\theta_2}$-st-LIM$^r y\neq\emptyset$ does not necessarily mean S_{θ_2}-LIM$^r y\neq\emptyset$. However, since $\mathscr{I}_2$ is an admissible ideal, S_{θ_2}-LIM$^r y\neq\emptyset$ implies $\mathscr{I}_{\theta_2}$-st-LIM$^r y\neq\emptyset$. In other words, when a sequence $y=\{y_{st}\}$ in (Y,ρ) is rough lacunary statistically convergent of roughness degree r, where $r\in W$ with $0<<r$ or $r=0$, then it is also rough $\mathscr{I}_{\theta_2}$-statistically convergent with a similar roughness degree r. Hence, when we denote all rough lacunary statistical convergent sequences in an CMS (Y,ρ) by S_{θ_2}-LIM$^r y$ and the set of all rough $\mathscr{I}_{\theta_2}$-statistical convergent sequences by $\mathscr{I}_{\theta_2}$-st-LIM$^r y$, then we have S_{θ_2}-LIM$^r y\subseteq\mathscr{I}_{\theta_2}$-$st$-LIM$^r y$.

A sequence $\{y_{st}\}$ in an CMS (Y,ρ) is named to be bounded when there is a $y^*\in Y$ and $r>0$ supplying $\rho\left(y_{st},y^*\right)<r$ for all $s,t\in\mathbb{N}$.

With this perspective, we define an $\mathscr{I}_2$-statistically bounded sequence in an CMS as follows:

Definition 3.6.
A sequence $y=\{y_{st}\}$ in an CMS (Y,ρ) is called to be $\mathscr{I}_{\theta_2}$-statistically bounded if there is a $y^*\in Y$ and $Q\in W$ with $0<<Q$ such that

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:Q-\rho\left(y_{st},y^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2.$$

Suppose that $\{y_{st}\}$ be bounded sequence in an CMS (Y,ρ), then there is a $z^* \in Y$ and $Q \in W$ with $0 << Q$ such that $\rho(z^*, y_{st}) << Q$ for all $s, t \in \mathbb{N}$. This means that $Q - \rho(z^*, y_{st}) \in intP$ for all $s, t \in \mathbb{N}$. Therefore

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : Q - \rho(z^*, y_{st}) \notin intP\right\}\right| \geq \kappa\right\} = \emptyset \in \mathscr{I}_2.$$

Hence, $\{y_{st}\}$ is $\mathscr{I}_{\theta_2}$-statistically bounded. However, as illustrated in Example 3.1, the reverse may not be accurate. For instance, if we choose $y^* = 2$ and $(0 <<) Q = (5,6)$, then we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : Q - \rho(y_{st}, y^*) \notin intP\right\}\right| \geq \kappa\right\} \subset \{1^2, 2^2, 3^2, ...\}$$

which gives that

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : Q - \rho(y_{st}, y^*) \notin intP\right\}\right| \geq \kappa\right\} \in \mathscr{I}_2.$$

Therefore, the sequence under consideration is $\mathscr{I}_2$-lacunary statistically bounded.

Theorem 3.1.
Take $\mathscr{I}_2$ as an admissible ideal of $\mathbb{N}^2$. Then, a sequence $y = \{y_{st}\}$ in an CMS (Y,ρ) is $\mathscr{I}_{\theta_2}$-statistically bounded iff there exists some $r \in W$ with $0 << r$ or $r = 0$ such that $\mathscr{I}_{\theta_2}$-st-$LIM^r y \neq \emptyset$.

Proof. Suppose that the sequence $y = \{y_{st}\}$ be $\mathscr{I}_{\theta_2}$-statistically bounded. Then, there is a $y^* \in Y$ and $(0 <<) r \in W$ such that the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : r - \rho(y_{st}, y^*) \notin intP\right\}\right| \geq \kappa\right\} \in \mathscr{I}_2.$$

Take $(0 <<) \sigma \in W$ (i.e., $\sigma \in intP$). So

$$\begin{aligned}&\left\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : r + \sigma - \rho(y_{st}, y^*) \notin intP\right\}\right| \geq \kappa\right\} \\ &\subseteq \left\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : r - \rho(y_{st}, y^*) \notin intP\right\}\right| \geq \kappa\right\} \in \mathscr{I}_2.\end{aligned}$$

Assume

$$(m,n) \in \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : r + \sigma - \rho(y_{st}, y^*) \notin intP\right\}\right| \geq \kappa\right\}.$$

Then, we get $r + \sigma - \rho(y_{st}, y^*) \notin intP$. So, $r - \rho(y_{st}, y^*) \notin intP$, hence we have

$$(m,n) \in \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : r - \rho(y_{st}, y^*) \notin intP\right\}\right| \geq \kappa\right\}.$$

So, we obtain $y^* \in \mathscr{I}_{\theta_2}$-$st$-LIM$^r y$.

Conversely, assume $\mathscr{I}_{\theta_2}$-st-LIM$^r y \neq \emptyset$ for some $r \in W$ with $0 << r$ or $r = 0$ and $z^* \in \mathscr{I}_{\theta_2}$-$st$-LIM$^r y$. So, for any $(0 <<) \sigma \in W$ (i.e., $\sigma \in intP$) the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : r + \sigma - \rho(y_{st}, z^*) \notin intP\right\}\right| \geq \kappa\right\} \in \mathscr{I}_2.$$

Now $r + \sigma \in intP$ for each $\sigma \in intP$. Hence, let $Q = r + \sigma \in intP$ (i.e., $0 << Q$), we get

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : Q - \rho(y_{st}, z^*) \notin intP\right\}\right| \geq \kappa\right\} \in \mathscr{I}_2.$$

As a result, $y = \{y_{st}\}$ is $\mathscr{I}_{\theta_2}$-statistically bounded. □

Theorem 3.2.
An $\mathscr{I}_{\theta_2}$-statistically bounded sequence $y = \{y_{st}\}$ in an CMS (Y,ρ) contains a subsequence that is rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r for some $(0 <<) r \in W$.

Proof. Let's assume a sequence $y=\{y_{st}\}$ in a CMS (Y,ρ) is $\mathscr{I}_{\theta_2}$-statistically bounded. So, there is a $x^*\in Y$ and $(0<<)\,Q\in W$ so that the set

$$A=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:Q-\rho\left(y_{st},x^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2,$$

i.e.,

$$A^c=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:Q-\rho\left(y_{st},x^*\right)\in intP\right\}\right|<\kappa\right\}\in\mathscr{F}(\mathscr{I}_2).$$

Upon examining the subsequence $\{y_{st}\}_{s,t\in A^c}$, we observe its lacunary statistical boundedness. Similar to any lacunary statistically bounded sequence $y=\{y_{st}\}$, S_{θ_2}-LIM$^r y\neq\emptyset$ for some $(0<<)\,r\in W$. Consequently, the subsequence $\{y_{st}\}_{s,t\in A^c}$ is rough lacunary statistically convergent with a roughness degree r ($(0<<)\,r\in W$). Thus, we have $\{y_{st}\}_{s,t\in A^c}$ is also rough $\mathscr{I}_{\theta_2}$-statistically convergent with roughness degree r ($(0<<)\,r\in W$). □

Theorem 3.3.
Take $y=\{y_{st}\}$ as a sequence in an CMS which is $\mathscr{I}_{\theta_2}$-statistically convergent to y^*. If $z=\{z_{st}\}$ is another sequence in (Y,ρ) such that $\rho\left(y_{st},z_{st}\right)\leq r$ for some $(0<<)\,r\in W$ and for all $s,t\in\mathbb{N}$, then, $z=\{z_{st}\}$ is rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r to y^*.

Proof. Take $y=\{y_{st}\}$ as a sequence in an CMS which is $\mathscr{I}_{\theta_2}$-statistically convergent to y^*. For $(0<<)\,\sigma\in W$ the set

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\sigma-\rho\left(y_{st},y^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2,$$

i.e.,

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\sigma-\rho\left(y_{st},y^*\right)\in intP\right\}\right|<\kappa\right\}\in\mathscr{F}(\mathscr{I}_2).$$

Also

$$\rho\left(z_{st},y^*\right)\leq\rho\left(z_{st},y_{st}\right)+\rho\left(y_{st},y^*\right)<r+\rho\left(y_{st},y^*\right).$$

This implies that $r+\rho\left(y_{st},y^*\right)-\rho\left(z_{st},y^*\right)\in P$. So, if $\sigma-\rho\left(y_{st},y^*\right)\in intP$, then

$$\left(r+\rho\left(y_{st},y^*\right)-\rho\left(z_{st},y^*\right)\right)+\left(\sigma-\rho\left(y_{st},y^*\right)\right)=r+\sigma-\rho\left(z_{st},y^*\right)\in intP.$$

Hence, the set

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:r+\sigma-\rho\left(z_{st},y^*\right)\in intP\right\}\right|<\kappa\right\}\in\mathscr{F}(\mathscr{I}_2).$$

As a result

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:r+\sigma-\rho\left(z_{st},y^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2$$

that implies $z=\{z_{st}\}$ is rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r to y^*. □

Theorem 3.4.
Take $y=\{y_{st}\}$ as a sequence in an CMS which is rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r for some $(0<<)\,r\in W$. Then, there does not exist x^*, $z^*\in\mathscr{I}_{\theta_2}$-$st$-LIM$y$ such that $nr<\rho(x^*,z^*)$, where $n(\in\mathbb{R})>2$.

Proof. Suppose, on the contrary, that there exist x^*, $z^*\in\mathscr{I}_{\theta_2}$-$st$-LIM$y$ such that $nr<\rho(x^*,z^*)$, where $n(\in\mathbb{R})>2$. Let's assume $(0<<)\sigma$ is arbitrarily selected in W. Since x^*, $z^*\in\mathscr{I}_{\theta_2}$-$st$-LIM$y$, we have

$$K_1=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:r+\frac{\sigma}{2}-\rho\left(y_{st},x^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2$$

and

$$K_2=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:r+\frac{\sigma}{2}-\rho\left(y_{st},z^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2.$$

Then, $K_1^c \in \mathscr{F}(\mathscr{I}_2)$ and $K_2^c \in \mathscr{F}(\mathscr{I}_2)$. Take $(u,v) \in K_1^c \cap K_2^c$. So, we have

$$r + \frac{\sigma}{2} - \rho\left(y_{uv}, x^*\right) \in intP \text{ and } r + \frac{\sigma}{2} - \rho\left(y_{uv}, z^*\right) \in intP.$$

Hence

$$\begin{aligned}&\left(r + \tfrac{\sigma}{2} - \rho\left(y_{uv}, x^*\right)\right) + \left(r + \tfrac{\sigma}{2} - \rho\left(y_{uv}, z^*\right)\right)\\ &= 2r + \sigma - \left(\rho\left(y_{uv}, x^*\right) + \rho\left(y_{uv}, z^*\right)\right) \in intP.\end{aligned}$$

Now

$$\rho\left(x^*, z^*\right) \leq \rho\left(y_{uv}, x^*\right) + \rho\left(y_{uv}, z^*\right),$$

so

$$\rho\left(y_{uv}, x^*\right) + \rho\left(y_{uv}, z^*\right) - \rho\left(x^*, z^*\right) \in P.$$

then, we obtain

$$\begin{aligned}&\left(2r + \sigma - \left(\rho\left(y_{uv}, x^*\right) + \rho\left(y_{uv}, z^*\right)\right)\right) + \left(\rho\left(y_{uv}, x^*\right) + \rho\left(y_{uv}, z^*\right) - \rho\left(x^*, z^*\right)\right)\\ &= 2r + \sigma - \rho\left(x^*, z^*\right) \in intP.\end{aligned}$$

Once again, according to our presumption $\rho(x^*, z^*) - nr \in P$. Hence

$$2r + \sigma - \rho\left(x^*, z^*\right) + \rho\left(x^*, z^*\right) - nr = 2r + \sigma - nr \in intP.$$

In other words, $\sigma - r(n-2) \in intP$. However, if we choose $\sigma = r(n-2)$, then we obtain $0 \in intP$, which leads to a contradiction. Therefore, the result is conclusive. □

Theorem 3.5.
Suppose $y = \{y_{st}\}$ be a sequence in an CMS which is rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r. Then, $\{y_{st}\}$ is also rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r_1 for any r_1 with $r < r_1$.

Proof. The proof is trivial and hence is omitted. □

In the light of the above theorem, we obtain the following consequence.

Corollary 3.1.
Let's assume $y = \{y_{st}\}$ is a rough $\mathscr{I}_{\theta_2}$-statistically convergent sequence in (Y,ρ) with a roughness degree r. Then, for a $(0<<)\, r_1$ with $r < r_1$, $LIM^r y \subset LIM^{r_1} y$.

Definition 3.7.
An element $\gamma \in Y$ is referred to as an $\mathscr{I}_{\theta_2}$-statistical cluster point of a double sequence $y = \{y_{st}\}$ in Y, provided that for any $(0<<)\sigma$, the set

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}}\left|\left\{(s,t) \in I_{mn} : \sigma - \rho\left(y_{st}, \gamma\right) \in intP\right\}\right| < \kappa\right\} \notin \mathscr{I}_2.$$

Theorem 3.6.
Take (Y,ρ) as an CMS. $\gamma \in Y$ and $(0<<)r$ be such that for any $y^* \in Y$ either $\rho\left(y^*, \gamma\right) \leq r$ or $r << \rho\left(y^*, \gamma\right)$. If γ is $\mathscr{I}_{\theta_2}$-statistical cluster point of a double sequence $y = \{y_{st}\}$ then $\mathscr{I}_{\theta_2}$-st-LIM$^r y \subset \overline{B_r}(\gamma)$, where $\overline{B_r}(\gamma) = \{y^* \in Y : \rho\left(y^*, \gamma\right) \leq r\}$.

Proof. If possible, let's assume that there is $x^* \in \mathscr{I}_{\theta_2}$-$st$-LIM$^r y$ but $x^* \notin \overline{B_r}(\gamma)$. According to our assumption, $r << \rho\left(x^*, \gamma\right)$. Take $(0<<)\sigma_1 = \rho\left(x^*, \gamma\right) - r$. Then, $\rho\left(x^*, \gamma\right) = r + \sigma_1$. Assume $(0<<)\sigma = \frac{\sigma_1}{2}$. Then, we have $\rho\left(x^*, \gamma\right) = r + 2\sigma$. Furthermore, we obtain $B_{r+\sigma}(x^*) \cap B_\sigma(\gamma) = \emptyset$. For, if $\alpha \in B_{r+\sigma}(x^*) \cap B_\sigma(\gamma)$ then $\rho(\alpha, x^*) << r + \sigma$ and $\rho(\alpha, \gamma) << \sigma$. So $r + \sigma - \rho(\alpha, x^*) \in intP$ and $\sigma - \rho(\alpha, \gamma) \in intP$. Hence

$$\left(r + \sigma - \rho\left(\alpha, x^*\right)\right) + \left(\sigma - \rho\left(\alpha, \gamma\right)\right) = r + 2\sigma - \left(\rho\left(\alpha, x^*\right) + \rho\left(\alpha, \gamma\right)\right) \in intP. \tag{1}$$

Since $\rho\left(x^*, \gamma\right) \leq \rho\left(x^*, \alpha\right) + \rho\left(\alpha, \gamma\right)$, therefore

$$\rho\left(x^*, \alpha\right) + \rho\left(\alpha, \gamma\right) - \rho\left(x^*, \gamma\right) \in P. \tag{2}$$

As a result from (1) and (2) we get

$$r+2\sigma-\left(\rho\left(\alpha,x^*\right)+\rho\left(\alpha,\gamma\right)\right)+\rho\left(x^*,\alpha\right)+\rho\left(\alpha,\gamma\right)-\rho\left(x^*,\gamma\right)=r+2\sigma-\rho\left(x^*,\gamma\right)=0\in intP,$$

which leads to a contradiction. Hence $B_{r+\sigma}(x^*)\cap B_\sigma(\gamma)=\emptyset$. Since $x^*\in\mathscr{I}_2$-st-LIM$^r y$, so the set

$$A=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:r+\sigma-\rho\left(y_{st},x^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2.$$

So, $A^c=\mathbb{N}^2\setminus A\in\mathscr{F}(\mathscr{I}_2)$. Once again, as γ is a $\mathscr{I}_{\theta_2}$-statistical cluster point of $y=\left\{y_{st}\right\}$ for $(0<<)\sigma$, one can write

$$B=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\sigma-\rho\left(y_{st},\gamma\right)\in intP\right\}\right|<\kappa\right\}\notin\mathscr{I}_2.$$

It is obvious that $B\not\subseteq A$. For example, if

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\sigma-\rho\left(y_{st},\gamma\right)\in intP\right\}\right|<\kappa\right\}\subset A,$$

we get

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\sigma-\rho\left(y_{st},\gamma\right)\in intP\right\}\right|<\kappa\right\}\in\mathscr{I}_2,$$

which contradicts the fact that γ is an $\mathscr{I}_{\theta_2}$-statistical cluster point of $y=\left\{y_{st}\right\}$. Here, we consider an element $(k,l)\in A^c$. So

$$(k,l)\in\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\sigma-\rho\left(y_{st},\gamma\right)\in intP\right\}\right|<\kappa\right\}.$$

Now, $(k,l)\in A^c$ means $r+\sigma-\rho\left(y_{kl},x^*\right)\in intP$. Hence, $\rho\left(y_{kl},x^*\right)<<r+\sigma$, which means $\left\{y_{kl}\right\}\in B_{r+\sigma}(x^*)$. Additionally

$$(k,l)\in\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\sigma-\rho\left(y_{st},\gamma\right)\in intP\right\}\right|<\kappa\right\}$$

gives $\sigma-\rho\left(y_{kl},\gamma\right)\in intP$. So $\rho\left(y_{kl},\gamma\right)<<\sigma$ which further means that $\left\{y_{kl}\right\}\in B_r(\gamma)$. As a consequence, we obtain $\left\{y_{kl}\right\}\in B_{r+\sigma}(x^*)\cap B_r(\gamma)$ which is a contradiction. So, we can deduce that assumption is incorrect and $x^*\in\overline{B_r(\gamma)}$. □

Theorem 3.7.
Assume $y=\left\{y_{st}\right\}$ be a rough $\mathscr{I}_{\theta_2}$-statistically convergence of roughness degree r in an CMS (Y,ρ) and $q=\left\{q_{st}\right\}$ be a $\mathscr{I}_{\theta_2}$-statistically convergent sequence in $\mathscr{I}_{\theta_2}$-st-LIM$^r y$ which is $\mathscr{I}_{\theta_2}$-statistically convergent to x^*. Then $x^*\in\mathscr{I}_{\theta_2}$-$st$-LIM$^r y$.

Proof. Suppose $(0<<)\sigma$ is taken. Since the sequence $q=\left\{q_{st}\right\}$ is $\mathscr{I}_{\theta_2}$-statistically convergent to x^*, for $(0<<)\sigma$ the set

$$A=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\frac{\sigma}{2}-\rho\left(q_{st},x^*\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2.$$

So, $A^c=\mathbb{N}^2\setminus A\in\mathscr{F}(\mathscr{I}_2)$. Take $(k,l)\in A^c$. Then $\frac{\sigma}{2}-\rho\left(q_{kl},x^*\right)\in intP$, and hence

$$\rho\left(q_{kl},x^*\right)<<\frac{\sigma}{2}. \quad (3)$$

In addition, as $q=\left\{q_{st}\right\}$ is a sequence in $\mathscr{I}_{\theta_2}$-st-LIM$^r y$, take $q_{kl}\in\mathscr{I}_{\theta_2}$-$st$-LIM^r. So, the set

$$B=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:r+\frac{\sigma}{2}-\rho\left(y_{st},q_{kl}\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2.$$

It is clear its complement $B^c=\mathbb{N}^2\setminus B\in\mathscr{F}(\mathscr{I}_2)$. Let us select an element $(h,j)\in B^c\,(\in\mathscr{F}(\mathscr{I}_2))$. So, $r+\frac{\sigma}{2}-\rho\left(y_{hj},q_{kl}\right)\in intP$, and hence

$$\rho\left(y_{hj},q_{kl}\right)<<r+\frac{\sigma}{2}. \quad (4)$$

Also for all $(s,t)\in\mathbb{N}^2$ we get

$$\rho\left(y_{st},x^*\right)\leq\rho\left(y_{st},q_{kl}\right)+\rho\left(q_{kl},x^*\right).$$

So

$$\rho\left(y_{st},q_{kl}\right)+\rho\left(q_{kl},x^{*}\right)-\rho\left(y_{st},x^{*}\right)\in P, \text{ for all } (s,t)\in\mathbb{N}^{2}.$$

Especially

$$\rho\left(y_{hj},q_{kl}\right)+\rho\left(q_{kl},x^{*}\right)-\rho\left(y_{hj},x^{*}\right)\in P. \tag{5}$$

According to (3) and (4) utilizing the Theorem 2.1 we obtain

$$\left(\frac{\sigma}{2}-\rho\left(q_{kl},x^{*}\right)\right)+\left(r+\frac{\sigma}{2}-\rho\left(y_{hj},q_{kl}\right)\right)=r+\sigma-\left(\rho\left(q_{kl},x^{*}\right)+\rho\left(y_{hj},q_{kl}\right)\right)\in intP. \tag{6}$$

Applying the Theorem 2.1 once more, we obtain from (5) and (6)

$$\begin{aligned}&\left(\rho\left(y_{hj},q_{kl}\right)+\rho\left(q_{kl},x^{*}\right)-\rho\left(y_{hj},x^{*}\right)\right)+\left(r+\sigma-\left(\rho\left(q_{kl},x^{*}\right)+\rho\left(y_{hj},q_{kl}\right)\right)\right)\\&=r+\sigma-\rho\left(y_{hj},x^{*}\right)\in intP.\end{aligned}$$

Now as $\mathscr{I}_2$ is arbitrarily selected from B^c, we have

$$\left\{(m,n)\in\mathbb{N}^{2}:\frac{1}{h_{mn}}\left|\left\{(h,j)\in I_{mn}:r+\sigma-\rho\left(y_{hj},x^{*}\right)\notin intP\right\}\right|\geq\kappa\right\}\subset B$$

and so

$$\left\{(m,n)\in\mathbb{N}^{2}:\frac{1}{h_{mn}}\left|\left\{(h,j)\in I_{mn}:r+\sigma-\rho\left(y_{hj},x^{*}\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_{2}.$$

Hence $x^{*}\in\mathscr{I}_{\theta_2}\text{-}st\text{-LIM}^{r}y$. □

Theorem 3.8.
When $y=\{y_{st}\}$ and $q=\{q_{st}\}$ are two sequences in an CMS (Y,ρ) such that for any $(0<<)\sigma$ the set

$$\left\{(m,n)\in\mathbb{N}^{2}:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\rho\left(y_{st},q_{st}\right)>\sigma\right\}\right|\geq\kappa\right\}\in\mathscr{I}_{2}.$$

Then, $y=\{y_{st}\}$ is rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r to x^{*} iff $q=\{q_{st}\}$ is rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r to x^{*}.

Proof. Suppose that $y=\{y_{st}\}$ be rough $\mathscr{I}_{\theta_2}$-statistically convergent of roughness degree r to x^{*}. Let $(0<<)\sigma$ given. Then, we have

$$A=\left\{(m,n)\in\mathbb{N}^{2}:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:r+\frac{\sigma}{2}-\rho\left(y_{st},x^{*}\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_{2}.$$

Also, by our assumption we get

$$B=\left\{(m,n)\in\mathbb{N}^{2}:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:\rho\left(y_{st},q_{st}\right)>\frac{\sigma}{2}\right\}\right|\geq\kappa\right\}\in\mathscr{I}_{2}.$$

$A^{c},B^{c}\in\mathscr{F}(\mathscr{I}_2)$ and hence $A^{c}\cap B^{c}\in\mathscr{F}(\mathscr{I}_2)$. Let us select an element $(k,l)\in\mathbb{N}^{2}$ so that $(k,l)\in A^{c}\cap B^{c}$. So

$$r+\frac{\sigma}{2}-\rho\left(y_{kl},x^{*}\right)\in intP \text{ and } \rho\left(y_{kl},q_{kl}\right)\leq\frac{\sigma}{2} \text{ i.e., } \frac{\sigma}{2}-\rho\left(y_{kl},q_{kl}\right)\in P.$$

Therefore

$$\left(r+\frac{\sigma}{2}-\rho\left(y_{kl},x^{*}\right)\right)+\left(\frac{\sigma}{2}-\rho\left(y_{kl},q_{kl}\right)\right)=r+\sigma-\left(\rho\left(y_{kl},x^{*}\right)+\rho\left(y_{kl},q_{kl}\right)\right)\in intP. \tag{7}$$

In addition for all $(s,t)\in\mathbb{N}^{2}$,

$$\rho\left(q_{st},x^{*}\right)\leq\rho\left(y_{st},q_{st}\right)+\rho\left(y_{st},x^{*}\right)$$

i.e.,

$$\rho\left(y_{st},q_{st}\right)+\rho\left(y_{st},x^{*}\right)-\rho\left(q_{st},x^{*}\right)\in P.$$

Especially

$$\rho\left(y_{kl},q_{kl}\right)+\rho\left(y_{kl},x^{*}\right)-\rho\left(q_{kl},x^{*}\right)\in P. \tag{8}$$

So from (7) and (8) we obtain

$$\begin{aligned}&\left(r+\sigma-\left(\rho\left(y_{kl},x^{*}\right)+\rho\left(y_{kl},q_{kl}\right)\right)\right)+\left(\rho\left(y_{kl},q_{kl}\right)+\rho\left(y_{kl},x^{*}\right)-\rho\left(q_{kl},x^{*}\right)\right)\\&=r+\sigma-\rho\left(q_{kl},x^{*}\right)\in intP.\end{aligned}$$

As a result, we get

$$\left\{(m,n)\in\mathbb{N}^{2}:\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn}:r+\sigma-\rho\left(q_{st},x^{*}\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_{2},$$

which means that $q=\{q_{st}\}$ is rough $\mathscr{I}_{\theta_2}$-statistically convergent to x^{*}. □

Definition 3.8.
A sequence $y = \{y_{st}\}$ in an CMS (Y,ρ) is called to be $\mathscr{I}^*_{\theta_2}$-statistical convergent to x^* if, there is a set $L \in \mathscr{F}(\mathscr{I}_2)$ (i.e., $\mathbb{N}^2 \setminus L \in \mathscr{I}_2$) such that the subsequence $\{y_{st}\}_{(s,t)\in L}$ is lacunary statistically convergent to x^* for any $\sigma > 0$ with $(0 <<)\sigma$ there is a $(s,t) \in I_{mn}$ such that

$$\lim_{m,n\to\infty} \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : \sigma - \rho\left(y_{st}, y^*\right) \notin intP\right\}\right| = 0.$$

We write $y_{st} \overset{\mathscr{I}^*_{\theta_2}-st}{\to} y^*$.

Definition 3.9.
A sequence $y = \{y_{st}\}$ in an CMS (Y,ρ) is said to be rough $\mathscr{I}^*_{\theta_2}$-statistical convergent with roughness degree r to x^* if, there is a set $L \in \mathscr{F}(\mathscr{I}_2)$ (i.e., $\mathbb{N}^2 \setminus L \in \mathscr{I}_2$) such that the subsequence $\{y_{st}\}_{(s,t)\in L}$ is rough lacunary statistically convergent with roughness degree r to x^* for some $(0 << r) \in W$ or $r = 0$ i.e., for any $\sigma > 0$ with $(0 <<)\sigma$ there is a $(s,t) \in I_{mn}$ so that

$$\lim_{m,n\to\infty} \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : r + \sigma - \rho\left(y_{st}, y^*\right) \notin intP\right\}\right| = 0.$$

We write $y_{st} \overset{r-\mathscr{I}^*_{\theta_2}-st}{\to} y^*$.

For $r = 0$ we get the definition of ordinary $\mathscr{I}^*_{\theta_2}$-statistical convergence of sequences in CMS. Obviously the rough $\mathscr{I}^*_{\theta_2}$-statistical limit of a sequence in general not unique. We can denote the set of all rough $\mathscr{I}^*_{\theta_2}$-statistical limit of a sequence $y = \{y_{st}\}$ by

$$\mathscr{I}^*_{\theta_2}\text{-}st\text{-LIM}^r y := \left\{ y^* \in Y : y_{st} \overset{r-\mathscr{I}^*_{\theta_2}-st}{\to} y^* \right\}.$$

of roughness degree r.

Theorem 3.9.
When $y_{st} \overset{r-\mathscr{I}^*_{\theta_2}-st}{\to} y^*$, then it is also $y_{st} \overset{r-\mathscr{I}_{\theta_2}-st}{\to} y^*$.

Proof. Let us presume that $y_{st} \overset{r-\mathscr{I}^*_{\theta_2}-st}{\to} y^*$. So, according to the definition there is a set $L \in \mathscr{F}(\mathscr{I}_2)$ (i.e., $Z = \mathbb{N}^2 \setminus L \in \mathscr{I}_2$) such that the subsequence $\{y_{st}\}_{(s,t)\in L}$ is rough lacunary statistically convergent with roughness degree r to x^* for some $(0 << r) \in W$ or $r = 0$ i.e., for any $\sigma > 0$ with $(0 <<)\sigma$ there exists a $(s,t) \in \mathbb{N}^2$ such that

$$\lim_{m,n\to\infty} \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : \left(r + \sigma - \rho\left(y_{st}, x^*\right)\right) \notin intP\right\}\right| = 0.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\rho\left(y_{st}, y^*\right) \ll r + \sigma$ then for all s,t such that $(s,t) \in L$ and $s,t \geq n_0$. Then

$$\begin{aligned} A(\sigma,\gamma) = &\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(s,t) \in I_{mn} : \left(r + \sigma - \rho\left(y_{st}, x^*\right)\right) \notin intP\right\}\right| \gg \kappa\right\} \\ &\subset Z \cup (L \cap ((\{1,2,\dots,(n_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\dots,(n_0-1)\}))). \end{aligned}$$

Now

$$Z \cup (L \cap ((\{1,2,\dots,(n_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\dots,(n_0-1)\}))) \in \mathscr{I}_2.$$

This demonstrates that $A(\sigma,\gamma) \in \mathscr{I}_2$. Therefore $y_{st} \overset{r-\mathscr{I}_{\theta_2}-st}{\to} y^*$. □

Theorem 3.10.
If an ideal $\mathscr{I}_2$ has the property $(AP2)$ then a sequence $y = \{y_{st}\}$ in an CMS (Y,ρ) which is rough $\mathscr{I}_{\theta_2}$-statistical convergent with roughness degree r to x^* is also rough $\mathscr{I}^*_{\theta_2}$-statistical convergent with roughness degree r to x^*.

Proof. Assume $\mathscr{I}_2$ be an ideal in $\mathbb{N}^2$ which supply the property $(AP2)$. Assume $y_{st} \overset{r-\mathscr{I}_{\theta_2}-st}{\to} y^*$. Then, for any $(0<<)\sigma \in W$ and for all $\kappa>0$, the set

$$T := \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\{(s,t)\in I_{mn} : (r+\sigma-\rho(y_{st},x^*))\notin intP\}\right| \geq \kappa\right\}\in\mathscr{I}_2.$$

So, we obtain

$$T^c := \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\{(s,t)\in I_{mn} : (r+\sigma-\rho(y_{st},x^*))\in intP\}\right| < \kappa\right\}\in\mathscr{F}(\mathscr{I}_2).$$

Take $(0<<)\eta\in W$. Now, let

$$A_i = \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn} : \rho(y_{st},x^*) << r+\frac{\eta}{i}\right\}\right| < \kappa\right\}\in\mathscr{F}(\mathscr{I}_2),$$

where $i=1,2,\ldots$. Since $\mathscr{I}_2$ has the property $(AP2)$, so there is a set $B\subset\mathbb{N}$ so that $B\in\mathscr{F}(\mathscr{I}_2)$ and $B\setminus A_i$ is finite for $i=1,2,\ldots$. Now take $(0<<)\sigma\in W$, then there is a $j\in\mathbb{N}$ so that $\frac{\eta}{j}<<\sigma$. Since $B\setminus A_j$ is finite, so there is a $t=t(j)\in\mathbb{N}$ such that $(m,n)\in B\cap A_j$ for all $m,n\geq t$. Hence $\rho(y_{st},x^*)<<r+\frac{\eta}{j}<<r+\sigma$ for all $(m,n)\in B$ and $m,n\geq t$. As a result, the subsequence $\{y_{st}\}_{s,t\in B}$ is rough lacunary statistically convergent to x^*, i.e.,

$$\lim_{m,n\to\infty}\frac{1}{h_{mn}}\left|\{(s,t)\in I_{mn} : r+\sigma-\rho(y_{st},y^*)\notin intP\}\right| = 0.$$

Hence, $y_{st} \overset{r-\mathscr{I}^*_{\theta_2}-st}{\to} x^*$. □

Theorem 3.11.
If $y'=\left\{y_{s_p t_q}\right\}_{p,q\in\mathbb{N}}$ be a subsequence of the sequence $y=\{y_{st}\}$, then $\mathscr{I}_{\theta_2}$-st-LIM$^r y\subset\mathscr{I}_{\theta_2}$-st-LIM$^r y'$.

Proof. If possible assume $x^*\in\mathscr{I}_{\theta_2}$-$st$-LIM$^r y$. Then, for any $(0<<)\sigma\in W$ and for all $\kappa>0$, the set

$$T := \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\{(s,t)\in I_{mn} : (r+\sigma-\rho(y_{st},x^*))\notin intP\}\right| \geq \kappa\right\}\in\mathscr{I}_2.$$

For the subsequence $y'=\left\{y_{s_p t_q}\right\}_{p,q\in\mathbb{N}}$, since

$$\begin{aligned}&\left\{(m,n)\in\mathbb{N}^2 : \tfrac{1}{h_{mn}}\left|\left\{(s_p,t_q)\in I_{mn} : \left(r+\sigma-\rho\left(y_{s_p t_q},x^*\right)\right)\notin intP\right\}\right|\geq\kappa\right\}\\ &\subset\left\{(m,n)\in\mathbb{N}^2 : \tfrac{1}{h_{mn}}\left|\{(s,t)\in I_{mn} : (r+\sigma-\rho(y_{st},x^*))\notin intP\}\right|\geq\kappa\right\}\end{aligned}$$

and

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\{(s_p,t_q)\in I_{mn} : (r+\sigma-\rho(y_{st},x^*))\notin intP\}\right|\geq\kappa\right\}\in\mathscr{I}_2,$$

so

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn} : \left(r+\sigma-\rho\left(y_{s_p t_q},x^*\right)\right)\notin intP\right\}\right|\geq\kappa\right\}\in\mathscr{I}_2.$$

Hence, the set

$$W = \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\{(s_p,t_q)\in I_{mn} : (r+\sigma-\rho(y_{st},x^*))\in intP\}\right|\geq\kappa\right\}\in\mathscr{F}(\mathscr{I}_2).$$

Take $y'=\left\{y_{s_p t_q}\right\}_{s_p,t_q\in W}$. Then, we have

$$\lim_{m,n\to\infty}\frac{1}{h_{mn}}\left|\left\{(s,t)\in I_{mn} : r+\sigma-\rho\left(y_{s_p t_q},x^*\right)\notin intP\right\}\right| = 0.$$

and so $r-S_{\theta_2}-\lim y_{s_p t_q}=y^*$. Therefore, we get $y_{s_p t_q} \overset{r-\mathscr{I}^*_{\theta_2}-st}{\to} x^*$. So, we obtain $x^*\in\mathscr{I}_2$-st-LIM$^r y'$. Therefore, we have $\mathscr{I}_{\theta_2}$-st-LIM$^r y\subset\mathscr{I}_{\theta_2}$-$st$-LIM$^r y'$. □

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