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Elastodynamic Problem of Co-planar Griffith Cracks by Anti-Plane Shear Wave

Research Article

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Abstract: The present work deals with the study of stress distribution and displacement by two coplanar Griffith cracks, which are moving with a constant velocity v, under anti-plane shear at the interface of two different orthotropic media. Two problems with practical significance are considered. In the first problem, cracks are situated at two bonded different semi-infinite orthotropic media and in the second problem, the propagation of the cracks at the interface of an orthotropic elastic layer of finite depth overlaying another semi infinite orthotropic medium has been considered. For both the cases, the boundary value problems have been reduced to the solution of triple integral equations. Finite Hilbert Transform technique has been applied to solve it. The analytical expression of the physical interest, i.e. crack opening displacement (COD) and Stress intensity factors(SIF) at the crack tips have been derived. Numerical results are presented graphically for different orthotropic materials.

MSC: 74G15 • 45B05

Keywords: Stress Intensity Factor • Crack Opening Displacement • Elliptic Integral • Fredholm Integral equation • Finite Hilbert Transform Method

1. Introduction

In recent years, Fracture mechanics has acquired its place as an independent discipline that helps to determine the conditions under which structural elements or machine obtain uncontrollable failure by crack propagation. When applied to design, the objective of fracture mechanics analysis is to limit the operating stress level so that a pre-existing crack should not grow to a critical size during the service life of the structure. Due to light weight and strong nature, orthotropic materials are frequently used in crack problems. Crack problem in elastic media have their immense importance in studying integrity and safety of structural components. Fracture mechanics is a relatively new field that enabled the safe use of high strength alloys, in the jet aircraft and other sophisticated applications like fabrication, joining and design requirements. Fracture mechanics provides the engineering basis to quantitatively predict the effects of these cracks. How much a given crack size will affect strength and how long it will take for a flaw to grow to a crack size that is dangerous for the operation can be answered for many materials. Many research articles have been dedicated to the study of crack problems in orthotropic material, mostly because of the fast development in construction engineering. In civil engineering, crack problems has great importance. It has a wide reach of structural designing applications for planning load-bearing parts for vehicles and mainly in building structure.

At the very beginning, Atkinson [1] has considered plane-strain problem of crack propagation in an finite elastic body.

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Erdogan et. al [2] have derived the expression of SIF and strain energy release rate of a penny shaped interface crack attached to half space and an elastic layer. The problem of crack propagation and crack arrest in stress gradient in infinite and semi infinite has been solved by Melville [3]. Atkinson and List [4] have considered the steady state problem of crack propagation into elastic media with different elastic moduli. Later, Dhwan and Dhaliwal [5] in their work have considered three co-planar cracks and obtain the closed form expressions for various physical quantities. Danyluk et al. [6] also derived the closed form solution of a finite crack moving in orthotropic layer. A problem of co-planar Griffith crack in an orthotropic layer sandwiched between two elastic half plane was solved by Itou [7]. Das et al. [8] obtained the expressions for COD and SIF for the problem of two co-planar crack attached between different two elastic media under anti-plane shear. De and Patra apllied integral transformation method to find out the solution of the two collinear Griffith crack in an orthotropic strip [9]. The expression of the effective SIF and stresses were derived by Beghini and Bertini [10]. Das et al. [11] in their research work have examined the effect of punch on the SIF and solved the problem approximately. The crack which is opened by the interaction of plane elastic waves, was considered by Debnath and Das [12]. Das et al. [13] have solved the problem of diffusion of SH-waves by Griffith crack in elastic media and they have also investigated the fact about interaction of three moving crack positioned between two bonded heterogeneous elastic media [14]. The problems of moving cracks in bonded dissimilar media have been considered by Mukherjee et al. [15] while Ma et al. [16] gave the overview of propagation of moving crack in non-homogeneous orthotropic medium. In 2006, Das [17] has used Fourier transform and Hilbert transform technique to solve dynamic problem of interface collinear Griffith crack under the application of anti-plane shear. Fotuhi et al. [18] analysed the crack problem in orthotropic half plane. The method of the determination of SIF of multiple collinear crack was briefly discussed by Jin et al. [19]. Itou [20] has studied the effect of time-harmonic disturbance by three collinear Griffith cracks. Mishra and Das [21] determined crack energy of two collinear Griffith cracks in thermo-elastic media by using Jacobi polynomial. The expression of crack opening potential and crack opening displacement was derived along with local intensity factors in piezoelectric media by Singh et al. [22] considering modified strip saturated models. Multiple moving cracks in orthotropic layer has been considered by Bagheshtani et al [23]. By using successive approximation method. Mondal et al. [24] solved dynamic crack problem in orthotropic strip. Problems of crack propagation in orthotropic interface was studied by Mandal et al. [25, 26]. Elfakhakhre et al. [27] gave numerical solution of circular arc cracks acting under uniaxial tension. Analysis of thermoelastic response in a functionally graded infinite space subjected to a mode-I crack was done by Sur and Kanoria [28]. In the research articles of Patra et. al. [29] an internal crack problem in an infinite transversely isotropic elastic layer was discussed. In recent times, researchers have dedicated many articles and papers in crack problems as [30] [31], [32], [33] and many more.
In the present work, we have focused on two diffrent cases of crack propagation under anti-plane shear. One is when the co-planar Griffith cracks are moving along the interface of two semi-infinite bonded dissimilar orthotropic media and another case is the propagation of two co-planar Griffith crack attached to the interface of a orthotropic layer of thickness l and a semi-infinite medium. Applying the Fourier cosine transformation, the problem reduced to the solution of a triple integral equations which was solved by Finite Hilbert Transform method. Expressions for stress intensity factors and crack opening displacement are derived for the first case where as for the second case, analytical expressions are obtained up to $O(l^{-4})$. Numerical computations are carried out for two different orthotropic materials viz, Cadmium (Cd) and Magnesium (Mg) and results ae presented graphically.

2. Formulation of the problem

We have considered the interaction of two coplanar Griffith cracks of finite length with anti-plane shear wave positioned between two different orthotropic layer. Referring to the principle co-ordinate system as (X,Y,Z) cracks are located on the line segments $(-b,-a)$ and (a,b) which are in motion with the constant speed v' along the positive x-axis. At the outer edge of the co-planar cracks co-ordinates (x,y,z) are counted as dimensionless. In the dynamic problem of anti-plane shear, there exist a single non vanishing component of displacement in the z-direction in the media $y>0$ and $y<0$ respectively.

The non-trivial stress components are,

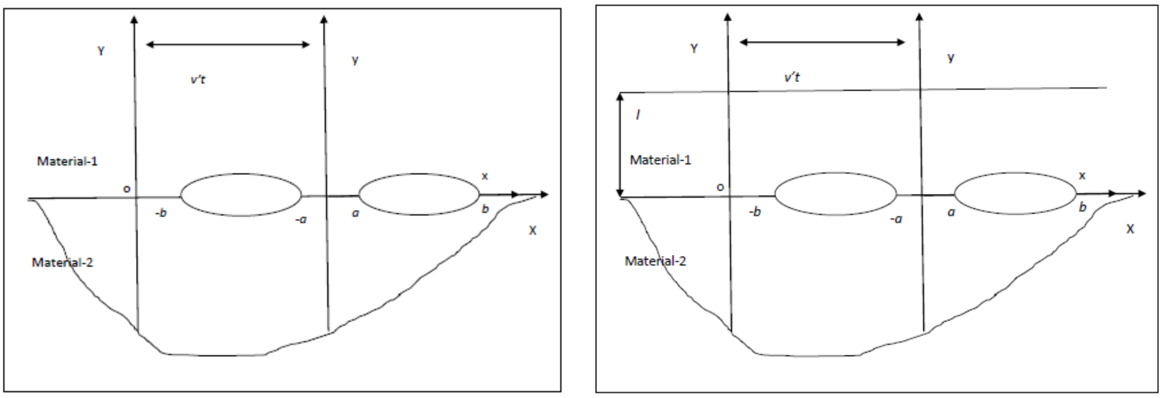
$$\tau_{yz} = C_{44}^{i}\frac{\partial W_i}{\partial Y},$$

$$\tau_{xy} = C_{55}^{i}\frac{\partial W_i}{\partial X} \qquad (i=1,2)$$

The equation of motion for orthotropic material takes the form,

$$C_{55}^{i}\frac{\partial^2 W_i}{\partial X^2} + C_{44}^{i}\frac{\partial^2 W_i}{\partial Y^2} = \rho_i\frac{\partial^2 W_i}{\partial t^2} \qquad (i=1,2) \tag{1}$$

where C_{55}^{i} and C_{44}^{i}, $i=1,2$ are dimensionless elastic constants and ρ_i represent the densities of two different orthotropic material-1 and material-2. Applying Galilean transformation,



(a) Problem-I (b) Problem-II

Fig. 1. Geometry of the two collinear Griffith cracks

$x = X - vt,\ y = Y,\ t = t$
equation (1) becomes independent of t and reduces to,

$$r_i^2 \frac{\partial^2 W_i}{\partial x^2} + \frac{\partial^2 W_i}{\partial y^2} = 0 \qquad (i = 1,2) \tag{2}$$

where

$$r_i^2 = \frac{C_{55} - \rho_i v'^2}{C_{44}^i} \qquad (i = 1,2) \tag{3}$$

2.1. Problem I

In the first problem, cracks are situated in the middle of two jointed dissimilar elastic half-spaces. The cracks are moving along the interface of these orthotropic media. An incident anti-plane shear wave is acting normally on the crack. The boundary conditions are,

$$[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 = -p_0, \qquad a < |x| < b \tag{4a}$$

$$[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2, \qquad 0 \le |x| < a, |x| > b \tag{4b}$$

$$W_1(x,0) = W_2(x,0), \qquad 0 \le |x| < a, |x| > b \tag{4c}$$

2.2. problem II

In the second case, we have considered two co-planar Griffith cracks of constant length at the interface of an elastic layer of thickness l overlaying a semi-infinite orthotrpic medium. Assuming the fact that the cracks are moving with a constant velocity v' between these bonded orthotropic layer under the anti-plane shear, the boundary conditions can be taken as,

$$[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2 = -p_0, \qquad a < |x| < b \tag{5a}$$

$$[\tau_{yz}(x,0)]_1 = [\tau_{yz}(x,0)]_2, \qquad 0 \le |x| < a, |x| > b \tag{5b}$$

$$W_1(x,0) = W_2(x,0), \qquad 0 \le |x| < a, |x| > b \tag{5c}$$

$$[\tau_{yz}(x,l)]_1 = 0, \qquad -\infty < |x| < \infty \tag{5d}$$

3. Solution of the problem I

Applying the Fourier cosine transformation in the equation of motion, the solution of equation (2) are obtained as,

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty M_1(\zeta)e^{-r_1\zeta y}\cos\zeta x\, d\eta, \qquad y>0 \tag{6}$$

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty M_2(\zeta)e^{r_2\zeta y}\cos\zeta x\, d\eta, \qquad y<0 \tag{7}$$

where r_i is the positive root of (3) and $M_i(\zeta)$ $(i=1,2)$ are unknown functions to be determined.
Now Using (6) and (7) we get,

$$[\tau_{yz}(x,y)]_1 = -\frac{2\mu_1 r_1}{\pi}\int_0^\infty \zeta M_1(\zeta)e^{-r_1\zeta y}\cos\zeta x\, d\zeta, \qquad y>0 \tag{8}$$

$$[\tau_{yz}(x,y)]_2 = \frac{2\mu_2 r_2}{\pi}\int_0^\infty \zeta M_2(\zeta)e^{-r_2\zeta y}\cos\zeta x\, d\zeta, \qquad y<0 \tag{9}$$

where $\mu_i = C_{44}^i$. Using the boundary conditions (4a) ad (4b) we get,

$$M_2(\zeta) = -\frac{\mu_1 r_1}{\mu_2 r_2}M_1(\zeta). \tag{10}$$

The crack opening displacement (COD) $\Delta W(x)$ can be derived as,

$$\begin{aligned}\Delta W(x) &= W_1(x,0+) - W_2(x,0-)\\ &= \frac{2M}{\pi}\int_0^\infty M_1(\zeta)\cos\zeta x\, d\zeta, \qquad a\le x\le b\\ &= 0, \qquad 0\le x<a,\ x>b\end{aligned} \tag{11}$$

where

$$M = \frac{\mu_1 r_1}{\mu_2 r_2} + 1. \tag{12}$$

From (8) and (4a) we get,

$$\int_0^\infty \zeta M_1(\zeta)\cos\zeta x\, d\zeta = \frac{p_0\pi}{2\mu_1 r_1}. \tag{13}$$

Let us take

$$M_1(\zeta) = \frac{1}{\zeta}\int_a^b g(t^2)\sin\zeta t\, dt. \tag{14}$$

From (11) and (14) we see that,

$$\int_a^b g(t^2) = 0. \tag{15}$$

Now substituting $M_1(\zeta)$ in (13) we have,

$$\int_0^\infty \frac{t g(t^2)\, dt}{x^2-t^2} = \frac{p_0\pi}{2\mu_1 r_1}, \qquad a<x<b. \tag{16}$$

Using Finite Hilbert Transformation we get the solution of (16) in the form,

$$\begin{aligned}g(t^2) = -\frac{2p_0}{\pi\mu_1 r_1}\frac{\sqrt{(t^2-a^2)}}{\sqrt{(b^2-t^2)}}\int_a^b \frac{\sqrt{(b^2-x^2)}}{\sqrt{(x^2-a^2)}}\frac{x}{x^2-t^2}\, dx\\ + \frac{C''}{\sqrt{(t^2-a^2)(b^2-t^2)}}\end{aligned} \tag{17}$$

where the unknown constant

$$C'' = \frac{p_0}{\mu_1 r_1}\left(a^2 - b^2\frac{E}{F}\right) \tag{18}$$

where, $F = F(\frac{\pi}{2}, \kappa)$ and $E = E(\frac{\pi}{2}, \kappa)$ are complete elliptic integrals of the first and second kind respectively and $\kappa^2 = \frac{b^2-a^2}{b^2}$.

Crack opening displacement and stress components at the interface are expressed as,

$$\Delta W(x) = M\int_x^b g(t^2)\,dt, \qquad a < x < b \tag{19}$$

$$[\tau_{yz}(x,0)]_1 = \frac{2\mu_1 r_1}{\pi}\int_x^b \frac{tg(t^2)}{x^2-t^2}\,dt, \qquad 0 \le x < a, x > b \tag{20}$$

Substituting the value of $g(t^2)$ from (17) into (19) and (20) we have,

$$\Delta W(x) = \frac{Mp_0}{\mu_1 r_1}[bE(\alpha,\kappa) - \frac{b^2 E(\alpha,\kappa)}{F(\alpha,\kappa)}] \tag{21}$$

where

$$\alpha = sin^{-1}[\frac{\sqrt{b^2-x^2}}{\sqrt{b^2-a^2}}]. \tag{22}$$

Using the formula,

$$\int_a^b \frac{t\,dt}{(x^2-t^2)\sqrt{(t^2-a^2)(b^2-x^2)}} = \begin{cases} -\frac{\pi}{2\sqrt{(a^2-x^2)(b^2-x^2)}} & \text{for } 0 < x < a \\ 0 & \text{for } a < x < b \\ \frac{\pi}{2\sqrt{(x^2-a^2)(x^2-b^2)}} & \text{for } x > b \end{cases}$$

we have the expression of stresses,

$$[\tau_{yz}(x,0)]_1 = p_0[\frac{\sqrt{x^2-a^2}}{\sqrt{x^2-b^2}} - 1 - \frac{\frac{b^2E}{F} - a^2}{\sqrt{(x^2-a^2)(x^2-b^2)}}] \text{ for } x > b \tag{23}$$

$$[\tau_{yz}(x,0)]_1 = p_0[\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-a^2}} - 1 + \frac{\frac{b^2E}{F} - a^2}{\sqrt{(a^2-x^2)(b^2-x^2)}}] \text{ for } x < a. \tag{24}$$

The expression dfor stress intensity factor at the crack tips $x = a$ and $x = b$ are,

$$\begin{aligned} K_b &= \lim_{x\to b}\sqrt{2(x-b)}[\tau_{yz}(x,0)]_1 \\ &= \frac{p_0 b^2(\frac{E}{F}-1)}{\sqrt{b(b^2-a^2)}} \end{aligned} \tag{25}$$

$$\begin{aligned} K_a &= \lim_{x\to a}\sqrt{2(a-x)}[\tau_{yz}(x,0)]_1 \\ &= -\frac{p_0(b^2\frac{E}{F}-a^2)}{\sqrt{a(b^2-a^2)}} \end{aligned} \tag{26}$$

4. Solution of the problem II

Apply the Fourier cosine transformation in the equation of motion, we have the solution of (2) as,

$$W_1(x,y) = \frac{2}{\pi}\int_0^\infty [N_1(\zeta)e^{-r_1\zeta y} + N_2(\zeta)e^{r_1\zeta y}]\cos\zeta x\,d\zeta, \qquad 0 \le y \le l \tag{27}$$

$$W_2(x,y) = \frac{2}{\pi}\int_0^\infty N_3(\zeta)e^{r_2\zeta y}\cos\zeta x\,d\zeta, \qquad y < 0 \tag{28}$$

where r_i is the positive root of (3) and $N_i(\zeta)$ are unknown functions to be determined. the stress components are

$$[\tau_{yz}(x,y)]_1 = -\frac{2\mu_1 r_1}{\pi}\int_0^\infty [N_1(\zeta)e^{-r_1\zeta y} + N_2(\zeta)e^{r_1\zeta y}]\cos\zeta x\,d\zeta, \qquad 0 \le y \le h \tag{29}$$

$$[\tau_{yz}(x,y)]_2 = \frac{2\mu_2 r_2}{\pi}\int_0^\infty N_3(\zeta)e^{r_2\zeta y}\cos\zeta x\,d\zeta, \qquad y < 0. \tag{30}$$

Applying the boundary conditions (5a), (5b) and (5d) we have,

$$N_3(\zeta) = \frac{\mu_1 r_1}{\mu_2 r_2}[N_2(\zeta) - N_1(\zeta)] \tag{31}$$

$$N_2(\zeta) = N_1(\zeta)e^{-2\zeta h r_1}. \tag{32}$$

Therefore, the crack opening displacement is,

$$\begin{aligned}\Delta W(x) &= W_1(x,0+) - W_2(x,0-)\\ &= \frac{2M}{\pi}\int_0^\infty \chi(\zeta x)\cos\zeta x\, d\zeta, \qquad a \le x \le b\end{aligned}$$

where

$$\chi(\zeta) = N_1(\zeta)[1 + \frac{\mu_2 r_2 - \mu_1 r_1}{\mu_2 r_2 + \mu_1 r_1}e^{-2\zeta h r_1}]. \tag{33}$$

Therefore,

$$\begin{aligned}\Delta W(x) &= W_1(x,0+) - W_2(x,0-)\\ &= \frac{2M}{\pi}\int_0^\infty \chi_1(\zeta x)\cos\zeta x\, d\zeta, \qquad a \le x \le b\\ &= 0, \qquad 0 \le x \le a,\ x > b.\end{aligned} \tag{34}$$

Therefore from (33) we get

$$\int_0^\infty \chi(\zeta)\cos\zeta x\, d\zeta = 0, \qquad 0 \le x \le a,\ x > b. \tag{35}$$

Also using the boundary condition $[\tau_{yz}(x,0)]_1 = -p_0$ we get,

$$\int_0^\infty \zeta\chi(\zeta)[D(\zeta l) + 1]\cos\zeta x\, d\zeta = \frac{p_0\pi}{2\mu_1 r_1}, \qquad a < x < b \tag{36}$$

where

$$D(\zeta l) = \frac{-e^{-2\zeta l r_1} + 1}{[1 + \frac{\mu_2 r_2 - \mu_1 r_1}{\mu_2 r_2 + \mu_1 r_1}e^{-2\zeta l r_1}]} - 1. \tag{37}$$

Now we assume,

$$\chi(\zeta) = \frac{1}{\zeta}\int_a^b g(t^2)\sin\zeta t\, dt. \tag{38}$$

It is obtained from the equations (35) and (36) that $g(x^2)$ is the solution of the following Fredholm integral equation:

$$g(x^2) + \int_a^b g(t^2)K(x^2, t)\, dt = G(x^2), \qquad a < x < b \tag{39}$$

satisfying the condition

$$\int_a^b g(x^2)\, dx = 0 \tag{40}$$

where,

$$K(x^2, t) = -\frac{4}{\pi^2}\frac{\sqrt{x^2 - a^2}}{\sqrt{b^2 - x^2}}\int_a^b \frac{y}{y^2 - a^2}\frac{\sqrt{b^2 - y^2}}{\sqrt{y^2 - a^2}}K_1(y, t)\, dy \tag{41}$$

with

$$K_1(y, t) = \int_0^\infty D(\zeta l)\cos(\zeta y)\sin\zeta t\, d\zeta \tag{42}$$

and

$$\begin{aligned}G(x^2) = -\frac{2p_0}{\pi\mu_1 r_1}\frac{\sqrt{x^2 - a^2}}{\sqrt{b^2 - x^2}}\int_a^b \frac{y}{y^2 - x^2}\frac{\sqrt{b^2 - y^2}}{\sqrt{y^2 - a^2}}\, dy\\ + \frac{C''}{\sqrt{(x^2 - a^2)(b^2 - x^2)}}\end{aligned} \tag{43}$$

where $C^{"}$ is an arbitrary constant which can be determined from the equation (40). Now if we take $l >> 1$ then by substituting $\lambda = \zeta l$ in (42) we have,

$$K_1(y,t) = \frac{F_0 t}{h^2} + \frac{F_1 t}{h^4}(t^2 + 3y^2) + o(h^{-6}) \tag{44}$$

where

$$F_j = \frac{(-1)^j}{(2j+1)!}\int_0^\infty \lambda^{2j+1} D(\lambda)\, d\lambda, \qquad j = 0,1. \tag{45}$$

Substituting these value of (44) in (41) we have,

$$K(x^2,t) = \frac{2\sqrt{x^2-a^2}}{\pi\sqrt{b^2-x^2}}[F_0 t l^{-2} + F_1 t l^{-4}(t^2 + 3x^2_{\frac{3}{2}}\kappa^2)] + o(l^-6) \tag{46}$$

where $\kappa^2 = \frac{b^2-a^2}{b^2}$. Integrating both side of (39) with respect to x from a to b and using the condition (40) we obtain

$$C^{"} = \frac{p_0}{\mu_1 r_1}[\frac{b^2 E}{F} - a^2] + \frac{b}{F}\int_a^b g(t^2) k^{'}(t)\, dt \tag{47a}$$

with

$$k^{'}(t) = \frac{2}{\pi}[F_0 t l^{-2}(\frac{(b^2E - a^2F}{b}) + F_1 t l^{-4}\{(t^2 - \frac{3}{2}\kappa^2)(\frac{(b^2E - a^2F}{b}) \\ -a^2 b(E+F) + 2b^3E\}] + o(l^{-6}) \tag{47b}$$

where, $F = F(\frac{\pi}{2},\kappa)$ and $E = E(\frac{\pi}{2},\kappa)$ are complete elliptic integrals of the first and second kind respectively and $\kappa^2 = \frac{b^2-a^2}{b^2}$, Hence, $g(x^2)$ must satisfy the integral equation,

$$g(x^2) = \int_a^b g(t^2) L(x^2,t)\, dt = R(x^2) \tag{48}$$

where,

$$L(x^2,t) = \frac{2\pi}{t\sqrt{(x^2-a^2)(b^2-x^2)}}[\frac{F_0}{l^2}(x^2 - \frac{b^2E}{F}) + \frac{F_1}{l^4}\{(t^2 - \frac{3}{2}\kappa^2) \\ (x^2 - \frac{b^2E}{F} + 3x^2(x^2-a^2) + \frac{a^2b^2E}{F} + a^2b^2 - \frac{2b^4E}{F})\}] \tag{49}$$

and

$$R(x^2) = \frac{p_0(x^2 - \frac{b^2E}{F})}{\mu_1 r_1\sqrt{(x^2-a^2)(b^2-x^2)}} \tag{50}$$

Since, $l >> 1$ and $|L(x^2,t)| < 1$, the solution of (48) can be written as,

$$g(x^2) = g_0(t^2) + l^{-2} g_1(x^2) + l^{-4} g_2(x^2) + o(l^{-6}) \tag{51}$$

where

$$g_0(x^2) = \frac{p_0(x^2 - \frac{b^2E}{F})}{\mu_1 r_1\sqrt{(x^2-a^2)(b^2-x^2)}} \tag{52}$$

$$g_1(x^2) = \frac{p_0 F_0 \epsilon_0 (x^2 - \frac{b^2E}{F})}{\mu_1 r_1\sqrt{(x^2-a^2)(b^2-x^2)}} \tag{53}$$

$$g_2(x^2) = -\frac{p_0\epsilon_0}{4\mu_1 r_1\sqrt{(x^2-a^2)(b^2-x^2)}}[F_0^2\epsilon_0 g_1(x^2)(x^2 - \frac{b^2E}{F} \\ -2F_1(3x^4 + \epsilon_1 x^2 + \epsilon_2))] \tag{54}$$

Here, $\epsilon_0 = a^2 + b^2 - \frac{2b^2E}{F}$
$\epsilon_1 = \frac{b^4\kappa^4}{4\epsilon_0} + \frac{1}{2}(a^2+b^2) - \frac{3}{2}(\kappa^2 + 2a^2)$

$\epsilon_2 = a^2b^2 - \frac{b^2E}{F}\{\epsilon_1 + 2(a^2+b^2)\}$.
The relevant crack opening displacement and stress component at the interface are,

$$\Delta W(x) = M\int_a^b g(t^2)\,dt, \qquad a \le x \le b \tag{55}$$

$$[\tau_{yz}(x,0)]_1 = -\frac{2\mu_1 r_1}{\pi}[\int_a^b \frac{tg(t^2)\,dt}{t^2-x^2} + \int_a^b g(t^2)k_1(x,t)\,dt], \qquad 0 \le x \le a, x > b. \tag{56}$$

Equations (51)-(54) together with (43) yield

$$\int_a^b g(t^2)k_1(x,t)\,dt = \frac{p_0\pi}{4\mu_1 r_1}[\frac{F_0\epsilon_0}{l^2} - \frac{F_0^2\epsilon_0^2}{2l^4} - \frac{F_1}{l^4}\{3x^2\epsilon_0 - \frac{(a^2+b^2)^2}{2} + \frac{(a^2-b^2)^2}{8} - \frac{b^2E}{F}(a^2+b2)\}] \tag{57}$$

and

$$\int_a^b \frac{tg(t^2)}{t^2-x^2}\,dt = \frac{p_0\pi}{2\mu_1 r_1}[\{1 - \frac{F_0\epsilon_0}{l^2} - \frac{F_0^2\epsilon_0^2}{l^4}\}\{(x^2 - \frac{b^2E}{F}T) + 1\}] - \frac{\epsilon_0 F_1 p_0\pi}{2\mu_1 r_1}[3\{\frac{(a^2+b^2)}{2} + x^2 + x^4T\} + \epsilon_1(1+x^2T) + \epsilon_2 T] \tag{58}$$

where

$$T = \begin{cases} -\frac{1}{\sqrt{(a^2-x^2)(b^2-x^2)}} & \text{for } 0 < x < a \\ \frac{1}{\sqrt{(x^2-a^2)(x^2-b^2)}} & \text{for } x > b \end{cases}$$

Using equations (50)-(53) and integrating (54) we have, the crack opening displacement as,

$$\Delta W(x) = \frac{p_0 M}{\mu_1 r_1}[\{1 - \frac{F_0\epsilon_0}{2l^2} + \frac{F_0^2\epsilon_0^2 + 2F_1\epsilon_0(\epsilon_1 - \frac{\kappa^4}{2\epsilon_0})}{4l^4}\} \times \{E(\alpha,\kappa) - \frac{E}{F}F(\alpha,\kappa)\} - \frac{2F_1\epsilon_0}{4l^4} \times \sqrt{(x^2-a^2)(x^2-b^2)}] + o(l^{-6}) \tag{59}$$

where $\alpha = sin^{-1}\frac{b^2-x^2}{b^2-a^2}$, $M = \frac{\mu_2 r_2 + \mu_1 r_1}{\mu_2 r_2}$. Substituting (57) and (58) on the right hand side of (55) we will have the expression of the stress.

The stress intensity factor at the crack tips $x = b$ and $x = a$ are given by

$$\begin{aligned} k_b &= \lim_{x\to b}\sqrt{2(x-b)}[\tau_{yz}(x,0)]_1 \\ &= \frac{p_0 b^2}{\sqrt{b(b^2-a^2)}}[(\frac{E}{F}-1)\{1 - \frac{F_0\epsilon_0}{2l^2} + \frac{\epsilon_0^2F_0^2}{4l^4}\} + \frac{F_1\epsilon_0}{2l^4}\{3b^4 + \epsilon_1 b^2 + \epsilon_2\}] \end{aligned} \tag{60}$$

$$\begin{aligned} k_a &= \lim_{x\to a}\sqrt{2(a-x)}[\tau_{yz}(x,0)]_1 \\ &= \frac{p_0}{\sqrt{a(b^2-a^2)}}[(a^2 - b^2\frac{E}{F})\{1 - \frac{F_0\epsilon_0}{2l^2} + \frac{\epsilon_0^2F_0^2}{4l^4}\} + \frac{F_1\epsilon_0}{2l^4}\{3a^4 + \epsilon_1 a^2 + \epsilon_2\}] \end{aligned} \tag{61}$$

5. Numerical Discussion

In this section, we consider two different orthotropic materials, cadmium (Cd) and magnesium (Md) to represent the expression of Stress Intensity Factors (SIF) and Crack Opening Displacement(COD) graphically. The mechanical properties of considered elastic media have listed in Table 1 The effect of crack speed with stress intensity factor at the crack tip $x = b$ has been presented in Fig. 7. For the fixed value of $l = 3$ the graphs are plotted for different values of $a = 0.3$, 0.5 and0.7. In general, it is observed that the graphs of $\frac{K_b}{p_0}$ tend to a straight line up to a certain point and then grow rapidly. Fig. 8 depicts the nature of stress intensity factor at the crack tip $x = a$ with the growing value of crack speed for different crack length. As a result, we can see that the nature of the graphs are similar with Fig. 7. Taking a larger value of $l = 8$ and for the same crack lengths it has been observed from Fig. 9 that nature of the graphs are very close to Fig. 7 and Fig. 8. First they tend to straight lines then a rapid growth is seen for the increasing value of crack speed. Fig. 10 represents the variation of $\frac{K_a}{p_0}$ vs. $\sqrt{\frac{\rho_1}{\mu_1}} \times v'$ for $l = 8$.
The variation of crack opening displacement (COD) with x has been plotted in Fig. 11 for different value of crack speed. In each case, it is seen that the graphs of COD start from zero, increase rapidly and reach the pick value then decrease gradually until it meet the point zero. From the graphical view it can be stated that with the increasing values of crack speed COD also increases.
Numerical results of Stress Intensity Factors at both the crack tips have been presented in Table 2 and Table 3 for different values of crack lengths.

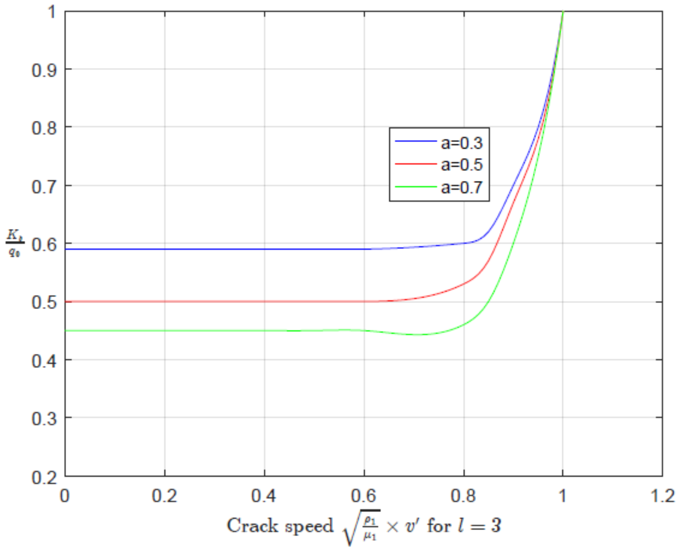


Fig. 2. Stress Intensity Factor K_b vs. Crack speed

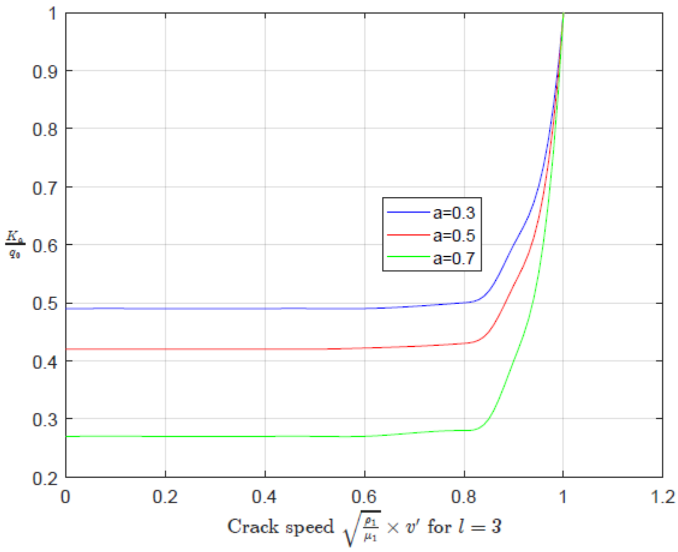


Fig. 3. Stress Intensity Factor K_a vs. Crack speed

Table 1. Elastic constants:

Type	Material	μ_i	C_{55}^i	ρ_i
Material-1	Magnesium	0.194	0.1842	1.738
Material-2	Cadmium	1.985	1.845	8.7

Table 2. value of K_b

a	$b = 0.9$	$b = 1$
0.2	-0.61915	-0.665142
0.3	-0.563033	-0.610872
0.4	-0.508268	-0.558945
0.5	-0.451349	-0.506244
0.6	-0.389089	-0.450403
0.7	-0.316813	-0.388702

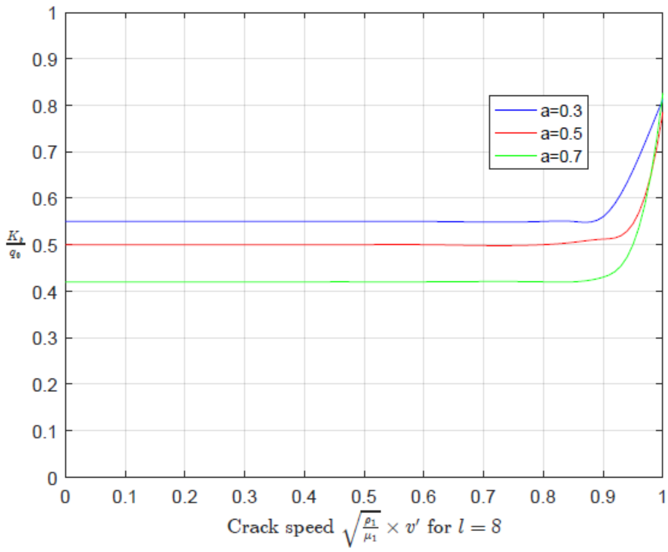


Fig. 4. Stress Intensity Factor K_b vs. Crack speed

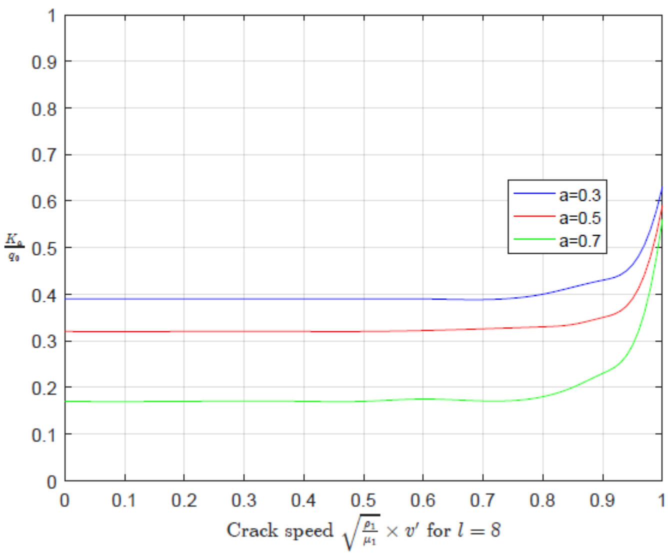


Fig. 5. Stress Intensity Factor K_b vs. Crack speed

Table 3. value of K_a

a	$b = 0.9$	$b = 1$
0.2	-0.648726	-0.703588
0.3	-0.573991	-0.626351
0.4	-0.512353	-0.565368
0.5	-0.452753	-0.508808
0.6	-0.38949	-0.451327
0.7	-0.316891	-0.388976

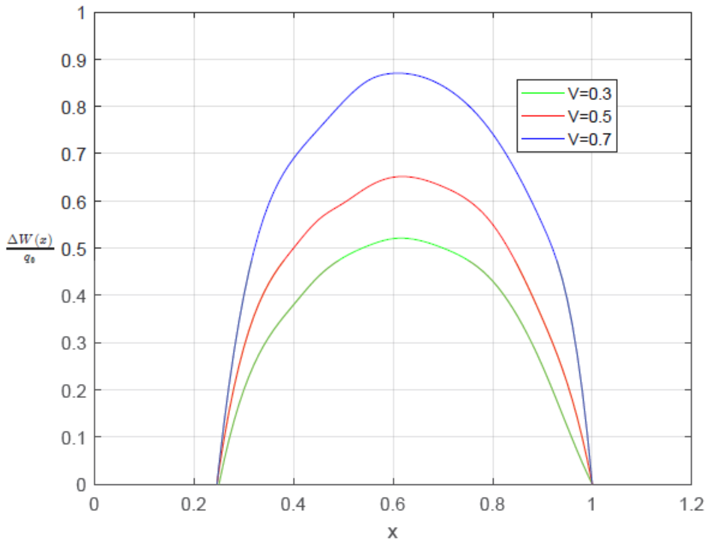


Fig. 6. COD $W(x)$ vs. Displacement x

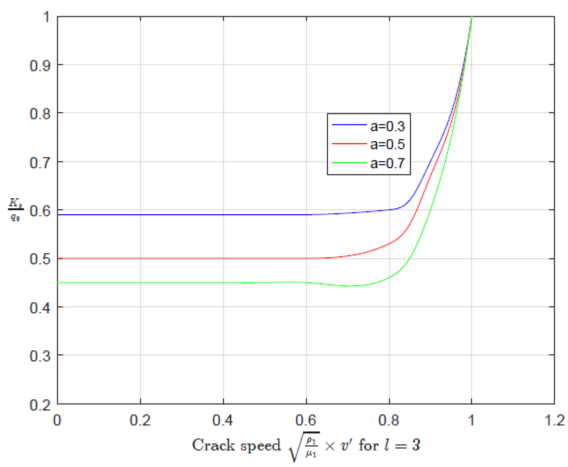


Fig. 7. Stress Intensity Factor K_b vs. Crack speed

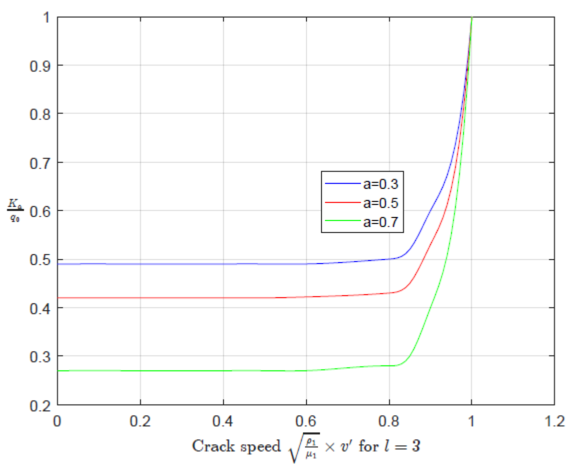


Fig. 8. Stress Intensity Factor K_a vs. Crack speed

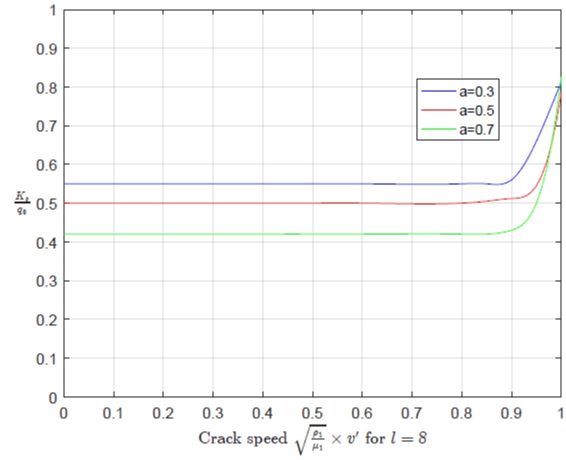


Fig. 9. Stress Intensity Factor K_b vs. Crack speed

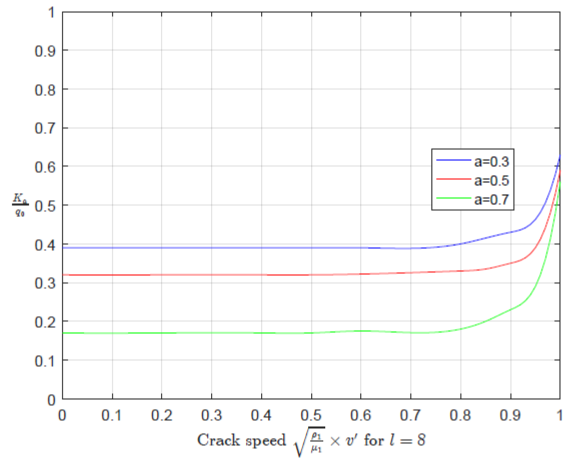


Fig. 10. Stress Intensity Factor K_b vs. Crack speed

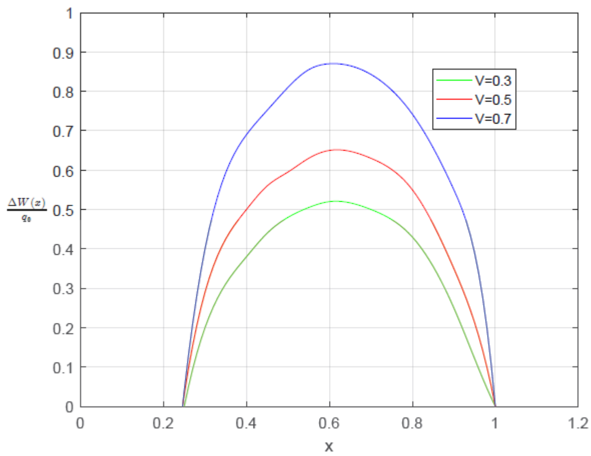


Fig. 11. COD $W(x)$ vs. Displacement x

6. Conclusion

In the present article we have studied the effect of antiplane shear wave to the co-planar Griffith cracks situated at the interface of two different orthotropic elatic media. Two cases of physical importance have been discussed briefly. To solve the boundary value problems Fourier cosine transformation and Finite Hilbert Transform method have employed.The closed form expression of dynamic stress intensity factors have been obtained for the Problem-I where the cracks are placed along the two heterogeneous orthotropic half-space. Expressions of SIF and COD for the Problem-II, where the cracks are situated at the interface of an orthotropic elastic solid of finite depth l and a semi-infinite orthotropic medium, are derived analytically. Graphical plots of SIF and COD are presented for two different orthotropic materials Magnesium (Mg) and Cadmium (Cd). It is been noticed that both the physical quantities (SIF and COD) depend on the crack length as well as crack speed. It has also observed that the graph of COD for different crack velocity start from zero, then reach the maximum value then again meet the point zero.

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