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Solving Fredholm Integral Equations of the First Kind with Symmetric Kernels

Research Article

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Abstract: We consider Fredholm integral equations of the first kind with symmetric kernels and use Tikhonov's regularization property to transform them into second kind Fredholm equations. We then apply the successive approximations and Adomian decomposition methods to these equations and compare the results with those from other methods used in the literature. Results indicate that the successive approximations method generally performs better than other methods as it leads to exact solutions.

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1. Introduction

Fredholm integral equations of the first kind (FIE-1s) are a type of inverse problem that arises in many areas of science and engineering such as image processing, signal processing, electromagnetics, geophysics, spectroscopy, backward heat equation, medical imaging (CT scanning, electrocardiography, etc.), image deblurring (e.g., astronomy, crime scene investigations, etc.), geographical prospecting (e.g., for oil, land-mines, etc.), to mention but a few [1–3]. However, FIE-1s are ill-posed, meaning that very tiny adjustments in the data cause very large changes in the results obtained. A problem is ill-posed if it has no solution, or if the solution exists but is not unique or does not depend continuously on the forcing function [2, 4]. Thus, an ill-posed problem is inherently unstable.

For this reason, FIE-1s generally need to be transformed into the well-posed FIE of the second kind using the regularization method (RM) which makes the equations stable and solvable. Many analytical and numerical methods have been devised in the literature for dealing with FIE-1s. Wazwaz [4, 5] proposed the regularization method of Tikhonov and Phillips as an effective means of converting Fredholm integral equations of the first-kind into equations of the second kind which can then be solved using available methods such as the direct computation method, Picard's successive approximations method (SAM) and the Adomian decomposition method (ADM). The SAM was also used to find approximate solutions to the Korteweg-de Vries and viscid Burgers equations [6] and the ADM was applied by Hendi and Al-Qarni [7] to find numerical solutions of nonlinear mixed Volterra-Fredholm integral equations with Carleman, logarithmic and Cauchy kernels. Rostamy and Maleknejad [3] solved Fredholm integral equations of the

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first kind by using B-spline wavelet bases via a regularization approach and proved the convergence of the method. By also converting first-kind Fredholm equations into those of the second kind by means of regularization, Kumar et al. [1] used the Legendre wavelet collocation method to find approximate solutions to Fredholm integral equations of the first kind.

Altürk [8] found numerical solutions of linear and nonlinear Fredholm integral equations using a semi-analytic method based on the weighted mean-value theorem. Altürk and Coşgun [9] used the Lavrentiev regularization method to find one or infinitely many solutions for first-kind Fredholm integral equations with separable kernels. Adibi and Assari [10] proposed a method that uses Chebyshev wavelets constructed on the unit interval as a basis in the Galerkin method and transforms the Fredholm integral equation into a system of algebraic equations. Maleknejad et al. [11] solved first-kind Fredholm integral equations using the sinc collocation method, while Almousa and Ismail [12] used the optimal homotopy asymptotic method (OHAM) which they found to combine simplicity with effectiveness. Adawi et al. [13] solved linear Volterra and Fredholm integral equations using the homotopy analysis method (HAM) which provides an analytical solution in terms of an infinite power series. For other methods, see the review paper [14] and references therein.

In this paper, we revisit the methods in [4, 5] and apply them to FIE-1s with symmetric kernels. A symmetric kernel is one that satisfies $K(x,t) = K(t,x)$. The rest of the paper is structured as follows: Section 2 identifies the problem to be solved, Section 3 outlines the proposed methods of solution, Section 4 gives numerical examples and Section 5 concludes the paper.

2. Mathematical Formulation

The most general form of a linear integral equation is

$$f(x) = cu(x) + \lambda \int_a^{b(x)} K(x,t)u(t)dt, \quad x \in [a,b], \tag{1}$$

where λ is a parameter, $a, c \in \mathbb{R}$ are constants, $K(x,t)$ is the kernel and $f(x)$ the forcing function of the integral equation. Both K and f are known continuous functions. In this study, $x \in [0,\infty)$, so that $K : [0,\infty) \to \mathbb{R}$ and $f : [0,\infty) \to \mathbb{R}$, while $u : [0,\infty) \to \mathbb{R}$ is the unknown function to be determined. If $c = 1$ then (1) is an equation of the second kind, while $c = 0$ makes it a first-kind equation. If $b(x) = x$ then (1) is a Volterra equation, while if $b(x) = b$, then it is a Fredholm integral equation. If the kernel $K(x,t) = K(x,t,u(t))$, then the FIE-1 is nonlinear. This paper focuses on linear Fredholm integral equations of the first kind which take the general form

$$f(x) = \lambda \int_a^{b} K(x,t)u(t)dt, \quad x \in [a,b], \tag{2}$$

In probabilistic terms, a FIE-1 equation can be interpreted as follows: given a probability density function $f(x)$ on $\mathbb{X} \subset \mathbb{R}$ and a non-negative kernel $K(x,t)$ which is a probability density function in x for each t, we seek to find a probability density function $u(t)$ satisfying equation (2). The main point of departure of FIE-1s from Volterra equations is that both limits of integration are constants whereas in Volterra equations at least one of the limits is a variable [4].

3. Proposed Methods of Solution

In this section, we present the proposed method of solution which is a hybrid of the regularization method and the successive approximations and Adomian decomposition methods.

3.1. Regularization Method

Assume that a solution exists to the linear ill-posed problem (2). Then by the regularization method (RM), independently established by Tinkhonov and Phillips [4], we convert (2) into a well-posed Fredholm integral equation of the second kind. The RM is a very reliable and most commonly used method for solving ill-posed problems such as FIE-1s. There are different techniques for regularizing FIE-1s (see [4, 5, 14]. In this paper we use the regularization method proposed by Wazwaz [4, 5]. Thus, equation (2) transforms into the second-kind FIE

$$\varepsilon u_\varepsilon(x) = f(x) - \lambda \int_a^{b} K(x,t)u_\varepsilon(t)dt, \quad x \in [a,b], \tag{3}$$

where ε is a small positive value referred to as a regularization parameter. The approximation Fredholm equation (3) is now a well-posed FIE of the second kind and is easier to solve than (2). Equivalently, (3) can be written as

$$u_\varepsilon(x) = \frac{1}{\varepsilon} f(x) - \frac{\lambda}{\varepsilon} \int_a^{b} K(x,t)u_\varepsilon(t)dt, \quad x \in [a,b], \tag{4}$$

or

$$u_\varepsilon(x) = g(x) - \frac{\lambda}{\varepsilon}\int_a^b K(x,t)u_\varepsilon(t)dt, \quad x \in [a,b], \tag{5}$$

where $g(x) = \frac{1}{\varepsilon}f(x)$ is the forcing function of the resulting second-kind FIE. Now, instead of solving (2), we solve equation (4). Also, it is known from [4, 5] that the solution $u_\varepsilon(x)$ of (4) converges to the solution $u(x)$ of (2) as $\varepsilon \to 0$, i.e.,

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x) \tag{6}$$

3.2. Successive Approximations Method

The second part of the solution involves application of existing methods for second-kind FIEs. In this paper, we use Picard's successive approximations method (SAM) and the standard Adomian decomposition method (ADM). Picard's method [15] provides a scheme for solving initial value problems or integral equations. The method solves any problem through successive approximations by starting with an initial guess or zeroth approximation $u_0(x)$. The initial guess can be any real-valued function and can be used in a recurrence relation to determine subsequent approximations, leading to the exact solution of FIE-1s via the resulting more stable FIE-2s. In Picard's method the n-th approximation for the regularized FIE-2 (4) is given recursively by

$$u_{\varepsilon_n}(x) = \frac{1}{\varepsilon}f(x) - \frac{\lambda}{\varepsilon}\int_a^b K(x,t)u_{\varepsilon_{n-1}}(t)dt, \quad x \in [a,b], \tag{7}$$

where the initial guess $u_{\varepsilon_0}(x)$ commonly takes the forms 0, 1 or x. With initial guess $u_{\varepsilon_0}(x)$, we have the approximations

$$\begin{aligned} u_{\varepsilon_1}(x) &= \frac{1}{\varepsilon}f(x) - \frac{\lambda}{\varepsilon}\int_a^b K(x,t)u_{\varepsilon_0}(t)dt \\ u_{\varepsilon_2}(x) &= \frac{1}{\varepsilon}f(x) - \frac{\lambda}{\varepsilon}\int_a^b K(x,t)u_{\varepsilon_1}(t)dt \\ u_{\varepsilon_3}(x) &= \frac{1}{\varepsilon}f(x) - \frac{\lambda}{\varepsilon}\int_a^b K(x,t)u_{\varepsilon_2}(t)dt \end{aligned}$$

and so on.

Remark 3.1.
If $f(x)$, $K(x,t)$ and $u_{\varepsilon_0}(x)$ are continuous, then so is $u_{\varepsilon_1}(x)$ and the rest of the approximations.

Remark 3.2.
With the selection of $u_{\varepsilon_0}(x) = 0$ as initial guess, the first approximation is $u_{\varepsilon_1}(x) = \frac{1}{\varepsilon}f(x)$ and the final solution $u_\varepsilon(x)$ is obtained by

$$u_\varepsilon(x) = \lim_{n\to\infty} u_{\varepsilon_n}(x),$$

so that the resulting solutions $u_\varepsilon(x)$ of the FIE-2 and $u(x)$ of the FIE-1 are independent of the choice of initial guess $u_{\varepsilon_0}(x)$.

3.3. Adomian Decomposition Method

In the standard ADM the components $u_{\varepsilon_0}(x)$, $u_{\varepsilon_1}(x)$, $u_{\varepsilon_2}(x)$,… of the solution $u_\varepsilon(x)$ of the resulting FIE-2 (4) are completely determined by the recursive relation

$$\begin{aligned} u_{\varepsilon_0}(x) &= g(x) = \frac{1}{\varepsilon}f(x) \\ u_{\varepsilon_{n+1}}(x) &= -\frac{\lambda}{\varepsilon}\int_a^b K(x,t)u_{\varepsilon_n}(t)dt, \ n \geq 1 \end{aligned} \tag{8}$$

The solution $u_\varepsilon(x)$ is then completely determined by the decomposition series

$$u_\varepsilon(x) = \sum_{n=0}^{\infty} u_{\varepsilon_n}(x). \tag{9}$$

For a full description of the Adomian decomposition method, see [16] and [17].

4. Numerical Experiments

In this section, numerical experiments are performed based on a selection of examples of FIE-1s with symmetric kernels. As pointed out in [4], the necessary condition to guarantee a solution to the first-kind FIE is that the forcing function $f(x)$ must contain components that are matched by the corresponding x components of the kernel $K(x,t)$. Due to similarity of these two methods (SAM and ADM), we will apply them alternately to the integral equations considered in the examples that follow. All the computations associated with these examples were performed using Samsung Series 3 PC with an Intel Celeron CPU 847 at 1.10GHz with 6.0GB internal memory and 64-bit operating system (Windows 8). The figures were constructed using MATLAB R2016a.

Example 4.1.
Consider the FIE-1 [12]:

$$\frac{x^2}{4}=\frac{5}{2}\int_0^1 x^2t^2u(t)dt \tag{10}$$

with exact solution $u(x)=\frac{1}{2}x^2$. This FIE can be rewritten as

$$x^2=10\int_0^1 x^2t^2u(t)dt,$$

and by the RM the FIE-1 converts to the second-kind equation

$$u_\varepsilon(x)=\frac{1}{\varepsilon}x^2-\frac{10}{\varepsilon}\int_0^1 x^2t^2u_\varepsilon(t)dt. \tag{11}$$

By the SAM with initial guess $u_{\varepsilon_0}(x)=0$, we have the approximations

$$\begin{aligned}
u_{\varepsilon_1}(x) &= \frac{1}{\varepsilon}x^2\\
u_{\varepsilon_2}(x) &= \frac{1}{\varepsilon}x^2-\frac{10}{\varepsilon}\int_0^1 x^2t^2\left(\frac{1}{\varepsilon}t^2\right)dt=\frac{1}{\varepsilon}x^2-\frac{2}{\varepsilon^2}x^2\\
u_{\varepsilon_3}(x) &= \frac{1}{\varepsilon}x^2-\frac{10}{\varepsilon}\int_0^1 x^2t^2\left(\frac{1}{\varepsilon}-\frac{2}{\varepsilon^2}\right)t^2dt=\frac{1}{\varepsilon}x^2-\frac{2}{\varepsilon^2}x^2+\frac{4}{\varepsilon^3}x^2\\
u_{\varepsilon_4}(x) &= \frac{1}{\varepsilon}x^2-\frac{10}{\varepsilon}\int_0^1 x^2t^2\left(\frac{1}{\varepsilon}-\frac{2}{\varepsilon^2}+\frac{4}{\varepsilon^3}\right)t^2dt=\frac{1}{\varepsilon}x^2-\frac{2}{\varepsilon^2}x^2+\frac{4}{\varepsilon^3}x^2-\frac{8}{\varepsilon^4}x^2
\end{aligned}$$

and so on. Thus, the solution of (11) is

$$u_\varepsilon(x)=\frac{1}{\varepsilon}\left(1-\frac{2}{\varepsilon}+\frac{4}{\varepsilon^2}-\frac{8}{\varepsilon^3}+\cdots\right)x^2=\frac{1}{\varepsilon}\left(\frac{\varepsilon}{\varepsilon+2}\right)x^2=\frac{1}{\varepsilon+2}x^2,$$

leading to the solution

$$u(x)=\lim_{\varepsilon\to 0}u_\varepsilon(x)=\frac{1}{2}x^2$$

of the given FIE-1. This happens to be the exact solution which Almousa and Ismail [12] also obtained using the optimal homotopy asymptotic method (OHAM). The results are compared in Table 1 and Figure 1(a).

Example 4.2.
Consider the FIE-1 [12]:

$$\frac{1}{2}\sin x=\int_0^{\frac{\pi}{2}}\frac{2}{\pi}\sin x\sin t\,u(t)dt \tag{12}$$

having exact solution $u(x)=\sin x$. Rewriting the FIE-1 yields

$$\sin x=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin x\sin t\,u(t)dt.$$

The RM transforms equation (12) into a Fredholm equation of the second kind

$$u_\varepsilon(x)=\frac{1}{\varepsilon}\sin x-\frac{4}{\varepsilon\pi}\int_0^{\frac{\pi}{2}}\sin x\sin t\,u_\varepsilon(t)dt. \tag{13}$$

Table 1. Comparison of approximate and exact solutions from Picard's SAM and Optimal Homotopy Asymptotic Method (OHAM) for Example 4.1

x	$u(x)$	$u_{OHAM}(x)$	$u_{SAM}(x)$	Absolute Error
0	0	0	0	0
0.1	0.005	0.005	0.005	0
0.2	0.02	0.02	0.02	0
0.3	0.045	0.045	0.045	0
0.4	0.08	0.08	0.08	0
0.5	0.125	0.125	0.125	0
0.6	0.18	0.18	0.18	0
0.7	0.245	0.245	0.245	0
0.8	0.32	0.32	0.32	0
0.9	0.405	0.405	0.405	0
1.0	0.5	0.5	0.5	0

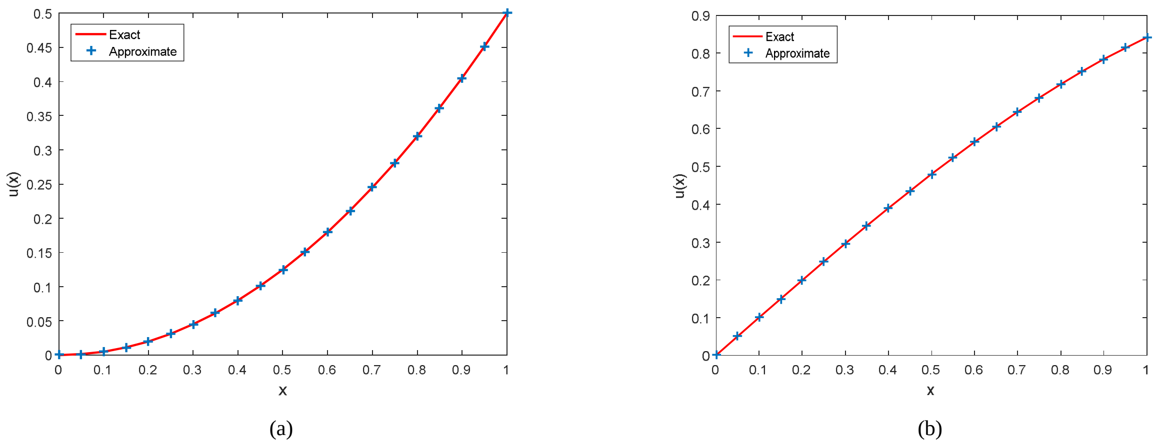

Fig. 1. Results for (a) Example 4.1 (b) Example 4.1

We will solve this equation using the ADM which requires the use of the forcing function $g(x)$ of the resulting FIE-2 as the initial approximant $u_{\varepsilon_0}(x)$. The rest of the approximants are based only on the integral portion of the FIE-2. Thus, if we let $g(x) = \frac{1}{\varepsilon}\sin x$, then by the ADM we have the recursive scheme

$$\begin{aligned} u_{\varepsilon_0}(x) &= g(x) \\ u_{\varepsilon_{n+1}}(x) &= -\frac{4}{\varepsilon\pi}\int_0^{\frac{\pi}{2}} \sin x \sin t\, u_{\varepsilon_n}(t)\,dt,\ n \geq 0 \end{aligned}$$

i.e.,

$$\begin{aligned} u_{\varepsilon_0}(x) &= \frac{1}{\varepsilon}\sin x \\ u_{\varepsilon_1}(x) &= -\frac{4}{\varepsilon\pi}\int_0^{\frac{\pi}{2}} \sin x \sin t\left(\frac{1}{\varepsilon}\sin t\right)dt = -\frac{4}{\varepsilon^2\pi}\cdot\frac{\pi}{4}\sin x = -\frac{1}{\varepsilon^2}\sin x \\ u_{\varepsilon_2}(x) &= -\frac{4}{\varepsilon\pi}\int_0^{\frac{\pi}{2}} \sin x \sin t\left(-\frac{1}{\varepsilon^2}\sin t\right)dt = \frac{4}{\varepsilon^3\pi}\cdot\frac{\pi}{4}\sin x = \frac{1}{\varepsilon^3}\sin x \\ u_{\varepsilon_3}(x) &= -\frac{4}{\varepsilon\pi}\int_0^{\frac{\pi}{2}} \sin x \sin t\left(\frac{1}{\varepsilon^3\pi}\sin t\right)dt = -\frac{4}{\varepsilon^4\pi}\cdot\frac{\pi}{4}\sin x = -\frac{1}{\varepsilon^4}\sin x \end{aligned}$$

and so on. The solution is given by the decomposition series

$$u_\varepsilon(x) = \sum_{n=0}^{\infty} u_{\varepsilon_n}(x) = \frac{1}{\varepsilon}\left(\frac{\varepsilon}{\varepsilon+1}\right)\sin x = \frac{1}{\varepsilon+1}\sin x$$

and so

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = \sin x,$$

the exact solution of the given FIE-1. Almousa and Ismail [12] solved this equation also but they used the OHAM and obtained the same result. The ADM and OHAM results are compared in Table 2 and Figure 1(b) and show that both the OHAM and ADM are very good methods for obtaining exact solutions.

Table 2. Comparison of approximate and exact solutions from the ADM and Optimal Homotopy Asymptotic Method (OHAM) for Example 4.2

x	$u(x)$	$u_{\text{OHAM}}(x)$	$u_{\text{ADM}}(x)$	Absolute Error
0	0	0	0	0
0.1	0.09983341665	0.09983341665	0.09983341665	0
0.2	0.19866933080	0.19866933080	0.19866933080	0
0.3	0.29552020666	0.29552020666	0.29552020666	0
0.4	0.38941834231	0.38941834231	0.38941834231	0
0.5	0.47942553860	0.47942553860	0.47942553860	0
0.6	0.56464247340	0.56464247340	0.56464247340	0
0.7	0.64421768724	0.64421768724	0.64421768724	0
0.8	0.71735609090	0.71735609090	0.71735609090	0
0.9	0.78332690963	0.78332690963	0.78332690963	0
1.0	0.84147098481	0.84147098481	0.84147098481	0

Example 4.3.
Consider the FIE-1 [13]:

$$\frac{1}{2}(\mathrm{e}-1)\mathrm{e}^x = \int_0^{\frac{1}{2}} \mathrm{e}^{x+t} u(t)\,dt \tag{14}$$

whose exact solution is $u(x) = \mathrm{e}^x$. Rewriting (14) gives

$$(\mathrm{e}-1)\mathrm{e}^x = 2\int_0^{\frac{1}{2}} \mathrm{e}^{x+t} u(t)\,dt$$

and using the RM we obtain the FIE of the second kind

$$u_\varepsilon(x) = \frac{\mathrm{e}-1}{\varepsilon}\mathrm{e}^x - \frac{2}{\varepsilon}\int_0^{\frac{1}{2}} \mathrm{e}^{x+t} u_\varepsilon(t)\,dt \tag{15}$$

We will solve (15) by the SAM. Let the initial guess be $u_{\varepsilon_0}(x) = 0$. Then the successive approximations are

$$\begin{aligned}
u_{\varepsilon_1}(x) &= \frac{\mathrm{e}-1}{\varepsilon}\mathrm{e}^x \\
u_{\varepsilon_2}(x) &= \frac{\mathrm{e}-1}{\varepsilon}\mathrm{e}^x - \frac{2}{\varepsilon}\int_0^{\frac{1}{2}} \mathrm{e}^{x+t}\left(\frac{\mathrm{e}-1}{\varepsilon}\mathrm{e}^t\right)dt = \frac{\mathrm{e}-1}{\varepsilon}\mathrm{e}^x - \frac{(\mathrm{e}-1)^2}{\varepsilon^2}\mathrm{e}^x \\
u_{\varepsilon_3}(x) &= \frac{\mathrm{e}-1}{\varepsilon}\mathrm{e}^x - \frac{2}{\varepsilon}\int_0^{\frac{1}{2}} \mathrm{e}^{x+t}\left(\frac{\mathrm{e}-1}{\varepsilon} - \frac{(\mathrm{e}-1)^2}{\varepsilon^2}\right)\mathrm{e}^t\,dt = \frac{\mathrm{e}-1}{\varepsilon}\mathrm{e}^x - \frac{(\mathrm{e}-1)^2}{\varepsilon^2}\mathrm{e}^x + \frac{(\mathrm{e}-1)^3}{\varepsilon^3}\mathrm{e}^x
\end{aligned}$$

and so on. Thus,

$$u_\varepsilon(x) = \frac{\mathrm{e}-1}{\varepsilon}\left(1 - \frac{\mathrm{e}-1}{\varepsilon} + \frac{(\mathrm{e}-1)^2}{\varepsilon^2} - \cdots\right)\mathrm{e}^x = \frac{\mathrm{e}-1}{\varepsilon}\left(\frac{\varepsilon}{\varepsilon+\mathrm{e}-1}\right)\mathrm{e}^x = \frac{\mathrm{e}-1}{\varepsilon+\mathrm{e}-1}\mathrm{e}^x,$$

and the solution to the FIE-1 is

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = \mathrm{e}^x,$$

which is the exact solution. We now compare this with the results obtained by Adawi et al. [13] using the homotopy analysis method (see Table 3 and Figure 3). The absolute errors for the HAM and SAM are given in Table 3 and show the superior performance of Picard's SAM over the HAM as it gives the exact solution of Fredholm equations of the first kind.

Table 3. Comparison of approximate and exact solutions from Picard's SAM and Homotopy Analysis Method (HAM) for Example 4.3

x	$u(x)$	$u_{HAM}(x)$	$u_{SAM}(x)$	e_{HAM}	e_{SAM}
0	1.000000000000	0.999999996924	1.000000000000	3.076E−9	0
0.1	1.105170918076	1.105170914677	1.105170918076	3.399E−9	0
0.2	1.221402758160	1.221402754403	1.221402758160	3.757E−9	0
0.3	1.349858807576	1.349858803425	1.349858807576	4.151E−9	0
0.4	1.491824697641	1.491824693052	1.491824697641	4.589E−9	0
0.5	1.648721270700	1.648721265629	1.648721270700	5.071E−9	0

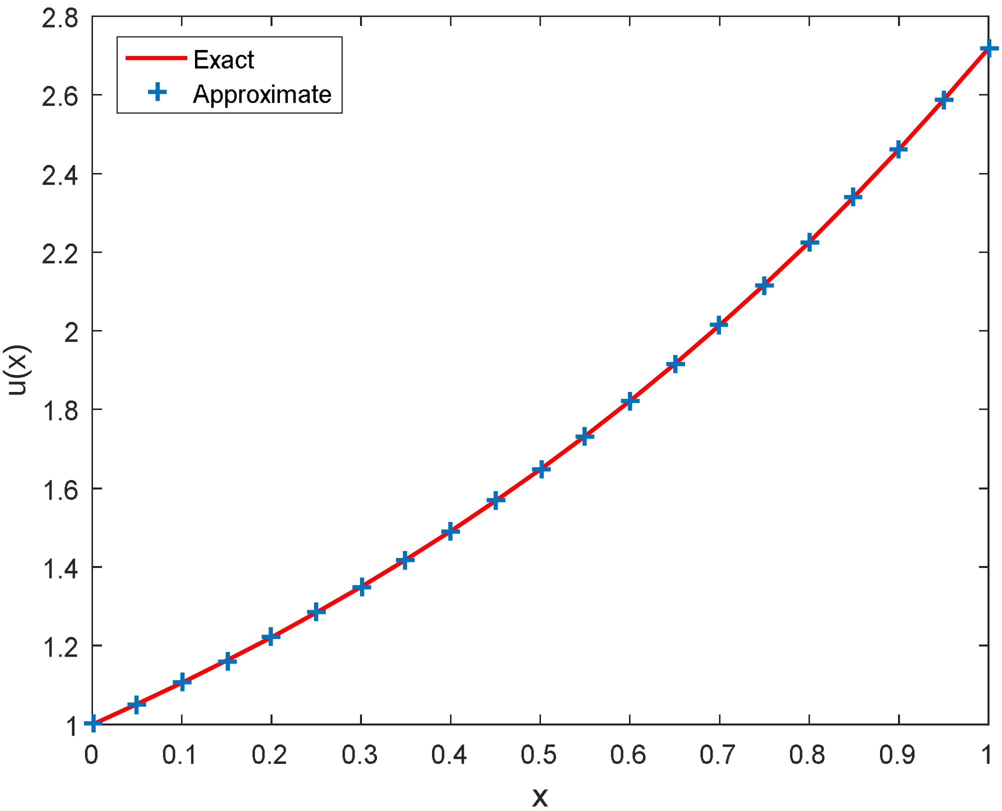

Fig. 2. Results for Example 4.3

Example 4.4.
Consider the FIE-1 [4]:

$$2x = \int_0^1 xtu(t)dt \tag{16}$$

having exact solution $u(x) = 6x$. Wazwaz [4] solved this problem using Picard's SAM but we will use the ADM for comparison purposes. Using the RM, (16) converts to the FIE of the second kind

$$u_\varepsilon(x) = \frac{2}{\varepsilon}x - \frac{1}{\varepsilon}\int_0^1 xtu_\varepsilon(t)dt \tag{17}$$

Let $g(x) = \frac{2}{\varepsilon}x$. Then by the ADM we have the recursive scheme

$$\begin{aligned} u_{\varepsilon_0}(x) &= g(x) \\ u_{\varepsilon_{n+1}}(x) &= -\frac{1}{\varepsilon}\int_0^1 xtu_{\varepsilon_n}(t)dt, \ n \geq 1 \end{aligned}$$

i.e.,

$$
\begin{aligned}
u_{\varepsilon_0}(x) &= \frac{2}{\varepsilon}x\\
u_{\varepsilon_1}(x) &= -\frac{1}{\varepsilon}\int_0^1 xt\left(\frac{2}{\varepsilon}t\right)dt = -\frac{2}{3\varepsilon^2}x\\
u_{\varepsilon_2}(x) &= -\frac{1}{\varepsilon}\int_0^1 xt\left(-\frac{2}{3\varepsilon^2}t\right)dt = \frac{2}{9\varepsilon^3}x\\
u_{\varepsilon_3}(x) &= -\frac{1}{\varepsilon}\int_0^1 xt\left(\frac{2}{9\varepsilon^3}t\right)dt = -\frac{2}{27\varepsilon^4}x
\end{aligned}
$$

and so on. The solution is

$$u_\varepsilon(x) = \frac{2}{\varepsilon}\left(1 - \frac{1}{3\varepsilon} + \frac{1}{9\varepsilon^2} - \frac{1}{27\varepsilon^3} + \cdots\right)x = \frac{2}{\varepsilon}\left(\frac{3\varepsilon}{3\varepsilon+1}\right)x = \frac{6}{3\varepsilon+1}x,$$

and

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x) = 6x,$$

which was also obtained using the SAM by Wazwaz [4] who pointed out that, being an ill-posed problem, (16) also has the solution $u(x) = 4x^2 + 5x^3$. These results are compared in Table 4 and Figure 3(a).

Table 4. Comparison of approximate and exact solutions from the ADM and Picard's SAM for Example 4.4

x	$u(x)$	$u_{\text{SAM}}(x)$	$u_{\text{ADM}}(x)$	Absolute Error
0	0	0	0	0
0.2	1.2	1.2	1.2	0
0.4	2.4	2.4	2.4	0
0.6	3.6	3.6	3.6	0
0.8	4.8	4.8	4.8	0
1.0	6.0	6.0	6.0	0

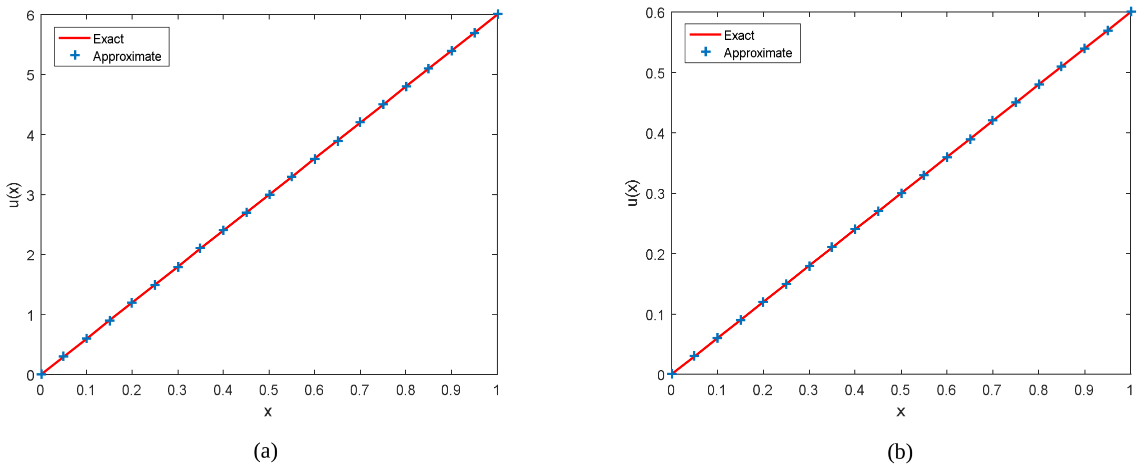

Fig. 3. Results for (a) Example 4.4 (b) Example 4.5

Example 4.5.
Consider the FIE-1 [4]:

$$\frac{1}{5}x = \int_0^1 xtu(t)dt \tag{18}$$

with exact solution $u(x) = \frac{3}{5}x$. This problem was solved by means of the ADM in [4]; here we use Picard's SAM for comparison purposes. By the RM, (17) becomes a second-kind FIE

$$u_\varepsilon(x) = \frac{1}{5\varepsilon}x - \frac{1}{\varepsilon}\int_0^1 xt u_\varepsilon(t)dt \tag{19}$$

Let $u_{\varepsilon_0}(x) = 0$ be the initial guess. Then the first few approximations by the SAM are

$$\begin{aligned}
u_{\varepsilon_1}(x) &= \frac{1}{5\varepsilon}x\\
u_{\varepsilon_2}(x) &= \frac{1}{5\varepsilon}x - \frac{1}{\varepsilon}\int_0^1 xt\left(\frac{1}{5\varepsilon}t\right)dt = \frac{1}{5\varepsilon}x - \frac{1}{15\varepsilon^2}x\\
u_{\varepsilon_3}(x) &= \frac{1}{5\varepsilon}x - \frac{1}{\varepsilon}\int_0^1 xt\left(\frac{1}{5\varepsilon} - \frac{1}{15\varepsilon^2}\right)tdt = \frac{1}{5\varepsilon}x - \frac{1}{15\varepsilon^2}x + \frac{1}{45\varepsilon^3}x\\
u_{\varepsilon_4}(x) &= \frac{1}{5\varepsilon}x - \frac{1}{\varepsilon}\int_0^1 xt\left(\frac{1}{5\varepsilon}x - \frac{1}{15\varepsilon^2}x + \frac{1}{45\varepsilon^3}\right)tdt = \frac{1}{5\varepsilon}x - \frac{1}{15\varepsilon^2}x + \frac{1}{45\varepsilon^3}x - \frac{1}{135\varepsilon^4}x
\end{aligned}$$

and so on. Thus, the solution is

$$u_\varepsilon(x) = \frac{1}{5\varepsilon}\left(1 - \frac{1}{3\varepsilon} + \frac{1}{9\varepsilon^2} - \frac{1}{27\varepsilon^3} + \cdots\right)x = \frac{1}{5\varepsilon}\left(\frac{3\varepsilon}{3\varepsilon+1}\right)x = \frac{1}{5}\left(\frac{3}{3\varepsilon+1}\right)x$$

and the solution of the FIE (17) is

$$u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x) = \frac{3}{5}x,$$

the same result obtained by Wazwaz [4] using the Adomian decomposition method. According to [4], $u(x) = x^3$ is also a solution to (17), underscoring the ill-posedness of FIE-1s.

Table 5. Comparison of approximate and exact solutions from Picard's SAM and the ADM for Example 4.5

x	$u(x)$	$u_{\text{SAM}}(x)$	$u_{\text{ADM}}(x)$	Absolute Error
0	0	0	0	0
0.2	0.12	0.12	0.12	0
0.4	0.24	0.24	0.24	0
0.6	0.36	0.36	0.36	0
0.8	0.48	0.48	0.48	0
1.0	0.60	0.60	0.60	0

5. Conclusion

This paper has shown by numerical examples that Picard's SAM and the ADM are excellent methods for solution of Fredholm integral equations of the first kind. The numerical experiments conducted have validated the accuracy of these two methods and shown that they lead to results which rapidly converge to the exact solutions. Future work might require using these methods to solve other types of integral and integrodifferential equations, as well as exploring other methods for solving FIE-1s.

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