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More on Quasi-Relaxation transforms and their applications

Research Article

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Abstract: In past researches and papers on the formulation and creation of a quasi-relaxation transforms pair, the use of the hereditary integrals is fundamental in la construction of these transforms pair and where the inheritance of the relaxation and creep functions are fundamental in the contribution to the characterization of the corresponding transforms, likewise also the energy contributions of the stress and strain tensors are present in the analytical process of these hereditary integrals. The goal of this paper is a review on the properties of the quasi-relaxation transforms and some new applications. Likewise we have the integral transforms pair

$$\Psi_c(\tau) = QX\{\phi(t)\} = \int_0^\infty \phi(t)e^{-\frac{t}{\tau}}\,dt,$$

$$\phi_c(t) = Q^{-1}X\{\Psi(\tau)\} = \int_{-\infty}^{\infty} \Psi(\tau)e^{\frac{t}{\tau}}\,dt,$$

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1. Introduction to the Problem of the Quasi-relaxation

Metallic specimens are submitted in the conventional machines (stress-deformation) through of several essays. Likewise, the specimen is previously loaded up to an initial level of the stress. After the motorized system of the machine is disconnected, which is observed a spontaneous fall of stress, however the length of specimen is conserved. This defines a quasi-relaxation state in the test-specimen [1,2].

Likewise, the kinetic of the fall of the stress is registered during all the process of the essays [1]. Then a similar experiment must be executed in a programmed specially machine, in which during the essay of automatic manage stays constant the longitude of the specimen, is to say, the condition of the essay in regime of quasi-relaxation can be expressed in the following form

$$l = cte, \tag{1}$$

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Nomenclature

$\mathscr{Q}X$	Direct quasi-relaxation transform operator.
$\Psi_c(\tau)$	Direct quasi-relaxation transform function on time domain τ.
$\phi(t)$	Relaxation function.
$\Psi(t)$	Creep funcion.
$\phi_c(t)$	Inverse quasi-relaxation transform function on time domain t.
$\mathscr{Q}^{-1}X$	Inverse quasi-relaxation transform operator.
$\mathscr{E}_0$	Initial Deformation on a material.
σ_0	Initial stress on a material.
t	Relaxation time.
$\mathscr{E}$	Deformation tensor applied in the quasi-relaxation process in the machine-specimen.
σ	Stress tensor applied in the quasi-relaxation process in the machine-specimen.
b	Burgers number, which represents the magnitude and direction of the lattice distortion of dislocation in a crystal lattice.
$\Gamma(\sigma - \mathscr{E}, t)$	Energy functional of the quasi-relaxation process.
$\mathscr{E}(t)$	Hereditary integral of the deformation tensor.
$\mathscr{E}(t)$	Hereditary integral of the stress tensor.
C	Carson transform operator.
$\mathscr{L}$	Laplace transform operator.
$E(t)$	Relaxation modulus.
E	Elsaki transform operator.
$J(t)$	The creep compliance function for the Kelvin-Voigt model.
$e^{Et/\tau}$	Integrating factor to the differential equation of Kelvin-Voigt model.
η	The argument of the exponential involves the viscosity .
E	and the relaxation modulus.,
$\tau = \eta/E$	Constants in the time in the corresponding hereditary integrals to the Maxwell and Kelvin-Voigt models. .

or well, in terms of constant deformation

$$\frac{d\mathscr{E}}{dt} = 0, \tag{2}$$

The condition (2) defines the meta-stability as a state of constant deformation only in their plastic characteristics in the initial process of dislocations [2-4], where the energy of the nano-crystals accumulate the enough energy to maintain the specimen in a stable range of recovering to the original state, in a very short time interval [3, 4].

Def. I. 1. The limit before of the dislocations in the system described by quasi-relaxation (in a plastic work) on $0 \le s \le t$, comes given by the linear hereditary integrals:[1]

$$\mathscr{E}(t) = \int_{-\infty}^{t} \frac{d\sigma(s)}{ds} \Psi(t-s)\,ds, \tag{3}$$

$$\sigma(t) = \int_{-\infty}^{t} \frac{d\mathscr{E}(s)}{ds} \phi(t-s)\,ds, \tag{4}$$

where the functions $\phi(t)$, and $\Psi(t)$, are the relaxation and creep functions respectively, and here $\phi(-\infty) = \Psi(-\infty) = 0$.

Likewise, is necessary realize a deep study of the trace of deformation tensor in function of the stress tensor corresponding to the plastic deformation and use a functional of energy [5], consigning to our problem of quasi-relaxation phenomena in the following theorem.

Theorem 1.1.
(F. Bulnes-Y. Yermishkin) The quantity of accumulated energy ΔG, during the quasi-relaxation is determined by the work of plastic deformation during the application of the system machine-specimen and $\Delta G = \Gamma(\sigma - \mathscr{E}, t)t$.

[1] *We take the initial condition to the differential equation to the Maxwell model $\sigma(-\infty) = 0$. Also the corresponding to the differential equation to Kelvin model, $\mathscr{E}(-\infty) = 0$.*

Proof. [5], [6] □

This result represents the medullar and vertebral part of the meaning of quasi-relaxation and its relation with the energy accumulated as plastic deformation, and the quasi-relaxation transform derives of the natural way from its energy functional [2, 6], which can be seen in the details of the demonstration of the theorem.

In the following lemma, which was demonstrated in [5] establishes that all function in the quasi-relaxation phenomena is of Laplace type.

Lemma 1.1.
Whole quasi-relaxation experiment has function of physical system dynamics of Laplace type.

Proof. [6] □

Likewise, we can see that the behavior of the quasirelaxation curves in the experiments comply with the

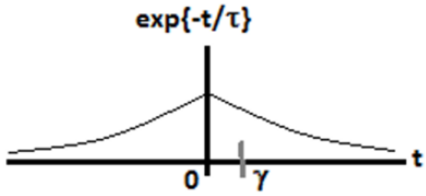

Fig. 1. Curve of whole development of quasi-relaxation experiment.

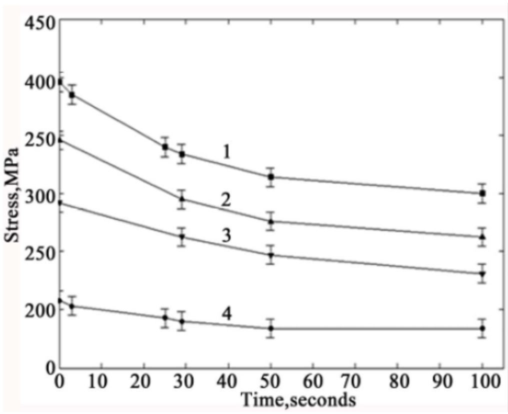

Fig. 2. Experimental quasi-relaxation curves [7].

Lemma 1.2.
The nucleus $K(t,\tau)$*, defined to operator* $I_{t\tau}$*, whose integral is*

$$I_{t\tau} = \int_{specimen} \xi(\tau) K(t,\tau) d\tau, \tag{5}$$

verifies

$$\int_{specimen} | K(t,\tau) | d\sigma(\tau) \le C_q || \Omega || \le 1, \tag{6}$$

Proof. [6] □

2. Quasi-Relaxation Transforms

After of analyze the semi-duality of the relaxation and creep functions and complies the fact that

$$\int_{-\infty}^{\infty} \phi(t)\Psi(\tau) - \Phi(\tau)\psi(t)dt = 0, \tag{7}$$

to the relaxation and creep functions as well as their corresponding transforms, further of others important facts as that exists the quasi-relaxation transform $\Psi_c(\tau)$,for all $\tau > 1/\gamma, \gamma \neq 0$, then we establish the formal definition of the quasi-relaxation transform.

Def. 2. 1. Let $\phi(t)$, be an integrable function in $t \geq 0$, with $\tau \neq 0$. Then the integral transform $\Psi_c(\tau)$, of $\phi(t)$, is defined by

$$\Psi_c(\tau) = QX\{\phi(t)\} = \int_0^{\infty} \phi(t)e^{-\frac{t}{\tau}}dt, \tag{8}$$

which exists in the space $\mathscr{D}_{\Psi_c(\tau)} = \{\tau \mid \tau > 1/\gamma, \gamma \neq 0\}$. Without of this space the integral transform $\Psi_c(\tau)$, don't exist.

In (8) we have that the quasi-relaxation transform is the transformation of the hereditary integral of its stress history $\mathscr{E}(t)$ due to the relaxation $\phi(t)$, along $(0,\infty)$. In resume we can to say that the quasi-relaxation transform is the relaxation in the meta-stable state.

Example 2.1.
Determine the quasi-relaxation transform considering the load $U(t-t')$.

We consider the definition of inverse transform and the corresponding hereditary integral to obtain

$$\begin{gathered}\Psi c(\tau) = \int_{-\infty}^{\infty} \phi(t)e^{-t/\tau}dt = \int_{-\infty}^{\infty} \mathscr{E}(t)e^{-t/\tau}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{t} \frac{d\sigma(t')}{dt'}\Psi(t-t')dt'e^{-t/\tau}d\tau = \\ \int_{-\infty}^{\infty}\int_{-\infty}^{t} \sigma_0\delta(t')\Psi(t-t')dt'e^{t/\tau}d\tau = \sigma_0\int_{-\infty}^{\infty}\Psi(t)U(t-t')e^{-t/\tau}d\tau\end{gathered} \tag{9}$$

where will be necessary apply the following functional analysis property

$$\int_{-\infty}^{t} \Psi(s)\delta(s-t)ds = \Psi(t)U(t-s), \tag{10}$$

and also consider the property,

$$\int_{-\infty}^{t} \Psi(t')\delta(t'-t_0)dt' = U(t-t_0)\int_{t_0}^{t}\Psi(t')dt', \tag{11}$$

Then the extreme right of (9) takes the form

$$\mathscr{Q}X\{\phi(t)\} = \sigma_0 U(t-t')\int_{-\infty}^{\infty}\Psi(t)e^{-t/\tau}d\tau = \sigma_0 U(t-t')\phi(\tau), \tag{12}$$

□

If we consider $t' = 0$, the direct quasi-relaxation transform reduces to the load case $U(t)$.

The lemma 1. 2, and the unicity of the quasi-relaxation transform can permit us enunciate the inverse quasi-relaxation transform as the recovering of the quasi-relaxation function through of the creep function:

$$\phi_c(t) = \frac{\alpha}{b\sqrt{\rho_{dym}}}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}\Psi(\tau)e^{t/\tau}d\tau, \tag{13}$$

The before integral relation can prove it that is a Bromwich equality. The criteria of complex contour are satisfied and we can observe from the material science the following fact:

$$\alpha \approx \frac{1}{2\pi} = 0.1587101, \tag{14}$$

and we consider a hypothetical value (only as functional coefficient of contour) of the density $\rho_{dym} = -1$. We can establish that the invariant value $\frac{1}{2\pi}$, appears in the analysis of cuasi-elaxation on materials (see the table [9]).

Table 1. Physical meaning of parameters in internal (13) and model dislocation integration with barriers [9].

α in equation (1)	Meaning of ρ	Interaction mechanical of dislocation
$\frac{1}{2\pi}$	Dislocation in the limits of crystalline lattice	The sliding dislocations surpass the resistance generated by the dislocations that generate periodic networks of dislocations
$\frac{1}{2\pi}$	Density of sliding dislocations	The sliding dislocations surpass the effect of containment of the !maws generated on them as a result of the intersection with the dislocation nucleus
$\frac{\sqrt{n}}{2\pi}, (n = 25, \alpha = 0.795)$	Concentration and dislocation density. n is the number of dislocations in planar concentration	The sliding dislocations overcome the long range stresses arising from the planar concentration of dislocations
$\frac{1}{\pi}$	Density of the dislocation forest	Deflection of the sliding dislocations in the dislocation forest
$\frac{1}{2\pi}$	Density of the dislocation forest	Resistance to the sliding dislocations from the intersection of the dislocation forest forming steps
$\frac{1}{2\pi}$	Total dislocation density	Overcoming the mutual attractive forces of the dislocations for a retains. displacement

and realize the contour construct considering in this little neighborhood that (without loss generality) the Burger's coefficient $b = 1$,[2]

The evaluation of the inverse transform is so hard, and involves non-elemental functions[3]. In some cases could be evaluated only by numerical methods. Likewise the integral of its inverse transform involves the term $e^{t/\tau}$, which is its nucleus whose integration is:

$$\int_0^\infty e^{t/\tau} d\tau = \left\{ e^{t/\tau}\Big|_0^\infty - Ei\left(\frac{t}{\tau}\right) \right\}, \tag{15}$$

Example 2.2.
Determine the inverse quasi-relaxation transform of the creep function considered in the example 2. 1.

We consider the definition of inverse transform and the corresponding hereditary integral to obtain

$$\begin{aligned} \phi_c(t) &= \int_{-\infty}^{\infty} \Psi(\tau) e^{t/\tau} d\tau = \int_{-\infty}^{\infty} \sigma(\tau) e^{t/\tau} d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{t} \frac{d\mathscr{E}(t')}{dt'} \phi(t-t') dt' e^{t/\tau} d\tau, \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{t} \mathscr{E}_0 \delta(t) \phi(t-t') dt' e^{t/\tau} d\tau = \mathscr{E}_0 \int_{-\infty}^{\infty} \phi(t') U(t-t') e^{t/\tau} d\tau, \end{aligned} \tag{16}$$

where has been considered the functional analysis property

[2] *The Burger's coefficient is given by the empirical formula $b = \frac{R_D}{L}$, where L, is the horizontal length scale and R_D, is the Rossby deformation radius.*

[3] *$Ei(t)$, is the exponential integral. $Ei(t) = -\int_t^\infty \frac{e^{-t}}{t} dt$*

$$\int_{-\infty}^{t} \phi(\tau)\delta(\tau - t)d\tau = \phi(t)U(t-\tau), \tag{17}$$

Then considering the following property to,

$$\int_{-\infty}^{t} \phi(t')\delta(t'-t_0)dt' = U(t-t_0)\int_{t_0}^{t} \phi(t')dt', \tag{18}$$

Then the extreme right of (40) takes the form

$$\begin{aligned} \mathscr{E}_0 \int_{-\infty}^{\infty} \phi(t')U(t-t')e^{t/\tau}d\tau = \mathscr{E}_0 U(t-t')\int_{-\infty}^{\infty} \phi(t')e^{t/\tau}d\tau = \mathscr{E}_0 U(t-t')\phi(t')\int_{-t_0}^{t} e^{t/\tau}d\tau \\ \mathscr{E}_0\phi(t)\int_{-t_0}^{t} e^{t/\tau}d\tau, \end{aligned} \tag{19}$$

But

$$\int_{-t_0}^{t} e^{t/\tau}d\tau = e^{t/\tau}\tau - Ei\left(\frac{t}{\tau}\right)\Big|_{t_0}^{t} = et - Ei(1) - [e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)], \tag{20}$$

Finally, the inverse quasi-relaxation transform is[4]

$$\phi_c(t) = E_0\phi(t)\left\{et - Ei(1) - e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)\right\}, \tag{21}$$

where see the footnote[5] to the value $Ei(1)$.

□

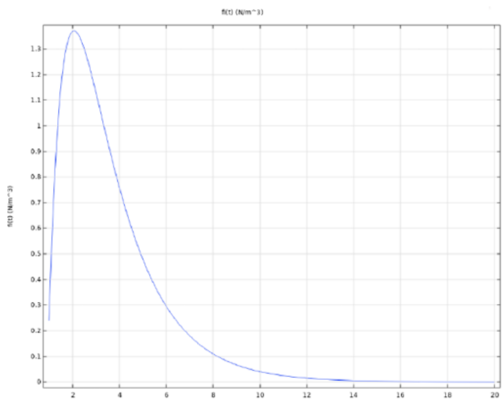

Fig. 3. Anti-transform of the creep function.

We have calculated and determined more direct and inverse quasi-relaxations transform for more load functions [2, 5, 6].

[4] *We can verify* $\Psi_c(\tau) = \sqrt{\Lambda T^{-1}}(\frac{1}{\sqrt{\Lambda}}T(\Psi_c(\tau))) = \frac{1}{\sqrt{\Lambda}}\sqrt{\Lambda}\mathscr{Q}X^{-1}\{\mathscr{E}_0\phi(t)\left\{et - Ei(1) - e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)\right\}\}.$

[5] $Ei(1) = 1.89511781.....$

3. Resolution of Kelvin-Voigt model

Important properties to the quasi-relaxation transform are considered [5]:

i. $\mathscr{Q}X\{\phi'(t)\} = \tau\Psi(\tau) - \phi(0)$

ii. $\mathscr{Q}X\{\phi''(t)\} = \tau^2\Psi(\tau) - \tau\phi(0) - \phi'(0)$

iii. $\mathscr{Q}X\{\phi^n(t)\} = \tau^n\Psi(\tau) - \sum_{k=0}^{n-1}\tau^{1-n+k}\phi^{(k)}(0)$

We will demonstrate the properties.

Proof (i). We consider

$$\mathscr{Q}X\{\phi'(t)\} = \int_0^\infty \phi'(t)e^{-t/\tau}dt, \tag{22}$$

Then integrating by parts we have:

$$\int_0^\infty \phi'(t)e^{-t/\tau}dt = e^{-t/\tau}\phi(t)\Big|_0^\infty + \tau\Psi(\tau), \tag{23}$$

where finally

$$\mathscr{Q}X\{\phi'(t)\} = \tau\Psi(\tau) - \phi(0), \tag{24}$$

Proof (ii). Let $\lambda(t) = \phi''(t)$, then

$$\mathscr{Q}X\{\lambda(t)\} = \tau\mathscr{Q}X\{\lambda'(t)\} - \lambda(0), \tag{25}$$

Therefore

$$\mathscr{Q}X\{\phi''(t)\} = \tau[\zeta\Psi(\tau) - \phi(0)] - \lambda(0) = \tau^2\Psi(\tau) - \tau\phi(0) - \lambda(0), \tag{26}$$

The proof of (iii) can be proved by induction on n.

Also we have the quasi-relaxation of integrals:

(iv).$\mathscr{Q}X\{\int_0^t \phi(u)du\} = \tau\Psi(t)$

Indeed, we have integrating by parts that

$$\mathscr{Q}X\{\int_0^t \phi(u)du\} = \int_0^\infty e^{-t/\tau}\Big[\int_0^t \phi(u)du\Big]dt = \int_0^t \phi(u)du\Big(-\tau e^{-t/\tau}\Big)\Big|_0^\infty + \tau\int_0^\infty \phi(t)e^{-t/\tau}dt = \tau\Psi(\tau), \tag{27}$$

Some relations with other integral transforms are the following:

(a). $\tau\mathscr{Q}X\{\phi(t)\} = \mathscr{E}\{\phi(t)\}$, where $\mathscr{E}\{\phi(t)\}$, is the Elzaki transform.

(b).If $f(t) = \phi(t)$, (i. e, if the function is a relaxation function) and $p = -1/\tau$, then the Laplace transform is a quasi-relaxation transform.

(c).Let be

$$\mathscr{C}\{\phi(t)\} = p\int_0^\infty e^{-pt}dt, \tag{28}$$

the Carson transform of the relaxation function $\phi(t)$. We consider that the Carson transform has the following relation with the Laplace transform:

$$\mathscr{C}\{\phi(t)\} = p\Phi(p) = p\mathscr{L}\{\phi(t)\} \tag{29}$$

If we elect $p = \frac{1}{\tau}$, then

$$\mathscr{C}\{\phi(t)\} = \frac{1}{\tau}\phi(t)e^{-t/\tau}dt = \frac{1}{\tau}\Psi(\tau) = \frac{1}{\tau}\mathscr{L}\{\phi\{t\}\}, \tag{30}$$

Now we consider a differential equations problem. In a before paper accepted in [5] was solved the differential equation in the Maxwell model in the viscoelasticity problem to a relaxation function on the interval $[0, \tilde{t}]$. Now we consider the differential equation in the Kelvin-Voigt model in the viscoelasticity problem to obtain the response to the Kelvin low creep function [9]

$$\frac{d\mathscr{E}}{dt} + \frac{E}{\eta}\mathscr{E}(t) = \frac{1}{\eta}\sigma(t), \tag{31}$$

Firstly, we observe that the equation is a first order differential equation which can be solved using the standard integrating factor. The initial condition from the hereditary integrals to $\sigma(t)$, can be taken as $\mathscr{E}(-\infty) = \mathscr{E}(0) = 0$, where we consider $\mathscr{E}(0) = 0$, to apply the direct quasi-relaxation transform. Solving by the integrating factor $e^{Et/\eta}$, we have:

$$\frac{d}{dt}(e^{-Et/\eta}\Psi) = \frac{1}{\eta}e^{-Et/\eta}\frac{d\sigma(t)}{dt}, \tag{32}$$

Where integrating both sides on $[0, \tau]$, we have

$$(e^{-Et/\eta}\mathscr{E})\Big|_{\mathscr{F}} - (e^{-Et/\eta}\mathscr{E})\Big|_{-\infty} = \frac{1}{\eta}e^{-Et/\eta}\frac{d\sigma(t)}{dt}, \tag{33}$$

Finally[6]

$$\mathscr{E}(t) = \int_{-\infty}^{\tau}\frac{1}{\eta}e^{-Et/\eta}\frac{d\sigma(t)}{dt}dt = \frac{\sigma_0}{E} + \int_{-\infty}^{\tau}J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau, \tag{34}$$

where $J(t)$, the creep compliance function for the Kelvin-Voigt model [9], is

$$J(t) = \frac{1}{E}(1 - e^{-Et/\eta}), \tag{35}$$

Then (34) is the solution to differential equation on said interval, where usually this solutions is written as

$$\mathscr{E}(t) = \frac{\sigma_0}{E}(1 - e^{-Et/\eta}), \tag{36}$$

Now we use our quasi-relaxations transforms considering $\tau = \eta/E$, and remembering by hereditary integrals, with $\mathscr{E}(-\infty) = 0$, that

$$\sigma(\tau) = \Psi_c(\tau) = \int_0^{\infty}\phi(t)e^{-Et/\eta}dt = \int_0^{\infty}\mathscr{E}(t)e^{-Et/\eta}dt, \tag{37}$$

Then by the derivatives properties

$$\mathscr{Q}X\left\{\frac{d\mathscr{E}}{dt}\right\} + \frac{E}{\eta}\mathscr{Q}X\{\mathscr{E}(t)\} = \mathscr{Q}X\left\{\frac{1}{\eta}\frac{d\sigma}{dt}\right\}, \tag{38}$$

or equivalently (using (34))

$$\sigma(\tau) = \frac{\eta}{\eta\tau - \eta + E}\int_0^{\infty}\frac{1}{\eta}\frac{d\sigma}{dt}e^{-Et/\eta}dt, \tag{39}$$

[6] *In the Kelvin model* $\frac{1}{\eta} = \frac{\sigma_0}{E}$.

Now we apply the respective inverse transform[7]

$$\Psi_c(t) = \mathcal{Q}X^{-1}\{\sigma(\tau)\} = \int_{-\infty}^{\tau} \left\{\frac{\eta}{\tau\eta - \eta + E}\int_0^{\infty} \frac{1}{\eta}\frac{d\sigma}{dt} e^{-Et/\eta} dt\right\} e^{Et/\eta} d\tau \tag{40}$$

Then the last integral (40) takes the form finally:

$$\begin{aligned}\Psi_c(t) &= \frac{\sigma_0}{E} + \int_{-\infty}^{\tau} J(t-\tau)\frac{d\sigma(\tau)}{d\tau} d\tau \\ &= \int_0^{\tau} \sigma(\tau) e^{Et/\eta} d\tau,\end{aligned} \tag{41}$$

The solution is the same solution obtained by classic methods to a first order differential equation. We consider the following short table 2.

Table 2. Short table of Quasi-relaxation transform [5].

Function of Load Strain	Direct Quasi-relaxation Transform	Inverse Quasi-relaxation Transform
$U(t)$	$\Psi_c(\tau) = \sigma_0\phi(\tau)$	$\phi_c(\tau) = \phi(t')e^{t/\tau}$
$U(t-t')$	$\Psi_c(\tau) = \sigma_0 U(t-t')\phi(\tau)$	$\phi_c(t) = \varepsilon_0\phi(t)\left\{et - Ei(1) - e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)\right\}$
$\delta(t)$	$\Psi_c(\tau) = \sigma_0\phi(t)\{e^{-1}\Psi(0) - \phi(\tau)\}$	$\phi_c(t) = \varepsilon_0\phi(0)\left\{et - Ei\left(\frac{t}{t_0}\right) - \phi(t)\right\}$
$\phi(\tau)$	$\Psi_c(t)$	$\phi_c(\tau)$
$\Psi(t)$	$\phi_c(\tau)$	$\Psi_c(t)$

NOTES. Has been used the properties of convolution with delta function inside hereditary integrals:

a. $\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)\delta(t-\tau)d\tau = f(t).$

b. $\mathcal{E}(t) = \sigma(0)J(t) + \hat{\sigma}J(t-T).$

c. We use $\sigma(t) = \hat{\sigma}U(t-a)$ in $\mathcal{E}(t) = \sigma(0)J(t) + \int_0^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau.$

4. Experimental proofs

We consider the creep function extracted from the Table 1:

$$\Psi_c(\tau) = \sigma_0 U(t-t_0)\phi(\tau), \tag{42}$$

and where we consider

$$\phi(\tau) = Ee^{-E\tau/\eta}, \tag{43}$$

---

7 *We demonstrate the following identity of the terms in the integration process of the integrals (40)* $J(t-\tau)\frac{d\sigma(\tau)}{d\tau} = \sigma(\tau)e^{-Et/\eta}$. *we start with the implication ⇒).*

$$\frac{d}{d\tau}\left\{\sigma(\tau)e^{-Et/\eta}\right\} = e^{-Et/\eta}\frac{d\sigma(\tau)}{d\tau} = e^{E(\tilde{t}-\tau)/\eta}\frac{d\sigma(\tau)}{d\tau} = e^{-E(\tau-\tilde{t})/\eta}\frac{d\sigma(\tau)}{d\tau},$$

$$t = \tilde{t} - \tau \qquad e^{E(\tilde{t}-\tau)/\eta} = e^{-E(\tau-\tilde{t})/\eta}$$

If we consider $\frac{1}{\eta} = \frac{\sigma_0}{E} = e^{\tilde{t}}$, *and also* $\sigma_0 = EJ(\tilde{t})$, *then* $e^{-E(\tau-\tilde{t})/\eta} = e^{\tilde{t}}e^{-E(t-\tau)/\eta}$. *Then*

$$e^{-E(\tau-\tilde{t})/\eta}\frac{d\mathcal{E}(\tau)}{d\tau} = \frac{\sigma_0}{E}e^{-E(t-\tau)/\eta} = \frac{1}{\eta}e^{-E(t-\tau)/\eta} = J(\tilde{t})e^{-E(t-\tau)/\eta} = J(t-\tau).$$

being E, the Young's modulus and η, is a viscosity of the material. We consider the following values, $E = 1$, and $\eta = 2$, in the material submitted under the load of $\sigma_0 = 5kg/mm^2$. Then we have in particular the quasi-relaxation transform:

$$\Psi_c(\tau) = 5U(t-5)e^{-\tau/2}, \tag{44}$$

whose curve is given in the figure 4.

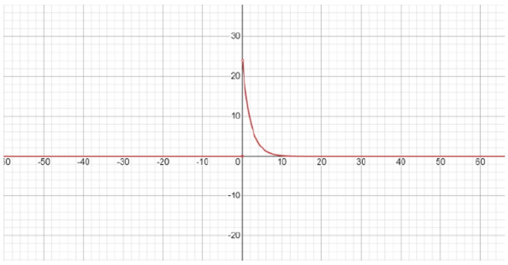

Fig. 4. Quasi relaxation transform with the load function $U(t-5)$.

Now, we consider the following creep spectra or inverse quasi-relaxation transform:

$$\phi_c(t) = 6e^{-3\tau/2}(e\tau - 1.8951178 - e^{\tau/5}),$$

where we have considered previously the function extracted from the Table 2,
$\phi_c(t) = \mathscr{E}_0\phi(t)\left\{et - Ei(1) - e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)\right\}$. We have took the value $Ei(1)$,expressed in the footnote 5,and the logarithmic expression approximation of the exponential integral given in the footnote 3 to $Ei\left(\frac{t}{t_0}\right)$. Then we have the following spectra curve $\phi_c(t)$ in the figure 5.

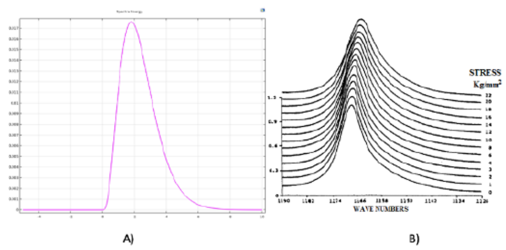

Fig. 5. A). Inverse quasi-relaxation function or spectra curve $\phi_c(t)$. B). Molecular stress distribution function determining the stress relaxation modulus or creep compliances. Also remember that the figure 3, in the section 2 of this paper.

In a tension von Mises Modulus model, the spectra given in the figure 5, can be viewed as the figure 6. which we can observer with the experimental serious studies in material sciences realized in [10], to spectroscopy of polymers, where the goal was obtain the molecular stress distribution function determining the stress relaxation modulus or creep compliances.

5. Conclusions

The hereditary integrals and their use is fundamental in la construction of the quasi-relaxation transforms pair, since in the process to obtain the corresponding transforms, both direct and inverse, the inheritance of the relaxation and creep functions respectively remains and contribute to the characterization of the corresponding quasi-relaxation transforms, considering also the energy contributions of the stress and strain tensors which were studied through its energy functional, which are present in the analytical process of the hereditary integrals. In the classical viscoelasticity phenomena the involving of other transforms such that as Laplace transform, Elzaki transform, Carson transform and their closely relation with the quasi-relaxation transform establish possible prospective studies on their generalization to the complex domain $\Psi(p)$, and meromorphic domains [2, 11, 12] where singularities of anti-quasi-relaxation transforms can be determined through criteria on complex spaces. Then can result interesting analyze non-linear aspects in viscoelastic systems [13, 14] where meromorphicity can be evidenced and studied through of a frame-work more adecuate, for example see [15]. Here only was analyzed the real part of its meaning.

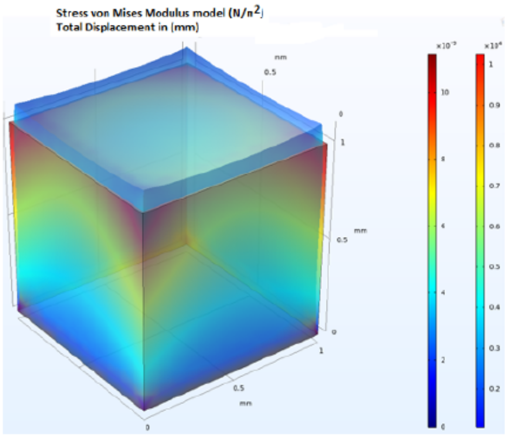

Fig. 6. Tension von Mises modulus model to the spectra given in the figure 5. The displacement is given in millimeters during 5 seconds.

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