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Constant Proportion Portfolio Insurance under Mean Version Stochastic Differential Equation and Risk-Measure Practical Implications

Research Article

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Abstract: We solve the dynamics of risk assets and cushion by the Mean Version stochastic differential equation under Geometric Brownian Motion. The CPPI are attractive investment strategies that give a guaranteed minimum return while focusing on maintaining a risk asset exposure continually equal to a constant multiple of the cushion. We determine risk measures such as the shortfall probability and the expected shortfall and discuss criteria that ensure that the gap risk does not increase to a level that contradicts the original intention of portfolio insurance. We estimate measures of the risk complicated in the practical implementation of discrete-time rebalancing rules directing the CPPI product. It is remarked that frequently rebalancing the portfolio will always lead to a high payoff and thus reduce the probability of losses in conjunction with a lower multiple.

MSC: 76A05 • 76D10

Keywords: Portfolio Insurance • Risk Measure • Mean Version Stochastic differential equation • Constant Proportion Portfolio Insurance

1. Introduction

Portfolio insurance is a strong tool that is created to protect investors against adverse market variation, while still engaging in case of upward market opportunities. Several portfolio insurance strategies are extensively studied Option-based portfolio insurance (OBPI), Stop loss, as analyzed by [1] and [2]. We study one popular example of portfolio insurance strategy the constant proportion portfolio insurance (CPPI). CPPI is a widely used investment strategy that admits to maintaining exposure to the upside potential of a risky asset while in the process providing a guarantee against the downside risk. This dynamic strategy is made up of setting a floor equal to the risky asset as we will describe below. Normally the floor percentage is picked by the investor depending on the risk tolerance of the client, risk preference clients choose a lower floor percentage and vice versa. The exposure is equal to the product of a cushion which establishes as the excess of the portfolio value over the floor and a prearranged multiple [3]. Suitable,

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multiple turns on the investor's risk tolerance. When the cushion increases exposure increases and when the portfolio value decreases, exposure will decreases and approach zero. Consequently, asset allocation should merely remain a risk-free asset when the portfolio value gets to the floor value. Still, the rise in the cushion will eventually put all the portfolio value entirely into risky assets. The indigenous CPPI model is formulated in continuous time and undertakes instantaneous trading and smooth price changes. However, in practice, this management is violated, this introduces the notion of gap risk. Here we examine discrete trading using different ranges of rebalancing frequencies and multiplier values to disclose the effect on payoff CPPI value in conjunction with the Expected terminal payoff, probability of loss, Expected losses, and volatility.
Initiated by [4] for equity instruments and [5] and [6] for fixed-income tools, CPPI is a portfolio insurance strategy that grants maintaining exposure to the upside potential of a risky asset while still providing a guarantee in case of the downside risk. Over an idealized setting under continuity assumptions imposed both on the trading frequency and the dynamics of the risky asset, the risk of breaking the floor of the CPPI is zero. Nevertheless, in reality, these assumptions are violated. This introduces the concept of gap risk the risk that the portfolio value will not reach the guarantee at maturity. Discontinuities in the price of the risky asset, trading frictions, and a lack of liquidity all put up to gap risk. The subject has been widely documented that price trajectories contain Jumps. [7], and [8], [9], motivate a risk of violating the floor even if it is undergoing continuous-time trading. [10] scrutinize the impact of price jumps using historical parameter estimates and find that in spite there is some gap risk, it is moderately low. [11] differentiates rebalancing strategies and ranks their effect for an investor with the power utility. She finds that the optimal rebalancing rule should turn on both transaction costs and the changes in the risky asset.
In a connected study [12] discover that in a setting where the risky asset is driven by a geometric Brownian motion, an individual with Hara utility and susilence level will find the CPPI strategy optimal among different investment strategies with an attainable guarantee attached. A comprehensive study on the problem under discrete-time trading is given by [13] and also [14] look into the effects of discrete-time trading restrictions on the CPPI. They discover that the discrete-time performance of the CPPI is exceptionally sensitive to an increase in the volatility of the underlying. The issue of discrete-time classical CPPI with a fixed-growth floor is studied by [15]. The literature also embraces hedging strategies with artificial assets to model jump and price gap risk. [3] try the greatest value theory to allow higher multiplier values when a quantile hedging approach is taken. The use of historical data from large indices was also considered by [16] and [17] . While [18], [17] and [16] investigate alternative strategies whereby rebalancing is triggered by movements in the risky asset.
To end [19] investigate the statistical properties and main issues in the implementation of the CPPI strategy. They show that when considering a realistic level of volatility the CPPI does not perform well in comparison to a risk-less investment and gap-less portfolio. CPPI returns are highly skewed and in certain cases, fat-tailed.

1.1. Outline of CPPI Strategy

A constant is defined as Proportion Portfolio Insurance was first introduced by [4] for equity instruments and [5] and [6] for fixed income instruments, It guaranteed the minimum amount G to be paid at maturity T to be not less than the desired floor F while providing a guarantee against the downside risk. The family of constant proportion portfolio insurance consists of investments for which the amount necessary for guaranteeing a repayment of fixed amount G at maturity T is invested in a risk-free way, typically a bond, B and only exceeding amount will be invested in one or more risky assets, S_t.
The product manager will take larger risks when the market is performing well. But if the market is going down he will reduce the risk rapidly. The following factors play a key role in the risk strategies an investor will take:

- Price: The current value of the CPPI. The value at time $t \in [0,T]$ will be denoted as V_t
- Floor: The reference level to which the CPPI is compared. This level will guarantee the possibility of repaying the fixed amount G at maturity T, hence it could be seen as the present value of G at maturity. Typically this is a zero-coupon bond and its price at time will be denoted as B_t.
- Cushion: The cushion is defined as the difference between the price and the floor,
  Cushion=Price-Floor.
- Cushion%= Cushion/Price.
- Multiplier: The multiplier is a fixed value which represents the amount of leverage an investor is willing to take.
- Investment level: is the percentage invested in the risky asset portfolio; this also known as the exposure and is for each step fixed at:
  e= Multiplier× Cushion%

Table 1. CPPI evolution strategy when m=3, r=5%, $\mu = 0.1$, $\sigma = 20\%$

T=t	F_t	S_t	C_t	e_t	B_t	V_t
1	95.12	100.00	10.83	32.50	73.46	105.96
2	90.48	110.00	23.71	71.14	43.06	114.19
3	86.07	120.00	39.31	117.93	7.45	125.38
4	81.87	130.00	58.47	175.41	-35.07	140.34
5	77.88	140.00	82.25	246.74	-86.62	160.13
6	74.08	150.00	111.97	335.91	-149.86	186.05
7	70.47	160.00	149.29	447.86	-228.10	219.75
8	67.03	170.00	196.26	588.78	-325.49	263.29
9	63.76	180.00	255.46	766.38	-447.16	319.22
10	60.65	190.00	330.08	990.25	-599.51	390.74

- ”gap” risk: is the probability that the CPPI value will fall under the Floor, see Cont and Tankov (2007).
  The level of risk an investor will take is equal to the investment level as long as the value of the CPPI exceeds the floor. For any time t the future investment decision will be made according to the following rule:
- if $V_t \leq Floor = B_t$, we will invest the complete portfolio into a zero-coupon bond,
- $V_t >$ Floor, we will invest an amount equal to e in the risky asset portfolio.

The floor F_t which is usually defined as a percentage p of the initial investment

$$F_t = GV_0. \tag{1}$$

CPPI portfolio must be managed to keep the portfolio value V_t, at any time $t \leq T$ above the determined floor value F_t with

$$F_t = Ge^{-r(T-t)} \tag{2}$$

$$dF_t = rF_t dt \tag{3}$$

To ensure this guarantee, First, the CPPI determines the amount to invest in risk asset S_t and allocates the remaining funds to a risk-free asset B_t. Which follows the same dynamics as the floor

$$B_t = V_t - E_t \tag{4}$$

$$dB_t = rB_t dt \tag{5}$$

The floor value F_t is used to calculate the cushion C_t, the difference between portfolio value V_t and the floor F_t.

$$V_t = F_t + C_t \tag{6}$$

$$C_t = V_t - F_t \tag{7}$$

The exposure e_t must be continuously adjusted to a constant multiple m.

$$e_t = m(V_t - F_t) = mC_t \tag{8}$$

To support these we illustrate the working, by considering an example, where $m = 3$ for a unit investment in a CPPI with a 5-year maturity, constant risk-free rate of 5%, and guarantee equal to the initial capital. At time $t = 0$, the initial value is (by assumption) $V_0 = 100$ which is also the guaranteed amount G the investor will receive at maturity. The initial floor is

$$F_0 = 100\%.e^{-0.05.5} = 77.88\%,$$

that gives an initial cushion of

$$C_0 = 100\% - 77.88\% = 22.12\%$$

with a chosen multiplier of $m = 3$, the initial exposure is

$$E_0 = 3.22.12\% = 66.36\%.$$

Hence at the start of the investment,

$$B_0 = 100\% - 66.36 = 33.64\%$$

of the initial capital is allocated to the risk-free asset and the remaining

$$E_0 = 66.36$$

is invested in a risky asset.

Table 1 provides an illustration of a simple CPPI strategy over ten years where the price of the underlying asset is assumed to always increases, with a multiplier of 3 and risk-free asset of 5%. Over time, if the risky asset performs well the investor increases exposure to the risky asset and switches out of the risk-free asset following poor performance, as it is observed from periods 4 to 10. At a certain point in time, the asset allocations to the portfolio need to be rebalanced in proportion to the investor's preference. A price change of the risky underlying will affect V_t so that the basic relationship holds again.

Recall m increases or decreases the amount allocated to the exposure, it is therefore by increasing the amount of m, we shift more capital from the risk-free asset to the risk-asset subject to a restriction on the amount of leverage allowed. Leverage constraints are often imposed on the CPPI to prevent additional capital from being borrowed outside of the initial investment. When leverage is not permitted then the portfolio is said to be unlevered or self-funding.

In supporting the above statements if we set $m = 6$ in the previous example, the exposure becomes equal to 132.72% which exceeds the initial capital and so this would be capped at 100%. Normally the value of the multiplier is determined according to the investor preference and risk tolerance and therefore remains constant through the life of the investment. The higher the multiplier value the greater the exposure to the risky asset and the greater the potential for larger gains and losses to be realized.

Since the risk-free rate is constant, the entire progress of the CPPI depends on the stochastic nature of the risky asset. As the return on the risky assets increases and is greater than the risk-free rate r, the cushion will value increases, hence, more capital will be allocated to the risky asset. This process is done by borrowing some additional funds from a risk-free asset held. Moreover, the cushion shrinks when the risky asset performs less than the risk-free in the period of $[0, T]$.

Considering the number of rebalances that are to take place throughout the investment period gives an understandable view of the terminal future fund. Concerning different multiplier values the behaviour of the CPPI is as follows. On the assumption that $m = 1$, the CPPI can be regarded as gapless in that there no rebalancing is needed; hence, there is a 100% confidence that the portfolio will not gap. Basically, with $m = 1$ the portfolio becomes a buy-and-hold investment since the exposure is equal to the cushion and risk-free investment is equal to the floor which grows to meet the guarantee at maturity. The performance of the gapless portfolio is determined wholly by the terminal asset price S_T and is not path dependent. Though it is for $m > 1$ that the CPPI is a fascinating strategy and this is also when the risk of gapping is introduced. Figure 1 shows how the CPPI portfolio value changes, with the different number of

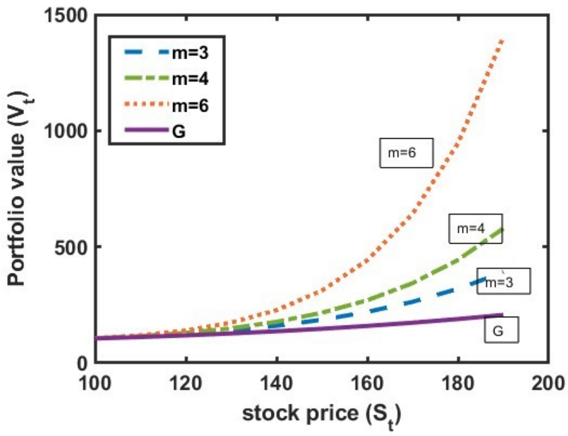

Fig. 1. CPPI payoff when $S_0 = 100, V_0 = 100, \mu = 0.1, T = 10$, and $\sigma = 0.02$

multiplier m. The payoff gets steeper the higher the multiplier m is, and therefore that the strategy does not fall below the floor value G.

2. Model setup

All stochastic processes are defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T^*]}, P)$ which satisfies the usual hypotheses. We consider two investment possibilities: a risky asset S and a risk-less bond B which grows with constant interest rate r, i.e.

$$\begin{cases} dB_t = B_t r dt \\ B_0 = b. \end{cases} \tag{9}$$

The evolution of the risky asset S, a stock or benchmark index, is given by a Geometric Brownian motion (GBM), i.e.

$$\begin{cases} dS_t = S_t(\mu dt + \sigma dW_t), \\ S_0 = s, \end{cases} \tag{10}$$

where $W = (W_t)_{t\in[0,T]}$ denotes a standard Brownian motion with respect to the real world measure P and μ, σ are constants with $\mu > r \geq 0$ and $\sigma > 0$. A continuous– time investment strategy or saving plan for the interval $[0, T]$ can be represented by a predictable process $(\alpha_t)_{t\in[0,T]}$ where α_t denotes the fraction of the portfolio value at time t which is invested in the risky asset S. If there are no additional borrowing restrictions, we can restrict ourselves to strategies that are self–financing, that is, strategies where money is neither injected nor withdrawn during the trading period $[0, T]$ [23][24]. Thus, the amount which is invested at date t in the risk-less bond B is given in terms of the fraction $1-\alpha_t . V = (V_t)_{t\in[0,T]}$ denotes the portfolio value process which is associated with the strategy α, i.e. V_t is the solution of

$$\begin{cases} dV_t(\alpha) = V_t\left(\alpha_t \frac{dS_t}{s_t}\right) + (1-\alpha_t)\frac{dB_t}{B_t} \\ V_0 = x. \end{cases} \tag{11}$$

Notice that there are alternative possibilities for portfolio insurance. Let T denote the terminal trading date. For example, one might think of T as the retirement day. The minimal wealth which must be obtained is denoted by G. The guaranteed amount is assumed to be less than the terminal value of a pure bond investment, i.e. we assume $G < e^{rT} V_0$. Besides a pure bond investment, a trivial possibility is given by a static trading strategy where at the initial time $t = 0$ the present value of the guarantee, i.e. Ge^{-rT} is invested in the bond B and the remaining part, i.e. the surplus $V_0 - e^{-rT}G$,
is invested in the risky asset S. Thus, although

$$\alpha_t = \frac{(V_0 - e - rTG)}{Vt}\frac{St}{S_0} \tag{12}$$

is stochastic, the strategy is static in the sense that there are no re-balancing decisions involved during the interval $[0, T]$. Abstracting from stochastic interest rates, the above strategy honors the guarantee G independent of the stochastic process generating the asset prices.
In the following, we concentrate on the CPPI approach. It is worth mentioning that even without a utility-based justification, the CPPI is an important strategy in practice. We fix the notation and review the basic form and properties of continuous–time CPPI strategies. Recall that the basic idea of the CPPI approach is to invest the amount of portfolio value that is above the present value of the guarantee in the risky asset S. Normally, the symbol F is used to denote the floor. The floor is defined by $F_t := e^{-r(T-t)}G$ and thus denotes the present value of the guarantee G. This is equivalent to

$$dF_t = F_t r dt$$

with $F_0 = e^{-rT}G$.
The surplus is called cushion and denoted by C, i.e. $C_t := V_t - F_t$. If the cushion is monitored in continuous time, it is even possible to invest a multiple of the cushion in the risky asset. Let m denote the multiplier, then the fraction α of a CPPI strategy is given by

$$\alpha_t := \frac{mC_t}{V_t}. \tag{13}$$

We call a continuous-time CPPI strategy which satisfies the above form simple. Notice that a simple CPPI strategy is given in terms of the guarantee G and the multiplier $m \geq 1$. In addition to the protection feature, this ensures that the value of the CPPI strategy is convex in the asset price, at least in a continuous-time setup with continuous asset paths. Throughout this paper, the guarantee is given exogenous, i.e. it is the minimal value of wealth which is needed at T. We review some basic properties of the continuous-time CPPI technique. First, consider the cushion process $(C_t)_{t\in[0,T]}$. We use the notation C_T for the cushion process in continuous time and likewise V_T for the value process in continuous time in order to distinguish from several discrete-time cushion and value processes.

Lemma 2.1.
If the asset price dynamic is lognormal, i.e. if it satisfies equation (10), the cushion process $(C_T)_{t\in[0,T]}$ of a simple CPPI is lognormal, too. In particular, it holds

$$dC_T = C_T((r + m(\mu - r)dt + \sigma m dW_t). \tag{14}$$

Proof. Notice that $C_T = V_T - F_t$ implies

$$dC_T = d\Big(V_T - F_t\Big)$$

$$= V_T\Big(\frac{mC_T}{V_T}\frac{dS_t}{S_t} + \Big(1 - \frac{mC_T}{V_T}\Big)\frac{dB_t}{B_t}\Big) - F_t\frac{dBt}{Bt}$$

$$= C_T\Big(m\frac{dS_t}{S_t} - (m-1)rd_t\Big) \tag{15}$$

We present the dynamics of CPPI strategy when the underlying risky asset follows a Mean Reversion Model as it adapted from [24]

$$dS(t) = kS(t)[\mu - \ln(S(t))]dt + S(t)\sigma dW(t). \tag{16}$$

And the risk-free asset B evolves according to a constant rate of return r

$$dB_t = B_t r dt. \tag{17}$$

The variations of CPPI portfolio value can be written as follows:

$$dV_t = (V_t - e_t)\frac{dB_t}{B_t} + e_t\frac{dS_t}{S_t}. \tag{18}$$

And the variations of the cushion are given by

$$dC_t = (V_t - e_t)\frac{dB_t}{B_t} + \frac{dS_t}{s_t} - dF_t. \tag{19}$$

Knowing that $V_t = C_t + F_t$ and $e_t = mC_t$, the previous equation can be written as follows:

$$dC_t = (C_t + F_t - mC_t)\frac{dB_t}{B_t} + (mCt))\frac{dS_t}{S_t} - \frac{dB_t}{B_t}F_t. \tag{20}$$

And because both of the risk-free asset and the floor evolve in the same deterministic way

$$\frac{dB_t}{B_t} = \frac{dF_t}{F_t} = rdt \tag{21}$$

we can rewrite equation (18) as

$$dC_t = (C_t - mC_t)\frac{dB_t}{B_t} + (mC_t)\frac{dS_t}{S_t} \tag{22}$$

which can also be written as

$$dC_t = C_t[r - mr]dt + C_t m(k[\mu - \ln S(t)]dt + C_t\sigma m dW(t)). \tag{23}$$

Finally, the variation of the cushion can be written as follows:

$$\frac{dC_t}{C_t} = (r - mr)dt + m(k[\mu - \ln S(t)]dt + \sigma m dW(t)). \tag{24}$$

By Applying the Ito's lemma

$$d\ln(C_t) = r(1-m)dt + m(k[\mu - \ln(C_t)])dt + \sigma m dW(t) - \frac{1}{2}[r(1-m)dt + m(k[\mu - \ln(C_t)])dt + m\sigma dW(t)]^2$$

$$= r(1-m)dt + m(k[\mu - \ln(C_t)])dt + \sigma m dW(t) - \frac{1}{2}m^2\sigma^2 dt \tag{25}$$

Let $P = \ln(C_t)$ then

$$d\ln(C_t) = [m(k\mu - r) + r - mkp - \frac{m^2\sigma^2}{2}]dt + m\sigma dW(t) \tag{26}$$

Integrating on both side, so the time t value of the cushion, C_t is given by

$$C_t = C_0 \exp[(m(k\mu - r) + r - mkp - \frac{m^2\sigma^2}{2})t + m\sigma W(t)] \tag{27}$$

Equation (27) shows that the cushion C_t has a log-normal distribution written on expected mean return

$$(m(k\mu - r) + r - mkp - \frac{m^2\sigma^2}{2})$$

and a volatility $m\sigma$.
Since the underlying risky asset S follows a mean reversion model

$$d\ln(S(t)) = k[\mu - P(t)]dt + \sigma dW(t) - \frac{1}{2}k^2\sigma^2 dt$$

$$= k[\mu - P(t) - \frac{k\sigma^2}{2}]dt + \sigma dw(t) \tag{28}$$

and hence

$$S_T = S_0 \exp\left[k[\mu - P(t) - \frac{\sigma^2}{2k}]T + \sigma W_T\right] \tag{29}$$

W_T can be written as follows

$$W_T = \frac{1}{\sigma}\left[log\left(\frac{S_T}{S_0}\right) - k(\mu - P(t) - \frac{\sigma^2}{2k})T\right] \tag{30}$$

Substituting this into equation (27) we have

$$C_t = C_0 \exp\left[(m(k\mu - r) + r - mkp - \frac{m^2\sigma^2}{2})t + m\sigma\left(\frac{1}{\sigma}[log\left(\frac{S_T}{S_0}\right) - k(\mu - P(t) - \frac{\sigma^2}{2k}\right)T\right] \tag{31}$$

$$= C_0 \exp\left[(m(k\mu - r) + r - mkP(t) - \frac{m^2\sigma^2}{2})t + \left(\left(\frac{S_T}{S_0}\right)^m - mk(\mu - P(t)) - \frac{\sigma^2}{2k}\right)T\right] \tag{32}$$

$$= C_0 \exp\left[(mk\mu - mr + r - mkP(t) - \frac{m^2\sigma^2}{2})t + \left(\left(\frac{S_T}{S_0}\right)^m - mk\mu - mkP(t)) + mk\frac{\sigma^2}{2k}\right)T\right] \tag{33}$$

$$= C_0 \exp\left[(-mr + r - \frac{m^2\sigma^2}{2})t + \left(\left(\frac{S_T}{S_0}\right)^m + mk\frac{\sigma^2}{2k}\right)T\right] \tag{34}$$

$$= C_0\left(\frac{S_t}{S_0}\right)^m \exp\left\{(r - m(r - \frac{\sigma^2}{2}) - \frac{m^2\sigma^2}{2})t\right\} \tag{35}$$

and hence

$$V_t = F_t + (V_0 - F_0)\left(\frac{S_t}{S_0}\right)^m \exp\left\{(r - m(r - \frac{\sigma^2}{2}) - \frac{m^2\sigma^2}{2})t\right\}. \tag{36}$$

□

We obtain equation (36), which is the same result when the risky asset follows a normal geometric Brownian motion (GBM). It illustrates, the basic property of a simple CPPI. The t–value of the strategy consists of the present value of the guarantee G, i.e. the floor at t, and a non–negative part which is proportional to $(\frac{S_t}{S_0})^m$.The volatility of the risky asset per year is denoted by σ. Now the cushion procedure will never become negative and thus the value of the portfolio at no time falls below the floor and the portfolio never gaps. Thus, the value process of a simple CPPI strategy is path independent. The payoff above the guarantee is linear for $m = 1$ and it is strictly convex for $m > 1$. In financial terms, the payoff of a CPPI strategy with $m > 1$ can be interpreted as a power claim. The portfolio protection is efficient with probability one, i.e. the terminal value of the strategy is higher than the guarantee with probability one. Notice that the lognormality of the asset price process implies the lognormality of the cushion process. Therefore, it is immediately clear that the strategy does not fall below the floor in all scenarios where the asset price dynamic is lognormal
Still, the cushion C_t follows a GBM under the mean version stochastic differential equation the expected value is

$$E[V_t - F_t] = C_0 \exp\{((1-m)r + m\mu)t\}$$

$$E[V_t] = F_t + (V_0 - F_0)\exp\{((1-m)r + m\mu)t\}$$

and hence

$$E[V_t] = F_t + (V_0 - F_0)\exp\{(r + m(\mu - r))t\}. \tag{37}$$

and the variance

$$Var[V_t] = (V_0 - F_0)^2 \exp\{((1-m)r + m\mu)t\}(\exp\{m^2\sigma^2 t\} - 1)$$

and therefore

$$Var[V_t] = (V_0 - F_0)^2 \exp\{2(r + m(\mu - r)t\}(\exp\{m^2\sigma^2 t\} - 1). \tag{38}$$

This gives the expected payoff of the CPPI as an expanding function of m and independent of volatility. According to Hull (2005,p282) the expected value and variance of the risky asset are stated as

$$E[S_t] = S_0 \exp\{\mu t\} \tag{39}$$

$$Var[S_t] = S_0^2 \exp\{2\mu t\}(\exp\{\sigma^2 t\} - 1). \tag{40}$$

3. Trading Restriction

We assume now that trading is restricted to a discrete set of dates and define a discrete-time version of the simple CPPI strategy satisfying the following three conditions. Firstly, the value process of the discrete-time version converges in distribution to the value process of the simple continuous-time CPPI strategy. Secondly, the discrete-time version is a self-financing strategy. This means, that after the initial investment $V_0 = x$, there is no outflow of funds. Thirdly, the strategy does not allow for negative asset exposure. Notice that the first condition implies that the cushion process of the discrete-time version converges to a lognormal process in distribution. However, the cushion process concerning a discrete-time set of trading dates may also be negative. Therefore, to avoid a negative asset exposure, this must be captured by the definition of the discrete–time version.

Let τ^n denote a sequence of equidistant refinements of the interval $[0, T]$, i.e.

$$\tau^n = \{t_0^n = 0, t_1^n < ...t_{n-1}^n < t_n^n = T\} \tag{41}$$

, where

$$t_{k+1}^n - t_k^n = \frac{T}{n} f or k = 0, ..., n-1 \tag{42}$$

. To simplify the notation, we drop the superscript

n and denote the set of trading dates with τ instead of τ and τ^n . The restriction that trading is only possible immediately after $t_k \in \tau$ implies that the number of shares held in the risky asset is constant on the intervals $[t_i, t_{i+1}]$ for $i = 0, .., n-1$ of wealth that are invested in the assets change as asset prices fluctuate. Thus, it is necessary to consider the number of shares held in the risky asset η and the number of bonds β, i.e. the tuple $\phi = (\eta, \beta)$. Concerning the continuous–time simple CPPI strategies, it holds

$$\eta_t = \frac{\alpha_t V_t}{S_t} = \frac{mC_t}{S_t} \tag{43}$$

$$\beta_t = \frac{(1-\alpha_t)V_t}{B_t} = \frac{V_t - mC_t}{B_t}. \tag{44}$$

The following argumentation illustrates that a time-discretized strategy ϕ^τ which is defined by

$$\phi_t^\tau = \phi_{t_k}, t \in [t_k, t_{k+1}], k = 0, ..., n-1 \tag{45}$$

is in general not self-financing. The value process $V^\tau := V(\phi;\tau)$ which is associated with the discrete-time version of ϕ, i.e with ϕ^τ, is defined by $V_0^\tau := V_0$ and

$$V_t(\phi;\tau) := \eta_{t_k} S_t + \beta_{t_k} B_t, t \in (t_k, t_{k+1}) \tag{46}$$

$$V_t(\phi) = (\eta_t - \eta_{t_k})S_t - (\beta_t - \beta_{t_k})B_t, t \in]t_k, t_{k+1}], \tag{47}$$

where

$$V_t(\phi) := \eta_t S_t + \beta_t B_t$$

If ϕ is self-financing, this is not necessarily true for ϕ^τ. Notice that ϕ^τ is self-financing iff

$$\eta_{t_k} S_{t_{k+1}} + \beta_{t_k} B_{t_{k+1}} = \eta_{t_{k+1}} S_{t_{k+1}} + \beta_{t_{k+1}} B_{t_{k+1}}, k = 0, ..., n-1 \tag{48}$$

$$\Longleftrightarrow V_{tk+1}(\phi;\tau) = V_{tk+1}(\phi), k = 0, ..., n-1 \tag{49}$$

This is only true in the limit, i.e. for $n \to \infty$. It is worth mentioning that it is not even clear whether the above time–discretized version is mean–self–financing concerning the real-world measure, c.f., for example, Mahayni (2003). To specify a meaningful discrete–time version of a simple CPPI strategy, it is necessary to admit only self–financing strategies. This is equal to the condition that

$$\beta_t^\tau = \frac{1}{B_{t_k}}(V_{t_k}^\tau - \eta_t^\tau S_{t_k}), t \in]t_k, t_{k+1}]. \tag{50}$$

Finally, recall that constant proportion portfolio insurance means that the fraction of wealth α which is invested in the risky asset is given proportionally to the difference of the portfolio value and the floor, i.e. the cushion. Let C^τ denote the discrete-time version of the cushion process C, then

$$C_t^\tau := V_t^\tau - F_t. \tag{51}$$

In addition, we do not allow for short positions in the risky asset, i.e. the asset exposure is bounded below zero. Thus, it is necessary to consider the positive part of the cushion. The above reasoning gives the following definition.

Definition 3.1.
(Discrete-Time CPPI). A strategy $\phi^\tau = (\eta^\tau, \beta^\tau)$ where for $t \in]t_k, t_{k+1}]$ and $k = 0, ..., n-1$

$$\eta_t^\tau := \max\left\{\frac{mC_{t_k}^\tau}{S_{t_k}}, 0\right\} \tag{52}$$

$$\beta_t^\eta := \frac{1}{B_{t_k}}(V_{t_k}^\tau - \eta_t^\tau S_{t_k}) \tag{53}$$

Proposition 3.1.
(Discrete-time cushion process). Define

$$t_s := \min\left\{t_k \in \tau | V_{t_k(\alpha) - F_{t_k}}^\tau \leq 0\right\} \tag{54}$$

with $t_s = \infty$ if the minimum is not attained. It holds

$$V_{t_{k+1}}^\tau = F_{t_{k+1}} e^{(t_{k+1} - \min\{t_s, t_{k+1}\})}(V_{t_0}^\tau - F_{t_0}) \prod_{i=1}^{\min\{s,k+1\}} \left(m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1)e^{r\frac{T}{n}}\right). \tag{55}$$

Proof. Notice that

$$V^{\tau}_{t_{k+1}} = \max\left\{\frac{mC^{\tau}_{t_k}}{S_{t_k}},0\right\} S_{t_{k+1}} + \left(V^{\tau}_{t_k} - \max\left\{\frac{mC^{\tau}_{t_k}}{S_{t_k}},0\right\} S_{t_k}\right)\frac{B_{t_{k+1}}}{B_{t_k}} \tag{56}$$

$$= \begin{cases} F_{t_k}\frac{B_{t_{k+1}}}{B_{t_k}} + (V^{\tau}_{t_k} - F_{t_k})\Big(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)\frac{B_{t_{k+1}}}{B_{t_k}}\Big) \text{for} V^{\tau}_{t_k} - F_{t_k} > 0 \\ V^{\tau}_{t_k}\frac{B_{t_{k+1}}}{B_{t_k}} \text{for } V^{\tau}_{t_k} - F_{t_k} \le 0 \end{cases}$$

□

Together with $F_{t_k}\frac{B_{t_{k+1}}}{B_{t_k}} = F_{t_{k+1}}$ it follows

$$V^{\tau}_{t_{k+1}} - F_{t_{k+1}} = \begin{cases} \big(V^{\tau}_{t_k} - F_{tk}\big)\big(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{r\frac{T}{n}}\big) \text{for } V^{\tau}_{t_k} - F_{tk} > 0 \\ \big(V^{\tau}_{t_k} - F_{tk}\big)e^{r\frac{T}{n}} \text{for } V^{\tau}_{t_k} - F_{tk} \le 0 \end{cases}$$

for all $k = 0, ..., n-1$. In particular, we have

$$V^{\tau}_{T} = \begin{cases} V^{\tau}_{t_s}e^{r(T-t_s)} \text{for} t_s \le t_{n-1} \\ G_T + \big(V^{\tau}_{t_{n-1}} - F_{t_{n-1}}\big)\big(m\frac{S_{t_n}}{S_{n_{n-1}}} - (m-1)\frac{B_{t_n}}{B_{t_{n-1}}}\big) \text{for } t_s \ge t_n. \end{cases}$$

In the following, we take the view of an investor who uses the CPPI as a savings plan with portfolio protection. A CPPI strategy contradicts the original idea of the portfolio insurance if it results in a very high gap risk, i.e. if the shortfall probability and the expected shortfall are prohibitively high. The investor has to decide whether this additional risk is not too high in terms of portfolio insurance. In addition to the expected final value and its standard deviation, we consider the shortfall probability and the expected shortfall given default as the risk measures which determine the effectiveness of the discrete-time CPPI strategy. The shortfall probability is the probability that the final value of the discrete–time CPPI strategy is less or equal to the guaranteed amount G. Intuitively, one can also define a local shortfall probability (given that no prior shortfall happened before). Additionally, we use the expected shortfall given default to describe the amount which is lost if a shortfall occurs.

4. Risk Measures of Discrete-Time CPPI

Call back that the basic idea of a CPPI strategy is portfolio protection. Analytically, the usage of these strategies is described by an investor who wants to engage in bullish markets but he is avoiding the terminal value of the strategy not to end up below a guaranteed amount G. So, the investor is completely risk-averse to values below the floor. However, in a special case of static portfolio insurance strategies, there is a positive probability that the terminal value is below the designed amount. Specifically, this is true for CPPI and OBPI strategies which include a synthetic put. The use of such constrained strategies or strategies which comprise a gap risk can be described as follows. On the one hand, one might think of an investor who welcomes, because of market incompleteness, a strategy that offers a guaranteed amount with a certain success probability. Additionally, one might think of retail products which are rooted in the CPPI method and are thus also hedged by a CPPI strategy. Usually, the buyer of such a product gets the guaranteed amount even in the case that the strategy breaks to fulfill it. For this purpose, the issuer takes the gap risk and considers this in his product pricing. One and the other, the risk profile of the CPPI is of substantial interest. It is required to calculate risk measures that allow a characterization if the constrained CPPI is still effective in terms of portfolio insurance.

To catch the statistical properties of the CPPI from the perspective of the insurer, the following measures have been attempted. We look at an investor who uses the CPPI as a saving plan with portfolio protection. A CPPI strategy disputes the original idea of portfolio insurance if it results in a very excessive gap risk. The investor has to determine whether this additional risk is not too high in terms of portfolio insurance. In addition to the expected final value, we consider the probability of loss and the expected loss given default as the risk measures which determine the effectiveness of the discrete-time CPPI strategy. The shortfall probability is the probability that the final value of the discrete-time CPPI strategy is less or equal to the guaranteed amount G.

Definition 4.1.
(Risk measures)

$P^{SF} := P(V^{\tau}_T \le G) = P(V^{\tau}_T \le F_T)$ shortfall probability

$P^{LSF}_{t_i,t_{i+1}} := P\big(V^{\tau}_{t_{i+1}} \le F_{t_{i+1}} | V^{\tau}_{t_i} > F_{t_i}\big)$ local shortfall probability

$ES := E[G - V_T^\tau \le G]$ expected shortfall given default.

It turns out that, in contrast to a discrete-time option-based strategy with synthetic put, the calculation of the shortfall probability implied by a CPPI strategy is very simple. This is easily explained if one observes that the shortfall event is equivalent to the event that the stopping time which is defined in Proposition is before the terminal date. It is convenient to consider the following lemma.

Lemma 4.1.
Let $A_k := \left\{ \frac{S_{t_k}}{S_{t_{k-1}}} > \frac{m-1}{m} e^{r\frac{T}{n}} \right\}$ *for* $k = 1, ..., n$*, then it holds*

$$\{t_s > t_i\} = \bigcap_{j=1}^{i} A_j \, and \{t_s = t_i\} = A_i^C \cap \Big(\bigcap_{j=1}^{i-1} A_j \Big) for\ i = 1, ..., n. \tag{57}$$

Proof. According to the proof of Proposition, it holds

$$V_{t_{k+1}}^\tau - F_{t_{k+1}} = \begin{cases} \big(V_{t_k}^\tau - F_{tk}\big)\big(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{r\frac{T}{n}}\big) \text{for } V_{t_k}^\tau - F_{tk} > 0 \\ \big(V_{t_k}^\tau - F_{tk}\big)e^{r\frac{T}{n}} \text{ for } V_{t_k}^\tau - F_{tk} \le 0 \end{cases}$$

The rest of the proof follows immediately with the definition of the stopping time t_s and

$$m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{r\frac{T}{n}} > 0 \Longleftrightarrow \frac{S_{t_{k+1}}}{S_{t_k}} > \frac{(m-1)}{m} e^{r\frac{T}{n}}.$$

□

Lemma 4.2.
The local shortfall probability is independent of t_i *and* t_{i+1}*, i.e.*
$P_{t_i,t_{i+1}}^{LSF} = P^{LSF} = \mathcal{N}(-d_2)$
where $d_2 := \frac{\ln\frac{m}{m-1} + (\mu - r)\frac{T}{n} - \frac{1}{2}\sigma^2\frac{T}{n}}{\sigma\sqrt{\frac{T}{n}}}$

Proof. Notice that

$$P_{t_i,t_{i+1}}^{LSF} = P\big(V_{t_{i+1}}^\tau \le F_{t_{i+1}} | V_{t_i}^\tau > F_{t_i}\big)$$
$$= P(t_s = t_{i+1} | t_s > t_i) = P\Big(\frac{S_{t_1}}{S_{t_0}} \le \frac{m-1}{m} e^{r\frac{T}{n}}\Big)$$

where the last equality follows with Lemma 2 and the assumption that the asset price increments are independent and identically distributed (iid). □

Proposition 4.1.
The shortfall probability P^{SF} *is given in terms of the local shortfall probability* P^{LSF}*, i.e.*

$$P^{SF} = 1 - (1 - P^{LSF}). \tag{58}$$

Proof. The above lemma is a direct consequence of Lemma 2 and the independence of asset price increments.

$$P^{SF} = 1 - P(t_s = \infty) = 1 - (1 - P^{LSF})^n \tag{59}$$

□

Proposition 4.2 (Expected final value.).
It holds

$$E[V_T^\tau] = G + (V_0 - F_0)\left[E_1^n + e^{-r\frac{T}{n}} E_2 \frac{e^{rT} - E_1^n}{1 - E_1 e^{-r\frac{T}{n}}}\right] \tag{60}$$

where

$$E_1 := m e^{\mu \frac{T}{n}} \mathcal{N}(d_1) - e^{r\frac{T}{n}}(m-1)\mathcal{N}(d_2) \tag{61}$$

$$E_2 := e^{r\frac{T}{n}}\left[1 + m\left(e^{(\mu - r)\frac{T}{n} - 1}\right)\right] - E_1. \tag{62}$$

d_2 is the same as in Lemma 3 and $d_1 := d_2 + \sigma\sqrt{\frac{T}{n}}$.

Proof. See Balder and Mahayni 2005 □

Table 2. Sensitivity of risk measures. Symbol ↑ is for monotonically increasing and ↓ is for monotonically decreasing.

Risk measures	Strategy parameter	Model parameter
	G m	μ σ
Mean	↓ ↑	↑ ↑
Stdv.	↓ ↑	↑ ↑
P^{SF}	- ↑	↓ ↑
ESF	↓ ↑	↑ ↑

Before we study the effectiveness of the time-discretized CPPI in detail, we end this section with a sensitivity analysis of the risk measures. In order to avoid a lengthy discussion of all possible sensitivities, we summarize the main results in Table 2. The corresponding proofs are straightforward. Notice that the shortfall probability is independent of G, c.f. Proposition 2. Partial differentiation immediately yields that the shortfall probability is increasing in σ and m but decreasing in μ. In contrast, the sensitivity analysis of the other risk measures is tedious. for example, the monotonicity of the expected terminal value, i.e. $E[V_T^\tau]$, in σ. Similar arguments to the one presented here can also be used to show that the expected terminal payoff is also increasing in μ and m. Monotonicity in G and V is immanent. With respect to the standard deviation, it is intuitively clear that the volatility σ has a positive effect on the standard deviation, so does m. It is worth mentioning that both the shortfall probability and the expected shortfall are increasing in m and σ. This implies that a discrete-time CPPI is not effective in discrete time if either the standard deviation is too high in comparison to the multiplier or vice versa.

5. Effectiveness of the Discrete-Time CPPI Method

5.0.1. Description of insurance market

First, we consider the question wheather the discrete-time CPPI method gives a good approximation of the continuous-time CPPI for a finite number of rehedges n. Recall that the value process of the discrete-time CPPI converges to the value process of the continuous-time CPPI distribution. Since the cushion process of the continuous-time CPPI is lognormal, the payoff distribution of the continuous-time CPPI is described by its mean and its standard deviation. These numbers are summarized in Table 2. Table 3 also summarizes the moment and risk measures for various numbers of rehedges n. At first glance, it might be tempting to think that the shortfall probability is monotonically decreasing in the hedging frequency, i.e. the number of rehedges n. In general, this is only true after a sufficiently high n is reached. The effect that the shortfall probability is increasing for small n is more pronounced for high volatility's and high multipliers. Let n^* denote the number of rehedges such that the shortfall probability is increasing in n for $n \le n^*$ and decreasing for $n \ge n^*$. The critical level n^* is to be interpreted as a minimal number of rehedges which is necessary such that the CPPI method is effective for $m \ge 2$ in discrete time. As shown above, the effectiveness of the discrete–time CPPI method depends on the strategy parameters, i.e. the multiplier m, the number of rehedges n, and the guarantee G, as well as the model parameters μ and σ. The most important influences are caused by the multiplier m and the volatility σ. Therefore, all examples are considered for varying multipliers and volatility. As shown in Table 3 consider the shortfall probability. Observe, that in the case where $\sigma = 0.1$, a monthly CPPI-strategy ($n = 12$) with a multiplier $m = 12$ implies a shortfall probability of only 0.01. In contrast, a volatility of $\sigma = 0.2$ gives a shortfall

Table 3. Moments of discrete-time CPPI

n	m	Mean	Stdv.Dev	SFP	ESF
12	12	1077.53	125.04	0.0115	5.463
24	12	1077.77	132.01	0.0002	2.981
48	12	1077.90	135.88	0.0000	1.574
96	12	1077.97	137.92	0.0000	0.000

probability of more than 0.5. Thus, the monthly CPPI strategy ensures a significant protection level for $\sigma = 0.1$ while the concept of portfolio insurance is already impeded for $\sigma = 0.2$. Here (for $\sigma = 0.2$), even a weekly rehedging, i.e. $n = 48$ is not enough to achieve a shortfall probability of less than 0.05. This illustrates that the effectiveness of the discrete-time CPPI method is very much sensitive to the volatility of the asset price process.

To test the effect of different distributions and market frictions, simulations of the CPPI are undertaken with the following assumptions. The risk asset is liquid and can be traded on the rebalancing date at the current price presented by the simulation. For underlying such as prominent stock indices (e.g. FTSE 100), liquidity can be taken up to be daily since these are busily traded products. For less liquid underlying (e.g. hedge funds), lessen frequency in portfolio rebalances can be argued to be reflective of the fact that the underlying can be only rebalanced at that frequency. Both risky and risk-less assets can be traded in boundlessly divisible amounts. This is not impractical considering that most CPPI funds have capital in surplus of $100 million and the effect of having to buy or sell individual units of risky assets or bonds becomes negligible. The risk-less asset can be considered analogous to a bank account with less impact transaction costs, so they are ignored here. Although, they are considered the risky asset. Shorting of the risky asset is not allowed. A unit investment is considered, giving a terminal log-payoff $\ln(V_T)$ equal to the accumulated log return. Unless otherwise stated the model assumes a GBM for the underlying risky asset, 5-year maturity ($T = 5$), monthly rebalancing (i.e. 12 periods per year p=12 or 60 rebalances during the investment's life of 5 years: n=60), no leverage (i.e. h=1), no transaction costs, no management fees and volatility of 20% ($\sigma = 0.2$). The risk-free rate and the expected rate of return are always fixed at 5% and 10% respectively. These parameter values can be considered typical for a CPPI with a reasonably liquid underlying.

5.0.2. Expected terminal portfolio value.

Figure 2 shows that low trading frequencies, always achieve less payoff in the investment. Usually, this works in line with a lower multiplier m. In the figure, for $m \leq 2$, we easily observe a low expected terminal portfolio value. Moreover, as the number of rebalancing picks up there is a little additional change in the payoff. Recalling that the portfolio payoff always increases for $m \geq 5$, the expected portfolio value grows, even as the number of trading frequencies stays low this is because the exposure to risk-asset has already increased due to an increase in the multiplier. The best payoff can also be achieved with an increase in the number of rebalancing. As one can analyze in the figure as the number of trading n increases the expected terminal portfolio also increases, this is because frequent increase in the number of rebalancing often decreases the risk as we will see in the following risk measures.

5.0.3. Probability of losses

The probability of loss: this indicator represents the probability that the portfolio value drop below the guaranteed amount at maturity. It is computed by the average of the total number of times when the portfolio value ends below the guarantee G_T at maturity.

$$pr[L_T] = \frac{1}{I}\sum_{i=1}^{I} \mathbb{I}_{(V_{T},i<G_T)}, \tag{63}$$

Where I is the number of portfolios and $\mathbb{I}$ is a binary indicator function that returns 1 if the subscript term is contented and 0 otherwise. Frequently rebalancing portfolios reduces the portfolio of losses. Figure 3 shows that, as n increases beyond 5 which is regarded as yearly trading $Pr[L_T]$ is decreasing. This is because when $n = 5$, the value of the portfolio is only observed at maturity where it can be ascertained whether or not it has met the guarantee, and as the case the substantial number of losses are not realized during the investment period. However, $Pr[L_T]$ increases with the number of multipliers, for $m \geq 2$ the probability of losses is high with a low number of trading n. This is because more percentage has been invested into the risky asset with yearly trading rebalance as a result any violation to the floor could not be realized on time.

5.0.4. Expected value of losses and Volatility effect

- Moment of the log terminal portfolio value V_T of all portfolios and log terminal portfolio value V_T^L of those portfolios that experienced a loss. V_T^L is defined as:

$$V_T^L = [V_T | V_T < G_T]. \tag{64}$$

- Distribution of the losses L_T, where a loss is taken as the amount the portfolio value is below the guarantee at maturity, given that it is below the guarantee:

$$L_T = [G_T - V_T | V_T < G_T.] \tag{65}$$

- The expected portfolio value at maturity in case of loss, i.e. the expected portfolio value at maturity given that the floor guarantee is violated:

$$\mathbb{E}[V_T^L] = \mathbb{E}[V_T | V_T < G_T]. \tag{66}$$

- The expected loss: this indicator gives the expected value of loss in case of violation of the guarantee G_T at maturity:

$$\mathbb{E}[L_T] = \mathbb{E}[G_T - V_T | V_t < G_t] \tag{67}$$

This section explores the effect of differing volatility's on the CPPI. We are already familiar with the fact that increases in volatility harm the expected value of V_T, which is further raised by large values of m. For higher values of volatility's, increases in m produce little additional payoff. Figure 4 shows the expected value of losses revealing that high volatility does greatly increase the size of losses, indicating less volatility would be favored. This has a similar effect with the increase in the multiplier m, for $m \geq 6$ as observed in the figure also increases the size of losses. This suggests that it is more effective to trade more frequently with a low level of m and volatility's σ, however, this comes at the expense of a lower $E[V_T]$.

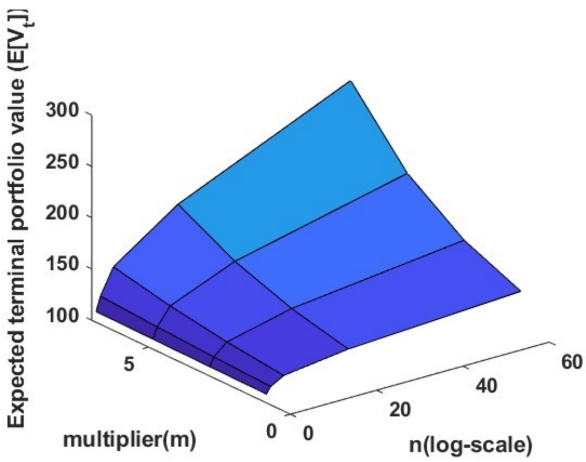

Fig. 2. Expected terminal payoff when $S_0 = 100, V_0 = 100, r = 0.05, T = 5, \mu = 0.1, n = 60$, and $\sigma = 0.2$

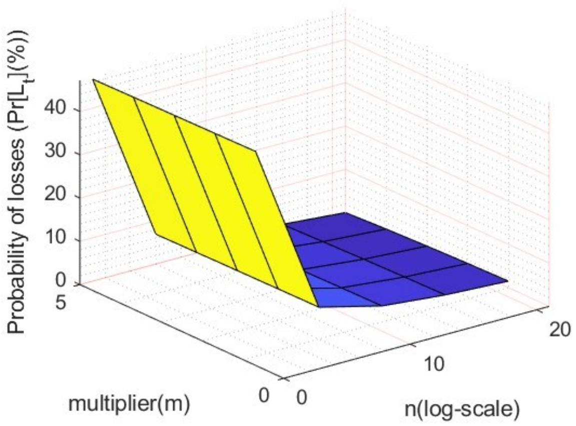

Fig. 3. The effects of probability of loss when $S_0 = 100, V_0 = 100, r = 0.05, \mu = 0.1, T = 5, n = 60$ and $\sigma = 0.2$

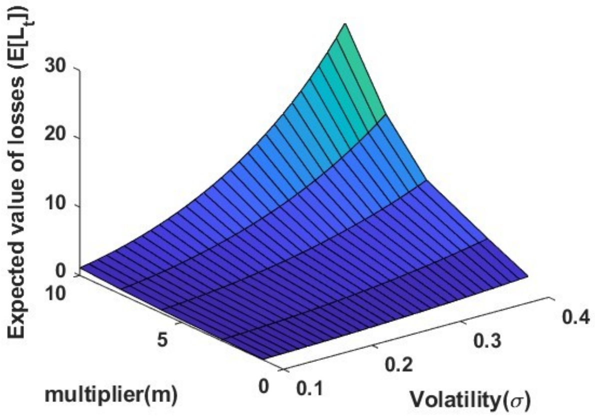

Fig. 4. The effects of Expected of loss and volatility when $S_0 = 100, V_0 = 100, r = 0.05, \mu = 0.1, T = 5, n = 60$ and $\sigma = 0.2$

6. Conclusion

The constant proportion portfolio insurance under Mean Version stochastic differential equation and its statistical test performance was the main goal of this paper. We develop cushion variations that lead us to the same conclusion as that CPPI under Geometric Brownian motion. The introduction of market incompleteness and model risk impedes the concept of dynamic portfolio insurance, i.e. the technique of constant proportion portfolio insurance. The introduction of trade restrictions is one possibility to model a gap risk in the sense that a CPPI strategy can not be adjusted adequately. Under discrete-time trading, it is assumed that the price of the risky asset is observed at intervals and that rebalancing of the portfolio only occurs at these times, except for static portfolio insurance strategies there is a positive probability that the terminal value is below the guaranteed amount. The analysis of the risk measure of a discrete-time CPPI strategy poses various problems which are to be considered. Basically, it is necessary to check the associated risk measures and to determine whether the strategy is still effective in terms of portfolio protection. For example, the protection feature is violated if the shortfall probability of the CPPI strategy under consideration exceeds the shortfall probability of a pure asset investment. Formally, the last on can be interpreted as a static CPPI. Intuitively, this explain the result that the shortfall probability of a discrete-time CPPI is only decreasing in the hedging frequency after a sufficiently high number of rehedges. Therefore with the implementation of the Statistical Test, we should be able to measure the risk that possibly leads to the payoff falling below the designed floor from the strong instrument probability of loss and expectation of loss. It was observed that, the higher the value of multiple m, the higher the expected terminal portfolio. And also the best payoff can be achieved with an increase in the number of rebalancing. Rebalancing a portfolio frequently reduces the portfolio's losses. The expected value of losses reveals that high volatility does greatly increase the size of losses, indicating less volatility would be preferred. Further study can be conducted to optimize different consumer appetites for risks and determine the optimal utility.

References

[1] P. Poncet and R. Portait, Assurance de Portefeulille in Y.Simon ed, Ency clopedie des maches Finaciers, Economica. (1997) 140-141.
[2] P. Bertrand and J. Prigent, Omega performance measure and portfolio insurance, Journal of Banking and Finance. 35 (2011) 1811-1823.
[3] P. Bertrand and J. Prigent, OBPI versus CPPI, GREQAM et Universite Montpellier. Universite de Cergy (2002).
[4] F. Black and R. Jones, Simplifying portfolio insurance, Journal of Portfolio Management. (1987) 48-51.
[5] F. Perold, Constant portfolio insurance. Harvard Business School (1986).
[6] A. R. Perold and W. Sharpe, Dynamic strategies for Asset Allocations, Financial Analyst Journal, 16-27.
[7] T. Andersen, L. Benzoni and J. Lund, An empirical investigation of continuous-time equity return models, J. Finance. 57 (2002) 1239-1284.
[8] O. Barndorff-Nielsen and N. Shephard, Econometrics of testing for jumps in financial economics using bipower variation, J. Financ Econom. 4 (2006) 1-30.
[9] Y. Ait-Sahalia, J. Cacho-Diaz and Laeven, Modeling financial contagion using mutually exciting jump process, J. Financ. Eco. 117 (2015) 585-606.
[10] R. Cont and P. Tankov, Constant Proportion insurance in the presence of jumps in asset prices, Mathematical Finance. 19

(2009) 379-401.

[11] C. Jessen, Constant proportion portfolio insurance: Discrete-time trading and gap risk coverage, J. Derivatives. 21 (2014) 36-53.

[12] N. Branger and J. Schneider, On the optimal design of insurance contracts with guarantees, Insur. Math. Eco. 46(2010) 485-492.

[13] S. Balder and A. Mahayni, Cash-lock comparison of portfolio insurance strategies. Duisburg: Univ. Duiburg-Essen (2010).

[14] S. Balder, M. Brandl and A. Mahayni, Effectiveness of CPPI Strategies under discrete time trading, Journal of Economics Dynamics and control. 33 (2010) 204-220.

[15] S. Balder, M. Brandl and A. Mahayni, Effectiveness of CPPI Strategies under discrete time trading, Department of Banking and Finance, University of Bonn (2005).

[16] D. Maringer and T, Ramtohul, GP-based rebalancing triggers for the CPPI, In computational Intelligence for Financial Engineering and Economics. (2011) 1-8.

[17] B. Do and B. Do, Accounting and Finance. 42 (2002) 279-296.

[18] H. Dichtl and W. Drobetz, Portfolio insurance and Prospect theory investors: Popularity and optimal design of capital protected financial products, Journal of Banking and Finance. 35 (2011) 1683-1697.

[19] A. Khuman, D. Maringer and N. Constantinou, Constant Proportion Portfolio Insurance, Statistical Properties and Practical Implications. (2008) 1-15

[20] K. Anil, M. Dietmar and C. Nick, Constant proportion portfolio insurance: Statistical properties and practical Implications. Centre for computational Finance and economic agents. CCFEA WP 023-08 (2008).

[21] J. Hull, Options, Futures and Other Derivatives. Prentice Hall, 6th edition (2005).

[22] D. Pain, Recent developments in portfolio insurance, Bank of England Quarterly Bulletin Q1 (Research and Analysis). (2008) 37-46.

[23] S. Jere, D. Mukonda, B. Kolosa, E. Moyo, An investment management with taxes and transaction costs under Vasicek and Constant Elasticity of Variance (CEV) models, Int. J. Adv. Appl. Math. and Mech. 11(4) (2024) 22 – 31.

[24] P. Nangolo, E. R. Offen and S. Jere, Optimal Portfolio Selection of a Constant Proportion Portfolio Insurance when Asset follows Hawkes-Jump-Diffusion, Int. J. Adv. Appl. Math. and Mech. 11(4) (2024) 51 – 70.