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Portfolio choice problem with trading fees, government taxes, dividends, stochastic wage income and inflation-adjusted wealth

Research Article

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Abstract: This paper addresses the portfolio choice problem for CRRA (Constant Relative Risk Aversion) investors in the presence of trading fees, government taxes, dividends, stochastic wage income, and inflation risk. These factors, influenced by fiscal and financial policy adjustments, are critical in real-world investment scenarios. We analyze an agent's investment strategy within a financial market comprising one risk-free asset (e.g., a money market account or bond) and one risky asset (e.g., a stock or stock index). The study's objective is to determine the optimal control strategy that maximizes the investor's objective function, ultimately leading to the value function. Utilizing the Dynamic Programming Principle (DPP), we derive the associated Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE). By solving this PDE, we establish both the value function and the optimal control strategy.

MSC: 76A06 • 76D09

Keywords: Portfolio choice • Trading fee · Government tax • Dividend • CRRA investor • Dynamic Programming Principle • HJB PDE

1. Introduction

In this paper, we address the portfolio choice problem for Constant Relative Risk Aversion (CRRA) investors by incorporating real-world complexities such as trading fees, government taxation, dividends, stochastic wage income, and inflation risk. These factors are critical in financial decision-making, as they represent the practical challenges faced by investors due to events like fiscal policy changes and financial market adjustments.

We focus on an investment scenario where the agent can allocate wealth between two financial instruments: a risk-free asset (such as a money market account or government bond) and a risky asset (such as an individual stock or a diversified stock index). The objective of this study is to identify the optimal investment strategy that maximizes the investor's expected utility of lifetime consumption and wealth, thus deriving the value function associated with this problem.

To achieve this, we employ the Dynamic Programming Principle (DPP) to formulate the Hamilton-Jacobi-Bellman (HJB) Partial Differential Equation (PDE) that characterizes the value function. By solving the HJB PDE, we obtain

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explicit expressions for both the value function and the corresponding optimal control strategy. This solution framework allows us to comprehensively analyze the impact of trading costs, taxation, dividends, income uncertainty, and inflation on the optimal portfolio choice, providing valuable insights for both theoretical research and practical applications in financial economics.

In practical modern finance under stochastic optimal control problems, realistic factors such as trading fees, government taxes, dividends, stochastic wage income, and inflation risk play a crucial role in portfolio management problems. These factors are significant concerns for both individual and institutional investors aiming to allocate wealth across various assets over a determined or indeterminate lifetime. Our study was driven by the need to effectively address issues related to trading fees, taxes, dividends, stochastic wage income, and inflation risk in portfolio management. To date, various authors have extended seminal Merton's work using methods such as the dynamic programming principle, the maximum principle, the viscosity solution concept, and backward stochastic differential equations (BSDE). There is a strong connection between viscosity solutions and BSDE through their PDE representation. This research further develops the renowned work of Merton, who solved the optimal control problem for an agent investing in one risk-free asset and one risky asset with constant interest rate and volatility.

The foundation work of [1] was the first to introduce Mean-Variance (MV) optimal portfolio selection problems in discrete time. Markowitz defined an optimal portfolio as an efficient frontier for any investor. The mean-variance (MV) model, formulated the portfolio choice problem as an optimization problem without accounting for the consumption of an investor. According to this study, the MV model minimizes the variance of the terminal wealth for a desired level of expectation. This study also showed that there are possibilities for different portfolios to have a different combination of return and risk. The second celebrated pioneer work of [2, 3] investigated portfolio optimization is based on the preferences of an investor as described by a utility function. Merton solved a stochastic control problem in continuous time for a financial market consisting of one risk-free asset and one risky asset with constant interest and volatility rates. The Hamilton-Jacobi-Bellman equation for the value function was determined by applying the dynamic programming principle. Merton showed that closed-form solutions exist for such non-linear partial differential equations. In the academic literature, there is a wide range of portfolio selection problems. Most common are problems formulated either in the Mean-Variance framework pioneered by [1] or problems of expected utility maximization type pioneered by [2, 3] for a diffusion-type model. In recent years jump models have become increasingly popular in academic and financial research to explain randomness. This is due to the shortcomings of the classical Brownian motion model developed in [4]. Studies so far have shown that stock market returns have higher peaks and heavier tails see [6, 7, 24]. Often jumps occur in the prices of stocks that cannot be explained by a Brownian motion model. These jumps also have large down movements in stock prices but not equally large up movements. Another feature often observed in stock price distributions is that large changes in prices are often followed by large changes and small changes tend to be followed by small changes. So far, researchers have solved these observations using the jump models which capture many of the empirical features of stock price returns. However, because of the independent increment property (i.e. the Markov property), jump processes may not model the effect of volatility clustering. On the other hand, jump models are generally assumed to have finite jumps during a finite time interval which represent rare events in real life. Thus our motivation to study diffusion-type models.

This study extends the pioneer work of Merton to include trading fees, government tax rates, dividend, inflation risk and stochastic wage income. The inflation risk is modeled as a Consumer Price Index (CPI). In real life, a financial market experience trading fees, Government tax rates, dividend, inflation risk and Stochastic wage due to uncertain events such as fiscal policy and financial policy adjustments. Inflation risk parameters as well as stochastic wage affect optimal investment decisions. We consider the stochastic control problem of a single investor with a portfolio consisting of one risky-free security (e.g. a money market account or bond) and one risky security (e.g. a stock or stock index). Our goal is to choose the optimal investment policy that maximizes terminal wealth. The investor preferences are modeled as a Constant Relative Risk Aversion (CRRA) function and trading takes place in a finite horizon.

The celebrated work of [2, 3] considered constant interest rate and constant volatility rate. However, such assumptions are not practical in modern finance. In this study, we extend Merton's work to include trading fee, government tax rates, dividend, stochastic wage income and inflation risk simultaneously. This results in sophisticated HJB PDEs. This paper outlines some new results in the field of Mathematics of Finance. The development of these new ideas was motivated by the need to adequately address challenges emanating from transaction costs, inflation and randomness and uncertainty in the portfolio management of modern finance. The most important contribution of this study is that we have extended Merton's problems with a unique mixture of trading fees, Government tax rates, dividends, stochastic wage income and inflation risk simultaneously.

2. Links to the literature

The problem of optimal investment has attracted a number of extensions. For instance, [8] investigated portfolio selection problems in a stochastic environment including inflation risk, and also apply Dynamic Programming Principle (DPP) to determine the value function and optimal policies. [9] applied a duality approach in solving a stochastic control problem. [10] used the duality approach to portfolio optimization problems with borrowing and short-sale

constraints. [11] considered an optimal investment for a pension fund under inflation risk by applying the Martingale method for a financial market consisting of a money account, a stock, and an inflation-linked bond. In the paper by [12], dynamic asset allocation under inflation was investigated by applying DPP to determine the value function and optimal policies. [13] investigated a stochastic control problem with stochastic volatility and constant interest rate. [15] considered optimal Investment-Consumption Strategy under Inflation in a Markovian Regime-Switching Market. [14] researched optimal portfolio selection with life insurance under Inflation Risk for CRRA investors. This paper analyzed how risk aversion, the correlation coefficient between inflation and the stock price, the inflation parameters, and the coefficient of utility affect the optimal investment and consumption strategy. [19] investigated optimal investment, consumption and insurance problems. They considered a market with a real zero coupon bond, the inflation-linked real money account and a risky share following a jump-diffusion process. They applied a backward stochastic differential equation (BSDE) with jumps to derive the explicit solutions. A paper by [17] was the first to extend Merton's work to include life insurance in a study titled optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. [18] extended Merton's work by adding life insurance but with constant labor income in the study titled optimal life insurance purchase and consumption/investment under uncertain lifetime. In their study, the agent has initial wealth but also receives an income continuously which can be terminated upon premature death. In this study, Merton's work is extended in a unique way by studying the stochastic control problem for an agent who faces inflation risk and stochastic wage income. Such assumptions are realistic and practical in the real financial world. Our goal is to allocate initial wealth between a risk-free asset account and a risky asset account to maximize the discounted expected utility of terminal wealth:
Portfolio choice with transaction cost starts with the seminal papers of [24, 25], and [26] , in the wake of the frictionless results of [27, 28]. From heuristic arguments, these early studies gleaned central insights that held up to subsequent formal proofs. First, an optimal portfolio entails a no-trade region, in which it is optimal to keep existing holdings in all assets optimal portfolio always remains within this region, and hence trading should merely take place at its boundaries. The no-trade region is wide, even for small transaction costs, implying that investors should accept wide fluctuations around the frictionless target. Second, the large no-trade region has a small welfare impact [24], because the displacement loss is small near the frictionless optimum, and a wide no-trade region minimizes the effect of transaction costs. Some numerical analysis of the no-trade region for multiple assets can be found in [21]. In addition, [22] employed a dual approach developed in [23] to derive several easy-to-compute heuristic trading strategies for the multi-asset portfolio problems with return predictability and transaction costs and provided the associated upper bound to asses the performance. The study by [29], consists of risk which was only due to stock price market, and [30] in their work include the risk associated with inflation.

The outline of this paper is as follows. Section 2, introduction. Section 3, literature review. Section 4, problem formulation. In section 5, the wealth model is determined. Section 6 Optimization criterion description. In section 7, the Hamilton-Jacobi-Bellman equation (HJB) for the value function is derived. In section 8, we investigate the value function and optimal policy. In Section 9, numerical examples and simulations are provided. Here, the effect of market parameters on the optimal investment policy is illustrated. In Section 10, the conclusion and suggested possible future research work are stated.

3. Problem formulation

Throughout this paper, we consider a financial market based on a continuous interval of $[0, T]$, where T is a positive finite constant and represent terminal time. Consider a financial market within a stochastic framework defined by the filtered complete probability space $(\Omega, \mathbb{F}, \mathscr{F}, \mathbb{P})$. This space is equipped with a filtration $(\mathscr{F}_t)_{0\leq t\leq T}$ that satisfies the usual conditions, including right-continuity and completeness with respect to the probability measure $\mathbb{P}$.

3.1. Financial Market Model

We investigate the stochastic control problem faced by a single investor whose portfolio comprises two types of securities: a risk-free security, such as a money market account or a bond denoted by $B(t)$, and a risky security, such as a stock or a stock index denoted by $S(t)$.

3.2. Risk-Free Security Dynamics

The price dynamics of the risk-free security $B(t)$ are governed by the following differential equation:

$$\begin{cases} dB(t) = r(t)B(t)dt, \\ B(0) = 1, \end{cases} \tag{1}$$

where $r(t)$ is the constant risk-free interest rate. This implies that $B(t)$ grows deterministically at the rate $r(t)$.

3.3. Stochastic Wage Income Dynamics

The investor's wage income, $\eta(t)$, is modeled as a stochastic process with dynamics given by:

$$\begin{cases} d\eta(t) = \mu_\eta(t)\eta(t)dt + \sigma_\eta(t)\eta(t)dW^\eta(t), \\ \eta(0) = \eta, \end{cases} \tag{2}$$

where $\mu_\eta(t)$ and $\sigma_\eta(t)$ are the drift and volatility functions of the wage income, respectively, and $W^\eta(t)$ is a standard Wiener process.

3.4. Inflation Risk Dynamics

To account for inflation risk, we introduce a price index $I(t)$, such as the Consumer Price Index (CPI), which reflects the value of a fixed basket of goods. The dynamics of $I(t)$ are described by:

$$\begin{cases} dI(t) = \mu_I(t)I(t)dt + \sigma_I(t)I(t)dW^I(t), \\ I(0) = i, \end{cases} \tag{3}$$

where $\mu_I(t)$ and $\sigma_I(t)$ represent the drift and volatility of the price index, and $W^I(t)$ is another standard Wiener process.

3.5. Risky Security Dynamics

The price dynamics of the risky security $S(t)$ follow a Geometric Brownian Motion (GBM) model with the potential for discontinuous jumps, modeled as a Geometric Lévy process:

$$\begin{cases} dS(t) = \mu_s S(t)dt + \sigma_s S(t)dW^S(t), \\ S(0) > 0, \end{cases} \tag{4}$$

where μ_s is the mean rate of return, σ_s is the volatility, and $W^S(t)$ is a standard Wiener process representing market risk.

So far have formulated a comprehensive financial market model for an investor's portfolio consisting of both a risk-free and a risky security. This model incorporates stochastic dynamics for wage income and inflation risk, providing a realistic framework for analyzing and optimizing investment strategies under uncertainty. The evolution of each component is defined by stochastic differential equations, which together describe the complex interplay between market factors and the investor's financial position.

4. The Wealth Model

Let θ be the trading fee, β be the government tax and γ represent the dividend. Let us assume an agent trading fees, government tax rates and dividend payment are charged on risky investments. Let $\eta(t)$ represents the stochastic wage income. Consider an investor who starts with an initial amount of money, denoted as $\mathscr{X}(0) > 0$, and is planning their investments over a specified time horizon, T. During this period, the investor will adjust their portfolio dynamically. Let $1-\pi(t)$ be the fraction of wealth invested in a risk-free security, and $\pi(t)$ represent the fraction of wealth invested in a risky asset, S. The investment strategy is represented by $\pi(t)$ [31, 32].

Lemma 4.1.
The dynamics of the investor's net wealth $\mathscr{X}(t)$, accounting for trading fee, Government tax, dividend, stochastic wage income, is described by the following differential equation:

$$\begin{cases} \frac{d\mathscr{X}(t)}{\mathscr{X}(t)} &= \left[1-\pi(t)\right]\frac{dB(t)}{B(t)} + \pi(t)\frac{dS(t)}{S(t)} - \theta\pi(t)dt - \beta\pi(t)dt - \gamma\pi(t)dt + \eta(t)dt \\ &= \left[\pi(t)(\mu_s - \theta - \beta - \gamma) + (1-\pi(t))r + \eta(t)\right]dt + \pi(t)\sigma_s dW^S(t), \\ \mathscr{X}(0) &> 0, \end{cases} \tag{5}$$

where $dB(t)$ and $dS(t)$ represent the changes in the values of the risk-free and risky assets, respectively, μ_s is the expected return on the risky asset, r is the risk-free rate, σ_s is the volatility of the risky asset, $dW^S(t)$ is a Wiener process, θ is the trading fee, β is the government tax, γ represent the dividend and $\eta(t)$ represents the stochastic wage income.

To account for inflation, we define the inflation-adjusted real wealth process at time t, denoted by $\bar{\mathscr{X}}(t)$, as:

$$\bar{\mathscr{X}}(t) = \frac{\mathscr{X}(t)}{I(t)}, \tag{6}$$

where $I(t)$ is the inflation index.

By applying the above lemma to this process, we derive the following model for real net wealth:

$$\begin{cases} d\bar{\mathscr{X}}(t) & = \frac{1}{I(t)} d\mathscr{X}(t) - \frac{\mathscr{X}(t)}{I(t)^2} dI(t) - \frac{d\mathscr{X}(t)dI(t)}{I(t)^2} + \frac{\mathscr{X}(t)}{I(t)^3} dI(t)^2 - \theta\pi(t)dt - \beta\pi(t)dt - \gamma\pi(t)dt + \eta(t)dt \\ & = \frac{\mathscr{X}(t)}{I(t)} \left(\frac{d\mathscr{X}(t)}{\mathscr{X}(t)} - \frac{dI(t)}{I(t)} + \frac{dI(t)^2}{I(t)^2} - \frac{d\mathscr{X}(t)dI(t)}{\mathscr{X}(t)I(t)} \right) - \theta\pi(t)dt - \beta\pi(t)dt - \gamma\pi(t)dt + \eta(t)dt, \end{cases} \tag{7}$$

which simplifies to:

$$\begin{cases} \frac{d\bar{\mathscr{X}}(t)}{\bar{\mathscr{X}}(t)} = \Big[\pi(t)(\mu_s - \theta - \beta - \gamma) + (1-\pi(t))r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 + \eta(t)\Big] dt - \sigma_I dW^I(t) + \pi(t)\sigma_s dW^S(t), \\ \bar{\mathscr{X}}(0) > 0, \end{cases} \tag{8}$$

where μ_I is the drift rate of the inflation index, σ_I is the volatility of inflation, ρ is the correlation between the risky asset and the inflation, $dW^I(t)$ is the Wiener process associated with inflation, and $\bar{\mathscr{X}}(t)$ is the inflation-adjusted real wealth. This model captures the complex dynamics of wealth dynamics considering investment decisions π, trading fees θ, government tax costs β, dividend γ and economic factors like inflation and wage income η.

Remark 4.1.
The correlation coefficient $\rho \in [-1,1]$ between W^S and W^η implies that $W^S W^\eta = \rho dt$, where ρ measures the degree of association between the stochastic processes driving the risky asset and wage income.

5. The Optimization Criterion

Definition 5.1.
An investment and consumption strategy $\pi(t)$ is deemed admissible if the following conditions are met:

(i) $\pi(t)$ is progressively $\mathscr{F}_t$-measurable, meaning it is adapted to the filtration $\mathscr{F}_t$ representing the information available up to time t.

(ii) The integral $\int_0^T \pi(t)^2 dt < \infty$ hold for all $T > 0$, ensuring investment is finite over any finite time horizon.

(iii) The wealth process $\bar{\mathscr{X}}(t)$, governed by the differential equation given earlier with initial wealth $\bar{\mathscr{X}}(0) > 0$, has a unique path-wise solution for the given strategy $\pi(t)$.

Remark 5.1.
The investor's goal is to maximize the expected discounted utility of their terminal real wealth and consumption over the investment horizon.

The objective function for this optimization problem is mathematically formulated as:

$$J(t,x;\pi(t)) = \mathbb{E}\left[\int_0^T e^{-\lambda T} U(\bar{\mathscr{X}}(T))\right], \tag{9}$$

where $\mathbb{E}$ denotes the expectation operator, λ is the subjective discount rate and $U(\bar{\mathscr{X}}(T))$ is the utility function for terminal wealth.

Definition 5.2.
The value function $V(t,x)$ is defined as:

$$V(t,x) = \sup_{\pi(t)} \mathbb{E}\left[\int_0^T e^{-\lambda T} U(\bar{\mathscr{X}}(T))\right], \tag{10}$$

with the boundary condition given by $V(T,x) = e^{-\lambda T} U(\bar{\mathscr{X}}(T))$.

This value function represents the maximum expected utility that can be achieved starting from time t with wealth x.

To conclude, the investor seeks to identify the optimal strategy $\pi(t)$ that maximizes their overall expected utility of the value of terminal wealth, while considering the dynamic nature of their wealth influenced by various economic factors. The value function encapsulates the best possible outcome in terms of utility given the constraints and dynamics of the investment environment.

6. The Hamilton-Jacobi-Bellman Equation

By leveraging the Dynamic Programming Principle (DPP), we derive the Hamilton-Jacobi-Bellman (HJB) equation for the value function associated with our stochastic control problem. The complete HJB partial differential equation (PDE) for this problem is a second-order nonlinear PDE given as follows:

$$V_t + \sup_{\pi}\Big[\big[\pi(\mu_s-\theta-\beta-\gamma)+(1-\pi)r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2+\eta(t)\big]\bar{\mathscr{X}}V_x+\Big[\frac{1}{2}\pi^2\sigma_s^2+\sigma_I^2-2\pi\sigma_s\sigma_I\rho\Big]\bar{\mathscr{X}}^2V_{xx} + \mu_\eta\eta(t)V_\eta+\frac{1}{2}\sigma_\eta^2\eta^2(t)V_{\eta\eta}+\pi\sigma_s\sigma_I\eta(t)\bar{\mathscr{X}}V_{x\eta}\Big]=0, \tag{11}$$

where V_t, V_x, V_{xx}, V_η, $V_{\eta\eta}$, and $V_{x\eta}$ represent partial derivatives with respect to their respective variables.
The HJB PDE simplifies to:

$$V_t + \sup_{\pi}\Big[\big[\pi(\mu_s-\theta-\beta-\gamma)+(1-\pi)r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2\big]\bar{\mathscr{X}}V_x+\Big[\frac{1}{2}\pi^2\sigma_s^2+\sigma_I^2-2\pi\sigma_s\sigma_I\rho\Big]\bar{\mathscr{X}}^2V_{xx}\Big]=0. \tag{12}$$

To find the optimal controls for this phase, we derive the following expression:

$$\pi^*(t) = -\frac{-(\mu_s-\theta-\beta-\gamma-r)\bar{\mathscr{X}}V_x-2\sigma_s\sigma_I\rho\bar{\mathscr{X}}^2V_{xx}}{\sigma_s^2\bar{\mathscr{X}}V_x}, \tag{13}$$

The utility function is specified as:

$$U(x)\frac{x^{1-\delta}}{1-\delta}, \quad \delta>0, \quad \delta\neq 1, \tag{14}$$

with δ being the risk aversion factor.
In summary, The HJB equation provides a framework for determining the value function in a stochastic control problem. The derived equations for optimal controls guide the investor on how to allocate their resources optimally.

7. The Value Function and Optimal Policies

Assuming that the solution V for the HJB PDE takes the form:

$$V(t,x)=\frac{x^{1-\delta}}{1-\delta}G(t), \quad G(T)=1. \tag{15}$$

The partial derivatives for V are then:

$$V_t=\frac{x^{1-\delta}}{1-\delta}G_t, \quad V_x=x^{-\delta}G, \quad V_{xx}=-\delta x^{-\delta-1}G. \tag{16}$$

Utilizing these forms, we simplify eq. (13) as:

$$\pi^*(t)=\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}, \tag{17}$$

Substituting eq. (16) and eq. (17) into eq. (12) gives:

$$\begin{aligned}
&\frac{x^{1-\delta}}{1-\delta}G_t+\sup_{\pi}\Bigg(\Bigg[\Bigg[\left(\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)\mu_s\\
&+\left(1-\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)r-\mu I-\sigma_s\sigma_I\rho+\sigma_I^2\Bigg]xx^{-\delta}G\\
&+\frac{1}{2}\Bigg[\left(\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)^2\sigma_s^2\\
&+\sigma_I^2-2\left(\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)\sigma_s\sigma_I\rho\Bigg]x^2(-\delta x^{-\delta-1}G)\Bigg)\\
&=0.
\end{aligned} \tag{18}$$

Equation (18) is still a Differential Equation (DE) which is difficult to solve. Inspired by [19, 20] , we assume G is given by:

$$G = \int_t^T \hat{G} du + \hat{G}. \tag{19}$$

This implieseq. (18) reduces to the following DE with well-defined solutions:

$$\begin{aligned} &\frac{x^{1-\delta}}{1-\delta}\hat{G}t + \sup_{\pi}\Bigg(\Bigg[\Bigg[\left(\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)\mu_s \\ &+\left(1-\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2\Bigg]xx^{-\delta}\hat{G} \\ &+\frac{1}{2}\Bigg[\Bigg[\left(\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)^2\sigma_s^2+\sigma^2 I \\ &-2\left(\frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)\sigma_s\sigma_I\rho\Bigg]x^2(-\delta x^{-\delta-1}\hat{G})\Bigg) \\ &=0. \end{aligned} \tag{20}$$

Simplifying further gives the following DE which has a well-defined solution:

$$\frac{\hat{G}t}{\hat{G}} = (1-\delta)\left[-\frac{[\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho]^2}{2\sigma_s^2\delta}-\left(r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2+\frac{\sigma_I^2\delta}{2}\right)\right]. \tag{21}$$

Solving eq. (21), we obtain the following solution:

$$\hat{G}(t) = \exp\left[(1-\delta)\left(-\frac{[\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho]^2}{2\sigma_s^2\delta}-\left[r-\mu I-\sigma_s\sigma_I\rho+\sigma_I^2+\frac{\sigma_I^2\delta}{2}\right]\right)(T-t)\right]. \tag{22}$$

Therefore, the value function V having solved eq. (21) is given by:

$$V(t,x) = \frac{x^{1-\delta}}{1-\delta}\exp\left[(1-\delta)\left(-\frac{[\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho]^2}{2\sigma_s^2\delta}-\left[r-\mu I-\sigma_s\sigma_I\rho+\sigma_I^2+\frac{\sigma_I^2\delta}{2}\right]\right)(T-t)\right]. \tag{23}$$

In addition, the optimal controls are given as follows:

$$\pi(t) = \frac{\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}, \tag{24}$$

where

$$\hat{G}(t) = \exp\left[(1-\delta)\left(-\frac{[\mu_s-\theta-\beta-\gamma-r+\sigma_s\sigma_I\rho]^2}{2\sigma_s^2\delta}-\left[r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2+\frac{\sigma_I^2\delta}{2}\right]\right)(T-t)\right]. \tag{25}$$

8. Simulations

In this section, we determine how parameters impact optimal investment control $\pi^*(t)$. Throughout our simulations, parameters will be estimated as follows: $\rho = 1; \delta = -1; \sigma_s = 1.2; \sigma_I = 1; \sigma_s^2 = 1.1; T = 1; \mu_s = 0.6; r = 0.5$; We first assess the impact of the trading fee θ on optimal investment $\pi^*(t)$ through curve analysis:

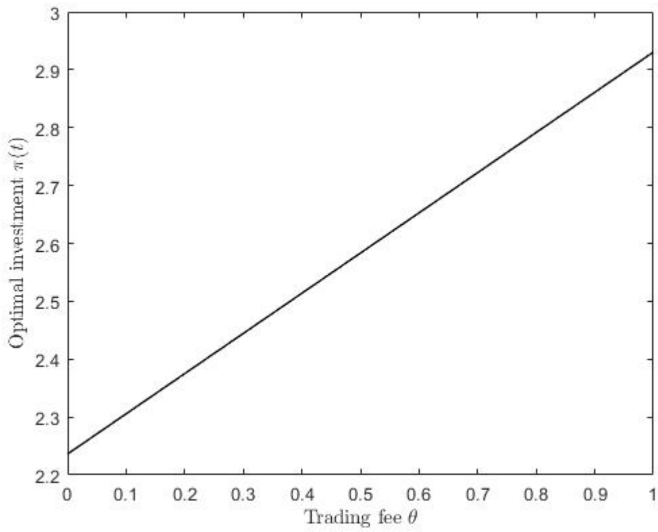

Fig. 1. The impact of the trading fee θ on optimal investment $\pi^*(t)$

Figure 1 shows the impact of the trading fee θ on the investment $\pi^*(t)$. The curve results indicate that the impact of the trading fee θ on investment is positive. The fact that the optimal investment and trading fee have a positive association shows that investors are prepared to spend more for potentially bigger profits. The perceived value of premium investment options, financial objectives, and risk tolerance are some of the elements that influence this behavior. Optimal investment and trading fee have a positive association because investors want to maximize their earnings and are prepared to trade off costs for possible rewards.

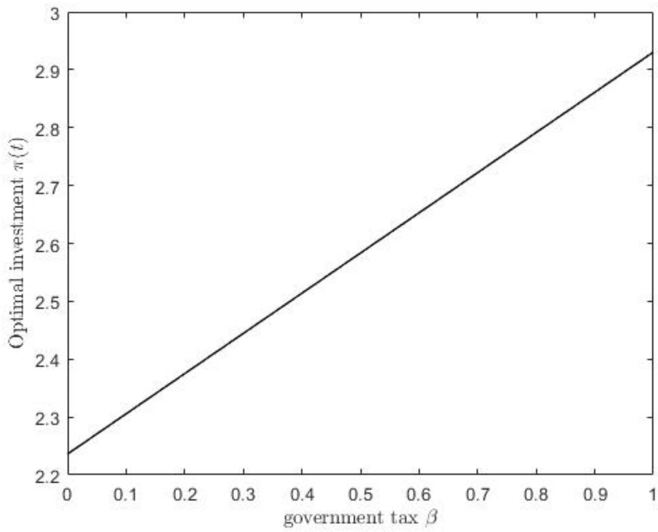

Fig. 2. The impact of government tax β on optimal investment $\pi^*(t)$

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In fig. 2 the optimal investment $\pi^*(t)$ increases with larger values of government tax β. The investor becomes vigorous in investing in risky assets. There are a number of reasons why the optimal investment increases when government tax values rise. This phenomena is influenced by government incentives, tax-loss harvesting, tax-deferred savings, tax-deferred investments, and the strategic conduct of investors. Investors will continue to adjust and react to changes in tax laws as they attempt to optimize their after-tax returns, which will increase the amount of optimal investments.

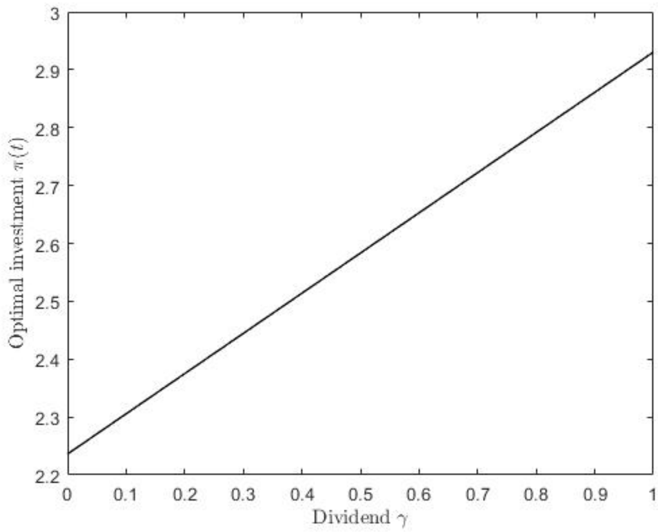

Fig. 3. The impact of dividend γ on optimal investment $\pi^*(t)$

Figure 3 shows the impact of dividend γ on optimal investment $\pi^*(t)$. The curve results indicate that dividend γ affects optimal investment in a positive way. Dividends have a very favorable effect on the optimal investing choices. Dividend-paying investments are an alluring option for investors because they show financial discipline, give a stable source of return, improve overall returns, hedge against inflation, and promote a long-term outlook. Achieving the best possible financial results can be greatly aided by dividend-focused investing as investors look to increase profits while lowering risk

9. Conclusion

Portfolio choice problem with trading fees, government taxes, dividends, stochastic wage income and inflation-adjusted wealth was investigated in this study. The goal was to allocate initial wealth between investment securities in order to maximize the expected discounted utilities derived from terminal wealth. This stochastic optimal control problem was linked to the non-linear second-order PDE called the HJB PDE via Bellman's optimality principle. Upon solving HJB PDE, the value function and optimal control was obtained. Simulation results indicate that the impact of investment $\pi^*(t)$ and trading fee θ have a positive association because investors want to maximize their earnings and are prepared to trade off costs for possible rewards. Furthermore, investment $\pi^*(t)$ increases with larger values of government tax β. This is so because the investor becomes vigorous in investing in risky assets to counter the effects of tax. Investors will continue to adjust and react to changes in tax laws as they attempt to optimize their after-tax returns, which will increase the amount of optimal investments. Finally, the impact of dividend γ on investment $\pi^*(t)$ is positive. Dividend-paying investments are an alluring option for investors because they show financial discipline, give a stable source of return, improve overall returns, hedge against inflation, and promote a long-term outlook. Achieving the best possible financial results can be greatly aided by dividend-focused investing as investors look to increase profits while lowering risk.

There are many related topics that may be worthy to study in the future, for example, extend the model to incorporate stochastic volatility and jump processes in asset prices, which could provide a more realistic representation of market dynamics and risk. This would require more advanced mathematical techniques for solving the resulting HJB equations. The latest publication by Mukonda et.al [33] and Malichi et.at [34] this year have motivated us to publish this paper in this journal.

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Conflicts of Interest

The authors declare no conflicts of interest.

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