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On Dual Generalized Woodall Numbers

Research Article

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Abstract: In this work, we introduce the generalized dual Woodall numbers. As special cases, we study with dual Woodall, dual modified Woodall, dual Cullen numbers and dual modified Cullen numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Also, we give Catalan's and Cassini's identities and present matrices related with these sequences.

MSC: 11B39 • 11B83

Keywords: Woodall numbers • Cullen numbers • dual numbers • Dual Woodall numbers • Dual Cullen numbers

1. Introduction

Firstly, we recall the definition and properties of generalized Woodall numbers.

The generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{1}$$

with the initial values W_0, W_1, W_2 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence eq. (1) holds for all integer n. For more details, see [34].

Now, Binet's formula of generalized Woodall sequence can be calculated using its characteristic equation which is given as

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2(x-1) = 0.$$

The roots of characteristic equation are

$$\alpha = \beta = 2, \ \gamma = 1.$$

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Using these roots and the recurrence relation, Binet's formula can be given as

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3 \gamma^n, \tag{2}$$
$$W_n = (A_1 + A_2 n) \times 2^n + A_3 \tag{3}$$

where

$$A_1 = -W_2 + 4W_1 - 3W_0,$$
$$A_2 = \frac{W_2 - 3W_1 + 2W_0}{2},$$
$$A_3 = W_2 - 4W_1 + 4W_0.$$

i.e.,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{4}$$

Now, the first few generalized Woodall numbers with positive subscript and negative subscript are given in the following table 1.

Table 1. A few generalized Woodall numbers.

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{4}(8W_0 - 5W_1 + W_2)$
2	W_2	$\frac{1}{4}(11W_0 - 9W_1 + 2W_2)$
3	$4W_0 - 8W_1 + 5W_2$	$\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$
4	$20W_0 - 36W_1 + 17W_2$	$\frac{1}{16}(57W_0 - 54W_1 + 13W_2)$
5	$68W_0 - 116W_1 + 49W_2$	$\frac{1}{64}(240W_0 - 233W_1 + 57W_2)$

Now, we define four specific cases of the sequence $\{W_n\}$.

The Woodall numbers $\{R_n\}$, sometimes called Riesel numbers, and also called Cullen numbers of the second kind, are numbers of the form

$$R_n = n \times 2^n - 1.$$

The first few Woodall numbers are:

$$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, \ldots$$

(sequence A003261 in the OEIS [29]]. Woodall numbers were first studied by Allan J. C. Cunningham and H. J. Woodall in [11] in 1917, inspired by James Cullen's earlier study of the similarly-defined Cullen numbers.

The Cullen numbers $\{C_n\}$ are numbers of the form

$$C_n = n \times 2^n + 1.$$

The first few Cullen numbers are:

$$1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, \ldots$$

(sequence A002064 in the OEIS). Woodall and Cullen sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [6, 7, 11, 14, 15, 19, 22–26] and references therein. Note that $\{R_n\}$ and $\{C_n\}$ hold the following relations:

$$R_n = 4R_{n-1} - 4R_{n-2} - 1,$$
$$C_n = 4C_{n-1} - 4C_{n-2} + 1.$$

Note also that the sequences $\{R_n\}$ and $\{C_n\}$ satisfy the following third order linear recurrences:

$$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \quad R_0 = -1, R_1 = 1, R_2 = 7 \tag{5}$$

and

$$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \quad C_0 = 1, C_1 = 3, C_2 = 9. \tag{6}$$

If we set $G_0 = 0$, $G_1 = 1$, $G_2 = 5$ then $\{G_n\}$ is the well-known modified Woodall sequence, if we set $H_0 = 3$, $H_1 = 5$, $H_2 = 9$ then $\{H_n\}$ is the well-known modified Cullen sequence. In other words, modified Woodall sequence $\{G_n\}_{n\geq 0}$ and modified Cullen sequence $\{H_n\}_{n\geq 0}$ are defined by the third-order recurrence relations

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \quad G_0 = 0, G_1 = 1, G_2 = 5, \tag{7}$$

$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \quad H_0 = 3, H_1 = 5, H_2 = 9. \tag{8}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
G_{-n} &= 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)},\\
H_{-n} &= 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)},\\
R_{-n} &= 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)},\\
C_{-n} &= 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences eqs. (5)–(8) hold for all integer n.

Using the initial conditions in eq. (4), we can obtain the Binet formulas of modified Woodall, modified Cullen, Woodall and Cullen sequences as follows:

$$\begin{aligned}
G_n &= (n-1)2^n + 1,\\
H_n &= 2^{n+1} + 1,\\
R_n &= n \times 2^n - 1,\\
C_n &= n \times 2^n + 1.
\end{aligned}$$

Now, we give the generating function and the Cassini identity for generalized Woodall numbers.

The generating function for generalized Woodall numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 5W_0)x + (W_2 - 5W_1 + 8W_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{9}$$

The Cassini identity for generalized Woodall numbers is

$$\begin{aligned}
W_{n+1}W_{n-1} - W_n^2 &= \frac{1}{4}2^n(A + B2^n + Cn).\\
A &= 4W_1^2 + W_2^2 - 4W_0W_1 + 4W_0W_2 - 5W_1W_2.\\
B &= -4W_0^2 - 9W_1^2 - W_2^2 + 12W_0W_1 - 4W_0W_2 + 6W_1W_2.\\
C &= 8W_0^2 + 12W_1^2 + W_2^2 - 20W_0W_1 + 6W_0W_2 - 7W_1W_2.
\end{aligned}$$

For further information about generalized Woodall numbers, see [34].

The hypercomplex numbers systems, [21] are extensions of real numbers. Some commutative examples of hypercomplex number systems; complex numbers, hyperbolic (double, split-complex) numbers [30] and dual numbers [13] are given below in order.

$$\begin{aligned}
\mathbb{C} &= \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},\\
\mathbb{H} &= \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},\\
\mathbb{D} &= \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.
\end{aligned}$$

Some non-commutative examples of hypercomplex number systems are quaternions, [18],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [5] and sedenions [31]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [8, 20, 27]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [18] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [10]. H. H. Cheng and S. Thompson [9] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [4] introduced dual hyperbolic numbers.

A dual number is a hyper-complex number and is defined by

$$q = a_0 + \varepsilon a_1$$

where a_0, a_1 are real numbers.

The set of all dual numbers are denoted by

$$\mathbb{D} = \{a_0 + \varepsilon a_1 : a_0, a_1 \in \mathbb{R}, \ \varepsilon^2 = 0, \ \varepsilon \neq 0\}.$$

The base elements $\{1, \varepsilon\}$ of dual numbers satisfy the following properties (commutative multiplications):

$$1.\varepsilon = \varepsilon, \ \varepsilon^2 = \varepsilon.\varepsilon = 0$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$).

Let m and n two dual numbers as $m = a_0 + \varepsilon a_1$ and $n = b_0 + \varepsilon b_1$; The addition and substraction of two dual numbers as m and n is

$$m \mp n = a_0 \mp b_0 + \varepsilon(a_1 \mp b_1),$$

then, the multiplication of two dual numbers as m and n is

$$mn = a_0 b_0 + \varepsilon(a_1 b_0 + a_0 b_1).$$

Now, we provide details about dual and some information related to dual hyperbolic sequences from the literature.

- Bród, Liana, Włoch [12] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

- Akar, Yüce and Şahin [4] presented the dual hyperbolic numbers.
- Soykan, Taşdemir and Okumuş [36] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Cockle [10] studied the Hyperbolic numbers with complex coefficients.
- Cihan, Azak, Güngör, Tosun, [1] studied dual hyperbolic Fibonacci and Lucas numbers given by,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$
$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Cheng and Thompson [9] introduced dual numbers with complex coefficients.
- Soykan, Gümüş, Göcen [33] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Halici [17] studied Dual Fibonacci Octonions as

$$p = \sum_{s=0}^{7} F_{n+s} e_s$$

where Fibonacci given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Gürses, Şentürk, Yüce [?] studied dual-generalized complex Fibonacci and Lucas numbers, respectively, as

$$\widetilde{\mathscr{F}}_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$
$$\widetilde{\mathscr{L}}_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3},$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1}+F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1}+L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Aydın [2] studied Dual Jacobsthal Quaternions as

$$QJ_{k;n} = J_{k;n} + i_1 J_{k;n+1} + i_2 J_{k;n+2} + i_3 J_{k;n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$.

- Nurkan, Guven, [28] studied Dual Fibonacci Quaternions as

$$\widetilde{Q}n = (F_n + F_{n+1}) + i(F_{n+1} + F_{n+2}) + j(F_{n+2} + F_{n+3}) + k(F_{n+3} + F_{n+4})$$

where Fibonacci given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Yüce, Aydın, [37] studied Generalized Dual Fibonacci Quaternions as

$$Q_{\mathbb{D}} = \left\{\mathbb{D}_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} : H_n \text{ is } n\text{-th generalized Fibonacci number}\right\}$$

where

$$i^2 = j^2 = k^2 = ijk = 0,\ ij = -ji = jk = -kj = ki = -ik = 0.$$

- Aydın, Köklü and Yüce, [3] presented Generalized dual Pell quaternions as

$$Q_{\mathbb{D}} = \left\{\mathbb{D}_n^P = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3} : P_n \text{ is } n\text{-th generalized Pell number}\right\}$$

where

$$i^2 = j^2 = k^2 = ijk = 0,\ ij = -ji = jk = -kj = ki = -ik = 0.$$

In this paper, we define the dual generalized Woodall numbers in the next section and give some properties of them.

2. Dual Generalized Woodall Numbers

In this section, we define dual generalized Woodall numbers and present generating functions and Binet's formulas for them.

We now define dual generalized Woodall numbers over $\mathbb{D}$. The nth dual generalized Woodall number is

$$\mathscr{D}W_n = W_n + \varepsilon W_{n+1}. \quad (10)$$

with the initial values $\mathscr{D}W_0, \mathscr{D}W_1, \mathscr{D}W_2$. eq. (10) can be written to negative subscripts by defining,

$$\mathscr{D}W_{-n} = W_{-n} + \varepsilon W_{-n+1}.$$

So identity eq. (10) holds for all integers n.

Some special cases, the nth dual modified Woodall, the nth dual modified Cullen, the nth dual Woodall and the nth dual Cullen numbers are given as

$$\mathscr{D}G_n = G_n + \varepsilon G_{n+1},$$
$$\mathscr{D}H_n = H_n + \varepsilon H_{n+1},$$
$$\mathscr{D}R_n = R_n + \varepsilon R_{n+1},$$
$$\mathscr{D}C_n = C_n + \varepsilon C_{n+1}.$$

It is clear that

$$\mathscr{D}W_n = 5\mathscr{D}W_{n-1} - 8\mathscr{D}W_{n-2} + 4\mathscr{D}W_{n-3}. \quad (11)$$

Table 2. A few dual generalized Woodall numbers.

n	$\mathscr{D}W_n$	$\mathscr{D}W_{-n}$
0	$\mathscr{D}W_0$	$\mathscr{D}W_0$
1	$\mathscr{D}W_1$	$\frac{1}{4}(8\mathscr{D}W_0-5\mathscr{D}W_1+\mathscr{D}W_2)$
2	$\mathscr{D}W_2$	$\frac{1}{4}(11\mathscr{D}W_0-9\mathscr{D}W_1+2\mathscr{D}W_2)$
3	$4\mathscr{D}W_0-8\mathscr{D}W_1+5\mathscr{D}W_2$	$\frac{1}{16}(52\mathscr{D}W_0-47\mathscr{D}W_1+11\mathscr{D}W_2)$
4	$20\mathscr{D}W_0-36\mathscr{D}W_1+17\mathscr{D}W_2$	$\frac{1}{16}(57\mathscr{D}W_0-54\mathscr{D}W_1+13\mathscr{D}W_2)$
5	$68\mathscr{D}W_0-116\mathscr{D}W_1+49\mathscr{D}W_2$	$\frac{1}{64}(240\mathscr{D}W_0-233\mathscr{D}W_1+57\mathscr{D}W_2)$

The sequence $\{\mathscr{D}W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathscr{D}W_{-n}=-2\mathscr{D}W_{-(n-1)}-\frac{5}{4}\mathscr{D}W_{-(n-2)}+\frac{1}{4}\mathscr{D}W_{-(n-3)}.$$

for $n=1,2,3,...$ respectively. Therefore, recurrence (eq. (11)) holds for all integer n.

The initial several dual generalized Woodall numbers with positive subscript and negative subscript are given in the following table 2.

Note that

$$\mathscr{D}W_0=W_0+\varepsilon W_1=W_0+\varepsilon W_1,$$
$$\mathscr{D}W_1=W_1+\varepsilon W_2=W_1+\varepsilon W_2,$$
$$\mathscr{D}W_2=W_2+\varepsilon W_3=W_2+\varepsilon(4\mathscr{D}W_0-8\mathscr{D}W_1+5\mathscr{D}W_2).$$

For dual modified Woodall numbers (taking $W_n=G_n$, $G_0=0$, $G_1=1$, $G_2=5$) we get

$$\mathscr{D}G_0=G_0+\varepsilon G_1=\varepsilon,$$
$$\mathscr{D}G_1=G_1+\varepsilon G_2=1+5\varepsilon,$$
$$\mathscr{D}G_2=G_2+\varepsilon G_3=5+17\varepsilon,$$

and for dual modified Cullen numbers (taking $W_n=H_n$, $H_0=3$, $H_1=5$, $H_2=9$) we get

$$\mathscr{D}H_0=H_0+\varepsilon H_1=3+5\varepsilon,$$
$$\mathscr{D}H_1=H_1+\varepsilon H_2=5+9\varepsilon,$$
$$\mathscr{D}H_2=H_2+\varepsilon H_3=9+17\varepsilon,$$

and for dual Woodall numbers (taking $W_n=R_n$, $R_0=-1$, $R_1=1$, $R_2=7$) we get

$$\mathscr{D}R_0=R_0+\varepsilon R_1=-1+\varepsilon,$$
$$\mathscr{D}R_1=R_1+\varepsilon R_2=1+7\varepsilon,$$
$$\mathscr{D}R_2=R_2+\varepsilon R_3=7+23\varepsilon,$$

and for dual Cullen numbers (taking $W_n=C_n$, $C_0=1$, $C_1=3$, $C_2=9$) we get

$$\mathscr{D}C_0=C_0+\varepsilon C_1=1+3\varepsilon,$$
$$\mathscr{D}C_1=C_1+\varepsilon C_2=3+9\varepsilon,$$
$$\mathscr{D}C_2=C_2+\varepsilon C_3=9+25\varepsilon.$$

A few dual modified Woodall numbers, dual modified Cullen numbers, dual Woodall numbers and dual Cullen numbers with positive subscript and negative subscript are given in the following tables 3–6.

Now, we will state Binet's formula for the dual generalized Woodall numbers and in the rest of the paper, we fix the following notations:

$$\widehat{\alpha}=1+2\varepsilon,$$
$$\widehat{\beta}=2\varepsilon,$$
$$\widehat{\gamma}=1+\varepsilon.$$

Note that we have the following identities:

$$\widehat{\alpha}^2=1+4\varepsilon,$$
$$\widehat{\beta}^2=0,$$
$$\widehat{\gamma}^2=1+2\varepsilon,$$
$$\widehat{\alpha}\widehat{\beta}=2\varepsilon,$$
$$\widehat{\alpha}\widehat{\gamma}=1+3\varepsilon,$$
$$\widehat{\beta}\widehat{\gamma}=2\varepsilon,$$
$$\widehat{\alpha}\widehat{\beta}\widehat{\gamma}=2\varepsilon.$$

Table 3. Dual modified Woodall numbers.

n	$\mathscr{D}G_n$	$\mathscr{D}G_{-n}$
0	ε	ε
1	$5\varepsilon+1$	0
2	$17\varepsilon+5$	$\frac{1}{4}$
3	$49\varepsilon+17$	$\frac{1}{4}\varepsilon+\frac{1}{2}$
4	$129\varepsilon+49$	$\frac{1}{2}\varepsilon+\frac{11}{16}$
5	$321\varepsilon+129$	$\frac{11}{16}\varepsilon+\frac{13}{16}$

Table 4. Dual modified Cullen numbers.

n	$\mathscr{D}H_n$	$\mathscr{D}H_{-n}$
0	$5\varepsilon+3$	$5\varepsilon+3$
1	$9\varepsilon+5$	$3\varepsilon+2$
2	$17\varepsilon+9$	$2\varepsilon+\frac{3}{2}$
3	$33\varepsilon+17$	$\frac{3}{2}\varepsilon+\frac{5}{4}$
4	$65\varepsilon+33$	$\frac{5}{4}\varepsilon+\frac{9}{8}$
5	$129\varepsilon+65$	$\frac{9}{8}\varepsilon+\frac{17}{16}$

2.1. Binet's Formula

Theorem 2.1.
(Binet's Formula) For any integer n, the nth dual generalized Woodall number is

$$\mathscr{D}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{12}$$

Proof. Using Binet's formula

$$W_n = (A_1 + A_2 n)2^n + A_3$$

of the generalized Woodall numbers, we obtain

$$\begin{aligned}
\mathscr{D}W_n &= W_n + \varepsilon W_{n+1} \\
&= (A_1 + A_2 n)2^n + A_3 + \varepsilon((A_1 + A_2(n+1))2^{n+1} + A_3) \\
&= A_1 2^n + A_2 n 2^n + A_3 \\
&\quad + \varepsilon A_1 2^{n+1} + \varepsilon A_2 n 2^{n+1} + \varepsilon A_2 2^{n+1} + \varepsilon A_3 \\
&= A_1 2^n(1+2\varepsilon) + A_2 n 2^n(1+2\varepsilon) + A_2 2^n(2\varepsilon) + A_3(1+\varepsilon) \\
&= A_1 2^n\widehat{\alpha} + A_2 n 2^n\widehat{\alpha} + A_2 2^n\widehat{\beta} + A_3\widehat{\gamma} \\
&= (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}.
\end{aligned}$$

This proves (eq. (12)). □

As special cases, for any integer n, the Binet's Formula of nth dual modified Woodall number, dual modified Cullen number, dual Woodall number and dual Cullen number are

- $\mathscr{D}G_n = (-\widehat{\alpha} + \widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\mathscr{D}G_n = 1 + (n-1)2^n + \varepsilon(1 + n2^{n+1})$.
- $\mathscr{D}H_n = (2\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\mathscr{D}H_n = 1 + 2^{n+1} + \varepsilon(1 + 2^{n+2})$.
- $\mathscr{D}R_n = (\widehat{\beta} + n\widehat{\alpha})2^n - \widehat{\gamma}$,
  $\mathscr{D}R_n = -1 + n2^n + \varepsilon(-1 + 2^{n+1} + n2^{n+1})$.
- $\mathscr{D}C_n = (\widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\mathscr{D}C_n = 1 + n2^n + \varepsilon(1 + 2^{n+1} + n2^{n+1})$.

Next, we present generating function.

Table 5. Dual Woodall numbers.

n	$\mathcal{D}R_n$	$\mathcal{D}R_{-n}$
0	$\varepsilon-1$	$\varepsilon-1$
1	$7\varepsilon+1$	$-\varepsilon-\frac{3}{2}$
2	$23\varepsilon+7$	$-\frac{3}{2}\varepsilon-\frac{3}{2}$
3	$63\varepsilon+23$	$-\frac{3}{2}\varepsilon-\frac{11}{8}$
4	$159\varepsilon+63$	$-\frac{11}{8}\varepsilon-\frac{5}{4}$
5	$383\varepsilon+159$	$-\frac{5}{4}\varepsilon-\frac{37}{32}$

Table 6. Dual Cullen numbers.

n	$\mathcal{D}C_n$	$\mathcal{D}C_{-n}$
0	$3\varepsilon+1$	$3\varepsilon+1$
1	$9\varepsilon+3$	$\varepsilon+\frac{1}{2}$
2	$25\varepsilon+9$	$\frac{1}{2}\varepsilon+\frac{1}{2}$
3	$65\varepsilon+25$	$\frac{1}{2}\varepsilon+\frac{5}{8}$
4	$161\varepsilon+65$	$\frac{5}{8}\varepsilon+\frac{3}{4}$
5	$385\varepsilon+161$	$\frac{3}{4}\varepsilon+\frac{27}{32}$

2.2. Generating Function

Theorem 2.2.
The generating function for the dual generalized Woodall numbers is

$$\sum_{n=0}^{\infty}\mathcal{D}W_nx^n=\frac{\mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2}{1-5x+8x^2-4x^3}. \tag{13}$$

Proof. Let

$$g(x)=\sum_{n=0}^{\infty}\mathcal{D}W_nx^n$$

be generating function of the dual generalized Woodall numbers. Then, using the definition of the dual generalized Woodall numbers, and substracting $xg(x)$, $x^2g(x)$ and $x^3g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}(1-5x+8x^2-4x^3)g(x) &= \sum_{n=0}^{\infty}\mathcal{D}W_nx^n-5x\sum_{n=0}^{\infty}\mathcal{D}W_nx^n+8x^2\sum_{n=0}^{\infty}\mathcal{D}W_nx^n-4x^3\sum_{n=0}^{\infty}\mathcal{D}W_nx^n\\
&= \sum_{n=0}^{\infty}\mathcal{D}W_nx^n-5\sum_{n=0}^{\infty}\mathcal{D}W_nx^{n+1}+8\sum_{n=0}^{\infty}\mathcal{D}W_nx^{n+2}-4\sum_{n=0}^{\infty}\mathcal{D}W_nx^{n+3}\\
&= \sum_{n=0}^{\infty}\mathcal{D}W_nx^n-5\sum_{n=1}^{\infty}\mathcal{D}W_{n-1}x^n+8\sum_{n=2}^{\infty}\mathcal{D}W_{n-2}x^n-4\sum_{n=3}^{\infty}\mathcal{D}W_{n-3}x^n\\
&= (\mathcal{D}W_0+\mathcal{D}W_1x+\mathcal{D}W_2x^2)-5(\mathcal{D}W_0x+\mathcal{D}W_1x^2)+8\mathcal{D}W_0x^2\\
&\quad+\sum_{n=3}^{\infty}(\mathcal{D}W_n-5\mathcal{D}W_{n-1}+8\mathcal{D}W_{n-2}-4\mathcal{D}W_{n-3})x^n\\
&= \mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2.\end{aligned}$$

Note that we used the recurrence relation $\mathcal{D}W_n=5\mathcal{D}W_{n-1}-8\mathcal{D}W_{n-2}+4\mathcal{D}W_{n-3}$. Rearranging above equation, we get

$$g(x)=\frac{\mathcal{D}W_0+(\mathcal{D}W_1-5\mathcal{D}W_0)x+(\mathcal{D}W_2-5\mathcal{D}W_1+8\mathcal{D}W_0)x^2}{1-5x+8x^2-4x^3}.$$

The proof is finished. □

As special cases, the generating functions for the dual modified Woodall, dual modified Cullen, dual Woodall and dual Cullen numbers are

$$\sum_{n=0}^{\infty}\mathcal{D}G_nx^n=\frac{\varepsilon+x}{1-5x+8x^2-4x^3},$$

$$\sum_{n=0}^{\infty}\mathcal{D}H_nx^n=\frac{5\varepsilon+3+(-16\varepsilon-10)x+(12\varepsilon+8)x^2}{1-5x+8x^2-4x^3},$$

$$\sum_{n=0}^{\infty}\mathcal{D}R_nx^n=\frac{-1+\varepsilon+(2\varepsilon+6)x+(-4\varepsilon-6)x^2}{1-5x+8x^2-4x^3}$$

and

$$\sum_{n=0}^{\infty} \mathscr{D}C_n x^n = \frac{3\varepsilon+1+(-6\varepsilon-2)x+(4\varepsilon+2)x^2}{1-5x+8x^2-4x^3}$$

respectively.

2.3. Obtaining Binet's Formula From Generating Function

We obtain Binet's formula of dual generalized Woodall number $\{\mathscr{D}W_n\}$ by the use of generating function for $\mathscr{D}W_n$.

Theorem 2.3.
(Binet's formula of dual generalized Woodall numbers)

$$\mathscr{D}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{14}$$

Proof. Let

$$\sum_{n=0}^{\infty} \mathscr{D}W_n x^n = \frac{\mathscr{D}W_0 + (\mathscr{D}W_1 - 5\mathscr{D}W_0)x + (\mathscr{D}W_2 - 5\mathscr{D}W_1 + 8\mathscr{D}W_0)x^2}{1-5x+8x^2-4x^3}.$$

Then we write

$$\frac{\mathscr{D}W_0 + (\mathscr{D}W_1 - 5\mathscr{D}W_0)x + (\mathscr{D}W_2 - 5\mathscr{D}W_1 + 8\mathscr{D}W_0)x^2}{(1-x)(1-2x)^2} = \frac{\mathscr{D}_1}{(1-x)} + \frac{\mathscr{D}_2}{(1-2x)} + \frac{\mathscr{D}_3}{(1-2x)^2}. \tag{15}$$

So

$$\mathscr{D}W_0 + (\mathscr{D}W_1 - 5\mathscr{D}W_0)x + (\mathscr{D}W_2 - 5\mathscr{D}W_1 + 8\mathscr{D}W_0)x^2 = (\mathscr{D}_1+\mathscr{D}_2+\mathscr{D}_3) + (-4\mathscr{D}_1 - 3\mathscr{D}_2 - \mathscr{D}_3)x + (4\mathscr{D}_1 + 2\mathscr{D}_2)x^2.$$

We get

$$\begin{aligned}
\mathscr{D}W_0 &= d_1 + d_2 + d_3,\\
\mathscr{D}W_1 - 5\mathscr{D}W_0 &= -4d_1 - 3d_2 - d_3,\\
\mathscr{D}W_2 - 5\mathscr{D}W_1 + 8\mathscr{D}W_0 &= 4d_1 + 2d_2.
\end{aligned}$$

If we solve these simultanious equations,

$$\begin{aligned}
d_1 &= 4\mathscr{D}W_0 - 4\mathscr{D}W_1 + \mathscr{D}W_2,\\
d_2 &= -4\mathscr{D}W_0 + \frac{11}{2}\mathscr{D}W_1 - \frac{3}{2}\mathscr{D}W_2,\\
d_3 &= \mathscr{D}W_0 - \frac{3}{2}\mathscr{D}W_1 + \frac{1}{2}\mathscr{D}W_2.
\end{aligned}$$

Thus (eq. (15)) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathscr{D}W_n x^n &= d_1\frac{1}{(1-x)} + d_2\frac{1}{(1-2x)} + d_3\frac{1}{(2x-1)^2},\\
&= d_1\sum_{n=0}^{\infty} x^n + d_2\sum_{n=0}^{\infty} 2^n x^n + d_3\sum_{n=0}^{\infty} 2^n(n+1)x^n,\\
&= \sum_{n=0}^{\infty} (d_1 + d_2 2^n + d_3 2^n(n+1))x^n,\\
&= \sum_{n=0}^{\infty} (4\mathscr{D}W_0 - 4\mathscr{D}W_1 + \mathscr{D}W_2 + (-4\mathscr{D}W_0 + \frac{11}{2}\mathscr{D}W_1 - \frac{3}{2}\mathscr{D}W_2)2^n\\
&\quad +(\mathscr{D}W_0 - \frac{3}{2}\mathscr{D}W_1 + \frac{1}{2}\mathscr{D}W_2)2^n(n+1))x^n,\\
&= \sum_{n=0}^{\infty} (4\mathscr{D}W_0 - 4\mathscr{D}W_1 + \mathscr{D}W_2 + (-4\mathscr{D}W_0 + \frac{11}{2}\mathscr{D}W_1 - \frac{3}{2}\mathscr{D}W_2)2^n\\
&\quad +(\mathscr{D}W_0 - \frac{3}{2}\mathscr{D}W_1 + \frac{1}{2}\mathscr{D}W_2)2^n + (\mathscr{D}W_0 - \frac{3}{2}\mathscr{D}W_1 + \frac{1}{2}\mathscr{D}W_2)2^n n)x^n,\\
&= \sum_{n=0}^{\infty} (4\mathscr{D}W_0 - 4\mathscr{D}W_1 + \mathscr{D}W_2 + (\mathscr{D}W_0 - \frac{3}{2}\mathscr{D}W_1 + \frac{1}{2}\mathscr{D}W_2)n2^n\\
&\quad +(-3\mathscr{D}W_0 + 4\mathscr{D}W_1 - \mathscr{D}W_2)2^n)x^n,\\
&= \sum_{n=0}^{\infty} ((-3\mathscr{D}W_0 + 4\mathscr{D}W_1 - \mathscr{D}W_2) + (\mathscr{D}W_0 - \frac{3}{2}\mathscr{D}W_1 + \frac{1}{2}\mathscr{D}W_2)n)2^n\\
&\quad +4\mathscr{D}W_0 - 4\mathscr{D}W_1 + \mathscr{D}W_2)x^n.
\end{aligned}$$

This gives

$$\mathscr{D}W_n = (\mathscr{D}A_1 + \mathscr{D}A_2 n)2^n + \mathscr{D}A_3$$

where

$$\mathscr{D}A_1 = -3\mathscr{D}W_0 + 4\mathscr{D}W_1 - \mathscr{D}W_2,$$
$$\mathscr{D}A_2 = \mathscr{D}W_0 - \frac{3}{2}\mathscr{D}W_1 + \frac{1}{2}\mathscr{D}W_2,$$
$$\mathscr{D}A_3 = 4\mathscr{D}W_0 - 4\mathscr{D}W_1 + \mathscr{D}W_2.$$

Note that the following equalities are true:

$$\begin{aligned} A_1\widehat{\alpha} + A_2\widehat{\beta} &= (-W_2 + 4W_1 - 3W_0)(1+2\varepsilon) + (\frac{W_2 - 3W_1 + 2W_0}{2})(2\varepsilon) \\ &= -3W_0 + 4W_1 - W_2 + \varepsilon(-4W_0 + 5W_1 - W_2). \end{aligned}$$

$$\begin{aligned} A_2\widehat{\alpha} &= \frac{W_2 - 3W_1 + 2W_0}{2}(1+2\varepsilon) \\ &= W_0 - \frac{3}{2}W_1 + \frac{1}{2}W_2 + \varepsilon(2W_0 - 3W_1 + W_2). \end{aligned}$$

$$A_3\widehat{\gamma} = W_2 - 4W_1 + 4W_0 + \varepsilon(W_2 - 4W_1 + 4W_0).$$

Therefore, we can write the following equalition:

$$\mathscr{D}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}.$$

The proof is finished. □

Next, using theorem 2.3, we present the Binet's formulas of dual modified Woodall, dual modified Cullen, dual Woodall and dual Cullen numbers.

3. Some Identities

We now present a few special identities for the dual generalized Woodall sequence $\{\mathscr{D}W_n\}$. The following theorem presents the Simpson's identity for the dual generalized Woodall numbers.

Theorem 3.1.
(Simpson's formula for dual generalized Woodall sequence)For all integers n we have

$$\begin{vmatrix} \mathscr{D}W_{n+2} & \mathscr{D}W_{n+1} & \mathscr{D}W_n \\ \mathscr{D}W_{n+1} & \mathscr{D}W_n & \mathscr{D}W_{n-1} \\ \mathscr{D}W_n & \mathscr{D}W_{n-1} & \mathscr{D}W_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} \mathscr{D}W_2 & \mathscr{D}W_1 & \mathscr{D}W_0 \\ \mathscr{D}W_1 & \mathscr{D}W_0 & \mathscr{D}W_{-1} \\ \mathscr{D}W_0 & \mathscr{D}W_{-1} & \mathscr{D}W_{-2} \end{vmatrix}.$$

Proof. For the proof we use mathematical induction. For $n = 0$ identity is true. Now we obtain is true for $n = k$. Hence we write the following identity

$$\begin{vmatrix} \mathscr{D}W_{k+2} & \mathscr{D}W_{k+1} & \mathscr{D}W_k \\ \mathscr{D}W_{k+1} & \mathscr{D}W_k & \mathscr{D}W_{k-1} \\ \mathscr{D}W_k & \mathscr{D}W_{k-1} & \mathscr{D}W_{k-2} \end{vmatrix} = 4^k \begin{vmatrix} \mathscr{D}W_2 & \mathscr{D}W_1 & \mathscr{D}W_0 \\ \mathscr{D}W_1 & \mathscr{D}W_0 & \mathscr{D}W_{-1} \\ \mathscr{D}W_0 & \mathscr{D}W_{-1} & \mathscr{D}W_{-2} \end{vmatrix}.$$

For $n = k+1$, we get

$$\begin{aligned} \begin{vmatrix} \mathscr{D}W_{k+3} & \mathscr{D}W_{k+2} & \mathscr{D}W_{k+1} \\ \mathscr{D}W_{k+2} & \mathscr{D}W_{k+1} & \mathscr{D}W_k \\ \mathscr{D}W_{k+1} & \mathscr{D}W_k & \mathscr{D}W_{k-1} \end{vmatrix} &= \begin{vmatrix} 5\mathscr{D}W_{k+2} - 8\mathscr{D}W_{k+1} + 4\mathscr{D}W_k & \mathscr{D}W_{k+2} & \mathscr{D}W_{k+1} \\ 5\mathscr{D}W_{k+1} - 8\mathscr{D}W_k + 4\mathscr{D}W_{k-1} & \mathscr{D}W_{k+1} & \mathscr{D}W_k \\ 5\mathscr{D}W_k - 8\mathscr{D}W_{k-1} + 4\mathscr{D}W_{k-2} & \mathscr{D}W_k & \mathscr{D}W_{k-1} \end{vmatrix} \\ &= 5\begin{vmatrix} \mathscr{D}W_{k+2} & \mathscr{D}W_{k+2} & \mathscr{D}W_{k+1} \\ \mathscr{D}W_{k+1} & \mathscr{D}W_{k+1} & \mathscr{D}W_k \\ \mathscr{D}W_k & \mathscr{D}W_k & \mathscr{D}W_{k-1} \end{vmatrix} - 8\begin{vmatrix} \mathscr{D}W_{k+1} & \mathscr{D}W_{k+2} & \mathscr{D}W_{k+1} \\ \mathscr{D}W_k & \mathscr{D}W_{k+1} & \mathscr{D}W_k \\ \mathscr{D}W_{k-1} & \mathscr{D}W_k & \mathscr{D}W_{k-1} \end{vmatrix} \\ &\quad +4\begin{vmatrix} \mathscr{D}W_k & \mathscr{D}W_{k+2} & \mathscr{D}W_{k+1} \\ \mathscr{D}W_{k-1} & \mathscr{D}W_{k+1} & \mathscr{D}W_k \\ \mathscr{D}W_{k-2} & \mathscr{D}W_k & \mathscr{D}W_{k-1} \end{vmatrix} \\ &= 4\begin{vmatrix} \mathscr{D}W_{k+2} & \mathscr{D}W_{k+1} & \mathscr{D}W_k \\ \mathscr{D}W_{k+1} & \mathscr{D}W_k & \mathscr{D}W_{k-1} \\ \mathscr{D}W_k & \mathscr{D}W_{k-1} & \mathscr{D}W_{k-2} \end{vmatrix} = 4^{k+1}\begin{vmatrix} \mathscr{D}W_2 & \mathscr{D}W_1 & \mathscr{D}W_0 \\ \mathscr{D}W_1 & \mathscr{D}W_0 & \mathscr{D}W_{-1} \\ \mathscr{D}W_0 & \mathscr{D}W_{-1} & \mathscr{D}W_{-2} \end{vmatrix}. \end{aligned}$$

Thus, the proof is finished. □

From prewious theorem, we get following corollary.

Corollary 3.1.
(Simpson's formula for dual generalized Woodall sequence's special cases)

(a) $\begin{vmatrix} \mathscr{D}G_{k+2} & \mathscr{D}G_{k+1} & \mathscr{D}G_{k} \\ \mathscr{D}G_{k+1} & \mathscr{D}G_{k} & \mathscr{D}G_{k-1} \\ \mathscr{D}G_{k} & \mathscr{D}G_{k-1} & \mathscr{D}G_{k-2} \end{vmatrix} = -4^{n-1}(9+9\varepsilon).$

(b) $\begin{vmatrix} \mathscr{D}H_{k+2} & \mathscr{D}H_{k+1} & \mathscr{D}H_{k} \\ \mathscr{D}H_{k+1} & \mathscr{D}H_{k} & \mathscr{D}H_{k-1} \\ \mathscr{D}H_{k} & \mathscr{D}H_{k-1} & \mathscr{D}H_{k-2} \end{vmatrix} = 0.$

(c) $\begin{vmatrix} \mathscr{D}R_{k+2} & \mathscr{D}R_{k+1} & \mathscr{D}R_{k} \\ \mathscr{D}R_{k+1} & \mathscr{D}R_{k} & \mathscr{D}R_{k-1} \\ \mathscr{D}R_{k} & \mathscr{D}R_{k-1} & \mathscr{D}R_{k-2} \end{vmatrix} = 4^{n-1}(9+9\varepsilon).$

(d) $\begin{vmatrix} \mathscr{D}C_{k+2} & \mathscr{D}C_{k+1} & \mathscr{D}C_{k} \\ \mathscr{D}C_{k+1} & \mathscr{D}C_{k} & \mathscr{D}C_{k-1} \\ \mathscr{D}C_{k} & \mathscr{D}C_{k-1} & \mathscr{D}C_{k-2} \end{vmatrix} = -4^{n-1}(9+9\varepsilon).$

Theorem 3.2.
(Catalan's identity) For all integers n and m, the following identity holds
$\mathscr{D}W_{n+m}\mathscr{D}W_{n-m}-\mathscr{D}W_n^2 = 2^{n-m}(-2^{m+n}m^2\widehat{\alpha}^2A_2^2+A_2A_3(-2^{m+1}\widehat{\beta}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}+2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma})+A_1A_3(\widehat{\alpha}\widehat{\gamma}-2^{m+1}\widehat{\alpha}\widehat{\gamma}+2^{2m}\widehat{\alpha}\widehat{\gamma})).$

Proof. Using the Binet's formula $\mathscr{D}W_n = (A_1\widehat{\alpha}+A_2\widehat{\beta}+A_2n\widehat{\alpha})2^n + A_3\widehat{\gamma}$, we get the required identity. □

As special cases of the above theorem, we give Catalan's identity of dual modified Woodall, dual modified Cullen, dual Woodall and dual Cullen numbers. Firstly, we present Catalan's identity of dual Woodall numbers.

Corollary 3.2.
(Catalan's identity for the dual modified Woodall numbers) For all integers n and m, the following identity holds

$$\begin{aligned}\mathscr{D}G_{n+m}\mathscr{D}G_{n-m}-\mathscr{D}G_n^2 = -2^{n-m}(&\widehat{\alpha}\widehat{\gamma}-\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\alpha}\widehat{\gamma}-2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\alpha}\widehat{\gamma}+2^{m+1}\widehat{\beta}\widehat{\gamma}+m\widehat{\alpha}\widehat{\gamma}-n\widehat{\alpha}\widehat{\gamma}\\ &+2^{m+n}m^2\widehat{\alpha}^2-2^{2m}m\widehat{\alpha}\widehat{\gamma}-2^{2m}n\widehat{\alpha}\widehat{\gamma}+2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n = G_n$ in theorem 3.2. □

Secondly, we give Catalan's identity of dual modified Cullen numbers.

Corollary 3.3.
(Catalan's identity for the dual modified Cullen numbers) For all integers n and m, the following identity holds

$$\mathscr{D}H_{n+m}\mathscr{D}H_{n-m}-\mathscr{D}H_n^2 = 2^{n-m}(2\widehat{\alpha}\widehat{\gamma}+2\times 2^{2m}\widehat{\alpha}\widehat{\gamma}-2\times 2^{m+1}\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = H_n$ in theorem 3.2. □

Third, we give Catalan's identity of dual Woodall numbers.

Corollary 3.4.
(Catalan's identity for the dual Woodall numbers) For all integers n and m, the following identity holds

$$\begin{aligned}\mathscr{D}R_{n+m}\mathscr{D}R_{n-m}-\mathscr{D}R_n^2 = -2^{n-m}(&\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}+2^{m+n}m^2\widehat{\alpha}^2+2^{2m}m\widehat{\alpha}\widehat{\gamma}\\ &+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n = R_n$ in theorem 3.2. □

Fourth, we give Catalan's identity of dual Cullen numbers.

Corollary 3.5.
(Catalan's identity for the dual Cullen numbers) For all integers n and m, the following identity holds

$$\begin{aligned}\mathscr{D}C_{n+m}\mathscr{D}C_{n-m}-\mathscr{D}C_n^2 = 2^{n-m}(&\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}-2^{m+n}m^2\widehat{\alpha}^2+2^{2m}m\widehat{\alpha}\widehat{\gamma}\\ &+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n = C_n$ in theorem 3.2. □

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the dual generalized Woodall sequence.

Corollary 3.6.

(Cassini's identity) For all integers n, the following identity holds

$$\mathscr{D}W_{n+1}\mathscr{D}W_{n-1} - \mathscr{D}W_n^2 = 2^{n-1}(A_2A_3(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + n\widehat{\alpha}\widehat{\gamma}) - 2^{n+1}A_2^2\widehat{\alpha}^2 + A_1A_3\widehat{\alpha}\widehat{\gamma}).$$

As special cases of Cassini's identity, we give Cassini's identity of dual modified Woodall, dual modified Cullen, dual Woodall and dual Cullen numbers. Firstly, we present Cassini's identity of dual modified Woodall numbers.

Corollary 3.7.

(Cassini's identity of dual modified Woodall numbers) For all integers n, the following identity holds

$$\mathscr{D}G_{n+1}\mathscr{D}G_{n-1} - \mathscr{D}G_n^2 = 2^{n-1}(2\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} - 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Secondly, we give Cassini's identity of dual modified Cullen numbers.

Corollary 3.8.

(Cassini's identity of dual modified Cullen numbers) For all integers n, the following identity holds

$$\mathscr{D}H_{n+1}\mathscr{D}H_{n-1} - \mathscr{D}H_n^2 = 2^n\widehat{\alpha}\widehat{\gamma}.$$

Fourth, we give Cassini's identity of dual Woodall numbers.

Corollary 3.9.

(Cassini's identity of dual Woodall numbers) For all integers n, the following identity holds

$$\mathscr{D}R_{n+1}\mathscr{D}R_{n-1} - \mathscr{D}R_n^2 = -2^{n-1}(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} + 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Third, we give Cassini's identity of dual Cullen numbers.

Corollary 3.10.

(Cassini's identity of dual Cullen numbers) For all integers n, the following identity holds

$$\mathscr{D}C_{n+1}\mathscr{D}C_{n-1} - \mathscr{D}C_n^2 = 2^{n-1}(3\widehat{\alpha}\widehat{\gamma} + \widehat{\beta}\widehat{\gamma} - 2^{n+1}\widehat{\alpha}^2 + n\widehat{\alpha}\widehat{\gamma}).$$

Theorem 3.3.

For all integers m, n, G_n is woodall numbers, the following identity is true:

$$\mathscr{D}W_{n+m} = \mathscr{D}W_nG_{m+1} + \mathscr{D}W_{n-1}(-8G_m + 4G_{m-1}) + 4\mathscr{D}W_{n-2}G_m.$$

Proof. The identity (theorem 3.3) can be proved by mathematical induction on m. First of all, we assume that $m \geq 0$ and $n \geq 0$. If $m = 0$ we get

$$\mathscr{D}W_n = \mathscr{D}W_nG_1 + \mathscr{D}W_{n-1}(-8G_0 + 4G_{-1}) + 4\mathscr{D}W_{n-2}G_0$$

which is true by seeing that $G_{-1} = 0$, $G_{-2} = \frac{1}{4}$, $G_{-3} = \frac{1}{2}$. We assume that the identity given holds for $m = k$. For $m = k+1$, we get

$$\begin{aligned}
\mathscr{D}W_{(k+1)+n} &= 5\mathscr{D}W_{n+k} - 8\mathscr{D}W_{n+k-1} + 4\mathscr{D}W_{n+k-2} \\
&= 5(\mathscr{D}W_nG_{k+1} + \mathscr{D}W_{n-1}(-8G_k + 4G_{k-1}) + 4\mathscr{D}W_{n-2}G_k) \\
&\quad -8(\mathscr{D}W_nG_k + \mathscr{D}W_{n-1}(-8G_{k-1} + 4G_{k-2}) + 4\mathscr{D}W_{n-2}G_{k-1}) \\
&\quad +4(\mathscr{D}W_nG_{k-1} + \mathscr{D}W_{n-1}(-8G_{k-2} + 4G_{k-3}) + 4\mathscr{D}W_{n-2}G_{k-2}) \\
&= \mathscr{D}W_n(5G_{k+1} - 8G_k + 4G_{k-1}) + \mathscr{D}W_{n-1}(-8(5G_k - 8G_{k-1} + 4G_{k-2}) \\
&\quad +4(5G_{k-1} - 8G_{k-2} + 4G_{k-3})) + 4\mathscr{D}W_{n-2}(5G_k - 8G_{k-1} + 4G_{k-2}) \\
&= \mathscr{D}W_nG_{k+2} + \mathscr{D}W_{n-1}(-8G_{k+1} + 4G_k) + 4\mathscr{D}W_{n-2}G_{k+1} \\
&= \mathscr{D}W_nG_{(k+1)+1} + \mathscr{D}W_{n-1}(-8G_{(k+1)} + 4G_{(k+1)-1}) + 4\mathscr{D}W_{n-2}G_{(k+1)}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (theorem 3.3). Similarly, we can show for $m < 0$ and $n < 0$. □

4. Linear Sums

In this section, we give the summation formulas of the dual generalized Woodall numbers with positive and negatif subscripts. Now, we present the summation formulas of the generalized Woodall numbers.

Proposition 4.1.
For the generalized Woodall numbers, we have the following formulas:

- $\sum_{k=0}^{n} W_k = \frac{1}{2}W_2(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_1(8n-2^{n+1}(3n-5)+2^{n+2}(3n-8)+22)+W_0(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9).$
- $\sum_{k=0}^{n} W_{k+1} = \frac{1}{2}W_2(2n+2^{n+3}(n-1)-2^{n+2}n+8)-\frac{1}{2}W_1(8n-2^{n+2}(3n-2)+2^{n+3}(3n-5)+30)+W_0(4n-2^{n+2}(n-1)+2^{n+3}(n-2)+12).$
- $\sum_{k=0}^{n} W_{k+2} = \frac{1}{2}W_2(2n-2^{n+3}(n+1)+2^{n+4}n+10)+W_0(4n+2^{n+4}(n-1)-2^{n+3}n+16)-\frac{1}{2}W_1(8n-2^{n+3}(3n+1)+2^{n+4}(3n-2)+40).$
- $\sum_{k=0}^{n} W_{k+3} = W_0(4n-2^{n+4}(n+1)+2^{n+5}n+20)-\frac{1}{2}W_1(8n+2^{n+5}(3n+1)-2^{n+4}(3n+4)+48)+\frac{1}{2}W_2(2n-2^{n+4}(n+2)+2^{n+5}(n+1)+10).$

Proof. For the proof, see Soykan [32]. □

Proposition 4.2.
For the generalized Woodall numbers, we have the following formulas:

- $\sum_{k=0}^{n} W_{2k} = \frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)-\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)+2^{2n+4}(6n-8)+120)+\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32).$
- $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)+2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64).$
- $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{9}W_0(36n-2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4)+2^{2n+6}(6n-2)+192)+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n+50).$
- $\sum_{k=0}^{n} W_{2k+3} = \frac{1}{18}W_2\left((18n-2^{2n+5}(2n+3)+2^{2n+7}(2n+1)+58\right)-\frac{1}{18}W_1(72n+2^{2n+7}(6n+1)-2^{2n+5}(6n+7)+240)+\frac{1}{9}W_0(36n-2^{2n+5}(2n+2)+2\times 2^{2n+7}n+100).$
- $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{18}W_2(18n-2^{2n+6}(2n+4)+2^{2n+8}(2n+2)+50)+\frac{1}{9}W_0(36n-2^{2n+6}(2n+3)+2^{2n+8}(2n+1)+116)-\frac{1}{18}W_1(72n+2^{2n+8}(6n+4)-2^{2n+6}(6n+10)+264).$

Proof. For the proof, see Soykan [32]. □

Proposition 4.3.
For the generalized Woodall numbers, we have the following formulas:

- $\sum_{k=0}^{n} W_{-k} = 4W_0(n+\frac{1}{2^{n+1}}(n+4)-\frac{1}{2^{n+2}}(n+3)-1)+2W_1(\frac{1}{2^{n+2}}(3n+8)-2n-\frac{1}{2^{n+1}}(3n+11)+\frac{7}{2})+2W_2(\frac{1}{2}n+\frac{1}{2^{n+1}}(n+3)-\frac{1}{2^{n+2}}(n+2)-1).$
- $\sum_{k=0}^{n} W_{-k+1} = 2W_2(\frac{1}{2}n+\frac{1}{2^n}(n+2)-\frac{1}{2^{n+1}}(n+1)-\frac{3}{2})+4W_0(n+\frac{1}{2^n}(n+3)-\frac{1}{2^{n+1}}(n+2)-2)+2W_1(\frac{1}{2^{n+1}}(3n+5)-2n-\frac{1}{2^n}(3n+8)+6).$
- $\sum_{k=0}^{n} W_{-k+2} = 2W_2(\frac{1}{2}n+2^{1-n}(n+1)-\frac{1}{2^n}n-\frac{3}{2})+4W_0(n-\frac{1}{2^n}(n+1)+2^{1-n}(n+2)-3)-2W_1(2n+2^{1-n}(3n+5)-\frac{1}{2^n}(3n+2)-8).$
- $\sum_{k=0}^{n} W_{-k+3} = 2W_2(\frac{1}{2}n+2^{2-n}n-2^{1-n}(n-1)+\frac{1}{2})+2W_1(2^{1-n}(3n-1)-2n-2^{2-n}(3n+2)+6)+4W_0(n-2^{1-n}n+2^{2-n}(n+1)-3).$

Proof. For the proof, see Soykan [section 5]. □

Proposition 4.4.
For the generalized Woodall numbers, we have the following formulas:

- $\sum_{k=0}^{n} W_{-2k} = \frac{8}{9}W_1(\frac{1}{2^{2n+4}}(6n+8) - \frac{9}{2}n - \frac{1}{2^{2n+2}}(6n+14)+3) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+2}}(2n+5) - \frac{1}{2^{2n+4}}(2n+3) - \frac{1}{2}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+2}}(2n+4) - \frac{1}{2^{2n+4}}(2n+2) - \frac{7}{8})$.
- $\sum_{k=0}^{n} W_{-2k+1} = \frac{8}{9}W_1(\frac{1}{2^{2n+3}}(6n+5) - \frac{9}{2}n - \frac{1}{2^{2n+1}}(6n+11)+6) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+1}}(2n+4) - \frac{1}{2^{2n+3}}(2n+2) - \frac{7}{4}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+1}}(2n+3) - \frac{1}{2^{2n+3}}(2n+1) - \frac{11}{8})$.
- $\sum_{k=0}^{n} W_{-2k+2} = \frac{8}{9}W_2(\frac{9}{8}n - \frac{2}{2^{2n+2}}n + \frac{1}{2^{2n}}(2n+2) - \frac{7}{8}) - \frac{16}{9}W_0(\frac{1}{2^{2n+2}}(2n+1) - \frac{9}{4}n - \frac{1}{2^{2n}}(2n+3) + \frac{11}{4}) + \frac{8}{9}W_1(\frac{1}{2^{2n+2}}(6n+2) - \frac{9}{2}n - \frac{1}{2^{2n}}(6n+8) + \frac{15}{2})$.
- $\sum_{k=0}^{n} W_{-2k+3} = \frac{8}{9}W_1(\frac{1}{2^{2n+1}}(6n-1) - \frac{9}{2}n - 2^{1-2n}(6n+5) + \frac{3}{2}) + \frac{8}{9}W_2(\frac{9}{8}n - \frac{1}{2^{2n+1}}(2n-1) + 2^{1-2n}(2n+1) + \frac{25}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{1-2n}(2n+2) - \frac{2}{2^{2n+1}}n - \frac{7}{4})$.
- $\sum_{k=0}^{n} W_{-2k+4} = \frac{8}{9}W_2(\frac{9}{8}n + 2\times 2^{2-2n}n - \frac{1}{2^{2n}}(2n-2) + \frac{137}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{2-2n}(2n+1) - \frac{1}{2^{2n}}(2n-1) + \frac{25}{4}) - \frac{8}{9}W_1(\frac{9}{2}n + 2^{2-2n}(6n+2) - \frac{1}{2^{2n}}(6n-4) + \frac{57}{2})$.

Proof. For the proof, see Soykan [32]. □

Next, we will introduce the formulas which give the summation of the dual generalized Woodall numbers in the following theorem.

Theorem 4.1.
For $n \geq 0$, dual generalized Woodall numbers have the following formulas:

(a) $\sum_{k=0}^{n} \mathcal{D}W_k = (3+n-3\times 2^n + 2^n n + 4\varepsilon + \varepsilon n - 2^{n+2}\varepsilon + 2^{n+1}\varepsilon n)W_2 + (-11-4n+11\times 2^n - 3\times 2^n n - 15\varepsilon - 4\varepsilon n + 2^{n+4}\varepsilon - 3\times 2^{n+1}\varepsilon n)W_1 + (9+4n-2^{n+3}+2^{n+1}n+12\varepsilon+4\varepsilon n - 3\times 2^{n+2}\varepsilon + 2^{n+2}\varepsilon n)W_0$.

(b) $\sum_{k=0}^{n} \mathcal{D}W_{2k} = (\frac{16}{9} + n - \frac{1}{9}2^{2n+4} + \frac{1}{3}2^{2n+2}n + \frac{20}{9}\varepsilon + \varepsilon n - \frac{5}{9}2^{2n+2}\varepsilon + \frac{1}{3}2^{2n+3}\varepsilon n)W_2 + (-\frac{20}{3} - 4n + \frac{5}{3}2^{2n+2} - 2^{2n+2}n - \frac{25}{3}\varepsilon + \frac{7}{3}2^{2n+2}\varepsilon - 4\varepsilon n - 2^{2n+3}\varepsilon n)W_1 + (\frac{53}{9} - \frac{11}{9}2^{2n+2} + 4n + \frac{1}{3}2^{2n+3}n + \frac{64}{9}\varepsilon - \frac{1}{9}2^{2n+6}\varepsilon + 4\varepsilon n + \frac{1}{3}2^{2n+4}\varepsilon n)W_0$.

(c) $\sum_{k=0}^{n} \mathcal{D}W_{2k+1} = (\frac{20}{9} - \frac{5}{9}2^{2n+2} + n + \frac{1}{3}2^{2n+3}n + \frac{25}{9}\varepsilon - \frac{1}{9}2^{2n+4}\varepsilon + \varepsilon n + \frac{1}{3}2^{2n+4}\varepsilon n)W_2 + (-\frac{25}{3} + \frac{7}{3}2^{2n+2} - 4n - 2^{2n+3}n + \frac{1}{3}2^{2n+5}\varepsilon - \frac{32}{3}\varepsilon - 4\varepsilon n - 2^{2n+4}\varepsilon n)W_1 + (\frac{64}{9} - \frac{1}{9}2^{2n+6} + 4n + \frac{1}{3}2^{2n+4}n + \frac{80}{9}\varepsilon - \frac{5}{9}2^{2n+4}\varepsilon + 4\varepsilon n + \frac{1}{3}2^{2n+5}\varepsilon n)W_0$.

Proof. Proof can be obtained by using proposition 4.4.

(a) We can derive the following using the formulas in proposition 4.1.

$$\sum_{k=0}^{n} \mathcal{D}W_k = \sum_{k=0}^{n} W_k + \varepsilon \sum_{k=0}^{n} W_{k+1}.$$

$$\begin{aligned}\sum_{k=0}^{n} \mathcal{D}W_k = {} & \frac{1}{2}W_2(2n - 2^{n+1}(n-1) + 2^{n+2}(n-2)+6) - \frac{1}{2}W_1(8n - 2^{n+1}(3n-5) + 2^{n+2}(3n-8)+22) \\ & + W_0(4n - 2^{n+1}(n-2) + 2^{n+2}(n-3)+9) \\ & + \varepsilon(\frac{1}{2}W_2(2n + 2^{n+3}(n-1) - 2^{n+2}n + 8) - \frac{1}{2}W_1(8n - 2^{n+2}(3n-2) + 2^{n+3}(3n-5)+30) \\ & + W_0(4n - 2^{n+2}(n-1) + 2^{n+3}(n-2)+12)).\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n} \mathcal{D}W_k = {} & (3+n-3\times 2^n + 2^n n + 4\varepsilon + \varepsilon n - 2^{n+2}\varepsilon + 2^{n+1}\varepsilon n)W_2 \\ & + (-11-4n+11\times 2^n - 3\times 2^n n - 15\varepsilon - 4\varepsilon n + 2^{n+4}\varepsilon - 3\times 2^{n+1}\varepsilon n)W_1 \\ & + (9+4n-2^{n+3}+2^{n+1}n + 12\varepsilon + 4\varepsilon n - 3\times 2^{n+2}\varepsilon + 2^{n+2}\varepsilon n)W_0.\end{aligned}$$

The proof is finished. □

(b) We can derive the following using the formulas in proposition 4.2.

$$\sum_{k=0}^{n} \mathcal{D}W_{2k} = \sum_{k=0}^{n} W_{2k} + \varepsilon \sum_{k=0}^{n} W_{2k+1}.$$

$$\sum_{k=0}^{n} \mathscr{D}W_{2k} = \frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)-\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)$$
$$+2^{2n+4}(6n-8)+120)+\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32)$$
$$+\varepsilon(\frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)$$
$$+2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)).$$

$$\sum_{k=0}^{n} \mathscr{D}W_{2k} = (\frac{16}{9}+n-\frac{1}{9}2^{2n+4}+\frac{1}{3}2^{2n+2}n+\frac{20}{9}\varepsilon+\varepsilon n-\frac{5}{9}2^{2n+2}\varepsilon+\frac{1}{3}2^{2n+3}\varepsilon n)W_2$$
$$+(-\frac{20}{3}-4n+\frac{5}{3}2^{2n+2}-2^{2n+2}n-\frac{25}{3}\varepsilon+\frac{7}{3}2^{2n+2}\varepsilon-4\varepsilon n-2^{2n+3}\varepsilon n)W_1$$
$$+(\frac{53}{9}-\frac{11}{9}2^{2n+2}+4n+\frac{1}{3}2^{2n+3}n+\frac{64}{9}\varepsilon-\frac{1}{9}2^{2n+6}\varepsilon+4\varepsilon n+\frac{1}{3}2^{2n+4}\varepsilon n)W_0.$$

The proof is completed. □

(c) We can derive the following using the formulas in proposition 4.4.

$$\sum_{k=0}^{n} \mathscr{D}W_{2k+1} = \sum_{k=0}^{n} W_{2k+1}+\varepsilon\sum_{k=0}^{n} W_{2k+2}.$$

$$\sum_{k=0}^{n} \mathscr{D}W_{2k+1} = \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)$$
$$+2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)$$
$$+\varepsilon(\frac{1}{9}W_0(36n-2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4)$$
$$+2^{2n+6}(6n-2)+192)+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n+50)).$$

$$\sum_{k=0}^{n} \mathscr{D}W_{2k+1} = (\frac{20}{9}-\frac{5}{9}2^{2n+2}+n+\frac{1}{3}2^{2n+3}n+\frac{25}{9}\varepsilon-\frac{1}{9}2^{2n+4}\varepsilon+\varepsilon n+\frac{1}{3}2^{2n+4}\varepsilon n)W_2$$
$$+(-\frac{25}{3}+\frac{7}{3}2^{2n+2}-4n-2^{2n+3}n+\frac{1}{3}2^{2n+5}\varepsilon-\frac{32}{3}\varepsilon-4\varepsilon n-2^{2n+4}\varepsilon n)W_1$$
$$+(\frac{64}{9}-\frac{1}{9}2^{2n+6}+4n+\frac{1}{3}2^{2n+4}n+\frac{80}{9}\varepsilon-\frac{5}{9}2^{2n+4}\varepsilon+4\varepsilon n+\frac{1}{3}2^{2n+5}\varepsilon n)W_0.$$

The proof is finished. □

As a first special case of the above theorem, we have the following summation formulas for dual Woodall numbers:

Corollary 4.1.
For $n \geq 0$, dual modified Woodall numbers have the following properties:

(a) $\sum_{k=0}^{n} \mathscr{D}G_k = 4+n+2^{n+1}n-2^{n+2}+\varepsilon(5-5\times 2^{n+2}+n+2^{n+4}+2^{n+2}n).$

(b) $\sum_{k=0}^{n} \mathscr{D}G_{2k} = \frac{20}{9}+n+\frac{2}{3}2^{2n+2}n+\frac{5}{3}2^{2n+2}-\frac{5}{9}2^{2n+4}+\varepsilon(\frac{25}{9}-\frac{4}{9}2^{2n+2}+n+\frac{2}{3}2^{2n+3}n).$

(c) $\sum_{k=0}^{n} \mathscr{D}G_{2k+1} = \frac{25}{9}+n+\frac{2}{3}2^{2n+3}n-\frac{4}{9}2^{2n+2}+\varepsilon(\frac{29}{9}-\frac{5}{9}2^{2n+4}+\frac{1}{3}2^{2n+5}+n+\frac{2}{3}2^{2n+4}n).$

As a second special case of the above theorem, we have the following summation formulas for dual modified Cullen numbers:

Corollary 4.2.
For $n \geq 0$, dual modified Cullen numbers have the following properties:

(a) $\sum_{k=0}^{n} \mathscr{D}H_k = -1+n-6\times 2^n n-3\times 2^{n+3}+3\times 2^{n+1}n+28\times 2^n+\varepsilon(-3-18\times 2^{n+2}+5\times 2^{n+4}+n-6\times 2^{n+1}n+3\times 2^{n+2}n).$

(b) $\sum_{k=0}^{n}\mathscr{D}H_{2k}=\frac{1}{3}+n-2^{2n+3}n+2^{2n+3}n+\frac{14}{3}2^{2n+2}-2^{2n+4}+\varepsilon(-\frac{1}{3}+\frac{20}{3}2^{2n+2}-\frac{1}{3}2^{2n+6}+n-2^{2n+4}n+2^{2n+4}n)$.

(c) $\sum_{k=0}^{n}\mathscr{D}H_{2k+1}=-\frac{1}{3}+n-2\times 2^{2n+3}n+2^{2n+4}n+\frac{20}{3}2^{2n+2}-\frac{1}{3}2^{2n+6}+\varepsilon(-\frac{5}{3}-\frac{8}{3}2^{2n+4}+\frac{5}{3}2^{2n+5}+n-2^{2n+5}n+2^{2n+5}n)$.

As a third special case of the above theorem, we have the following summation formulas for dual Woodall numbers:

Corollary 4.3.
For $n\geq 0$, dual Woodall numbers have the following properties:

(a) $\sum_{k=0}^{n}\mathscr{D}R_{k}=1-n+4\times 2^{n}n+2^{n+3}-2^{n+1}n-10\times 2^{n}+\varepsilon(1-2^{n+4}+2^{n+4}-n+2^{n+3}n-2^{n+2}n)$.

(b) $\sum_{k=0}^{n}\mathscr{D}R_{2k}=-\frac{1}{9}-n+\frac{4}{3}2^{2n+2}n-\frac{1}{3}2^{2n+3}n+\frac{26}{9}2^{2n+2}-\frac{7}{9}2^{2n+4}+\varepsilon(\frac{1}{9}-n-\frac{14}{9}2^{2n+2}+\frac{1}{9}2^{2n+6}+\frac{4}{3}2^{2n+3}n-\frac{1}{3}2^{2n+4}n)$.

(c) $\sum_{k=0}^{n}\mathscr{D}R_{2k+1}=\frac{1}{9}-n+\frac{4}{3}2^{2n+3}n-\frac{1}{3}2^{2n+4}n-\frac{14}{9}2^{2n+2}+\frac{1}{9}2^{2n+6}+\varepsilon(-\frac{1}{9}+\frac{1}{3}2^{2n+5}-\frac{2}{9}2^{2n+4}-n+\frac{4}{3}2^{2n+4}n-\frac{1}{3}2^{2n+5}n)$.

As a fourth special case of the above theorem, we have the following summation formulas for dual Cullen numbers:

Corollary 4.4.
For $n\geq 0$, dual Cullen numbers have the following properties.

(a) $\sum_{k=0}^{n}\mathscr{D}C_{k}=3+n-2^{n+3}+2^{n+1}n+6\times 2^{n}+\varepsilon(3+n+2^{n+2}n)$.

(b) $\sum_{k=0}^{n}\mathscr{D}C_{2k}=\frac{17}{9}+n+\frac{1}{3}2^{2n+3}n-\frac{2}{9}2^{2n+2}+\varepsilon(\frac{19}{9}+n+\frac{1}{9}2^{2n+3}+\frac{1}{3}2^{2n+4}n)$.

(c) $\sum_{k=0}^{n}\mathscr{D}C_{2k+1}=\frac{19}{9}+n+\frac{1}{3}2^{2n+4}n+\frac{1}{9}2^{2n+3}+\varepsilon(\frac{17}{9}+\frac{4}{9}2^{2n+4}+n+\frac{1}{3}2^{2n+5}n)$.

Next, we present the formulas which give the summation of the dual generalized Woodall numbers with negative subscripts.

Theorem 4.2.
For $n\geq 0$, dual generalized Woodall numbers have the following formulas:

(a) $\sum_{k=0}^{n}\mathscr{D}W_{-k}=(-2+\frac{2}{2^{n}}-3\varepsilon+n+\frac{3}{2^{n}}\varepsilon+\frac{1}{2\times 2^{n}}n+\varepsilon n+\frac{1}{2^{n}}\varepsilon n)W_{2}+(7-\frac{7}{2^{n}}+12\varepsilon-4n-\frac{11}{2^{n}}\varepsilon-\frac{3}{2\times 2^{n}}n-4\varepsilon n-\frac{3}{2^{n}}\varepsilon n)W_{1}+(-4+\frac{5}{2^{n}}-8\varepsilon+4n+\frac{8}{2^{n}}\varepsilon+\frac{1}{2^{n}}n+4\varepsilon n+\frac{2}{2^{n}}\varepsilon n)W_{0}$.

(b) $\sum_{k=0}^{n}\mathscr{D}W_{-2k}=(-\frac{7}{9}+\frac{7}{9\times 2^{2n}}-\frac{11}{9}\varepsilon+n+\frac{11}{9\times 2^{2n}}\varepsilon+\frac{1}{3\times 2^{2n}}n+\varepsilon n+\frac{2}{3\times 2^{2n}}\varepsilon n)W_{2}+(\frac{8}{3}-\frac{8}{3\times 2^{2n}}+\frac{16}{3}\varepsilon-4n-\frac{13}{3\times 2^{2n}}\varepsilon-\frac{1}{2^{2n}}n-4\varepsilon n-\frac{2}{2^{2n}}\varepsilon n)W_{1}+(-\frac{8}{9}+\frac{17}{9\times 2^{2n}}-\frac{28}{9}\varepsilon+4n+\frac{28}{9\times 2^{2n}}\varepsilon+\frac{2}{3\times 2^{2n}}n+4\varepsilon n+\frac{4}{3\times 2^{2n}}\varepsilon n)W_{0}$.

(c) $\sum_{k=0}^{n}\mathscr{D}W_{-2k+1}=(-\frac{11}{9}+\frac{11}{9\times 2^{2n}}-\frac{7}{9}\varepsilon+n+\frac{16}{9\times 2^{2n}}\varepsilon+\frac{2}{3\times 2^{2n}}n+\varepsilon n+\frac{4}{3\times 2^{2n}}\varepsilon n)W_{2}+(\frac{16}{3}-\frac{13}{3\times 2^{2n}}+\frac{20}{3}\varepsilon-4n-\frac{20}{3\times 2^{2n}}\varepsilon-\frac{2}{2^{2n}}n-4\varepsilon n-\frac{4}{2^{2n}}\varepsilon n)W_{1}+(-\frac{28}{9}+\frac{28}{9\times 2^{2n}}-\frac{44}{9}\varepsilon+4n+\frac{44}{9\times 2^{2n}}\varepsilon+\frac{4}{3\times 2^{2n}}n+4\varepsilon n+\frac{8}{3\times 2^{2n}}\varepsilon n)W_{0}$.

Proof. Proof can be obtained by using proposition 4.3.

(a) We can derive the following using the formulas in proposition 4.3.

$$\sum_{k=0}^{n}\mathscr{D}W_{-k}=\sum_{k=0}^{n}W_{-k}+\varepsilon\sum_{k=0}^{n}W_{-k+1}.$$

$$\begin{aligned}\sum_{k=0}^{n}\mathscr{D}W_{-k}&=4W_{0}(n+\frac{1}{2^{n+1}}(n+4)-\frac{1}{2^{n+2}}(n+3)-1)+2W_{1}(\frac{1}{2^{n+2}}(3n+8)-2n-\frac{1}{2^{n+1}}(3n+11)+\frac{7}{2})\\&+2W_{2}(\frac{1}{2}n+\frac{1}{2^{n+1}}(n+3)-\frac{1}{2^{n+2}}(n+2)-1)\\&+\varepsilon(2W_{2}(\frac{1}{2}n+\frac{1}{2^{n}}(n+2)-\frac{1}{2^{n+1}}(n+1)-\frac{3}{2})+4W_{0}(n+\frac{1}{2^{n}}(n+3)-\frac{1}{2^{n+1}}(n+2)-2)\\&+2W_{1}(\frac{1}{2^{n+1}}(3n+5)-2n-\frac{1}{2^{n}}(3n+8)+6)).\end{aligned}$$

$$\sum_{k=0}^{n}\mathscr{D}W_{-k} = (-2+\frac{2}{2^n}-3\varepsilon+n+\frac{3}{2^n}\varepsilon+\frac{1}{2\times 2^n}n+\varepsilon n+\frac{1}{2^n}\varepsilon n)W_2$$
$$+(7-\frac{7}{2^n}+12\varepsilon-4n-\frac{11}{2^n}\varepsilon-\frac{3}{2\times 2^n}n-4\varepsilon n-\frac{3}{2^n}\varepsilon n)W_1$$
$$+(-4+\frac{5}{2^n}-8\varepsilon+4n+\frac{8}{2^n}\varepsilon+\frac{1}{2^n}n+4\varepsilon n+\frac{2}{2^n}\varepsilon n)W_0.$$

This proves (a). We can be prove (b) and (c) similarly way using proposition 4.4. □

As a first special case of the above theorem, we have the following summation formulas for dual modified Woodall numbers:

Corollary 4.5.
For $n \geq 0$, dual modified Woodall numbers have the following properties:

(a) $\sum_{k=0}^{n}\mathscr{D}G_{-k} = -3+n+\frac{n+3}{2^n}+\varepsilon(-3+n+\frac{2n+4}{2^n})$.

(b) $\sum_{k=0}^{n}\mathscr{D}G_{-2k} = -\frac{11}{9}+n+\frac{11+6n}{9\times 2^{2n}}+\varepsilon(-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\mathscr{D}G_{-2k+1} = -\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}}+\varepsilon(\frac{25}{9}+n+\frac{20+24n}{9\times 2^{2n}})$.

As a second special case of the above theorem, we have the following summation formulas for dual modified Cullen numbers:

Corollary 4.6.
For $n \geq 0$, dual modified Cullen numbers have the following properties:

(a) $\sum_{k=0}^{n}\mathscr{D}H_{-k} = 5+n-\frac{2}{2^n}+\varepsilon(9-\frac{4}{2^n}+n)$.

(b) $\sum_{k=0}^{n}\mathscr{D}H_{-2k} = \frac{11}{3}+n-\frac{2}{3\times 2^{2n}}+\varepsilon(\frac{19}{3}-\frac{4}{3\times 2^{2n}}+n)$.

(c) $\sum_{k=0}^{n}\mathscr{D}H_{-2k+1} = \frac{19}{3}+n-\frac{4}{3\times 2^{2n}}+\varepsilon(\frac{35}{3}-\frac{8}{3\times 2^{2n}}+n)$.

As a third special case of the above theorem, we have the following summation formulas for dual Woodall numbers:

Corollary 4.7.
For $n \geq 0$, dual Woodall numbers have the following properties:

(a) $\sum_{k=0}^{n}\mathscr{D}R_{-k} = -3-n+\frac{2+n}{2^n}+\varepsilon(-1-n+\frac{2+2n}{2^n})$.

(b) $\sum_{k=0}^{n}\mathscr{D}R_{-2k} = -\frac{17}{9}-n+\frac{8+6n}{9\times 2^{2n}}+\varepsilon(-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\mathscr{D}R_{-2k+1} = -\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}}+\varepsilon(\frac{55}{9}-n+\frac{8+24n}{9\times 2^{2n}})$.

As a fourth special case of the above theorem, we have the following summation formulas for dual Cullen numbers:

Corollary 4.8.
For $n \geq 0$, dual Cullen numbers have the following properties:

(a) $\sum_{k=0}^{n}\mathscr{D}C_{-k} = -1+n+\frac{2+n}{2^n}+\varepsilon(1+\frac{2+2n}{2^n}+n)$.

(b) $\sum_{k=0}^{n}\mathscr{D}C_{-2k} = \frac{1}{9}+n+\frac{8+6n}{9\times 2^{2n}}+\varepsilon(\frac{17}{9}+\frac{10+12n}{9\times 2^{2n}}+n)$.

(c) $\sum_{k=0}^{n}\mathscr{D}C_{-2k+1} = \frac{17}{9}+n+\frac{10+12n}{9\times 2^{2n}}+\varepsilon(\frac{73}{9}+\frac{8+24n}{9\times 2^{2n}}+n)$.

5. Matrices related with Dual Generalized Woodall Numbers

In this section, we present matrices related with dual generalized Woodall numbers.
Now, $\{G_n\}$ defined by the third-order recurrence relation as follows

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3} \text{ with the initial conditions } G_0 = 0,\ G_1 = 1,\ G_2 = 5.$$

We present the square matrix A of order 3 as

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Then, we give the following Lemma.

Lemma 5.1.
For all integers n the following identity is true.

$$\begin{pmatrix} \mathcal{D}W_{n+2} \\ \mathcal{D}W_{n+1} \\ \mathcal{D}W_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix}.$$

Proof. First, we suppose that $n \geq 0$. (lemma 5.1) can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \mathcal{D}W_{k+2} \\ \mathcal{D}W_{k+1} \\ \mathcal{D}W_k \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix}$$

For $n = k+1$, we get

$$\begin{aligned} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix} &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathcal{D}W_2 \\ \mathcal{D}W_1 \\ \mathcal{D}W_0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{D}W_{k+2} \\ \mathcal{D}W_{k+1} \\ \mathcal{D}W_k \end{pmatrix} \\ &= \begin{pmatrix} 5\mathcal{D}W_{k+2} - 8\mathcal{D}W_{k+1} + 4\mathcal{D}W_k \\ \mathcal{D}W_{k+2} \\ \mathcal{D}W_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{D}W_{k+3} \\ \mathcal{D}W_{k+2} \\ \mathcal{D}W_{k+1} \end{pmatrix}. \end{aligned}$$

If we suppose that $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed.

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix}$$

For the proof see [35].

Theorem 5.1.
If we define the matrices $N_{\mathscr{D}W}$ and $E_{\mathscr{D}W}$ as follow.

$$N_{\mathscr{D}W}=\begin{pmatrix}\mathscr{D}W_2 & \mathscr{D}W_1 & \mathscr{D}W_0\\ \mathscr{D}W_1 & \mathscr{D}W_0 & \mathscr{D}W_{-1}\\ \mathscr{D}W_0 & \mathscr{D}W_{-1} & \mathscr{D}W_{-2}\end{pmatrix},\ E_{\mathscr{D}W}=\begin{pmatrix}\mathscr{D}W_{n+2} & \mathscr{D}W_{n+1} & \mathscr{D}W_n\\ \mathscr{D}W_{n+1} & \mathscr{D}W_n & \mathscr{D}W_{n-1}\\ \mathscr{D}W_n & \mathscr{D}W_{n-1} & \mathscr{D}W_{n-2}\end{pmatrix}.$$

then the following identity is true:

$$A^nN_{\mathscr{D}W}=E_{\mathscr{D}W}.$$

Proof. We can use the following identities for the proof.

$$A^nN_{\mathscr{D}W}=\begin{pmatrix}G_{n+1} & -8G_n+4G_{n-1} & 4G_n\\ G_n & -8G_{n-1}+4G_{n-2} & 4G_{n-1}\\ G_{n-1} & -8G_{n-2}+4G_{n-3} & 4G_{n-2}\end{pmatrix}\begin{pmatrix}\mathscr{D}W_2 & \mathscr{D}W_1 & \mathscr{D}W_0\\ \mathscr{D}W_1 & \mathscr{D}W_0 & \mathscr{D}W_{-1}\\ \mathscr{D}W_0 & \mathscr{D}W_{-1} & \mathscr{D}W_{-2}\end{pmatrix},$$
$$=\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}$$

where

$$\begin{aligned}
b_{11}&=\mathscr{D}W_2G_{n+1}+\mathscr{D}W_1(-8G_n+4G_{n-1})+\mathscr{D}W_04G_n,\\
b_{12}&=\mathscr{D}W_1G_{n+1}+\mathscr{D}W_0(-8G_n+4G_{n-1})+\mathscr{D}W_{-1}4G_n,\\
b_{13}&=\mathscr{D}W_0G_{n+1}+\mathscr{D}W_{-1}(-8G_n+4G_{n-1})+\mathscr{D}W_{-2}4G_n,\\
b_{21}&=\mathscr{D}W_2G_n+\mathscr{D}W_1(-8G_n+4G_{n-1})+\mathscr{D}W_04G_{n-1},\\
b_{22}&=\mathscr{D}W_1G_n+\mathscr{D}W_0(-8G_n+4G_{n-1})+\mathscr{D}W_{-1}4G_{n-1},\\
b_{23}&=\mathscr{D}W_0G_n+\mathscr{D}W_{-1}(-8G_n+4G_{n-1})+\mathscr{D}W_{-2}4G_{n-1},\\
b_{31}&=\mathscr{D}W_2G_{n-1}+\mathscr{D}W_1(-8G_n+4G_{n-1})+\mathscr{D}W_04G_{n-2},\\
b_{32}&=\mathscr{D}W_1G_{n-1}+\mathscr{D}W_0(-8G_n+4G_{n-1})+\mathscr{D}W_{-1}4G_{n-2},\\
b_{33}&=\mathscr{D}W_0G_{n-1}+\mathscr{D}W_{-1}(-8G_n+4G_{n-1})+\mathscr{D}W_{-2}4G_{n-2},
\end{aligned}$$

Using the (theorem 3.3) the proof is done. □

From theorem 5.1 , we can write the following corollary.

Corollary 5.1.
We have the following identity.

(a) *If we define $N_{\mathscr{D}G}$ and $E_{\mathscr{D}G}$ as follows.*

$$N_{\mathscr{D}G}=\begin{pmatrix}\mathscr{D}G_2 & \mathscr{D}G_1 & \mathscr{D}G_0\\ \mathscr{D}G_1 & \mathscr{D}G_0 & \mathscr{D}G_{-1}\\ \mathscr{D}G_0 & \mathscr{D}G_{-1} & \mathscr{D}G_{-2}\end{pmatrix},\ E_{\mathscr{D}G}=\begin{pmatrix}\mathscr{D}G_{n+2} & \mathscr{D}G_{n+1} & \mathscr{D}G_n\\ \mathscr{D}G_{n+1} & \mathscr{D}G_n & \mathscr{D}G_{n-1}\\ \mathscr{D}G_n & \mathscr{D}G_{n-1} & \mathscr{D}G_{n-2}\end{pmatrix},$$

then we get

$$A^nN_{\mathscr{D}G}=E_{\mathscr{D}G}.$$

(b) *If we define $N_{\mathscr{D}H}$ and $E_{\mathscr{D}H}$ as follows.*

$$N_{\mathscr{D}H}=\begin{pmatrix}\mathscr{D}H_2 & \mathscr{D}H_1 & \mathscr{D}H_0\\ \mathscr{D}H_1 & \mathscr{D}H_0 & \mathscr{D}H_{-1}\\ \mathscr{D}H_0 & \mathscr{D}H_{-1} & \mathscr{D}H_{-2}\end{pmatrix},\ E_{\mathscr{D}H}=\begin{pmatrix}\mathscr{D}H_{n+2} & \mathscr{D}H_{n+1} & \mathscr{D}H_n\\ \mathscr{D}H_{n+1} & \mathscr{D}H_n & \mathscr{D}H_{n-1}\\ \mathscr{D}H_n & \mathscr{D}H_{n-1} & \mathscr{D}H_{n-2}\end{pmatrix},$$

then we get

$$A^nN_{\mathscr{D}H}=E_{\mathscr{D}H}.$$

(c) *If we define $N_{\mathscr{D}R}$ and $E_{\mathscr{D}R}$ as follows.*

$$N_{\mathscr{D}R}=\begin{pmatrix}\mathscr{D}R_2 & \mathscr{D}R_1 & \mathscr{D}R_0\\ \mathscr{D}R_1 & \mathscr{D}R_0 & \mathscr{D}R_{-1}\\ \mathscr{D}R_0 & \mathscr{D}R_{-1} & \mathscr{D}R_{-2}\end{pmatrix},\ E_{\mathscr{D}R}=\begin{pmatrix}\mathscr{D}R_{n+2} & \mathscr{D}R_{n+1} & \mathscr{D}R_n\\ \mathscr{D}R_{n+1} & \mathscr{D}R_n & \mathscr{D}R_{n-1}\\ \mathscr{D}R_n & \mathscr{D}R_{n-1} & \mathscr{D}R_{n-2}\end{pmatrix}.$$

then we get

$$A^nN_{\mathscr{D}R}=E_{\mathscr{D}R}.$$

(d) *If we define $N_{\mathscr{D}C}$ and $E_{\mathscr{D}C}$ as follows.*

$$N_{\mathscr{D}C}=\begin{pmatrix}\mathscr{D}C_2 & \mathscr{D}C_1 & \mathscr{D}C_0\\ \mathscr{D}C_1 & \mathscr{D}C_0 & \mathscr{D}C_{-1}\\ \mathscr{D}C_0 & \mathscr{D}C_{-1} & \mathscr{D}C_{-2}\end{pmatrix},\ E_{\mathscr{D}C}=\begin{pmatrix}\mathscr{D}C_{n+2} & \mathscr{D}C_{n+1} & \mathscr{D}C_n\\ \mathscr{D}C_{n+1} & \mathscr{D}C_n & \mathscr{D}C_{n-1}\\ \mathscr{D}C_n & \mathscr{D}C_{n-1} & \mathscr{D}C_{n-2}\end{pmatrix},$$

then we get

$$A^n N_{\mathscr{D}C}=E_{\mathscr{D}C}.$$

Declarations
Conflict of interest The author declares that there is no conflict of interest.

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