Int. J. Adv. Appl. Math. and Mech. 12(2) (2024) pages 35–128 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

Generalized Tetranacci Polynomials and Numbers

Research Article

Yüksel Soykan *

Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey

Volume 12, Issue 02
Received: 05 October 2024
Accepted (in revised version): 06 November 2024

Abstract: In this paper, we investigate the generalized Tetranacci polynomials and we deal with, in detail, two special cases which we call them $(r,s,t,u)-$Tetranacci and $(r,s,t,u)-$Tetranacci-Lucas polynomials. We also introduce and investigate a new sequence and its two special cases namely the generalized co-Tetranacci, $(r,s,t,u)-$co-Tetranacci and $(r,s,t,u)-$co-Tetranacci-Lucas polynomials, respectively. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these polynomial sequences. Moreover, we give some identities and matrices related to these polynomials. Furthermore, we evaluate the infinite sums of special cases of $(r,s,t,u)-$Tetranacci and $(r,s,t,u)-$Tetranacci-Lucas polynomials.

MSC: 11B37 • 11B39 • 11B83

Keywords: Tetranacci polynomials • Tetranacci-Lucas polynomials • Tetranacci numbers • co-Tetranacci polynomials

1. Introduction: Generalized Tetranacci Polynomials

Recently, there have been so many studies of the sequences of numbers and polynomials in the literature and they were widely used in many research areas, such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}$ is defined by the second-order linear recurrence sequence

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

with initial conditions $F_0 = 0$, $F_1 = 1$. A generalization of the sequence $\{F_n\}$ is sequence of Fibonacci polynomials which are defined by the second-order linear recurrence sequence

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ F_0(x) = 0,\ F_1(x) = 1, \quad n \geq 2.$$

The Fibonacci numbers are recovered by evaluating the polynomials $F_n(x)$ at $x = 1$. The Fibonacci numbers and polynomials and their generalizations have many interesting properties and applications to almost every field. For some references on special cases of second-order linear recurrence sequences of polynomials and numbers, see for instance [3–5, 10, 24, 25] for papers and [1, 2, 7–9, 11, 23] for books.

In this paper, we investigate the fourth order generalization of Fibonacci numbers and polynomials.

* *E-mail address(es):* yuksel_soykan@hotmail.com

The generalized Tetranacci polynomials (or generalized $(r(x),s(x),t(x),u(x))-$Tetranacci polynomials or x-Tetranacci polynomials or generalized $(r(x),s(x),t(x),u(x))$-polynomials or 4-step Fibonacci polynomials)

$$\{W_n(W_0(x),W_1(x),W_2(x),W_3(x);r(x),s(x),t(x),u(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} W_n(x) &= r(x)W_{n-1}(x)+s(x)W_{n-2}(x)+t(x)W_{n-3}(x)+u(x)W_{n-4}(x), \\ W_0(x) &= a(x), W_1(x)=b(x), W_2(x)=c(x), W_3(x)=d(x) \quad n\geq 4 \end{aligned} \tag{1}$$

where $W_0(x),W_1(x),W_2(x),W_3(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$, $s(x)$, $t(x)$ and $u(x)$ are polynomials with real coefficients and $u(x)\neq 0$.

Special cases of this sequence has been studied by many authors. If $r(x)=r$, $s(x)=s$, $t(x)=t$ and $u(x)=u$ are real or complex numbers then this polynomials are called as the generalized Tetranacci numbers (or generalized $(r,s,t,u)-$Tetranacci numbers or 4-step Fibonacci numbers). For some references on special cases of generalized Tetranacci polynomials and numbers, see for example, [13–22].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = -\frac{t(x)}{u(x)}W_{-(n-1)}(x)-\frac{s(x)}{u(x)}W_{-(n-2)}(x)-\frac{r(x)}{u(x)}W_{-(n-3)}(x)+\frac{1}{u(x)}W_{-(n-4)}(x)$$

for $n=1,2,3,...$ when $u(x)\neq 0$. Therefore, recurrence eq. (1) holds for all integers n. Note that for $n\geq 1$, $W_{-n}(x)$ need not to be a polynomial in the ordinary sense.

Binet's formula of generalized Tetranacci polynomials, as $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), can be calculated using its characteristic equation (auxiliary equation, polynomial) which is given as

$$z^4-r(x)z^3-s(x)z^2-t(x)z-u(x)=0. \tag{2}$$

The roots of characteristic equation of $\{W_n\}$ will be denoted as $\alpha(x)=\alpha(x,r,s,t,u)$, $\beta(x)=\beta(x,r,s,t,u)$, $\gamma(x)=\gamma(x,r,s,t,u)$, $\delta(x)=\delta(x,r,s,t,u)$.

Remark 1.1.
For the sake of simplicity throughout the rest of the paper, we use

$$W_n,r,s,t,u,W_0,W_1,W_2,W_3,\alpha,\beta,\gamma,\delta,$$

instead of

$$W_n(x),r(x),s(x),t(x),u(x),W_0(x),W_1(x),W_2(x),W_3(x),\alpha(x),\beta(x),\gamma(x),\delta(x),$$

respectively, unless otherwise stated. For example, we write

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}, \quad W_0=a,W_1=b,W_2=c,W_3=d, \quad n\geq 4$$

for the equation eq. (2). Also we write U_n,U_0,U_1,U_2,U_3 instead of $U_n(x)$ with initial conditions $U_0(x)$, $U_1(x)$, $U_2(x)$, $U_3(x)$ for any subsequence $\{U_n(x)\}$ of $\{W_n\}$.

Remark 1.2.
If r,s,t,u are real or complex numbers, then the roots of characteristic equation of $\{W_n\}$ are

$$\begin{aligned} \alpha &= z_1 = -\frac{g_1}{2}+\sqrt{\frac{g_1^2}{4}-h_1}, \\ \beta &= z_2 = -\frac{g_1}{2}-\sqrt{\frac{g_1^2}{4}-h_1}, \\ \gamma &= z_3 = -\frac{g_2}{2}+\sqrt{\frac{g_2^2}{4}-h_2}, \\ \delta &= z_4 = -\frac{g_2}{2}-\sqrt{\frac{g_2^2}{4}-h_2}, \end{aligned}$$

where

$$
\begin{aligned}
g_1 &= \frac{-r}{2} - \sqrt{\frac{r^2}{4} + s + y_1}, \\
g_2 &= \frac{-r}{2} + \sqrt{\frac{r^2}{4} + s + y_1}, \\
h_1 &= \frac{y_1}{2} + \sum\nolimits_{sign} \sqrt{\frac{y_1^2}{4} + u}, \\
h_2 &= \frac{y_1}{2} - \sum\nolimits_{sign} \sqrt{\frac{y_1^2}{4} + u}, \\
\sum\nolimits_{sign} &= \begin{cases} 1 & , \text{ if } \frac{1}{2} r y_1 - t > 0 \\ -1 & , \quad \text{otherwise} \end{cases}
\end{aligned}
$$

and y_1 as the greatest real solution of the resolvent cubic equation

$$y^3 + sy^2 + (4u + rt)y + 4su - t^2 + r^2 u = 0.$$

We have the following identities between $\alpha, \beta, \gamma, \delta$ and r, s, t, u.

Lemma 1.1.
There are close relations between the roots of characteristic eq. (2) and r, s, t, u as follows.

(a) *Arbitrary Roots Case ($\alpha, \beta, \gamma, \delta$ are arbitrary) (including Four Distinct Roots Case, i.e.:$\alpha \neq \beta \neq \gamma \neq \delta$).*

$$\begin{cases} \alpha + \beta + \gamma + \delta = r, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -s, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = t, \\ \alpha\beta\gamma\delta = -u, \end{cases} \tag{3}$$

i.e.,

$$
\begin{aligned}
&r = \alpha + \beta + \gamma + \delta, \\
&s = -(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta), \\
&t = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta, \\
&u = -\alpha\beta\gamma\delta.
\end{aligned}
$$

(b) *Three Distinct Roots Case ($\alpha \neq \beta \neq \gamma = \delta$).*

$$
\begin{aligned}
&\alpha \neq \beta \neq \gamma = \delta \\
&\quad \Leftrightarrow \\
r &= \alpha + \beta + 2\gamma, \\
s &= -\gamma^2 - 2\gamma\left(\alpha + \beta\right) - \alpha\beta, \\
t &= \gamma(2\alpha\beta + \gamma(\alpha + \beta)), \\
u &= -\alpha\beta\gamma^2.
\end{aligned}
$$

(c) *Two Distinct Roots Case ($\alpha \neq \beta = \gamma = \delta$).*

$$
\begin{aligned}
&\alpha \neq \beta = \gamma = \delta \\
&\quad \Leftrightarrow \\
r &= \alpha + 3\beta, \\
s &= -3\beta(\alpha + \beta), \\
t &= \beta^2(3\alpha + \beta), \\
u &= -\alpha\beta^3.
\end{aligned}
$$

(d) *Single Root Case* ($\alpha=\beta=\gamma=\delta=\frac{r}{4}$).

$$\begin{aligned}\alpha &= \beta=\gamma=\delta\\ &\Leftrightarrow\\ r &= 4\alpha,\ s=-6\alpha^2,\ t=4\alpha^3,\ u=-\alpha^4\\ &\Leftrightarrow\\ \frac{r}{4} &= \alpha,\ s=-\frac{3}{8}r^2,\ t=\frac{1}{16}r^3,\ u=-\frac{1}{256}r^4.\end{aligned}$$

Proof. The identities in eq. (3) are well known. In fact, just compare

$$z^4-rz^3-sz^2-tz-u=0$$

and

$$\begin{aligned}&(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)=z^4-(\alpha+\beta+\gamma+\delta)z^3\\&+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)z^2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)z+\alpha\beta\gamma\delta\\&=0\end{aligned}$$

Use eq. (3) to prove (b),(c) and (d). □

Using the roots of characteristic equation and the recurrence relation of W_n, Binet's formula can be given as follows:

Theorem 1.1.
For all integers n, Binet's formula of generalized Tetranacci polynomials is given as follows.

(a) *(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta$) Binet's formula of generalized Tetranacci polynomials is*

$$\begin{aligned}W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\\&+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\\&=A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n\end{aligned} \tag{4}$$

where

$$\begin{aligned}p_1 &= W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0,\\ p_2 &= W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0,\\ p_3 &= W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0,\\ p_4 &= W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0,\end{aligned}$$

and

$$\begin{aligned}A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)},\\ A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},\\ A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},\\ A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},\end{aligned}$$

i.e.,

$$\begin{aligned}W_n &= \frac{p_1\alpha^{n+1}}{r\alpha^3+2s\alpha^2+3t\alpha+4u}+\frac{p_2\beta^{n+1}}{r\beta^3+2s\beta^2+3t\beta+4u}\\&+\frac{p_3\gamma^{n+1}}{r\gamma^3+2s\gamma^2+3t\gamma+4u}+\frac{p_4\delta^{n+1}}{r\delta^3+2s\delta^2+3t\delta+4u}\\&=A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n\end{aligned}$$

where

$$p_1 = W_3 - (r-\alpha)W_2 + (\alpha^2 - r\alpha - s)W_1 + \frac{u}{\alpha}W_0,$$
$$p_2 = W_3 - (r-\beta)W_2 + (\beta^2 - r\beta - s)W_1 + \frac{u}{\beta}W_0,$$
$$p_3 = W_3 - (r-\gamma)W_2 + (\gamma^2 - r\gamma - s)W_1 + \frac{u}{\gamma}W_0,$$
$$p_4 = W_3 - (r-\delta)W_2 + (\delta^2 - r\delta - s)W_1 + \frac{u}{\delta}W_0,$$

and

$$A_1 = \frac{\alpha p_1}{r\alpha^3 + 2s\alpha^2 + 3t\alpha + 4u},$$
$$A_2 = \frac{\beta p_2}{r\beta^3 + 2s\beta^2 + 3t\beta + 4u},$$
$$A_3 = \frac{\gamma p_3}{r\gamma^3 + 2s\gamma^2 + 3t\gamma + 4u},$$
$$A_4 = \frac{\delta p_4}{r\delta^3 + 2s\delta^2 + 3t\delta + 4u},$$

that is,

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n,$$

where

$$A_1 = \frac{(\alpha W_3 - \alpha(r-\alpha)W_2 + \alpha(\alpha^2 - r\alpha - s)W_1 + uW_0)}{r\alpha^3 + 2s\alpha^2 + 3t\alpha + 4u},$$
$$A_2 = \frac{(\beta W_3 - \beta(r-\beta)W_2 + \beta(\beta^2 - r\beta - s)W_1 + uW_0)}{r\beta^3 + 2s\beta^2 + 3t\beta + 4u},$$
$$A_3 = \frac{(\gamma W_3 - \gamma(r-\gamma)W_2 + \gamma(\gamma^2 - r\gamma - s)W_1 + uW_0)}{r\gamma^3 + 2s\gamma^2 + 3t\gamma + 4u},$$
$$A_4 = \frac{(\delta W_3 - \delta(r-\delta)W_2 + \delta(\delta^2 - r\delta - s)W_1 + uW_0)}{r\delta^3 + 2s\delta^2 + 3t\delta + 4u}.$$

i.e.,

$$W_n = A_5\alpha^{n-1} + A_6\beta^{n-1} + A_7\gamma^{n-1} + A_8\delta^{n-1},$$

where

$$A_5 = \frac{(\alpha W_3 - \alpha(r-\alpha)W_2 + \alpha(\alpha^2 - r\alpha - s)W_1 + uW_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)},$$
$$A_6 = \frac{(\beta W_3 - \beta(r-\beta)W_2 + \beta(\beta^2 - r\beta - s)W_1 + uW_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},$$
$$A_7 = \frac{(\gamma W_3 - \gamma(r-\gamma)W_2 + \gamma(\gamma^2 - r\gamma - s)W_1 + uW_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},$$
$$A_8 = \frac{(\delta W_3 - \delta(r-\delta)W_2 + \delta(\delta^2 - r\delta - s)W_1 + uW_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.$$

(b) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = \delta$) Binet's formula of generalized Tetranacci polynomials is*

$$W_n = A_1\alpha^n + A_2\beta^n + (A_3 + A_4 n)\gamma^n$$

where

$$A_1 = \frac{W_3 - (\beta + 2\gamma)W_2 + \gamma(2\beta + \gamma)W_1 - \beta\gamma^2 W_0}{(\alpha-\gamma)^2(\alpha-\beta)},$$
$$A_2 = \frac{-W_3 + (\alpha + 2\gamma)W_2 - \gamma(2\alpha + \gamma)W_1 + \alpha\gamma^2 W_0}{(\beta-\gamma)^2(\alpha-\beta)},$$

$$A_3 = \frac{1}{(\beta-\gamma)^2(\alpha-\gamma)^2}((\alpha+\beta-2\gamma)W_3 - (\alpha^2+\beta^2-3\gamma^2+\alpha\beta)W_2 + \gamma(2\alpha^2+2\beta^2+2\alpha\beta-3\alpha\gamma-3\beta\gamma)W_1 + \alpha\beta(3\gamma^2+\alpha\beta-2\alpha\gamma-2\beta\gamma)W_0),$$

$$A_4 = \frac{W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0}{\gamma(\beta-\gamma)(\alpha-\gamma)},$$

i.e.,

$$A_1 = \frac{\alpha W_3-\alpha(r-\alpha)W_2+\alpha(\alpha^2-r\alpha-s)W_1+uW_0}{r\alpha^3+2s\alpha^2+3t\alpha+4u},$$

$$A_2 = \frac{\beta W_3-\beta(r-\beta)W_2+\beta(\beta^2-r\beta-s)W_1+uW_0}{r\beta^3+2s\beta^2+3t\beta+4u},$$

$$A_3 = \frac{1}{8\gamma^8-5r\gamma^7+(r^2+s)\gamma^6+t\gamma^5-5u\gamma^4+2ru\gamma^3+u^2}((r-4\gamma)\gamma^4W_3-(-2\gamma^2-2r\gamma+r^2+s)\gamma^4W_2+\gamma^3(2r\gamma^3+2(r^2+6s)\gamma^2+11t\gamma+12u)W_1+(-3\gamma^6+2r\gamma^5+(s-7u)\gamma^4+2ru\gamma^3-u\gamma^2+u^2)W_0),$$

$$A_4 = \frac{\gamma^2W_3-(r-\gamma)\gamma^2W_2+(-r\gamma^3-2s\gamma^2-2t\gamma-3u)W_1+u\gamma W_0}{(2r^2+3s)\gamma^3+(3t+2rs)\gamma^2+2(u+rt)\gamma+2ur}.$$

(c) *(Two Distinct Roots Case: $\alpha\neq\beta=\gamma=\delta$) Binet's formula of generalized Tetranacci polynomials is*

$$W_n = A_1\alpha^n+(A_2+A_3n+A_4n^2)\beta^n$$

where

$$A_1 = \frac{W_3-3W_2\beta+3W_1\beta^2-W_0\beta^3}{(\alpha-\beta)^3},$$

$$A_2 = \frac{-W_3+3W_2\beta-3W_1\beta^2+\alpha(\alpha^2+3\beta^2-3\alpha\beta)W_0}{(\alpha-\beta)^3},$$

$$A_3 = \frac{(\alpha-3\beta)W_3-(\alpha^2-8\beta^2+\alpha\beta)W_2+\beta(4\alpha^2-5\beta^2-5\alpha\beta)W_1-\alpha\beta^2(3\alpha-5\beta)W_0}{2\beta^2(\alpha-\beta)^2},$$

$$A_4 = \frac{-W_3+(\alpha+2\beta)W_2-\beta(2\alpha+\beta)W_1+\alpha\beta^2W_0}{2\beta^2(\alpha-\beta)},$$

i.e.,

$$A_1 = \frac{W_3-3W_2\beta+3W_1\beta^2-W_0\beta^3}{(r-4\beta)^3},$$

$$A_2 = \frac{-W_3+3W_2\beta-3W_1\beta^2+(r-3\beta)(21\beta^2-9r\beta+r^2)W_0}{(r-4\beta)^3},$$

$$A_3 = \frac{(r-6\beta)W_3+(2\beta^2+5r\beta-r^2)W_2+\beta(46\beta^2-29r\beta+4r^2)W_1+(\beta^2(14\beta-3r)(r-3\beta))W_0}{2\beta^2(r-4\beta)^2},$$

$$A_4 = \frac{-W_3+(r-\beta)W_2+\beta(5\beta-2r)W_1+(r-3\beta)\beta^2W_0}{2\beta^2(r-4\beta)}.$$

(d) *(Single Root Case: $\alpha=\beta=\gamma=\delta=\frac{r}{4}$) Binet's formula of generalized Tetranacci polynomials is*

$$W_n = (A_1+A_2n+A_3n^2+A_4n^3)\alpha^n$$

where

$$A_1 = W_0,$$

$$A_2 = \frac{2W_3-9\alpha W_2+18\alpha^2W_1-11\alpha^3W_0}{6\alpha^3},$$

$$A_3 = \frac{-W_3+4\alpha W_2-5\alpha^2W_1+2\alpha^3W_0}{2\alpha^3},$$

$$A_4 = \frac{W_3-3\alpha W_2+3\alpha^2W_1-\alpha^3W_0}{6\alpha^3},$$

i.e.,

$$W_n = (A_1 + A_2 n + A_3 n^2 + A_4 n^3)\left(\frac{r}{4}\right)^n$$

where

$$A_1 = W_0,$$
$$A_2 = \frac{1}{6}\frac{128W_3 - 144rW_2 + 72r^2W_1 - 11r^3W_0}{r^3},$$
$$A_3 = \frac{-32W_3 + 32rW_2 - 10r^2W_1 + r^3W_0}{r^3},$$
$$A_4 = \frac{1}{6}\frac{64W_3 - 48rW_2 + 12r^2W_1 - r^3W_0}{r^3}.$$

Proof.

(a) If the roots $\alpha, \beta, \gamma, \delta$ of eq. (2) are distinct, i.e., $\alpha \neq \beta \neq \gamma \neq \delta$, then (the sequences $(\alpha^n)_{n\geq 0}$, $(\beta^n)_{n\geq 0}$, $(\gamma^n)_{n\geq 0}$ and $(\delta^n)_{n\geq 0}$ are solutions of eq. (1) and the general formula of W_n is in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n$$

where the coefficients A_1, A_2, A_3 and A_4 are determined by the system of linear equations

$$W_0 = A_1 + A_2 + A_3 + A_4$$
$$W_1 = A_1\alpha + A_2\beta + A_3\gamma + A_4\delta$$
$$W_2 = A_1\alpha^2 + A_2\beta^2 + A_3\gamma^2 + A_4\delta^2$$
$$W_3 = A_1\alpha^3 + A_2\beta^3 + A_3\gamma^3 + A_4\delta^3$$

Solving these four simultaneous equations for W_0, W_1, W_2 and W_3, we obtain the required result.

(b) If eq. (2) have three distinct roots i.e., $\alpha \neq \beta \neq \gamma = \delta$, then W_n is in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + (A_3 + A_4 n)\gamma^n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4 n\gamma^n$$

where A_1, A_2, A_3 and A_4 are the polynomials whose values are determined by the values W_0, W_1, W_2 and W_3. By using the values W_0, W_1, W_2 and W_3, we obtain

$$W_0 = A_1 + A_2 + A_3$$
$$W_1 = A_1\alpha + A_2\beta + (A_3 + A_4)\gamma = A_1\alpha + A_2\beta + A_3\gamma + A_4\gamma$$
$$W_2 = A_1\alpha^2 + A_2\beta^2 + (A_3 + 2A_4)\gamma^2 = A_1\alpha^2 + A_2\beta^2 + A_3\gamma^2 + 2A_4\gamma^2$$
$$W_3 = A_1\alpha^3 + A_2\beta^3 + (A_3 + 3A_4)\gamma^3 = A_1\alpha^3 + A_2\beta^3 + A_3\gamma^3 + 3A_4\gamma^3$$

Solving these four simultaneous equations for W_0, W_1, W_2 and W_3, we get the required result.

(c) If eq. (2) have two distinct roots i.e., $\alpha \neq \beta = \gamma = \delta$, then W_n is in the following form:

$$W_n = A_1\alpha^n + (A_2 + A_3 n + A_4 n^2)\beta^n = A_1\alpha^n + A_2\beta^n + A_3 n\beta^n + A_4 n^2\beta^n$$

where A_1, A_2, A_3 and A_4 are the polynomials whose values are determined by the values W_0, W_1, W_2 and W_3. By using the values W_0, W_1, W_2 and W_3, we obtain

$$W_0 = A_1 + A_2$$
$$W_1 = A_1\alpha + (A_2 + A_3 + A_4)\beta = A_1\alpha + A_2\beta + A_3\beta + A_4\beta$$
$$W_2 = A_1\alpha^2 + (A_2 + 2A_3 + 4A_4)\beta^2 = A_1\alpha^2 + A_2\beta^2 + 2A_3\beta^2 + 4A_4\beta^2$$
$$W_2 = A_1\alpha^3 + (A_2 + A_3 \times 3 + A_4 \times 3^2)\beta^3 = A_1\alpha^3 + A_2\beta^3 + 3A_3\beta^3 + 9A_4\beta^3$$

Solving these four simultaneous equations for W_0, W_1, W_2 and W_3, we get the required result.

(d) If the roots $\alpha,\beta,\gamma,\delta$ of eq. (2) are equal, i.e., $\alpha=\beta=\gamma=\delta=\frac{r}{4}$), then W_n is in the following form:

$$W_n=(A_1+A_2n+A_3n^2+A_4n^3)\alpha^n=A_1\alpha^n+A_2n\alpha^n+A_3n^2\alpha^n+A_4n^3\alpha^n$$

where A_1, A_2, A_3 and A_4 are the polynomials whose values are determined by the values W_0, W_1, W_2 and W_3. By using the values W_0, W_1, W_2 and W_3, we obtain

$$\begin{aligned}
W_0&=A_1\\
W_1&=(A_1+A_2+A_3+A_4)\alpha=A_1\alpha+A_2\alpha+A_3\alpha+A_4\alpha\\
W_2&=(A_1+2A_2+4A_3+8A_4)\alpha^2=A_1\alpha^2+2A_2\alpha^2+4A_3\alpha^2+8A_4\alpha^2\\
W_3&=(A_1+3A_2+9A_3+27A_4)\alpha^3=A_1\alpha^3+3A_2\alpha^3+9A_3\alpha^3+27A_4\alpha^3
\end{aligned}$$

Solving these four simultaneous equations for W_0, W_1, W_2 and W_3, we get the required result. □

Remark 1.3.
Note that the Binet form of a sequence satisfying theorem 1.1 for non-negative integers is valid for all integers n (see [6], this result of Howard and Saidak [6] is even true in the case of higher-order recurrence relations).

Note that (a), (b), (c) and (d) of the above theorem 1.1 can be given as follows:

$$W_n=\begin{cases}A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n & ,\ \text{if}\quad \alpha\neq\beta\neq\gamma\neq\delta \text{ (Four Distinct Roots Case)}\\ A_1\alpha^n+A_2\beta^n+(A_3+A_4n)\gamma^n & ,\ \text{if}\quad \alpha\neq\beta\neq\gamma=\delta \text{ (Three Distinct Roots Case)}\\ A_1\alpha^n+(A_2+A_3n+A_4n^2)\beta^n & ,\ \text{if}\quad \alpha\neq\beta=\gamma=\delta \text{ (Two Distinct Roots Case)}\\ (A_1+A_2n+A_3n^2+A_4n^3)\alpha^n & ,\ \text{if}\quad \alpha=\beta=\gamma=\delta=\frac{r}{4} \text{ (Single Root Case)}\end{cases} \tag{5}$$

where each quadruple A_1, A_2, A_3, A_4 are given as in theorem 1.1 (a), (b), (c) and (d), respectively.

Lemma 1.2.

(a) *(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta$). If A_1, A_2, A_3 and A_4 are as in theorem 1.1 (a), then we have*

$$A_1+A_2+A_3+A_4=W_0.$$

(b) *(Three Distinct Roots Case: $\alpha\neq\beta\neq\gamma=\delta$). If A_1, A_2, A_3 and A_4 are as in theorem 1.1 (b), then we have*

$$A_1+A_2+A_3+A_4=\frac{W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\gamma^2(\alpha+\beta-\gamma)W_0}{\gamma(\beta-\gamma)(\alpha-\gamma)}.$$

(c) *(Two Distinct Roots Case: $\alpha\neq\beta=\gamma=\delta$). If A_1, A_2, A_3 and A_4 are as in theorem 1.1 (c), then we have*

$$\begin{aligned}A_1+A_2+A_3+A_4&=\frac{-W_3+3\beta W_2+(\alpha^2-2\beta^2-2\alpha\beta)W_1+\beta^3W_0}{\beta(\alpha-\beta)^2}\\&=\frac{-W_3+3\beta W_2+(13\beta^2-8r\beta+r^2)W_1+\beta^3W_0}{\beta(r-4\beta)^2}.\end{aligned}$$

(d) *(Single Root Case: $\alpha=\beta=\gamma=\delta=\frac{r}{4}$). If A_1, A_2, A_3 and A_4 are as in theorem 1.1 (d), then we have*

$$A_1+A_2+A_3+A_4=\frac{W_1}{\alpha}=\frac{4}{r}W_1.$$

Proof. Use theorem 1.1. □
theorem 1.1 can be given in the following form:

Theorem 1.2.
Binet's formula of generalized Tetranacci polynomials is given as follows according to the roots of characteristic equation eq. (2):

(a) *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) Binet's formula of generalized Tetranacci polynomials is*

$$
\begin{aligned}
W_n = {} & \frac{(W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\
& + \frac{(W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
& + \frac{(W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} \\
& + \frac{(W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
W_n = {} & \frac{(W_3-(r-\alpha)W_2+(\alpha^2-r\alpha-s)W_1+\frac{u}{\alpha}W_0)\alpha^{n+1}}{r\alpha^3+2s\alpha^2+3t\alpha+4u} \\
& + \frac{(W_3-(r-\beta)W_2+(\beta^2-r\beta-s)W_1+\frac{u}{\beta}W_0)\beta^{n+1}}{r\beta^3+2s\beta^2+3t\beta+4u} \\
& + \frac{(W_3-(r-\gamma)W_2+(\gamma^2-r\gamma-s)W_1+\frac{u}{\gamma}W_0)\gamma^{n+1}}{r\gamma^3+2s\gamma^2+3t\gamma+4u} \\
& + \frac{(W_3-(r-\delta)W_2+(\delta^2-r\delta-s)W_1+\frac{u}{\delta}W_0)\delta^{n+1}}{r\delta^3+2s\delta^2+3t\delta+4u}.
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
W_n = {} & \frac{(\alpha W_3-\alpha(r-\alpha)W_2+\alpha(\alpha^2-r\alpha-s)W_1+uW_0)\alpha^n}{r\alpha^3+2s\alpha^2+3t\alpha+4u} \\
& + \frac{(\beta W_3-\beta(r-\beta)W_2+\beta(\beta^2-r\beta-s)W_1+uW_0)\beta^n}{r\beta^3+2s\beta^2+3t\beta+4u} \\
& + \frac{(\gamma W_3-\gamma(r-\gamma)W_2+\gamma(\gamma^2-r\gamma-s)W_1+uW_0)\gamma^n}{r\gamma^3+2s\gamma^2+3t\gamma+4u} \\
& + \frac{(\delta W_3-\delta(r-\delta)W_2+\delta(\delta^2-r\delta-s)W_1+uW_0)\delta^n}{r\delta^3+2s\delta^2+3t\delta+4u}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
W_n = {} & \frac{(\alpha W_3-\alpha(r-\alpha)W_2+\alpha(\alpha^2-r\alpha-s)W_1+uW_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^{n-1} \\
& + \frac{(\beta W_3-\beta(r-\beta)W_2+\beta(\beta^2-r\beta-s)W_1+uW_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^{n-1} \\
& + \frac{(\gamma W_3-\gamma(r-\gamma)W_2+\gamma(\gamma^2-r\gamma-s)W_1+uW_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^{n-1} \\
& + \frac{(\delta W_3-\delta(r-\delta)W_2+\delta(\delta^2-r\delta-s)W_1+uW_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^{n-1}
\end{aligned}
$$

(b) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = \delta$) Binet's formula of generalized Tetranacci polynomials is*

$W_n = \frac{1}{(\alpha-\gamma)^2(\alpha-\beta)}(W_3-(\beta+2\gamma)W_2+\gamma(2\beta+\gamma)W_1-\beta\gamma^2W_0)\alpha^n + \frac{1}{(\beta-\gamma)^2(\alpha-\beta)}(-W_3+(\alpha+2\gamma)W_2-\gamma(2\alpha+\gamma)W_1+\alpha\gamma^2W_0)\beta^n + (\frac{1}{(\beta-\gamma)^2(\alpha-\gamma)^2}((\alpha+\beta-2\gamma)W_3-(\alpha^2+\beta^2-3\gamma^2+\alpha\beta)W_2+\gamma(2\alpha^2+2\beta^2+2\alpha\beta-3\alpha\gamma-3\beta\gamma)W_1+\alpha\beta(3\gamma^2+\alpha\beta-2\alpha\gamma-2\beta\gamma)W_0)+\frac{1}{\gamma(\beta-\gamma)(\alpha-\gamma)}(W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0)n)\gamma^n$

i.e.,

$W_n = \frac{1}{r\alpha^3+2s\alpha^2+3t\alpha+4u}(\alpha W_3-\alpha(r-\alpha)W_2+\alpha(\alpha^2-r\alpha-s)W_1+uW_0)\alpha^n + \frac{1}{r\beta^3+2s\beta^2+3t\beta+4u}(\beta W_3-\beta(r-\beta)W_2+\beta(\beta^2-r\beta-s)W_1+uW_0)\beta^n + (\frac{1}{8\gamma^8-5r\gamma^7+(r^2+s)\gamma^6+t\gamma^5-5u\gamma^4+2ru\gamma^3+u^2}((r-4\gamma)\gamma^4W_3-(-2\gamma^2-2r\gamma+r^2+s)\gamma^4W_2+\gamma^3(2r\gamma^3+2(r^2+6s)\gamma^2+11t\gamma+12u)W_1+(-3\gamma^6+2r\gamma^5+(s-7u)\gamma^4+2ru\gamma^3-u\gamma^2+u^2)W_0)+\frac{1}{(2r^2+3s)\gamma^3+(3t+2rs)\gamma^2+2(u+rt)\gamma+2ur}(\gamma^2W_3-(r-\gamma)\gamma^2W_2+(-r\gamma^3-2s\gamma^2-2t\gamma-3u)W_1+u\gamma W_0)n)\gamma^n$

(c) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = \delta$) Binet's formula of generalized Tetranacci polynomials is*

$W_n = \frac{1}{(\alpha-\beta)^3}(W_3-3W_2\beta+3W_1\beta^2-W_0\beta^3)\alpha^n+(\frac{1}{(\alpha-\beta)^3}(-W_3+3W_2\beta-3W_1\beta^2+\alpha(\alpha^2+3\beta^2-3\alpha\beta)W_0)+\frac{1}{2\beta^2(\alpha-\beta)^2}((\alpha-3\beta)W_3-(\alpha^2-8\beta^2+\alpha\beta)W_2+\beta(4\alpha^2-5\beta^2-5\alpha\beta)W_1-\alpha\beta^2(3\alpha-5\beta)W_0)n+\frac{1}{2\beta^2(\alpha-\beta)}(-W_3+(\alpha+2\beta)W_2-\beta(2\alpha+\beta)W_1+\alpha\beta^2W_0)n^2)\beta^n$

i.e.,

$W_n = \frac{1}{(r-4\beta)^3}(W_3-3W_2\beta+3W_1\beta^2-W_0\beta^3)\alpha^n+(\frac{1}{(r-4\beta)^3}(-W_3+3W_2\beta-3W_1\beta^2+(r-3\beta)(21\beta^2-9r\beta+r^2)W_0)+\frac{1}{2\beta^2(r-4\beta)^2}((r-6\beta)W_3+(2\beta^2+5r\beta-r^2)W_2+\beta(46\beta^2-29r\beta+4r^2)W_1+(\beta^2(14\beta-3r)(r-3\beta))W_0)n+\frac{1}{2\beta^2(r-4\beta)}(-W_3+(r-\beta)W_2+\beta(5\beta-2r)W_1+(r-3\beta)\beta^2W_0)n^2)\beta^n.$

(d) *(Single Root Case: $\alpha = \beta = \gamma = \delta = \frac{r}{4}$) Binet's formula of generalized Tetranacci polynomials is*

$W_n = (W_0+\frac{1}{6\alpha^3}(2W_3-9\alpha W_2+18\alpha^2W_1-11\alpha^3W_0)n+\frac{1}{2\alpha^3}(-W_3+4\alpha W_2-5\alpha^2W_1+2\alpha^3W_0)n^2+\frac{1}{6\alpha^3}(W_3-3\alpha W_2+3\alpha^2W_1-\alpha^3W_0)n^3)\alpha^n,$

i.e.,

$W_n = (W_0+\frac{1}{6r^3}(128W_3-144rW_2+72r^2W_1-11r^3W_0)n+\frac{1}{r^3}(-32W_3+32rW_2-10r^2W_1+r^3W_0)n^2+\frac{1}{6r^3}(64W_3-48rW_2+12r^2W_1-r^3W_0)n^3)\left(\frac{r}{4}\right)^n.$

Note that each of theorems 1.1 and 1.2 and eq. (5) have their advantages to present Binet's formula of generalized Tetranacci polynomials.

If some of the roots of characteristic equation is 1 then we get the following corollary as a special case of theorem 1.2.

Corollary 1.1.
Binet's formula of generalized Tetranacci polynomials is given as follows according to the roots of characteristic eq. (2):

(a) *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$)*

$$\begin{aligned} W_n = {} & \frac{(W_3-(\beta+\gamma+1)W_2+(\beta\gamma+\beta+\gamma)W_1-\beta\gamma W_0)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-1)} \\ & + \frac{(W_3-(\alpha+\gamma+1)W_2+(\alpha\gamma+\alpha+\gamma)W_1-\alpha\gamma W_0)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-1)} \\ & + \frac{(W_3-(\alpha+\beta+1)W_2+(\alpha\beta+\alpha+\beta)W_1-\alpha\beta W_0)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-1)} \\ & + \frac{(W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0)}{(1-\alpha)(1-\beta)(1-\gamma)} \end{aligned}$$

i.e.,

$$\begin{aligned} W_n = {} & \frac{(W_3-(r-\alpha)W_2+(\alpha^2-r\alpha-s)W_1+\frac{u}{\alpha}W_0)\alpha^{n+1}}{r\alpha^3+2s\alpha^2+3t\alpha+4u} \\ & + \frac{(W_3-(r-\beta)W_2+(\beta^2-r\beta-s)W_1+\frac{u}{\beta}W_0)\beta^{n+1}}{r\beta^3+2s\beta^2+3t\beta+4u} \\ & + \frac{(W_3-(r-\gamma)W_2+(\gamma^2-r\gamma-s)W_1+\frac{u}{\gamma}W_0)\gamma^{n+1}}{r\gamma^3+2s\gamma^2+3t\gamma+4u} \\ & + \frac{(W_3-(r-1)W_2+(1-r-s)W_1+uW_0)}{r+2s+3t+4u} \end{aligned}$$

i.e.,

$$\begin{aligned} W_n = {} & \frac{(\alpha W_3-\alpha(r-\alpha)W_2+\alpha(\alpha^2-r\alpha-s)W_1+uW_0)\alpha^n}{r\alpha^3+2s\alpha^2+3t\alpha+4u} \\ & + \frac{(\beta W_3-\beta(r-\beta)W_2+\beta(\beta^2-r\beta-s)W_1+uW_0)\beta^n}{r\beta^3+2s\beta^2+3t\beta+4u} \\ & + \frac{(\gamma W_3-\gamma(r-\gamma)W_2+\gamma(\gamma^2-r\gamma-s)W_1+uW_0)\gamma^n}{r\gamma^3+2s\gamma^2+3t\gamma+4u} \\ & + \frac{(W_3-(r-1)W_2+(1-r-s)W_1+uW_0)}{r+2s+3t+4u} \end{aligned}$$

i.e.,

$$W_n = \frac{(\alpha W_3 - \alpha(r-\alpha)W_2 + (-\alpha^2 + (r-1)\alpha - u)W_1 + uW_0)\alpha^n}{(r^2-r+2s)\alpha^2 + (r^2+rs-r+3t)\alpha + (4-r)u} + \frac{(\beta W_3 - \beta(r-\beta)W_2 + (-\beta^2 + (r-1)\beta - u)W_1 + uW_0)\beta^n}{(r^2-r+2s)\beta^2 + (r^2+rs-r+3t)\beta + (4-r)u} + \frac{(\gamma W_3 - \gamma(r-\gamma)W_2 + (-\gamma^2 + (r-1)\gamma - u)W_1 + uW_0)\gamma^n}{(r^2-r+2s)\gamma^2 + (r^2+rs-r+3t)\gamma + (4-r)u} + \frac{W_3 - (r-1)W_2 + (1-r-s)W_1 + uW_0}{r+2s+3t+4u}$$

i.e.,

$$W_n = \frac{(\alpha W_3 - \alpha(r-\alpha)W_2 + \alpha(\alpha^2 - r\alpha - s)W_1 + uW_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-1)}\alpha^{n-1} + \frac{(\beta W_3 - \beta(r-\beta)W_2 + \beta(\beta^2 - r\beta - s)W_1 + uW_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-1)}\beta^{n-1} + \frac{(\gamma W_3 - \gamma(r-\gamma)W_2 + \gamma(\gamma^2 - r\gamma - s)W_1 + uW_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-1)}\gamma^{n-1} + \frac{(W_3 - (r-1)W_2 + (1-r-s)W_1 + uW_0)}{(1-\alpha)(1-\beta)(1-\gamma)}.$$

(b) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = \delta = 1$)*

$W_n = \frac{1}{(\alpha-1)^2(\alpha-\beta)}(W_3 - (\beta+2)W_2 + (2\beta+1)W_1 - \beta W_0)\alpha^n + \frac{1}{(\beta-1)^2(\alpha-\beta)}(-W_3 + (\alpha+2)W_2 - (2\alpha+1)W_1 + \alpha W_0)\beta^n + (\frac{1}{(\beta-1)^2(\alpha-1)^2}((\alpha+\beta-2)W_3 - (\alpha^2+\beta^2-3+\alpha\beta)W_2 + (2\alpha^2+2\beta^2+2\alpha\beta-3\alpha-3\beta)W_1 + \alpha\beta(3+\alpha\beta-2\alpha-2\beta)W_0) + \frac{1}{(\beta-1)(\alpha-1)}(W_3 - (\alpha+\beta+1)W_2 + (\alpha\beta+\alpha+\beta)W_1 - \alpha\beta W_0)n)$

i.e.,

$W_n = \frac{1}{r\alpha^3+2s\alpha^2+3t\alpha+4u}(\alpha W_3 - \alpha(r-\alpha)W_2 + \alpha(\alpha^2 - r\alpha - s)W_1 + uW_0)\alpha^n + \frac{1}{r\beta^3+2s\beta^2+3t\beta+4u}(\beta W_3 - \beta(r-\beta)W_2 + \beta(\beta^2 - r\beta - s)W_1 + uW_0)\beta^n + (\frac{1}{r^2+2ru-5r+u^2-5u+s+t+8}((r-4)W_3 - (-2-2r+r^2+s)W_2 + (2r+2(r^2+6s)+11t+12u)W_1 + (-3+2r+(s-7u)+2ru-u+u^2)W_0) + \frac{1}{(2r^2+3s)+(3t+2rs)+2(u+rt)+2ur}(W_3 - (r-1)W_2 + (-r-2s-2t-3u)W_1 + uW_0)n)$

.

(c) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = \delta = 1$)*

$W_n = \frac{1}{(\alpha-1)^3}(W_3 - 3W_2 + 3W_1 - W_0)\alpha^n + (\frac{1}{(\alpha-1)^3}(-W_3 + 3W_2 - 3W_1 + \alpha(\alpha^2+3-3\alpha)W_0) + \frac{1}{2(\alpha-1)^2}((\alpha-3)W_3 - (\alpha^2-8+\alpha)W_2 + (4\alpha^2-5-5\alpha)W_1 - \alpha(3\alpha-5)W_0)n + \frac{1}{2(\alpha-1)}(-W_3 + (\alpha+2)W_2 - (2\alpha+1)W_1 + \alpha W_0)n^2)$

i.e.,

$W_n = \frac{1}{(r-4)^3}(W_3 - 3W_2 + 3W_1 - W_0)\alpha^n + (\frac{1}{(r-4)^3}(-W_3 + 3W_2 - 3W_1 + (r-3)(21-9r+r^2)W_0) + \frac{1}{2(r-4)^2}((r-6)W_3 + (2+5r-r^2)W_2 + (46-29r+4r^2)W_1 + ((14-3r)(r-3))W_0)n + \frac{1}{2(r-4)}(-W_3 + (r-1)W_2 + (5-2r)W_1 + (r-3)W_0)n^2).$

(d) *(Single Root Case: $\alpha = \beta = \gamma = \delta = 1 = \frac{r}{4}$)*

$W_n = \frac{1}{6}(n(n-1)(n-2)W_3 - 3n(n-1)(n-3)W_2 + 3n(n-2)(n-3)W_1 - (n-3)(n-1)(n-2)W_0).$

Proof. Use theorem 1.2. □

We have the following formula: for $n = 1,2,3,...$ we have

$$W_{-n} = \frac{1}{(-u)^n}\frac{D_1}{D_2}W_n$$

where

$D_1 = \beta^n\gamma^n\delta^n(\gamma-\delta)(\beta-\delta)(\beta-\gamma)p_1 - \alpha^n\gamma^n\delta^n(\gamma-\delta)(\alpha-\delta)(\alpha-\gamma)p_2 + \alpha^n\beta^n\delta^n(\beta-\delta)(\alpha-\delta)(\alpha-\beta)p_3 - \alpha^n\beta^n\gamma^n(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)p_4,$

$D_2 = \alpha^n(\gamma-\delta)(\beta-\delta)(\beta-\gamma)p_1 - \beta^n(\gamma-\delta)(\alpha-\delta)(\alpha-\gamma)p_2 + \gamma^n(\beta-\delta)(\alpha-\delta)(\alpha-\beta)p_3 - \delta^n(\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)p_4.$

We can also give Binet's formula of the generalized Tetranacci polynomials for the negative subscripts as follows: for $n = 1,2,3,...$ we have

$$W_{-n} = \frac{\alpha^3 - r\alpha^2 - s\alpha - t}{u(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} p_1 \alpha^{1-n} + \frac{\beta^3 - r\beta^2 - s\beta - t}{u(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} p_2 \beta^{1-n} + \frac{\gamma^3 - r\gamma^2 - s\gamma - t}{u(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} p_3 \gamma^{1-n} + \frac{\delta^3 - r\delta^2 - s\delta - t}{u(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} p_4 \delta^{1-n}.$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 1.3.
Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Tetranacci polynomials $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{1 - rz - sz^2 - tz^3 - uz^4}. \quad (6)$$

Proof. Using the definition of generalized Tetranacci polynomials and substracting $rz\sum_{n=0}^{\infty} W_n z^n$, $sz^2\sum_{n=0}^{\infty} W_n z^n$, $tz^3\sum_{n=0}^{\infty} W_n z^n$ and $uz^4\sum_{n=0}^{\infty} W_n z^n$ from $\sum_{n=0}^{\infty} W_n z^n$ we obtain

$$\begin{aligned}
&(1 - rz - sz^2 - tz^3 - uz^4)\sum_{n=0}^{\infty} W_n z^n \\
&= \sum_{n=0}^{\infty} W_n z^n - rz\sum_{n=0}^{\infty} W_n z^n - sz^2\sum_{n=0}^{\infty} W_n z^n - tz^3\sum_{n=0}^{\infty} W_n z^n - uz^4\sum_{n=0}^{\infty} W_n z^n \\
&= \sum_{n=0}^{\infty} W_n z^n - r\sum_{n=0}^{\infty} W_n z^{n+1} - s\sum_{n=0}^{\infty} W_n z^{n+2} - t\sum_{n=0}^{\infty} W_n z^{n+3} - u\sum_{n=0}^{\infty} W_n z^{n+4} \\
&= \sum_{n=0}^{\infty} W_n z^n - r\sum_{n=1}^{\infty} W_{n-1} z^n - s\sum_{n=2}^{\infty} W_{n-2} z^n - t\sum_{n=3}^{\infty} W_{n-3} z^n - u\sum_{n=4}^{\infty} W_{n-4} z^n \\
&= (W_0 + W_1 z + W_2 z^2 + W_3 z^3) - r(W_0 z + W_1 z^2 + W_2 z^3) - s(W_0 z^2 + W_1 z^3) - tW_0 z^3 \\
&\quad + \sum_{n=4}^{\infty} (W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4}) z^n \\
&= W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3.
\end{aligned}$$

Rearranging above equation, we obtain eq. (6). □

We next find Binet's formula of generalized Tetranacci polynomials $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.3.
(Binet's formula of generalized Tetranacci polynomials:Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$).

$$W_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \quad (7)$$

where

$$\begin{aligned}
q_1 &= W_0\alpha^3 + (W_1 - rW_0)\alpha^2 + (W_2 - rW_1 - sW_0)\alpha + (W_3 - rW_2 - sW_1 - tW_0), \\
q_2 &= W_0\beta^3 + (W_1 - rW_0)\beta^2 + (W_2 - rW_1 - sW_0)\beta + (W_3 - rW_2 - sW_1 - tW_0), \\
q_3 &= W_0\gamma^3 + (W_1 - rW_0)\gamma^2 + (W_2 - rW_1 - sW_0)\gamma + (W_3 - rW_2 - sW_1 - tW_0), \\
q_4 &= W_0\delta^3 + (W_1 - rW_0)\delta^2 + (W_2 - rW_1 - sW_0)\delta + (W_3 - rW_2 - sW_1 - tW_0).
\end{aligned}$$

Proof. Let

$$h(z) = 1 - rz - sz^2 - tz^3 - uz^4.$$

Then for some α, β, γ and δ we write

$$h(z) = (1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z)$$

i.e.,

$$1 - rz - sz^2 - tz^3 - tz^4 = (1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z) \tag{8}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(z)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{z}) = 1 - \frac{r}{z} - \frac{s}{z^2} - \frac{t}{z^3} - \frac{u}{z^4} = 0.$$

This implies $z^4 - rz^3 - sz^2 - tz - u = 0$. Now, by eq. (6) and eq. (8)), it follows that

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{(1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z)}.$$

Then we write

$$\begin{aligned} &\frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{(1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z)} \\ &= \frac{B_1}{(1-\alpha z)} + \frac{B_2}{(1-\beta z)} + \frac{B_3}{(1-\gamma z)} + \frac{B_4}{(1-\delta z)}. \end{aligned} \tag{9}$$

So

$$\begin{aligned} &W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3 \\ &= B_1(1-\beta z)(1-\gamma z)(1-\delta z) + B_2(1-\alpha z)(1-\gamma z)(1-\delta z) \\ &\quad + B_3(1-\alpha z)(1-\beta z)(1-\delta z) + B_3(1-\alpha z)(1-\beta z)(1-\gamma z). \end{aligned}$$

If we consider $z = \frac{1}{\alpha}$, we get $W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2} + (W_3 - rW_2 - sW_1 - tW_0)\frac{1}{\alpha^3} = B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$.This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3(W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2} + (W_3 - r2W_2 - sW_1 - t5W_0)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{W_0\alpha^3 + (W_1 - rW_0)\alpha^2 + (W_2 - rW_1 - sW_0)\alpha + (W_3 - rW_2 - sW_1 - tW_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{W_0\beta^3 + (W_1 - rW_0)\beta^2 + (W_2 - rW_1 - sW_0)\beta + (W_3 - rW_2 - sW_1 - tW_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{W_0\gamma^3 + (W_1 - rW_0)\gamma^2 + (W_2 - rW_1 - sW_0)\gamma + (W_3 - rW_2 - sW_1 - tW_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{W_0\delta^3 + (W_1 - rW_0)\delta^2 + (W_2 - rW_1 - sW_0)\delta + (W_3 - rW_2 - sW_1 - tW_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus eq. (9) can be written as

$$\sum_{n=0}^{\infty} W_n z^n = B_1(1-\alpha z)^{-1} + B_2(1-\beta z)^{-1} + B_3(1-\gamma z)^{-1} + B_4(1-\delta z)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} W_n z^n &= B_1 \sum_{n=0}^{\infty} \alpha^n z^n + B_2 \sum_{n=0}^{\infty} \beta^n z^n + B_3 \sum_{n=0}^{\infty} \gamma^n z^n + B_4 \sum_{n=0}^{\infty} \delta^n z^n \\ &= \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n) z^n. \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$W_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n$$

and then we get eq. (7). □

Note that from eq. (4) and eq. (7) we have (in the case of four distinct roots case: $\alpha \neq \beta \neq \gamma \neq \delta$)

$$\begin{aligned}
W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0 &= W_0\alpha^3+(W_1-rW_0)\alpha^2+(W_2-rW_1-sW_0)\alpha \\
&\quad +(W_3-rW_2-sW_1-tW_0), \\
W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0 &= W_0\beta^3+(W_1-rW_0)\beta^2+(W_2-rW_1-sW_0)\beta \\
&\quad +(W_3-rW_2-sW_1-tW_0), \\
W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0 &= W_0\gamma^3+(W_1-rW_0)\gamma^2+(W_2-rW_1-sW_0)\gamma \\
&\quad +(W_3-rW_2-sW_1-tW_0), \\
W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0 &= W_0\delta^3+(W_1-rW_0)\delta^2+(W_2-rW_1-sW_0)\delta \\
&\quad +(W_3-rW_2-sW_1-tW_0).
\end{aligned}$$

In this paper, we define and investigate, in detail, two special cases of the generalized Tetranacci (sequences of) polynomials $\{W_n\}$ which we call them (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas (sequences of) polynomials. (r,s,t,u)-Tetranacci (sequences of) polynomials $\{G_n\}_{n\geq 0}$ and (r,s,t,u)-Tetranacci-Lucas (sequences of) polynomials $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}
G_{n+4} &= rG_{n+3}+sG_{n+2}+tG_{n+1}+uG_n, \qquad (10)\\
& G_0=0, G_1=1, G_2=r, G_3=r^2+s,\\
H_{n+4} &= rH_{n+3}+sH_{n+2}+tH_{n+1}+uH_n, \qquad (11)\\
& H_0=4, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t.
\end{aligned}$$

Equations (10) and (11) can be written as:

$$\begin{aligned}
G_n &= rG_{n-1}+sG_{n-2}+tG_{n-3}+uG_{n-4},\\
H_n &= rH_{n-1}+sH_{n-2}+tH_{n-3}+uH_{n-4}.
\end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
G_{-n} &= -\frac{t}{u}G_{-(n-1)}-\frac{s}{u}G_{-(n-2)}-\frac{r}{u}G_{-(n-3)}+\frac{1}{u}G_{-(n-4)},\\
H_{-n} &= -\frac{t}{u}H_{-(n-1)}-\frac{s}{u}H_{-(n-2)}-\frac{r}{u}H_{-(n-3)}+\frac{1}{u}H_{-(n-4)},
\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences eq. (10)) and eq. (11) hold for all integers n.

Some special cases of (r,s,t,u) sequence $\{G_n(0,1,r,r^2+s;r,s,t,u)\}$ and Lucas (r,s,t,u) sequence $\{H_n(4,r,2s+r^2,r^3+3sr+3t;r,s,t,u)\}$ are as follows:

1. $G_n(0,1,1,2;1,1,1,1)=M_n$, Tetranacci sequence,
2. $H_n(4,1,3,7;1,1,1,1)=R_n$, Tetranacci-Lucas sequence,
3. $G_n(0,1,2,5;2,1,1,1)=P_n$, fourth-order Pell sequence,
4. $H_n(4,2,6,17;2,1,1,1)=Q_n$, fourth-order Pell-Lucas sequence.

Next, we present the first few values of the (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas (sequences of) polynomials with positive and negative subscripts:

Table 1. The first few values of the special fourth-order numbers with positive and negative subscripts.

n	0	1	2	3	4
G_n	0	1	r	r^2+s	$r^3+2sr+t$
G_{-n}		0	0	$\frac{1}{u}$	$-\frac{t}{u^2}$
H_n	4	r	$2s+r^2$	$r^3+3sr+3t$	$r^4+4r^2s+4tr+2s^2+4u$
H_{-n}		$-\frac{t}{u}$	$\frac{1}{u^2}(t^2-2su)$	$-\frac{1}{u^3}(t^3-3stu+3ru^2)$	$\frac{1}{u^4}(2s^2u^2-4st^2u+t^4+4rtu^2+4u^3)$

Note that (in the four distinct roots case: $\alpha \neq \beta \neq \gamma \neq \delta$) for all n we have

$$G_{-n}=\frac{\Lambda_1}{\Lambda_2}G_n, \quad n\geq 1,$$

where

$$\begin{aligned}
\Lambda_1 &= (\delta-\gamma)(\beta-\delta)(\beta-\gamma)\alpha^{2-n}+(\gamma-\delta)(\alpha-\delta)(\alpha-\gamma)\beta^{2-n}+(\delta-\beta)(\alpha-\delta)(\alpha-\beta)\gamma^{2-n}+(\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)\delta^{2-n},\\
\Lambda_2 &= (\delta-\gamma)(\beta-\delta)(\beta-\gamma)\alpha^{n+2}+(\gamma-\delta)(\alpha-\delta)(\alpha-\gamma)\beta^{n+2}+(\delta-\beta)(\alpha-\delta)(\alpha-\beta)\gamma^{n+2}+(\beta-\gamma)(\alpha-\gamma)(\alpha-\beta)\delta^{n+2}.
\end{aligned}$$

Lemma 1.3 gives the following results as particular examples (generating functions of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials).

Corollary 1.2.
Generating functions of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials are

$$\sum_{n=0}^{\infty} G_n x^n = \frac{x}{1-rx-sx^2-tx^3-ux^4},$$
$$\sum_{n=0}^{\infty} H_n x^n = \frac{4-3rx-2sx^2-tx^3}{1-rx-sx^2-tx^3-ux^4},$$

respectively.

Proof. In lemma 1.3, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r, G_3 = r^2+s$ and $W_n = H_n$ with $H_0 = 4, H_1 = r, H_2 = 2s+r^2, H_3 = r^3+3sr+3t$, respectively. □

For all integers n, Binet's formula of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials (using initial conditions in eqs. (10) and (11)) can be expressed as follows:

Theorem 1.4.

(a) *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$). For all integers n, Binet's formulas of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials are*

$$G_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$
$$= \frac{\alpha^{n+3}}{r\alpha^3+2s\alpha^2+3t\alpha+4u} + \frac{\beta^{n+3}}{r\beta^3+2s\beta^2+3t\beta+4u} + \frac{\gamma^{n+3}}{r\gamma^3+2s\gamma^2+3t\gamma+4u} + \frac{\delta^{n+3}}{r\delta^3+2s\delta^2+3t\delta+4u}$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n.$$

(b) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = \delta$) For all integers n, Binet's formulas of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials are*

$G_n = \frac{1}{(\alpha-\gamma)^2(\alpha-\beta)}((r^2+s)-(\beta+2\gamma)r+\gamma(2\beta+\gamma))\alpha^n + \frac{1}{(\beta-\gamma)^2(\alpha-\beta)}(-(r^2+s)+(\alpha+2\gamma)r-\gamma(2\alpha+\gamma))\beta^n + (\frac{1}{(\beta-\gamma)^2(\alpha-\gamma)^2}((\alpha+\beta-2\gamma)(r^2+s)-(\alpha^2+\beta^2-3\gamma^2+\alpha\beta)r+\gamma(2\alpha^2+2\beta^2+2\alpha\beta-3\alpha\gamma-3\beta\gamma)) + \frac{1}{\gamma(\beta-\gamma)(\alpha-\gamma)}((r^2+s)-(\alpha+\beta+\gamma)r+(\alpha\beta+\alpha\gamma+\beta\gamma))n)\gamma^n$

i.e.,

$G_n = \frac{1}{r\alpha^3+2s\alpha^2+3t\alpha+4u}(\alpha(r^2+s)-\alpha(r-\alpha)r+\alpha(\alpha^2-r\alpha-s))\alpha^n + \frac{1}{r\beta^3+2s\beta^2+3t\beta+4u}(\beta(r^2+s)-\beta(r-\beta)r+\beta(\beta^2-r\beta-s))\beta^n + (\frac{1}{8\gamma^8-5r\gamma^7+(r^2+s)\gamma^6+t\gamma^5-5u\gamma^4+2ru\gamma^3+u^2}((r-4\gamma)\gamma^4(r^2+s)-(-2\gamma^2-2r\gamma+r^2+s)\gamma^4 r+\gamma^3(2r\gamma^3+2(r^2+6s)\gamma^2+11t\gamma+12u)) + \frac{1}{(2r^2+3s)\gamma^3+(3t+2rs)\gamma^2+2(u+rt)\gamma+2ur}(\gamma^2(r^2+s)-(r-\gamma)\gamma^2 r+(-r\gamma^3-2s\gamma^2-2t\gamma-3u))n)\gamma^n$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n = \alpha^n + \beta^n + 2\gamma^n.$$

(c) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = \delta$) For all integers n, Binet's formulas of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials are*

$G_n = \frac{1}{(\alpha-\beta)^3}((r^2+s)-3r\beta+3\beta^2)\alpha^n + (\frac{1}{(\alpha-\beta)^3}(-(r^2+s)+3r\beta-3\beta^2) + \frac{1}{2\beta^2(\alpha-\beta)^2}((\alpha-3\beta)(r^2+s)-(\alpha^2-8\beta^2+\alpha\beta)r+\beta(4\alpha^2-5\beta^2-5\alpha\beta))n + \frac{1}{2\beta^2(\alpha-\beta)}(-(r^2+s)+(\alpha+2\beta)r-\beta(2\alpha+\beta))n^2)\beta^n$

i.e.,

$G_n = \frac{1}{(r-4\beta)^3}((r^2+s)-3r\beta+3\beta^2)\alpha^n + (\frac{1}{(r-4\beta)^3}(-(r^2+s)+3r\beta-3\beta^2) + \frac{1}{2\beta^2(r-4\beta)^2}((r-6\beta)(r^2+s)+(2\beta^2+5r\beta-r^2)r+\beta(46\beta^2-29r\beta+4r^2))n + \frac{1}{2\beta^2(r-4\beta)}(-(r^2+s)+(r-\beta)r+\beta(5\beta-2r))n^2)\beta^n$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n = \alpha^n + 3\beta^n.$$

(d) *(Single Root Case: $\alpha=\beta=\gamma=\delta=\frac{r}{4}$) For all integers n, Binet's formulas of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials are*

$$G_n=\frac{1}{6}n((7\alpha^2+s)n^2-3n(s+5\alpha^2)+14\alpha^2+2s)\alpha^{n-3}$$

i.e.,

$$G_n=\frac{2n}{3r^3}((16s+7r^2)n^2-3n(16s+5r^2)+32s+14r^2)\left(\frac{r}{4}\right)^n$$

and

$$H_n=\alpha^n+\beta^n+\gamma^n+\delta^n=4\alpha^n.$$

Proof. (a),(b),(c),(d): Use the equations in eq. (3) and theorem 1.1 (or theorem 1.2) and initial conditions in eqs. (10) and (11). (e) follows from (a), (b), (c) and (d). □

If some of the roots of characteristic equation is 1 then we get the following corollary as a special case of theorem 1.4
.

Corollary 1.3.
For all integers n, Binet's formulas of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials are given as follows:

(a) *(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta=1$).*

$$\begin{aligned}G_n&=\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-1)}+\frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-1)}\\&+\frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-1)}+\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)}\\&=\frac{\alpha^{n+3}}{r\alpha^3+2s\alpha^2+3t\alpha+4u}+\frac{\beta^{n+3}}{r\beta^3+2s\beta^2+3t\beta+4u}\\&+\frac{\gamma^{n+3}}{r\gamma^3+2s\gamma^2+3t\gamma+4u}+\frac{1}{r+2s+3t+4u}\end{aligned}$$

and

$$H_n=\alpha^n+\beta^n+\gamma^n+1.$$

(b) *(Three Distinct Roots Case: $\alpha\neq\beta\neq\gamma=\delta=1$).*

$$G_n=\frac{1}{(\alpha-1)^2(\alpha-\beta)}(s-2r+2\beta-r\beta+r^2+1)\alpha^n+\frac{1}{(\beta-1)^2(\alpha-\beta)}(2r-s-2\alpha+r\alpha-r^2-1)\beta^n+\frac{1}{(\beta-1)^2(\alpha-1)^2}(n(\beta-1)(\alpha-1)(r^2-r-r\alpha-r\beta+s+\alpha+\beta+\alpha\beta)+2\alpha^2-r\alpha^2+r^2\alpha+s\alpha-3\alpha+2\beta^2-r\beta^2+2\alpha\beta+r^2\beta+s\beta-r\alpha\beta-3\beta-2r^2+3r-2s)$$

i.e.,

$$G_n=\frac{1}{r\alpha^3+2s\alpha^2+3t\alpha+4u}(\alpha(r^2+s)-\alpha(r-\alpha)r+\alpha(\alpha^2-r\alpha-s))\alpha^n+\frac{1}{r\beta^3+2s\beta^2+3t\beta+4u}(\beta(r^2+s)-\beta(r-\beta)r+\beta(\beta^2-r\beta-s))\beta^n+(-5r+s+t-5u+2ru+r^2+u^2+8)^{-1}(3s+3t+2u+2rs+2rt+2ru+2r^2)^{-1}(n(s+2t+3u)(-r^2-u^2+5r+5u-2ru-s-t-8)+(4r+8s+11t+12u)(2r^2+2u+2rs+2rt+2ru+3s+3t))$$

$$H_n=\alpha^n+\beta^n+2.$$

(c) *(Two Distinct Roots Case: $\alpha\neq\beta=\gamma=\delta=1$).*

$$G_n=\frac{1}{(\alpha-1)^3}(r^2-3r+s+3)\alpha^n+\frac{1}{2(\alpha-1)^3}(n^2(\alpha-1)^2(2r-r^2-s-1+r\alpha-2\alpha)+n(\alpha-1)((4-r)\alpha^2+(r^2-r+s-5)\alpha+8r-3s-3r^2-5)+2(-r^2+3r-s-3))$$

i.e.,

$$G_n=\frac{1}{(r-4)^3}(r^2-3r+s+3)\alpha^n+\frac{1}{2(r-4)^3}(n^2(r-4)^2(5-3r-s)+n(r-4)(3r^2-27r+rs-6s+46)+2(-r^2+3r-s-3))$$

and

$$H_n=\alpha^n+3.$$

(d) *(Single Root Case: $\alpha=\beta=\gamma=\delta=1=\frac{r}{4}$).*

$$G_n=\frac{1}{6}n((s+7)n^2-3(s+5)n+2s+14)$$

and

$$H_n=4.$$

2. Generalized co-Tetranacci Polynomials

In this section, for r, s, t, u satisfying eq. (1), we define and investigate a new sequence and its two special cases, namely the generalized co-Tetranacci, $(r, s, t, u)-$co-Tetranacci and $(r, s, t, u)-$co-Tetranacci-Lucas polynomials.

For r, s, t, u satisfying eq. (1), the generalized co-Tetranacci polynomials (or generalized $(r, s, t, u)-$co-Tetranacci polynomials or generalized co-4-step Fibonacci polynomials)

$$\{Y_n(Y_0(x), Y_1(x), Y_2(x), Y_3(x); t, -su, ru^2, u^3)\}_{n\geq 0}$$

(or shortly $\{Y_n(x)\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} &Y_n(x) = tY_{n-1}(x) - suY_{n-2}(x) + ru^2 Y_{n-3}(x) + u^3 Y_{n-4}(x), \\ &Y_0(x) = d(x), Y_1(x) = e(x), Y_2(x) = f(x), Y_3(x) = g(x) \quad n \geq 4 \end{aligned} \tag{12}$$

where $Y_0(x), Y_1(x), Y_2(x), Y_3(x)$ are arbitrary complex (or real) polynomials with real coefficients.

The sequence $\{Y_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$Y_{-n}(x) = -\frac{ru^2}{u^3} Y_{-(n-1)}(x) - \frac{-su}{u^3} Y_{-(n-2)}(x) - \frac{t}{u^3} Y_{-(n-3)}(x) + \frac{1}{u^3} Y_{-(n-4)}(x), \text{ for } n = 1, 2, 3, ...$$

for $n = 1, 2, 3, ...$ when $u \neq 0$. Therefore, recurrence eq. (12) holds for all integer n. Note that for $n \geq 1$, $Y_{-n}(x)$ need not to be a polynomial in the ordinary sense.

Remark 2.1.
For simplicity, throughout the rest of the paper we denote

$$r_1 = t, s_1 = -su, t_1 = ru^2, u_1 = u^3$$

and write eq. (12) as

$$\begin{aligned} &Y_n(x) = r_1 Y_{n-1}(x) + s_1 Y_{n-2}(x) + t_1 Y_{n-3}(x) + y_1 Y_{n-4}(x), \\ &Y_0(x) = d(x), Y_1(x) = e(x), Y_2(x) = f(x), Y_3(x) = g(x), \quad n \geq 4 \end{aligned} \tag{13}$$

and

$$Y_{-n}(x) = -\frac{t_1}{u_1} Y_{-(n-1)}(x) - \frac{s_1}{u_1} Y_{-(n-2)}(x) - \frac{r_1}{u_1} Y_{-(n-3)}(x) + \frac{1}{u_1} Y_{-(n-4)}(x)$$

unless otherwise stated. So, we can easily use and modify the results which are given in section Introduction and forthcoming sections, just by setting (substituting) r_1, s_1, t_1, u_1 for r, s, t, u, respectively.

As $\{Y_n(x)\}$ is a fourth-order recurrence sequence (difference equation), it's characteristic equation is

$$y^4 - r_1 y^3 - s_1 y^2 - t_1 y - u_1 = 0, \tag{14}$$

i.e.,

$$y^4 - ty^3 + suy^2 - ru^2 y - u^3 = 0. \tag{15}$$

The roots of characteristic equation of $\{Y_n\}$ will be denoted as

$$\begin{aligned} \theta_1(x) &= \theta_1(x, r_1, s_1, t_1, u_1) = \theta_1(x, t, -su, ru^2, u^3), \\ \theta_2(x) &= \theta_2(x, r_1, s_1, t_1, u_1) = \theta_2(x, t, -su, ru^2, u^3), \\ \theta_3(x) &= \theta_3(x, r_1, s_1, t_1, u_1) = \theta_3(x, t, -su, ru^2, u^3), \\ \theta_4(x) &= \theta_4(x, r_1, s_1, t_1, u_1) = \theta_{43}(x, t, -su, ru^2, u^3). \end{aligned}$$

Remark 2.2.
As before, for the sake of simplicity throughout the rest of the paper, we use

$$Y_n, Y_0, Y_1, Y_2, Y_3, \theta_1, \theta_2, \theta_3, \theta_4,$$

instead of

$$Y_n(x), Y_0(x), Y_1(x), Y_2(x), Y_3(x), \theta_1(x), \theta_2(x), \theta_3(x), , \theta_4(x)$$

respectively, unless otherwise stated. For example, we write

$$Y_n = tY_{n-1} - suY_{n-2} + ru^2Y_{n-3} + u^3Y_{n-4}, \quad Y_0 = d, Y_1 = e, Y_2 = f, Y_3 = g, \ n \geq 4$$

and

$$Y_n = r_1Y_{n-1}(x) + s_1Y_{n-2} + t_1Y_{n-3} + y_1Y_{n-4}, \quad Y_0 = d, Y_1 = e, Y_2 = f, Y_3 = g, \ n \geq 4$$

and

$$Y_{-n} = -\frac{ru^2}{u^3}Y_{-(n-1)} - \frac{-su}{u^3}Y_{-(n-2)} - \frac{t}{u^3}Y_{-(n-3)}(x) + \frac{1}{u^3}Y_{-(n-4)}$$

and

$$Y_{-n} = -\frac{t_1}{u_1}Y_{-(n-1)} - \frac{s_1}{u_1}Y_{-(n-2)} - \frac{r_1}{u_1}Y_{-(n-3)} + \frac{1}{u_1}Y_{-(n-4)}$$

for the eqs. (12) and (13). Also we write V_n, V_0, V_1, V_2, V_3 instead of $V_n(x)$ with initial conditions $V_0(x), V_0(x), V_1(x), V_2(x), V_3(x)$ for any subsequence $\{V_n(x)\}$ of $\{Y_n\}$.

We have the following relations between the roots of characteristic equations of generalized Tetranacci and generalized co-Tetranacci polynomials:

Lemma 2.1.
There are the following relations between $\alpha,\beta,\gamma,\delta$ and $\theta_1,\theta_2,\theta_3,\theta_4$:

(a)

$\beta\gamma\delta,\ \alpha\gamma\delta,\ \alpha\beta\delta$ *and* $\alpha\beta\gamma$

are the roots of characteristic equation eq. (14) of generalized co-Tetranacci polynomials

(b) *We can choose $\theta_1,\theta_2,\theta_3,\theta_4$ as*

$$\theta_1 = \beta\gamma\delta, \tag{16}$$
$$\theta_2 = \alpha\gamma\delta, \tag{17}$$
$$\theta_3 = \alpha\beta\delta, \tag{18}$$
$$\theta_4 = \alpha\beta\gamma. \tag{19}$$

(c)

$$\theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4 \Leftrightarrow \alpha \neq \beta \neq \gamma \neq \delta.$$

(d)

$$\theta_1 \neq \theta_2 \neq \theta_3 = \theta_4 \Leftrightarrow \alpha \neq \beta \neq \gamma = \delta.$$

(e)

$$\theta_1 \neq \theta_2 = \theta_3 = \theta_4 \Leftrightarrow \alpha \neq \beta = \gamma = \delta.$$

(f)

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 \Leftrightarrow \alpha = \beta = \gamma = \delta.$$

We have the following identities between $\theta_1,\theta_2,\theta_3,\theta_4$ and r_1, s_1, t_1, u_1.

Lemma 2.2.
There are close relations between the roots of characteristic eq. (14) and r_1, s_1, t_1, u_1 as follows.

(a) *Arbitrary Roots Case ($\theta_1,\theta_2,\theta_3,\theta_4$ are arbitrary) (including Four Distinct Roots Case, i.e.:$\theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4$).*

$$\begin{cases} \theta_1+\theta_2+\theta_3+\theta_4 = r_1 = t, \\ \theta_1\theta_2+\theta_1\theta_3+\theta_1\theta_4+\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4 = -s_1 = su, \\ \theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_3\theta_4+\theta_2\theta_3\theta_4 = t_1 = ru^2, \\ \theta_1\theta_2\theta_3\theta_4 = -u_1 = -u^3, \end{cases} \tag{20}$$

i.e.,

$$r_1 = \theta_1+\theta_2+\theta_3+\theta_4,$$
$$s_1 = -(\theta_1\theta_2+\theta_1\theta_3+\theta_1\theta_4+\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4),$$
$$t_1 = \theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_3\theta_4+\theta_2\theta_3\theta_4,$$
$$u_1 = -\theta_1\theta_2\theta_3\theta_4.$$

(b) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = \theta_4$).*

$$\begin{aligned}
\theta_1 &\neq \theta_2 \neq \theta_3 = \theta_4 \\
&\Leftrightarrow \\
r_1 &= \theta_1 + \theta_2 + 2\theta_3, \\
s_1 &= -(\theta_3^2 + \theta_1\theta_2 + 2\theta_1\theta_3 + 2\theta_2\theta_3), \\
t_1 &= \theta_1\theta_3^2 + \theta_2\theta_3^2 + 2\theta_1\theta_2\theta_3, \\
u_1 &= -\theta_1\theta_2\theta_3^2.
\end{aligned}$$

(c) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = \theta_4$).*

$$\begin{aligned}
\theta_1 &\neq \theta_2 = \theta_3 = \theta_4 \\
&\Leftrightarrow \\
r_1 &= \theta_1 + 3\theta_2, \\
s_1 &= -(3\theta_2^2 + 3\theta_1\theta_2), \\
t_1 &= \theta_2^3 + 3\theta_1\theta_2^2, \\
u_1 &= -\theta_1\theta_2^3.
\end{aligned}$$

(d) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{r_1}{4}$).*

$$\begin{aligned}
\theta_1 &= \theta_2 = \theta_3 = \theta_4 \\
&\Leftrightarrow \\
r_1 &= 4\theta_1, \\
s_1 &= -6\theta_1^2, \\
t_1 &= 4\theta_1^3, \\
u_1 &= -\theta_1^4.
\end{aligned}$$

Using the roots of characteristic equation and the recurrence relation of Y_n, Binet's formula can be given as follows:

Theorem 2.1.
For all integers n, Binet's formula of generalized co-Tetranacci polynomials is given as follows.

(a) *(Four Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4$) Binet's formula of generalized co-Tetranacci polynomials is*

$$\begin{aligned}
Y_n &= \frac{p_5\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)} + \frac{p_6\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)} \\
&\quad + \frac{p_7\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)} + \frac{p_8\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)} \\
&= B_1\theta_1^n + B_2\theta_2^n + B_3\theta_3^n + B_4\theta_4^n
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
p_5 &= Y_3 - (\theta_2+\theta_3+\theta_4)Y_2 + (\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4)Y_1 - \theta_2\theta_3\theta_4 Y_0, \\
p_6 &= Y_3 - (\theta_1+\theta_3+\theta_4)Y_2 + (\theta_1\theta_3+\theta_1\theta_4+\theta_3\theta_4)Y_1 - \theta_1\theta_3\theta_4 Y_0, \\
p_7 &= Y_3 - (\theta_1+\theta_2+\theta_4)Y_2 + (\theta_1\theta_2+\theta_1\theta_4+\theta_2\theta_4)Y_1 - \theta_1\theta_2\theta_4 Y_0, \\
p_8 &= Y_3 - (\theta_1+\theta_2+\theta_3)Y_2 + (\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)Y_1 - \theta_1\theta_2\theta_3 Y_0,
\end{aligned}$$

and

$$\begin{aligned}
B_1 &= \frac{p_5}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)}, \\
B_2 &= \frac{p_6}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)}, \\
B_3 &= \frac{p_7}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)}, \\
B_4 &= \frac{p_8}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)},
\end{aligned}$$

i.e.,

$$\begin{aligned}
Y_n &= \frac{p_5\theta_1^{n+1}}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}+\frac{p_6\theta_2^{n+1}}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1} \\
&\quad +\frac{p_7\theta_3^{n+1}}{r_1\theta_3^3+2s_1\theta_3^2+3t_1\theta_3+4u_1}+\frac{p_8\theta_4^{n+1}}{r_1\theta_4^3+2s_1\theta_4^2+3t_1\theta_4+4u_1} \\
&= B_1\theta_1^n+B_2\theta_2^n+B_3\theta_3^n+B_4\theta_4^n
\end{aligned}$$

where

$$\begin{aligned}
p_5 &= Y_3-(r_1-\theta_1)Y_2+(\theta_1^2-r_1\theta_1-s_1)Y_1+\frac{u_1}{\theta_1}Y_0, \\
p_6 &= Y_3-(r_1-\theta_2)Y_2+(\theta_2^2-r_1\theta_2-s_1)Y_1+\frac{u_1}{\theta_2}Y_0, \\
p_7 &= Y_3-(r_1-\theta_3)Y_2+(\theta_3^2-r_1\theta_3-s_1)Y_1+\frac{u_1}{\theta_3}Y_0, \\
p_8 &= Y_3-(r_1-\theta_4)Y_2+(\theta_4^2-r_1\theta_4-s_1)Y_1+\frac{u_1}{\theta_4}Y_0,
\end{aligned}$$

and

$$\begin{aligned}
B_1 &= \frac{\theta_1 p_5}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}, \\
B_2 &= \frac{\theta_2 p_6}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}, \\
B_3 &= \frac{\theta_3 p_7}{r_1\theta_3^3+2s_1\theta_3^2+3t_1\theta_3+4u_1}, \\
B_4 &= \frac{\theta_4 p_8}{r_1\theta_4^3+2s_1\theta_4^2+3t_1\theta_4+4u_1},
\end{aligned}$$

that is,

$$Y_n = B_1\theta_1^n+B_2\theta_2^n+B_3\theta_3^n+B_4\theta_4^n,$$

where

$$\begin{aligned}
B_1 &= \frac{(\theta_1Y_3-\theta_1(r_1-\theta_1)Y_2+\theta_1(\theta_1^2-r_1\theta_1-s_1)Y_1+u_1Y_0)}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}, \\
B_2 &= \frac{(\theta_2Y_3-\theta_2(r_1-\theta_2)Y_2+\theta_2(\theta_2^2-r_1\theta_2-s_1)Y_1+u_1Y_0)}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}, \\
B_3 &= \frac{(\theta_3Y_3-\theta_3(r_1-\theta_3)Y_2+\theta_3(\theta_3^2-r_1\theta_3-s_1)Y_1+u_1Y_0)}{r_1\theta_3^3+2s_1\theta_3^2+3t_1\theta_3+4u_1}, \\
B_4 &= \frac{(\theta_4Y_3-\theta_4(r_1-\theta_4)Y_2+\theta_4(\theta_4^2-r_1\theta_4-s_1)Y_1+u_1Y_0)}{r_1\theta_4^3+2s_1\theta_4^2+3t_1\theta_4+4u_1},
\end{aligned}$$

i.e.,

$$Y_n = A_5\theta_1^{n-1}+A_6\theta_2^{n-1}+A_7\theta_3^{n-1}+A_8\theta_4^{n-1}$$

where

$$\begin{aligned}
A_5 &= \frac{(\theta_1Y_3-\theta_1(r_1-\theta_1)Y_2+\theta_1(\theta_1^2-r_1\theta_1-s_1)Y_1+u_1Y_0)}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)}, \\
A_6 &= \frac{(\theta_2Y_3-\theta_2(r_1-\theta_2)Y_2+\theta_2(\theta_2^2-r_1\theta_2-s_1)Y_1+u_1Y_0)}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)}, \\
A_7 &= \frac{(\theta_3Y_3-\theta_3(r_1-\theta_3)Y_2+\theta_3(\theta_3^2-r_1\theta_3-s_1)Y_1+u_1Y_0)}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)}, \\
A_8 &= \frac{(\theta_4Y_3-\theta_4(r_1-\theta_4)Y_2+\theta_4(\theta_4^2-r_1\theta_4-s_1)Y_1+u_1Y_0)}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)}.
\end{aligned}$$

(b) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = \theta_4$) Binet's formula of generalized co-Tetranacci polynomials is*

$$Y_n = B_1\theta_1^n + B_2\theta_2^n + (B_3 + B_4 n)\theta_3^n$$

where

$$B_1 = \frac{Y_3 - (\theta_2 + 2\theta_3)Y_2 + \theta_3(2\theta_2 + \theta_3)Y_1 - \theta_2\theta_3^2 Y_0}{(\theta_1 - \theta_3)^2(\theta_1 - \theta_2)},$$

$$B_2 = \frac{-Y_3 + (\theta_1 + 2\theta_3)Y_2 - \theta_3(2\theta_1 + \theta_3)Y_1 + \theta_1\theta_3^2 Y_0}{(\theta_2 - \theta_3)^2(\theta_1 - \theta_2)},$$

$$B_3 = \frac{1}{(\theta_2 - \theta_3)^2(\theta_1 - \theta_3)^2}((\theta_1 + \theta_2 - 2\theta_3)Y_3 - (\theta_1^2 + \theta_2^2 - 3\theta_3^2 + \theta_1\theta_2)Y_2 + \theta_3(2\theta_1^2 + 2\theta_2^2 + 2\theta_1\theta_2 - 3\theta_1\theta_3 - 3\theta_2\theta_3)Y_1 + \theta_1\theta_2(3\theta_3^2 + \theta_1\theta_2 - 2\theta_1\theta_3 - 2\theta_2\theta_3)Y_0),$$

$$B_4 = \frac{Y_3 - (\theta_1 + \theta_2 + \theta_3)Y_2 + (\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3)Y_1 - \theta_1\theta_2\theta_3 Y_0}{\theta_3(\theta_2 - \theta_3)(\theta_1 - \theta_3)},$$

i.e.,

$$B_1 = \frac{\theta_1 Y_3 - \theta_1(r_1 - \theta_1)Y_2 + \theta_1(\theta_1^2 - r_1\theta_1 - s_1)Y_1 + u_1 Y_0}{r_1\theta_1^3 + 2s_1\theta_1^2 + 3t_1\theta_1 + 4u_1},$$

$$B_2 = \frac{\theta_2 Y_3 - \theta_2(r_1 - \theta_2)Y_2 + \theta_2(\theta_2^2 - r_1\theta_2 - s_1)Y_1 + u_1 Y_0}{r_1\theta_2^3 + 2s_1\theta_2^2 + 3t_1\theta_2 + 4u_1},$$

$$B_3 = \frac{1}{8\theta_3^8 - 5r_1\theta_3^7 + (r_1^2 + s_1)\theta_3^6 + t_1\theta_3^5 - 5u_1\theta_3^4 + 2r_1u_1\theta_3^3 + u_1^2}((r_1 - 4\theta_3)\theta_3^4 Y_3 - (-2\theta_3^2 - 2r_1\theta_3 + r_1^2 + s_1)\theta_3^4 Y_2 + \theta_3^3(2r_1\theta_3^3 + 2(r_1^2 + 6s_1)\theta_3^2 + 11t_1\theta_3 + 12u_1)Y_1 + (-3\theta_3^6 + 2r_1\theta_3^5 + (s_1 - 7u_1)\theta_3^4 + 2r_1u_1\theta_3^3 - u_1\theta_3^2 + u_1^2)Y_0),$$

$$B_4 = \frac{\theta_3^2 Y_3 - (r_1 - \theta_3)\theta_3^2 Y_2 + (-r_1\theta_3^3 - 2s_1\theta_3^2 - 2t_1\theta_3 - 3u_1)Y_1 + u_1\theta_3 Y_0}{(2r_1^2 + 3s_1)\theta_3^3 + (3t_1 + 2r_1s_1)\theta_3^2 + 2(u_1 + r_1t_1)\theta_3 + 2u_1r_1}.$$

(c) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = \theta_4$) Binet's formula of generalized co-Tetranacci polynomials is*

$$Y_n = B_1\theta_1^n + (B_2 + B_3 n + B_4 n^2)\theta_2^n$$

where

$$B_1 = \frac{Y_3 - 3Y_2\theta_2 + 3Y_1\theta_2^2 - Y_0\theta_2^3}{(\theta_1 - \theta_2)^3},$$

$$B_2 = \frac{-Y_3 + 3Y_2\theta_2 - 3Y_1\theta_2^2 + \theta_1(\theta_1^2 + 3\theta_2^2 - 3\theta_1\theta_2)Y_0}{(\theta_1 - \theta_2)^3},$$

$$B_3 = \frac{1}{2\theta_2^2(\theta_1 - \theta_2)^2}((\theta_1 - 3\theta_2)Y_3 - (\theta_1^2 - 8\theta_2^2 + \theta_1\theta_2)Y_2 + \theta_2(4\theta_1^2 - 5\theta_2^2 - 5\theta_1\theta_2)Y_1 - \theta_1\theta_2^2(3\theta_1 - 5\theta_2)Y_0),$$

$$B_4 = \frac{-Y_3 + (\theta_1 + 2\theta_2)Y_2 - \theta_2(2\theta_1 + \theta_2)Y_1 + \theta_1\theta_2^2 Y_0}{2\theta_2^2(\theta_1 - \theta_2)},$$

i.e.,

$$B_1 = \frac{Y_3 - 3Y_2\theta_2 + 3Y_1\theta_2^2 - Y_0\theta_2^3}{(r_1 - 4\theta_2)^3},$$

$$B_2 = \frac{-Y_3 + 3Y_2\theta_2 - 3Y_1\theta_2^2 + (r_1 - 3\theta_2)(21\theta_2^2 - 9r_1\theta_2 + r_1^2)Y_0}{(r_1 - 4\theta_2)^3},$$

$$B_3 = \frac{1}{2\theta_2^2(r_1 - 4\theta_2)^2}((r_1 - 6\theta_2)Y_3 + (2\theta_2^2 + 5r_1\theta_2 - r_1^2)Y_2 + \theta_2(46\theta_2^2 - 29r_1\theta_2 + 4r_1^2)Y_1 + (\theta_2^2(14\theta_2 - 3r_1)(r_1 - 3\theta_2))Y_0),$$

$$B_4 = \frac{-Y_3 + (r_1 - \theta_2)Y_2 + \theta_2(5\theta_2 - 2r_1)Y_1 + (r_1 - 3\theta_2)\theta_2^2 Y_0}{2\theta_2^2(r_1 - 4\theta_2)}.$$

(d) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{r_1}{4}$) Binet's formula of generalized co-Tetranacci polynomials is*

$$Y_n = (B_1 + B_2 n + B_3 n^2 + B_4 n^3)\theta_1^n$$

where

$$B_1 = Y_0,$$
$$B_2 = \frac{2Y_3 - 9\theta_1 Y_2 + 18\theta_1^2 Y_1 - 11\theta_1^3 Y_0}{6\theta_1^3},$$
$$B_3 = \frac{-Y_3 + 4\theta_1 Y_2 - 5\theta_1^2 Y_1 + 2\theta_1^3 Y_0}{2\theta_1^3},$$
$$B_4 = \frac{Y_3 - 3\theta_1 Y_2 + 3\theta_1^2 Y_1 - \theta_1^3 Y_0}{6\theta_1^3},$$

i.e.,

$$Y_n = (B_1 + B_2 n + B_3 n^2 + B_4 n^3)\left(\frac{r_1}{4}\right)^n$$

where

$$B_1 = Y_0,$$
$$B_2 = \frac{1}{6}\frac{128Y_3 - 144r_1 Y_2 + 72r_1^2 Y_1 - 11r_1^3 Y_0}{r_1^3},$$
$$B_3 = \frac{-32Y_3 + 32r_1 Y_2 - 10r_1^2 Y_1 + r_1^3 Y_0}{r_1^3},$$
$$B_4 = \frac{1}{6}\frac{64Y_3 - 48r_1 Y_2 + 12r_1^2 Y_1 - r_1^3 Y_0}{r_1^3}.$$

Note that (a), (b), (c) and (d) of the above theorem 2.1 can be given as follows:

$$Y_n = \begin{cases} B_1\theta_1^n + B_2\theta_2^n + B_3\theta_3^n + B_4\theta_4^n & , \text{ if } \theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4 \text{ (Four Distinct Roots Case)} \\ B_1\theta_1^n + B_2\theta_2^n + (B_3 + B_4 n)\theta_3^n & , \text{ if } \theta_1 \neq \theta_2 \neq \theta_3 = \theta_4 \text{ (Three Distinct Roots Case)} \\ B_1\theta_1^n + (B_2 + B_3 n + B_4 n^2)\theta_2^n & , \text{ if } \theta_1 \neq \theta_2 = \theta_3 = \theta_4 \text{ (Two Distinct Roots Case)} \\ (B_1 + B_2 n + B_3 n^2 + B_4 n^3)\theta_1^n & , \text{ if } \theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{r_1}{4} \text{ (Single Root Case)} \end{cases} \tag{22}$$

where each quadruple B_1, B_2, B_3, B_4 are given as in theorem 2.1 (a), (b), (c) and (d), respectively.

Lemma 2.3.

(a) *(Four Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4$). If B_1, B_2, B_3 and B_4 are as in theorem 2.1 (a), then we have*

$$B_1 + B_2 + B_3 + B_4 = Y_0.$$

(b) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = \theta_4$). If B_1, B_2, B_3 and B_4 are as in theorem 2.1 (b), then we have*

$$B_1 + B_2 + B_3 + B_4 = \frac{Y_3 - (\theta_1 + \theta_2 + \theta_3)Y_2 + (\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3)Y_1 - \theta_3^2(\theta_1 + \theta_2 - \theta_3)Y_0}{\theta_3(\theta_2 - \theta_3)(\theta_1 - \theta_3)}.$$

(c) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = \theta_4$). If B_1, B_2, B_3 and B_4 are as in theorem 2.1 (c), then we have*

$$\begin{aligned} B_1 + B_2 + B_3 + B_4 &= \frac{-Y_3 + 3\theta_2 Y_2 + (\theta_1^2 - 2\theta_2^2 - 2\theta_1\theta_2)Y_1 + \theta_2^3 Y_0}{\theta_2(\theta_1 - \theta_2)^2} \\ &= \frac{-Y_3 + 3\theta_2 Y_2 + (13\theta_2^2 - 8r_1\theta_2 + r_1^2)Y_1 + \theta_2^3 Y_0}{\theta_2(r_1 - 4\theta_2)^2}. \end{aligned}$$

(d) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{r_1}{4}$). If B_1, B_2, B_3 and B_4 are as in theorem 2.1 (d), then we have*

$$B_1 + B_2 + B_3 + B_4 = \frac{Y_1}{\theta_1} = \frac{4}{r_1}Y_1.$$

theorem 2.1 can be given in the following form:

Theorem 2.2.
Binet's formula of generalized co-Tetranacci polynomials is given as follows according to the roots of characteristic eq. (14)):

(a) *(Four Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4$) Binet's formula of generalized co-Tetranacci polynomials is*

$$
\begin{aligned}
Y_n &= \frac{(Y_3-(\theta_2+\theta_3+\theta_4)Y_2+(\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4)Y_1-\theta_2\theta_3\theta_4Y_0)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)} \\
&+\frac{(Y_3-(\theta_1+\theta_3+\theta_4)Y_2+(\theta_1\theta_3+\theta_1\theta_4+\theta_3\theta_4)Y_1-\theta_1\theta_3\theta_4Y_0)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)} \\
&+\frac{(Y_3-(\theta_1+\theta_2+\theta_4)Y_2+(\theta_1\theta_2+\theta_1\theta_4+\theta_2\theta_4)Y_1-\theta_1\theta_2\theta_4Y_0)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)} \\
&+\frac{(Y_3-(\theta_1+\theta_2+\theta_3)Y_2+(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)Y_1-\theta_1\theta_2\theta_3Y_0)\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
Y_n &= \frac{(Y_3-(r_1-\theta_1)Y_2+(\theta_1^2-r_1\theta_1-s_1)Y_1+\frac{u_1}{\theta_1}Y_0)\theta_1^{n+1}}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1} \\
&+\frac{(Y_3-(r_1-\theta_2)Y_2+(\theta_2^2-r_1\theta_2-s_1)Y_1+\frac{u_1}{\theta_2}Y_0)\theta_2^{n+1}}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1} \\
&+\frac{(Y_3-(r_1-\theta_3)Y_2+(\theta_3^2-r_1\theta_3-s_1)Y_1+\frac{u_1}{\theta_3}Y_0)\theta_3^{n+1}}{r_1\theta_3^3+2s_1\theta_3^2+3t_1\theta_3+4u_1} \\
&+\frac{(Y_3-(r_1-\theta_4)Y_2+(\theta_4^2-r_1\theta_4-s_1)Y_1+\frac{u_1}{\theta_4}Y_0)\theta_4^{n+1}}{r_1\theta_4^3+2s_1\theta_4^2+3t_1\theta_4+4u_1}.
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
Y_n &= \frac{(\theta_1Y_3-\theta_1(r_1-\theta_1)Y_2+\theta_1(\theta_1^2-r_1\theta_1-s_1)Y_1+u_1Y_0)\theta_1^n}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1} \\
&+\frac{(\theta_2Y_3-\theta_2(r_1-\theta_2)Y_2+\theta_2(\theta_2^2-r_1\theta_2-s_1)Y_1+u_1Y_0)\theta_2^n}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1} \\
&+\frac{(\theta_3Y_3-\theta_3(r_1-\theta_3)Y_2+\theta_3(\theta_3^2-r_1\theta_3-s_1)Y_1+u_1Y_0)\theta_3^n}{r_1\theta_3^3+2s_1\theta_3^2+3t_1\theta_3+4u_1} \\
&+\frac{(\theta_4Y_3-\theta_4(r_1-\theta_4)Y_2+\theta_4(\theta_4^2-r_1\theta_4-s_1)Y_1+u_1Y_0)\theta_4^n}{r_1\theta_4^3+2s_1\theta_4^2+3t_1\theta_4+4u_1}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
Y_n &= \frac{(\theta_1Y_3-\theta_1(r_1-\theta_1)Y_2+\theta_1(\theta_1^2-r_1\theta_1-s_1)Y_1+u_1Y_0)}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)}\theta_1^{n-1} \\
&+\frac{(\theta_2Y_3-\theta_2(r_1-\theta_2)Y_2+\theta_2(\theta_2^2-r_1\theta_2-s_1)Y_1+u_1Y_0)}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)}\theta_2^{n-1} \\
&+\frac{(\theta_3Y_3-\theta_3(r_1-\theta_3)Y_2+\theta_3(\theta_3^2-r_1\theta_3-s_1)Y_1+u_1Y_0)}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)}\theta_3^{n-1} \\
&+\frac{(\theta_4Y_3-\theta_4(r_1-\theta_4)Y_2+\theta_4(\theta_4^2-r_1\theta_4-s_1)Y_1+u_1Y_0)}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)}\theta_4^{n-1}
\end{aligned}
$$

(b) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = \theta_4$) Binet's formula of generalized co-Tetranacci polynomials is*

$Y_n = \frac{1}{(\theta_1-\theta_3)^2(\theta_1-\theta_2)}(Y_3-(\theta_2+2\theta_3)Y_2+\theta_3(2\theta_2+\theta_3)Y_1-\theta_2\theta_3^2Y_0)\theta_1^n+\frac{1}{(\theta_2-\theta_3)^2(\theta_1-\theta_2)}(-Y_3+(\theta_1+2\theta_3)Y_2-\theta_3(2\theta_1+\theta_3)Y_1+\theta_1\theta_3^2Y_0)\theta_2^n+(\dfrac{1}{(\theta_2-\theta_3)^2(\theta_1-\theta_3)^2}((\theta_1+\theta_2-2\theta_3)Y_3-(\theta_1^2+\theta_2^2-3\theta_3^2+\theta_1\theta_2)Y_2+\theta_3(2\theta_1^2+2\theta_2^2+2\theta_1\theta_2-3\theta_1\theta_3-3\theta_2\theta_3)Y_1+\theta_1\theta_2(3\theta_3^2+\theta_1\theta_2-2\theta_1\theta_3-2\theta_2\theta_3)Y_0)+\frac{1}{\theta_3(\theta_2-\theta_3)(\theta_1-\theta_3)}(Y_3-(\theta_1+\theta_2+\theta_3)Y_2+(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)Y_1-\theta_1\theta_2\theta_3Y_0)n)\theta_3^n$

i.e.,

$Y_n = \frac{1}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}(\theta_1Y_3-\theta_1(r_1-\theta_1)Y_2+\theta_1(\theta_1^2-r_1\theta_1-s_1)Y_1+u_1Y_0)\theta_1^n$

$+\frac{1}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}(\theta_2 Y_3-\theta_2(r_1-\theta_2)Y_2+\theta_2(\theta_2^2-r_1\theta_2-s_1)Y_1+u_1Y_0)\theta_2^n$

$+(\dfrac{1}{8\theta_3^8-5r_1\theta_3^7+(r_1^2+s_1)\theta_3^6+t_1\theta_3^5-5u_1\theta_3^4+2r_1u_1\theta_3^3+u_1^2}((r_1-4\theta_3)\theta_3^4Y_3-(-2\theta_3^2-2r_1\theta_3+r_1^2+s_1)\theta_3^4Y_2+\theta_3^3(2r_1\theta_3^3+2(r_1^2+6s_1)\theta_3^2+11t_1\theta_3+12u_1)Y_1+(-3\theta_3^6+2r_1\theta_3^5+(s_1-7u_1)\theta_3^4+2r_1u_1\theta_3^3-u_1\theta_3^2+u_1^2)Y_0)+\frac{1}{(2r_1^2+3s_1)\theta_3^3+(3t_1+2r_1s_1)\theta_3^2+2(u_1+r_1t_1)\theta_3+2u_1r_1}(\theta_3^2Y_3-(r_1-\theta_3)\theta_3^2Y_2+(-r_1\theta_3^3-2s_1\theta_3^2-2t_1\theta_3-3u_1)Y_1+u_1\theta_3Y_0)n)\theta_3^n$

(c) *(Two Distinct Roots Case: $\theta_1\neq\theta_2=\theta_3=\theta_4$) Binet's formula of generalized co-Tetranacci polynomials is*

$Y_n=\frac{1}{(\theta_1-\theta_2)^3}(Y_3-3Y_2\theta_2+3Y_1\theta_2^2-Y_0\theta_2^3)\theta_1^n+(\frac{1}{(\theta_1-\theta_2)^3}(-Y_3+3Y_2\theta_2-3Y_1\theta_2^2+\theta_1(\theta_1^2+3\theta_2^2-3\theta_1\theta_2)Y_0)+\frac{1}{2\theta_2^2(\theta_1-\theta_2)^2}((\theta_1-3\theta_2)Y_3-(\theta_1^2-8\theta_2^2+\theta_1\theta_2)Y_2+\theta_2(4\theta_1^2-5\theta_2^2-5\theta_1\theta_2)Y_1-\theta_1\theta_2^2(3\theta_1-5\theta_2)Y_0)n+\frac{1}{2\theta_2^2(\theta_1-\theta_2)}(-Y_3+(\theta_1+2\theta_2)Y_2-\theta_2(2\theta_1+\theta_2)Y_1+\theta_1\theta_2^2Y_0)n^2)\theta_2^n$

i.e.,

$Y_n=\frac{1}{(r_1-4\theta_2)^3}(Y_3-3Y_2\theta_2+3Y_1\theta_2^2-Y_0\theta_2^3)\theta_1^n+(\frac{1}{(r_1-4\theta_2)^3}(-Y_3+3Y_2\theta_2-3Y_1\theta_2^2+(r_1-3\theta_2)(21\theta_2^2-9r_1\theta_2+r_1^2)Y_0)+\frac{1}{2\theta_2^2(r_1-4\theta_2)^2}((r_1-6\theta_2)Y_3+(2\theta_2^2+5r_1\theta_2-r_1^2)Y_2+\theta_2(46\theta_2^2-29r_1\theta_2+4r_1^2)Y_1+(\theta_2^2(14\theta_2-3r_1)(r_1-3\theta_2))Y_0)n+\frac{1}{2\theta_2^2(r_1-4\theta_2)}(-Y_3+(r_1-\theta_2)Y_2+\theta_2(5\theta_2-2r_1)Y_1+(r_1-3\theta_2)\theta_2^2Y_0)n^2)\theta_2^n.$

(d) *(Single Root Case: $\theta_1=\theta_2=\theta_3=\theta_4=\dfrac{r_1}{4}$) Binet's formula of generalized co-Tetranacci polynomials is*

$Y_n=(Y_0+\frac{1}{6\theta_1^3}(2Y_3-9\theta_1Y_2+18\theta_1^2Y_1-11\theta_1^3Y_0)n+\frac{1}{2\theta_1^3}(-Y_3+4\theta_1Y_2-5\theta_1^2Y_1+2\theta_1^3Y_0)n^2+\frac{1}{6\theta_1^3}(Y_3-3\theta_1Y_2+3\theta_1^2Y_1-\theta_1^3Y_0)n^3)\theta_1^n,$

i.e.,

$Y_n=(Y_0+\frac{1}{6r_1^3}(128Y_3-144r_1Y_2+72r_1^2Y_1-11r_1^3Y_0)n+\frac{1}{r_1^3}(-32Y_3+32r_1Y_2-10r_1^2Y_1+r_1^3Y_0)n^2+\frac{1}{6r_1^3}(64Y_3-48r_1Y_2+12r_1^2Y_1-r_1^3Y_0)n^3)\left(\frac{r_1}{4}\right)^n.$

If some of the roots of characteristic equation is 1 then we get the following corollary as a special case of theorem 2.2.

Corollary 2.1.
Binet's formula of generalized co-Tetranacci polynomials is given as follows according to the roots of characteristic equation eq. (14):

(a) *(Four Distinct Roots Case: $\theta_1\neq\theta_2\neq\theta_3\neq\theta_4=1$)*

$$\begin{aligned}Y_n=&\frac{(Y_3-(\theta_2+\theta_3+1)Y_2+(\theta_2\theta_3+\theta_2+\theta_3)Y_1-\theta_2\theta_3Y_0)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-1)}\\&+\frac{(Y_3-(\theta_1+\theta_3+1)Y_2+(\theta_1\theta_3+\theta_1+\theta_3)Y_1-\theta_1\theta_3Y_0)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-1)}\\&+\frac{(Y_3-(\theta_1+\theta_2+1)Y_2+(\theta_1\theta_2+\theta_1+\theta_2)Y_1-\theta_1\theta_2Y_0)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-1)}\\&+\frac{(Y_3-(\theta_1+\theta_2+\theta_3)Y_2+(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)Y_1-\theta_1\theta_2\theta_3Y_0)}{(1-\theta_1)(1-\theta_2)(1-\theta_3)}\end{aligned}$$

i.e.,

$$\begin{aligned}Y_n=&\frac{(Y_3-(r_1-\theta_1)Y_2+(\theta_1^2-r_1\theta_1-s_1)Y_1+\frac{u_1}{\theta_1}Y_0)\theta_1^{n+1}}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}\\&+\frac{(Y_3-(r_1-\theta_2)Y_2+(\theta_2^2-r_1\theta_2-s_1)Y_1+\frac{u_1}{\theta_2}Y_0)\theta_2^{n+1}}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}\\&+\frac{(Y_3-(r_1-\theta_3)Y_2+(\theta_3^2-r_1\theta_3-s_1)Y_1+\frac{u_1}{\theta_3}Y_0)\theta_3^{n+1}}{r_1\theta_3^3+2s_1\theta_3^2+3t_1\theta_3+4u_1}\\&+\frac{(Y_3-(r_1-1)Y_2+(1-r_1-s_1)Y_1+u_1Y_0)}{r_1+2s_1+3t_1+4u_1}\end{aligned}$$

i.e.,

$$
\begin{aligned}
Y_n = &\frac{(\theta_1 Y_3 - \theta_1(r_1-\theta_1)Y_2 + \theta_1(\theta_1^2 - r_1\theta_1 - s_1)Y_1 + u_1 Y_0)\theta_1^n}{r_1\theta_1^3 + 2s_1\theta_1^2 + 3t_1\theta_1 + 4u_1} \\
&+\frac{(\theta_2 Y_3 - \theta_2(r_1-\theta_2)Y_2 + \theta_2(\theta_2^2 - r_1\theta_2 - s_1)Y_1 + u_1 Y_0)\theta_2^n}{r_1\theta_2^3 + 2s_1\theta_2^2 + 3t_1\theta_2 + 4u_1} \\
&+\frac{(\theta_3 Y_3 - \theta_3(r_1-\theta_3)Y_2 + \theta_3(\theta_3^2 - r_1\theta_3 - s_1)Y_1 + u_1 Y_0)\theta_3^n}{r_1\theta_3^3 + 2s_1\theta_3^2 + 3t_1\theta_3 + 4u_1} \\
&+\frac{(Y_3 - (r_1-1)Y_2 + (1-r_1-s_1)Y_1 + u_1 Y_0)}{r_1 + 2s_1 + 3t_1 + 4u_1}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
Y_n = &\frac{(\theta_1 Y_3 - \theta_1(r_1-\theta_1)Y_2 + (-\theta_1^2 + (r_1-1)\theta_1 - u_1)Y_1 + u_1 Y_0)\theta_1^n}{(r_1^2 - r_1 + 2s_1)\theta_1^2 + (r_1^2 + r_1 s_1 - r_1 + 3t_1)\theta_1 + (4-r_1)u_1} \\
&+\frac{(\theta_2 Y_3 - \theta_2(r_1-\theta_2)Y_2 + (-\theta_2^2 + (r_1-1)\theta_2 - u_1)Y_1 + u_1 Y_0)\theta_2^n}{(r_1^2 - r_1 + 2s_1)\theta_2^2 + (r_1^2 + r_1 s_1 - r_1 + 3t_1)\theta_2 + (4-r_1)u_1} \\
&+\frac{(\theta_3 Y_3 - \theta_3(r_1-\theta_3)Y_2 + (-\theta_3^2 + (r_1-1)\theta_3 - u_1)Y_1 + u_1 Y_0)\theta_3^n}{(r_1^2 - r_1 + 2s_1)\theta_3^2 + (r_1^2 + r_1 s_1 - r_1 + 3t_1)\theta_3 + (4-r_1)u_1} \\
&+\frac{Y_3 - (r_1-1)Y_2 + (1-r_1-s_1)Y_1 + u_1 Y_0}{r_1 + 2s_1 + 3t_1 + 4u_1}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
Y_n = &\frac{(\theta_1 Y_3 - \theta_1(r_1-\theta_1)Y_2 + \theta_1(\theta_1^2 - r_1\theta_1 - s_1)Y_1 + u_1 Y_0)}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-1)}\theta_1^{n-1} \\
&+\frac{(\theta_2 Y_3 - \theta_2(r_1-\theta_2)Y_2 + \theta_2(\theta_2^2 - r_1\theta_2 - s_1)Y_1 + u_1 Y_0)}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-1)}\theta_2^{n-1} \\
&+\frac{(\theta_3 Y_3 - \theta_3(r_1-\theta_3)Y_2 + \theta_3(\theta_3^2 - r_1\theta_3 - s_1)Y_1 + u_1 Y_0)}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-1)}\theta_3^{n-1} \\
&+\frac{(Y_3 - (r_1-1)Y_2 + (1-r_1-s_1)Y_1 + u_1 Y_0)}{(1-\theta_1)(1-\theta_2)(1-\theta_3)}.
\end{aligned}
$$

(b) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = \theta_4 = 1$)*

$Y_n = \frac{1}{(\theta_1-1)^2(\theta_1-\theta_2)}(Y_3 - (\theta_2+2)Y_2 + (2\theta_2+1)Y_1 - \theta_2 Y_0)\theta_1^n + \frac{1}{(\theta_2-1)^2(\theta_1-\theta_2)}(-Y_3 + (\theta_1+2)Y_2 - (2\theta_1+1)Y_1 + \theta_1 Y_0)\theta_2^n + (\dfrac{1}{(\theta_2-1)^2(\theta_1-1)^2}((\theta_1+\theta_2-2)Y_3 - (\theta_1^2+\theta_2^2-3+\theta_1\theta_2)Y_2 + (2\theta_1^2+2\theta_2^2+2\theta_1\theta_2-3\theta_1-3\theta_2)Y_1 + \theta_1\theta_2(3+\theta_1\theta_2-2\theta_1-2\theta_2)Y_0) + \frac{1}{(\theta_2-1)(\theta_1-1)}(Y_3 - (\theta_1+\theta_2+1)Y_2 + (\theta_1\theta_2+\theta_1+\theta_2)Y_1 - \theta_1\theta_2 Y_0)n)$

i.e.,

$Y_n = \frac{1}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}(\theta_1 Y_3 - \theta_1(r_1-\theta_1)Y_2 + \theta_1(\theta_1^2 - r_1\theta_1 - s_1)Y_1 + u_1 Y_0)\theta_1^n$
$+\frac{1}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}(\theta_2 Y_3 - \theta_2(r_1-\theta_2)Y_2 + \theta_2(\theta_2^2 - r_1\theta_2 - s_1)Y_1 + u_1 Y_0)\theta_2^n$
$+(\frac{1}{r_1^2+2r_1u_1-5r_1+u_1^2-5u_1+s_1+t_1+8}((r_1-4)Y_3 - (-2-2r_1+r_1^2+s_1)Y_2 + (2r_1+2(r_1^2+6s_1)+11t_1+12u_1)Y_1 + (-3+2r_1+(s_1-7u_1)+2r_1u_1-u_1+u_1^2)Y_0) + \frac{1}{(2r_1^2+3s_1)+(3t_1+2r_1s_1)+2(u_1+r_1t_1)+2u_1r_1}(Y_3 - (r_1-1)Y_2 + (-r_1-2s_1-2t_1-3u_1)Y_1 + u_1 Y_0)n).$

(c) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = \theta_4 = 1$)*

$Y_n = \frac{1}{(\theta_1-1)^3}(Y_3 - 3Y_2 + 3Y_1 - Y_0)\theta_1^n + (\frac{1}{(\theta_1-1)^3}(-Y_3 + 3Y_2 - 3Y_1 + \theta_1(\theta_1^2+3-3\theta_1)Y_0) + \frac{1}{2(\theta_1-1)^2}((\theta_1-3)Y_3 - (\theta_1^2-8+\theta_1)Y_2 + (4\theta_1^2-5-5\theta_1)Y_1 - \theta_1(3\theta_1-5)Y_0)n + \frac{1}{2(\theta_1-1)}(-Y_3 + (\theta_1+2)Y_2 - (2\theta_1+1)Y_1 + \theta_1 Y_0)n^2)$

i.e.,

$Y_n = \frac{1}{(r_1-4)^3}(Y_3 - 3Y_2 + 3Y_1 - Y_0)\theta_1^n + (\frac{1}{(r_1-4)^3}(-Y_3 + 3Y_2 - 3Y_1 + (r_1-3)(21-9r_1+r_1^2)Y_0) + \frac{1}{2(r_1-4)^2}((r_1-6)Y_3 + (2+5r_1-r_1^2)Y_2 + (46-29r_1+4r_1^2)Y_1 + ((14-3r_1)(r_1-3))Y_0)n + \frac{1}{2(r_1-4)}(-Y_3 + (r_1-1)Y_2 + (5-2r_1)Y_1 + (r_1-3)Y_0)n^2).$

(d) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1 = \dfrac{r_1}{4}$)*

$Y_n = \frac{1}{6}(n(n-1)(n-2)Y_3 - 3n(n-1)(n-3)Y_2 + 3n(n-2)(n-3)Y_1 - (n-3)(n-1)(n-2)Y_0).$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} Y_n x^n$ of the sequence Y_n.

Lemma 2.4.
Suppose that $f_{Y_n}(x) = \sum_{n=0}^{\infty} Y_n x^n$ is the ordinary generating function of the generalized co-Tetranacci (sequence of) polynomials $\{Y_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} Y_n x^n$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} Y_n z^n &= \frac{Y_0 + (Y_1 - r_1 Y_0)z + (Y_2 - r_1 Y_1 - s_1 Y_0)z^2 + (Y_3 - r_1 Y_2 - s_1 Y_1 - t_1 Y_0)z^3}{1 - r_1 z - s_1 z^2 - t_1 z^3 - u_1 z^4} \\ &= \frac{Y_0 + (Y_1 - tY_0)z + (Y_2 - tY_1 - (-su)Y_0)z^2 + (Y_3 - tY_2 - (-su)Y_1 - ru^2 Y_0)z^3}{1 - tx - (-su)x^2 - ru^2 x^3 - u^3 x^4}.\end{aligned}$$

In this paper, we define and investigate, in detail, two special cases of the generalized co-Tetranacci (sequences of) polynomials $\{Y_n\}$ which we call them (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas (sequences of) polynomials. For r,s,t,u satisfying eq. (1), (r,s,t,u)-co-Tetranacci (sequences of) polynomials $\{U_n\}_{n\geq 0}$ and (r,s,t,u)-co-Tetranacci-Lucas (sequences of) polynomials $\{S_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}&U_n = r_1 U_{n-1} + s_1 U_{n-2} + t_1 U_{n-3} + u_1 U_{n-4}, \\ &U_0 = 0, U_1 = 1, U_2 = r_1, U_3 = r_1^2 + s_1,\end{aligned} \tag{23}$$

and

$$\begin{aligned}&S_n = r_1 S_{n-1} + s_1 S_{n-2} + t_1 S_{n-3} + u_1 S_{n-4}, \\ &S_0 = 4, S_1 = r_1, S_2 = 2s_1 + r_1^2, S_3 = r_1^3 + 3s_1 r_1 + 3t_1,\end{aligned} \tag{24}$$

i.e.,

$$\begin{aligned}&U_n = tU_{n-1} - suU_{n-2} + ru^2 U_{n-3} + u^3 U_{n-4}, \\ &U_0 = 0, U_1 = 1, U_2 = t, U_3 = t^2 - su\end{aligned}$$

and

$$\begin{aligned}&S_n = tS_{n-1} - suS_{n-2} + ru^2 S_{n-3} + u^3 S_{n-4}, \\ &S_0 = 4, S_1 = t, S_2 = t^2 - 2su, S_3 = t^3 - 3stu + 3ru^2.\end{aligned}$$

The sequences $\{U_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}U_{-n} &= -\frac{t_1}{u_1} U_{-(n-1)} - \frac{s_1}{u_1} U_{-(n-2)} - \frac{r_1}{u_1} U_{-(n-3)} + \frac{1}{u_1} U_{-(n-4)}, \\ S_{-n} &= -\frac{t_1}{u_1} S_{-(n-1)} - \frac{s_1}{u_1} S_{-(n-2)} - \frac{r_1}{u_1} S_{-(n-3)} + \frac{1}{u_1} S_{-(n-4)},\end{aligned}$$

i.e.,

$$\begin{aligned}U_{-n} &= -\frac{ru^2}{u^3} U_{-(n-1)} - \frac{-su}{u^3} U_{-(n-2)} - \frac{t}{u^3} U_{-(n-3)} + \frac{1}{u^3} U_{-(n-4)}, \\ S_{-n} &= -\frac{ru^2}{u^3} S_{-(n-1)} - \frac{-su}{u^3} S_{-(n-2)} - \frac{t}{u^3} S_{-(n-3)} + \frac{1}{u^3} S_{-(n-4)},\end{aligned}$$

for $n = 1,2,3,\ldots$ respectively. Therefore, recurrences eqs. (23) and (24) hold for all integers n.

Next, we present the first few values of the (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas (sequences of) polynomials with positive and negative subscripts (in terms of r_1, s_1, t_1, u_1):

We present the first few values of the (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials with positive and negative subscripts (in terms of r,s,t,u):

For all integers n, Binet's formula of (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials (using initial conditions in eqs. (23) and (24) can be expressed as follows:

Table 2. The first few values of the special fourth-order numbers with positive and negative subscripts.

n	0	1	2	3	4
U_n	0	1	r_1	$r_1^2+s_1$	$r_1^3+2s_1r_1+t_1$
U_{-n}		0	0	$\frac{1}{u_1}$	$-\frac{t_1}{u_1^2}$
S_n	4	r_1	$2s_1+r_1^2$	$r_1^3+3s_1r_1+3t_1$	$r_1^4+4r_1^2s_1+4t_1r_1+2s_1^2+4u_1$
S_{-n}		$-\frac{t_1}{u_1}$	$\frac{1}{u_1^2}(t_1^2-2s_1u_1)$	$-\frac{1}{u_1^3}(t_1^3-3s_1t_1u_1+3r_1u_1^2)$	$\frac{1}{u_1^4}(2s_1^2u_1^2-4s_1t_1^2u_1+t_1^4+4r_1t_1u_1^2+4u_1^3)$

Table 3. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4
U_n	0	1	t	t^2-su	$t^3-2stu+4\alpha u^2$
U_{-n}		0	0	$\frac{1}{u^3}$	$-\frac{r}{u^4}$
S_n	4	t	t^2-2su	$t^3-3stu+3ru^2$	$2s^2u^2-4st^2u+t^4+4rtu^2+4u^3$
S_{-n}		$-\frac{r}{u}$	$\frac{1}{u^6}\left(r^2u^4+2su^4\right)$	$-\frac{1}{u^3}(3t+3rs+r^3)$	$\frac{1}{u^4}(4u+4r^2s+4rt+r^4+2s^2)$

Theorem 2.3. **(a)** *(Four Distinct Roots Case: $\theta_1\neq\theta_2\neq\theta_3\neq\theta_4$). For all integers n, Binet's formulas of (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials are*

$$\begin{aligned}U_n &= \frac{\theta_1^{n+2}}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)}+\frac{\theta_2^{n+2}}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)}\\ &\quad+\frac{\theta_3^{n+2}}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)}+\frac{\theta_4^{n+2}}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)}\\ &=\frac{\theta_1^{n+3}}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}+\frac{\theta_2^{n+3}}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}\\ &\quad+\frac{\theta_3^{n+3}}{r_1\theta_3^3+2s_1\theta_3^2+3t_1\theta_3+4u_1}+\frac{\theta_4^{n+3}}{r_1\theta_4^3+2s_1\theta_4^2+3t_1\theta_4+4u_1}\end{aligned}$$

and

$$S_n=\theta_1^n+\theta_2^n+\theta_3^n+\theta_4^n.$$

(b) *(Three Distinct Roots Case: $\theta_1\neq\theta_2\neq\theta_3=\theta_4$) For all integers n, Binet's formulas of (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials are*

$U_n=\frac{1}{(\theta_1-\theta_3)^2(\theta_1-\theta_2)}((r_1^2+s_1)-(\theta_2+2\theta_3)r_1+\theta_3(2\theta_2+\theta_3))\theta_1^n+\frac{1}{(\theta_2-\theta_3)^2(\theta_1-\theta_2)}(-(r_1^2+s_1)+(\theta_1+2\theta_3)r_1-\theta_3(2\theta_1+\theta_3))\theta_2^n+(\frac{1}{(\theta_2-\theta_3)^2(\theta_1-\theta_3)^2}((\theta_1+\theta_2-2\theta_3)(r_1^2+s_1)-(\theta_1^2+\theta_2^2-3\theta_3^2+\theta_1\theta_2)r_1+\theta_3(2\theta_1^2+2\theta_2^2+2\theta_1\theta_2-3\theta_1\theta_3-3\theta_2\theta_3))+\frac{1}{\theta_3(\theta_2-\theta_3)(\theta_1-\theta_3)}((r_1^2+s_1)-(\theta_1+\theta_2+\theta_3)r_1+(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3))n)\theta_3^n$

i.e.,

$U_n=\frac{1}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}(\theta_1(r_1^2+s_1)-\theta_1(r_1-\theta_1)r_1+\theta_1(\theta_1^2-r_1\theta_1-s_1))\theta_1^n$
$+\frac{1}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}(\theta_2(r_1^2+s_1)-\theta_2(r_1-\theta_2)r_1+\theta_2(\theta_2^2-r_1\theta_2-s_1))\theta_2^n$
$+(\frac{1}{8\theta_3^8-5r_1\theta_3^7+(r_1^2+s_1)\theta_3^6+t_1\theta_3^5-5u_1\theta_3^4+2r_1u_1\theta_3^3+u_1^2}((r_1-4\theta_3)\theta_3^4(r_1^2+s_1)-(-2\theta_3^2-2r_1\theta_3+r_1^2+s_1)\theta_3^4r_1+\theta_3^3(2r_1\theta_3^3+2(r_1^2+6s_1)\theta_3^2+11t_1\theta_3+12u_1))$
$+\frac{1}{(2r_1^2+3s_1)\theta_3^3+(3t_1+2r_1s_1)\theta_3^2+2(u_1+r_1t_1)\theta_3+2u_1r_1}(\theta_3^2(r_1^2+s_1)-(r_1-\theta_3)\theta_3^2r_1+(-r_1\theta_3^3-2s_1\theta_3^2-2t_1\theta_3-3u_1))n)\theta_3^n$

and

$$S_n=\theta_1^n+\theta_2^n+\theta_3^n+\theta_4^n=\theta_1^n+\theta_2^n+2\theta_3^n.$$

(c) *(Two Distinct Roots Case: $\theta_1\neq\theta_2=\theta_3=\theta_4$) For all integers n, Binet's formulas of (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials are*

$U_n=\frac{1}{(\theta_1-\theta_2)^3}((r_1^2+s_1)-3r_1\theta_2+3\theta_2^2)\theta_1^n+(\frac{1}{(\theta_1-\theta_2)^3}(-(r_1^2+s_1)+3r_1\theta_2-3\theta_2^2)+\frac{1}{2\theta_2^2(\theta_1-\theta_2)^2}((\theta_1-3\theta_2)(r_1^2+s_1)-(\theta_1^2-8\theta_2^2+\theta_1\theta_2)r_1+\theta_2(4\theta_1^2-5\theta_2^2-5\theta_1\theta_2))n+\frac{1}{2\theta_2^2(\theta_1-\theta_2)}(-(r_1^2+s_1)+(\theta_1+2\theta_2)r_1-\theta_2(2\theta_1+\theta_2))n^2)\theta_2^n$

i.e.,

$$U_n = \frac{1}{(r_1-4\theta_2)^3}((r_1^2+s_1)-3r_1\theta_2+3\theta_2^2)\theta_1^n+(\frac{1}{(r_1-4\theta_2)^3}(-(r_1^2+s_1)+3r_1\theta_2-3\theta_2^2)+\frac{1}{2\theta_2^2(r_1-4\theta_2)^2}((r_1-6\theta_2)(r_1^2+s_1)+(2\theta_2^2+5r_1\theta_2-r_1^2)r_1+\theta_2(46\theta_2^2-29r_1\theta_2+4r_1^2))n+\frac{1}{2\theta_2^2(r_1-4\theta_2)}(-(r_1^2+s_1)+(r_1-\theta_2)r_1+\theta_2(5\theta_2-2r_1))n^2)\theta_2^n$$

and

$$S_n = \theta_1^n+\theta_2^n+\theta_3^n+\theta_4^n = \theta_1^n+3\theta_2^n.$$

(d) *(Single Root Case: $\theta_1=\theta_2=\theta_3=\theta_4=\frac{r_1}{4}$) For all integers n, Binet's formulas of (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials are*

$$U_n = \frac{1}{6}n((7\theta_1^2+s_1)n^2-3n(s_1+5\theta_1^2)+14\theta_1^2+2s_1)\theta_1^{n-3}$$

i.e.,

$$U_n = \frac{2n}{3r_1^3}((16s_1+7r_1^2)n^2-3n(16s_1+5r_1^2)+32s_1+14r_1^2)\left(\frac{r_1}{4}\right)^n$$

and

$$S_n = \theta_1^n+\theta_2^n+\theta_3^n+\theta_4^n = 4\theta_1^n.$$

If some of the roots of characteristic equation is 1 then we get the following corollary as a special case of theorem 2.3.

Corollary 2.2.
For all integers n, Binet's formulas of (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials are given as follows:

(a) *(Four Distinct Roots Case: $\theta_1\neq\theta_2\neq\theta_3\neq\theta_4=1$).*

$$\begin{aligned}U_n &= \frac{\theta_1^{n+2}}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-1)}+\frac{\theta_2^{n+2}}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-1)}\\ &\quad+\frac{\theta_3^{n+2}}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-1)}+\frac{1}{(1-\theta_1)(1-\theta_2)(1-\theta_3)}\\ &= \frac{\theta_1^{n+3}}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}+\frac{\theta_2^{n+3}}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}\\ &\quad+\frac{\theta_3^{n+3}}{r_1\theta_3^3+2s_1\theta_3^2+3t_1\theta_3+4u_1}+\frac{1}{r_1+2s_1+3t_1+4u_1}\end{aligned}$$

and

$$S_n = \theta_1^n+\theta_2^n+\theta_3^n+1.$$

(b) *(Three Distinct Roots Case: $\theta_1\neq\theta_2\neq\theta_3=\theta_4=1$).*

$$U_n = \frac{1}{(\theta_1-1)^2(\theta_1-\theta_2)}(s_1-2r_1+2\theta_2-r_1\theta_2+r_1^2+1)\theta_1^n+\frac{1}{(\theta_2-1)^2(\theta_1-\theta_2)}(2r_1-s_1-2\theta_1+r_1\theta_1-r_1^2-1)\theta_2^n+\frac{1}{(\theta_2-1)^2(\theta_1-1)^2}(n(\theta_2-1)(\theta_1-1)(r_1^2-r_1-r_1\theta_1-r_1\theta_2+s_1+\theta_1+\theta_2+\theta_1\theta_2)+2\theta_1^2-r_1\theta_1^2+r_1^2\theta_1+s_1\theta_1-3\theta_1+2\theta_2^2-r_1\theta_2^2+2\theta_1\theta_2+r_1^2\theta_2+s_1\theta_2-r_1\theta_1\theta_2-3\theta_2-2r_1^2+3r_1-2s_1)$$

i.e.,

$$U_n = \frac{1}{r_1\theta_1^3+2s_1\theta_1^2+3t_1\theta_1+4u_1}(\theta_1(r_1^2+s_1)-\theta_1(r_1-\theta_1)r_1+\theta_1(\theta_1^2-r_1\theta_1-s_1))\theta_1^n$$
$$+\frac{1}{r_1\theta_2^3+2s_1\theta_2^2+3t_1\theta_2+4u_1}(\theta_2(r_1^2+s_1)-\theta_2(r_1-\theta_2)r_1+\theta_2(\theta_2^2-r_1\theta_2-s_1))\theta_2^n+(-5r_1+s_1+t_1-5u_1+2r_1u_1+r_1^2+u_1^2+8)^{-1}(3s_1+3t_1+2u_1+2r_1s_1+2r_1t_1+2r_1u_1+2r_1^2)^{-1}(n(s_1+2t_1+3u_1)(-r_1^2-u_1^2+5r_1+5u_1-2r_1u_1-s_1-t_1-8)+(4r_1+8s_1+11t_1+12u_1)(2r_1^2+2u_1+2r_1s_1+2r_1t_1+2r_1u_1+3s_1+3t_1))$$

$$S_n = \theta_1^n+\theta_2^n+2.$$

(c) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = \theta_4 = 1$).*

$U_n = \frac{1}{(\theta_1-1)^3}(r_1^2-3r_1+s_1+3)\theta_1^n + \frac{1}{2(\theta_1-1)^3}(n^2(\theta_1-1)^2(2r_1-r_1^2-s_1-1+r_1\theta_1-2\theta_1)+n(\theta_1-1)((4-r_1)\theta_1^2+(r_1^2-r_1+s_1-5)\theta_1+8r_1-3s_1-3r_1^2-5)+2(-r_1^2+3r_1-s_1-3))$

i.e.,

$U_n = \frac{1}{(r_1-4)^3}(r_1^2-3r_1+s_1+3)\theta_1^n + \frac{1}{2(r_1-4)^3}(n^2(r_1-4)^2(5-3r_1-s_1)+n(r_1-4)(3r_1^2-27r_1+r_1s_1-6s_1+46)+2(-r_1^2+3r_1-s_1-3))$

and

$$S_n = \theta_1^n + 3.$$

(d) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1 = \frac{r_1}{4}$).*

$$U_n = \frac{1}{6}n((s_1+7)n^2 - 3(s_1+5)n + 2s_1 + 14)$$

and

$$S_n = 4.$$

Lemma 2.4 gives the following results as particular examples (generating functions of (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials).

Corollary 2.3.
Generating functions of (r,s,t,u)-co-Tetranacci and (r,s,t,u)-co-Tetranacci-Lucas polynomials are

$$\sum_{n=0}^{\infty} U_n z^n = \frac{U_0 + (U_1 - r_1U_0)z + (U_2 - r_1U_1 - s_1U_0)z^2 + (U_3 - r_1U_2 - s_1U_1 - t_1U_0)z^3}{1 - r_1z - s_1z^2 - t_1z^3 - u_1z^4}$$
$$= \frac{z}{1 - tz - (-su)z^2 - ru^2z^3 - u^3z^4},$$

and

$$\sum_{n=0}^{\infty} S_n z^n = \frac{S_0 + (S_1 - r_1S_0)z + (S_2 - r_1S_1 - s_1S_0)z^2 + (S_3 - r_1S_2 - s_1S_1 - t_1S_0)z^3}{1 - r_1z - s_1z^2 - t_1z^3 - u_1z^4}$$
$$= \frac{4 - 3tz + 2suz^2 - ru^2z^3}{1 - tz - (-su)z^2 - ru^2z^3 - u^3z^4},$$

respectively.

3. Connections between G_n, H_n and U_n, S_n

S_n can be given as follows.

Lemma 3.1.
($\alpha,\beta,\gamma,\delta;\theta_1,\theta_2,\theta_3,\theta_4$: arbitrary) For all integers n, we have the following formula for S_n.

$$S_n = \beta^n\gamma^n\delta^n + \alpha^n\gamma^n\delta^n + \alpha^n\beta^n\delta^n + \alpha^n\beta^n\gamma^n.$$

Proof. Use the identities

$$\theta_1 = \beta\gamma\delta,\ \theta_2 = \alpha\gamma\delta,\ \theta_3 = \alpha\beta\delta,\ \theta_4 = \alpha\beta\gamma$$

and

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n + \theta_4^n. \square$$

For the special cases of $\alpha,\beta,\gamma,\delta;\theta_1,\theta_2,\theta_3,\theta_4$, lemma 3.1 can be written as follows.

Lemma 3.2. **(a)** *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta, \theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4$).*

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n + \theta_4^n = \beta^n\gamma^n\delta^n + \alpha^n\gamma^n\delta^n + \alpha^n\beta^n\delta^n + \alpha^n\beta^n\gamma^n.$$

(b) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = \delta, \theta_1 \neq \theta_2 \neq \theta_3 = \theta_4$).*

$$S_n = \theta_1^n + \theta_2^n + 2\theta_3^n = \gamma^{2n}(\alpha^n + \beta^n) + 2\alpha^n\beta^n\gamma^n.$$

.

(c) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = \delta, \theta_1 \neq \theta_2 = \theta_3 = \theta_4$).*

$$S_n = \theta_1^n + 3\theta_2^n = \beta^{2n}(3\alpha^n + \beta^n).$$

(d) *(Single Root Case: $\alpha = \beta = \gamma = \delta, \theta_1 = \theta_2 = \theta_3 = \theta_4$).*

$$S_n = 4\theta_1^n = 4\alpha^{3n}.$$

We can present the relations between U_n, S_n and G_n, H_n as follows.

Lemma 3.3.
For all integers n, we have the following formulas:

(a) $U_n = -(-u)^n G_{-n-2}$ *and* $U_{-n} = -(-u)^{-n} G_{n-2}$.

(b) $S_n = (-u)^n H_{-n}$ *and* $S_{-n} = (-u)^{-n} H_n$.

(c) $S_n = \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)$.

Proof. Use theorems 1.4 and 2.3 and the identities

$$\alpha\beta\gamma\delta = -u,$$
$$\theta_1 = \beta\gamma\delta,\ \theta_2 = \alpha\gamma\delta,\ \theta_3 = \alpha\beta\delta,\ \theta_4 = \alpha\beta\gamma,$$
$$S_n = \beta^n\gamma^n\delta^n + \alpha^n\gamma^n\delta^n + \alpha^n\beta^n\delta^n + \alpha^n\beta^n\gamma^n.$$

(a) We only prove the four distinct roots case. The other cases can be proved similarly.

(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta, \theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4$). By using theorems 1.4 and 2.3 we get

$$\begin{aligned}
U_n &= \frac{\theta_1^{n+2}}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)} + \frac{\theta_2^{n+2}}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)} \\
&\quad + \frac{\theta_3^{n+2}}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)} + \frac{\theta_4^{n+2}}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)} \\
&= \frac{(\beta\gamma\delta)^{n+2}}{(\beta\gamma\delta-\alpha\gamma\delta)(\beta\gamma\delta-\alpha\beta\delta)(\beta\gamma\delta-\alpha\beta\gamma)} + \frac{(\alpha\gamma\delta)^{n+2}}{(\alpha\gamma\delta-\beta\gamma\delta)(\alpha\gamma\delta-\alpha\beta\delta)(\alpha\gamma\delta-\alpha\beta\gamma)} \\
&\quad + \frac{(\alpha\beta\delta)^{n+2}}{(\alpha\beta\delta-\beta\gamma\delta)(\alpha\beta\delta-\alpha\gamma\delta)(\alpha\beta\delta-\alpha\beta\gamma)} + \frac{(\alpha\beta\gamma)^{n+2}}{(\alpha\beta\gamma-\beta\gamma\delta)(\alpha\beta\gamma-\alpha\gamma\delta)(\alpha\beta\gamma-\alpha\beta\delta)} \\
&= \frac{(\beta\gamma\delta)^{n+2}}{-\beta^2\gamma^2\delta^2(\alpha-\delta)(\alpha-\gamma)(\alpha-\beta)} + \frac{(\alpha\gamma\delta)^{n+2}}{-\alpha^2\gamma^2\delta^2(\beta-\delta)(\beta-\gamma)(\beta-\alpha)} \\
&\quad + \frac{(\alpha\beta\delta)^{n+2}}{-\alpha^2\beta^2\delta^2(\gamma-\delta)(\beta-\gamma)(\alpha-\gamma)} + \frac{(\alpha\beta\gamma)^{n+2}}{-\alpha^2\beta^2\gamma^2(\gamma-\delta)(\beta-\delta)(\delta-\alpha)} \\
&= -(-u)^n\Big(\frac{\alpha^{-n}}{(\alpha-\delta)(\alpha-\gamma)(\alpha-\beta)} + \frac{\beta^{-n}}{(\beta-\delta)(\beta-\gamma)(\beta-\alpha)} \\
&\quad + \frac{\gamma^{-n}}{(\gamma-\delta)(\beta-\gamma)(\alpha-\gamma)} + \frac{\delta^{-n}}{(\gamma-\delta)(\beta-\delta)(\delta-\alpha)}\Big) \\
&= -(-u)^n G_{-n-2}.
\end{aligned}$$

(b) We only prove the four distinct roots case. The other cases can be proved similarly.

((Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$, $\theta_1 \neq \theta_2 \neq \theta_3 \neq \theta_4$).

$$\begin{aligned} S_n &= \theta_1^n + \theta_2^n + \theta_3^n + \theta_4^n \\ &= (\beta\gamma\delta)^n + (\alpha\gamma\delta)^n + (\alpha\beta\delta)^n + (\alpha\beta\gamma)^n \\ &= (-u\alpha^{-1})^n + (-u\beta^{-1})^n + (-u\gamma^{-1})^n + (-u\delta^{-1})^n \\ &= (-u)^n\alpha^{-n} + (-u)^n\beta^{-n} + (-u)^n\gamma^{-n} + (-u)^n\delta^{-n} \\ &= (-u)^n(\alpha^{-n} + \beta^{-n} + \gamma^{-n} + \delta^{-n}) \\ &= (-u)^n H_{-n}. \end{aligned}$$

(c) Using the formula $S_n = \beta^n\gamma^n\delta^n + \alpha^n\gamma^n\delta^n + \alpha^n\beta^n\delta^n + \alpha^n\beta^n\gamma^n$, we get

$$\begin{aligned} 6S_n &= 6(\alpha^n\beta^n\gamma^n + \alpha^n\beta^n\delta^n + \alpha^n\gamma^n\delta^n + \beta^n\gamma^n\delta^n) \\ &= (\alpha^n + \beta^n + \gamma^n + \delta^n)^3 + 2(\alpha^{3n} + \beta^{3n} + \gamma^{3n} + \delta^{3n}) \\ &\quad -3(\alpha^{2n} + \beta^{2n} + \gamma^{2n} + \delta^{2n})(\alpha^n + \beta^n + \gamma^n + \delta^n) \\ &= H_n^3 + 2H_{3n} - 3H_{2n}H_n \\ &\Rightarrow \\ S_n &= \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n). \ \square \end{aligned}$$

4. Simson's Formulas of Generalized Tetranacci Polynomials

The following theorem gives Simson's formula of the generalized Tetranacci polynomials $\{W_n\}$.

Theorem 4.1 (Simson's Formula of Generalized Tetranacci Polynomials).
For all integers n, we have

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{25}$$

Proof. eq. (25) can be proved by mathematical induction. For the proof of the case of generalized Tetranacci numbers, see Soykan [[12], Theorem 2.3]. For an alternative proof, proof by matrix method, see theorem 18.3 (b). $\square$

The previous theorem gives the following results as particular examples.

Corollary 4.1.
For all integers n, Simson formula of (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials are given as, respectively,

$$\begin{vmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{vmatrix} = (-1)^{n+1} u^{n-1},$$

$$\begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix} = (-1)^n u^{n-3} g(r,s,t,u),$$

where
$g(r,s,t,u) = 27r^4u^2 - 18r^3stu + 4r^3t^3 + 4r^2s^3u - r^2s^2t^2 + 144r^2su^2 - 6r^2t^2u - 80rs^2tu + 18rst^3 + 192rtu^2 + 16s^4u - 4s^3t^2 + 128s^2u^2 - 144st^2u + 27t^4 + 256u^3$.

Note also that eq. (25) can be written as

$$\begin{vmatrix} W_{n+m+3} & W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+2} & W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} & W_{n+m-2} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} & W_{n+m-3} \end{vmatrix} = (-1)^{n+m} u^{n+m} \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}$$

for all integers n, m.

We define

$\Lambda_W(n) = -W_{n+3}^4+(u+r^2s+rt)W_{n+2}^4+(st^2-u^2-rtu)W_{n+1}^4+u^3W_n^4+3rW_{n+2}W_{n+3}^3+2sW_{n+1}W_{n+3}^3+tW_nW_{n+3}^3+(-t-2rs+r^3)W_{n+2}^3W_{n+3}+(2rs^2+r^2t+ru+st)W_{n+2}^3W_{n+1}+(r^2u-2su+t^2+rst)W_{n+2}^3W_n+(-r^2u-2su-t^2+rst)W_{n+1}^3W_{n+3}+(rt^2+2s^2t+tu-rsu)W_{n+1}^3W_{n+2}+(-ru^2+t^3+2stu)W_{n+1}^3W_n+ru^2W_n^3W_{n+3}+2su^2W_n^3W_{n+2}+3tu^2W_n^3W_{n+1}+(s-3r^2)W_{n+3}^2W_{n+2}^2+(2u-rt-s^2)W_{n+3}^2W_{n+1}^2-suW_{n+3}^2W_n^2+s(3u+3rt+s^2)W_{n+2}^2W_{n+1}^2+u(-2u+rt+s^2)W_{n+2}^2W_n^2+u(su+3t^2)W_{n+1}^2W_n^2+(3t-4rs)W_{n+3}^2W_{n+2}W_{n+1}+2(2u-rt)W_{n+3}^2W_{n+2}W_n-(3ru+st)W_{n+3}^2W_{n+1}W_n+2(-2u+r^2s-2rt-s^2)W_{n+2}^2W_{n+3}W_{n+1}+(r^2t-5ru-st)W_{n+2}^2W_{n+3}W_n+(rt^2+s^2t+tu+5rsu)W_{n+2}^2W_{n+1}W_n+(rs^2+r^2t+ru-5st)W_{n+1}^2W_{n+3}W_{n+2}+(rt^2-5tu+rsu)W_{n+1}^2W_{n+3}W_n+2(st^2+s^2u+2u^2+2rtu)W_{n+1}^2W_{n+2}W_n+u(-3t+rs)W_n^2W_{n+3}W_{n+2}+2u(-2u+rt)W_n^2W_{n+3}W_{n+1}+u(3ru+4st)W_n^2W_{n+2}W_{n+1}+(3r^2u-4su-3t^2+rst)W_{n+3}W_{n+2}W_{n+1}W_n$

then

$\Lambda_W(0) = -W_3^4+(u+r^2s+rt)W_2^4+(st^2-u^2-rtu)W_1^4+u^3W_0^4+3rW_2W_3^3+2sW_1W_3^3+tW_0W_3^3+(-t-2rs+r^3)W_2^3W_3+(2rs^2+r^2t+ru+st)W_2^3W_1+(r^2u-2su+t^2+rst)W_2^3W_0+(-r^2u-2su-t^2+rst)W_1^3W_3+(rt^2+2s^2t+tu-rsu)W_1^3W_2+(-ru^2+t^3+2stu)W_1^3W_0+ru^2W_0^3W_3+2su^2W_0^3W_2+3tu^2W_0^3W_1+(s-3r^2)W_3^2W_2^2+(2u-rt-s^2)W_3^2W_1^2-suW_3^2W_0^2+s(3u+3rt+s^2)W_2^2W_1^2+u(-2u+rt+s^2)W_2^2W_0^2+u(su+3t^2)W_1^2W_0^2+(3t-4rs)W_3^2W_2W_1+2(2u-rt)W_3^2W_2W_0-(3ru+st)W_3^2W_1W_0+2(-2u+r^2s-2rt-s^2)W_2^2W_3W_1+(r^2t-5ru-st)W_2^2W_3W_0+(rt^2+s^2t+tu+5rsu)W_2^2W_1W_0+(rs^2+r^2t+ru-5st)W_1^2W_3W_2+(rt^2-5tu+rsu)W_1^2W_3W_0+2(st^2+s^2u+2u^2+2rtu)W_1^2W_2W_0+u(-3t+rs)W_0^2W_3W_2+2u(-2u+rt)W_0^2W_3W_1+u(3ru+4st)W_0^2W_2W_1+(3r^2u-4su-3t^2+rst)W_3W_2W_1W_0.$

For the special cases of W_n we have

$\Lambda_G(n) = -G_{n+3}^4+(u+r^2s+rt)G_{n+2}^4+(st^2-u^2-rtu)G_{n+1}^4+u^3G_n^4+3rG_{n+2}G_{n+3}^3+2sG_{n+1}G_{n+3}^3+tG_nG_{n+3}^3+(-t-2rs+r^3)G_{n+2}^3G_{n+3}+(2rs^2+r^2t+ru+st)G_{n+2}^3G_{n+1}+(r^2u-2su+t^2+rst)G_{n+2}^3G_n+(-r^2u-2su-t^2+rst)G_{n+1}^3G_{n+3}+(rt^2+2s^2t+tu-rsu)G_{n+1}^3G_{n+2}+(-ru^2+t^3+2stu)G_{n+1}^3G_n+ru^2G_n^3G_{n+3}+2su^2G_n^3G_{n+2}+3tu^2G_n^3G_{n+1}+(s-3r^2)G_{n+3}^2G_{n+2}^2+(2u-rt-s^2)G_{n+3}^2G_{n+1}^2-suG_{n+3}^2G_n^2+s(3u+3rt+s^2)G_{n+2}^2G_{n+1}^2+u(-2u+rt+s^2)G_{n+2}^2G_n^2+u(su+3t^2)G_{n+1}^2G_n^2+(3t-4rs)G_{n+3}^2G_{n+2}G_{n+1}+2(2u-rt)G_{n+3}^2G_{n+2}G_n-(3ru+st)G_{n+3}^2G_{n+1}G_n+2(-2u+r^2s-2rt-s^2)G_{n+2}^2G_{n+3}G_{n+1}+(r^2t-5ru-st)G_{n+2}^2G_{n+3}G_n+(rt^2+s^2t+tu+5rsu)G_{n+2}^2G_{n+1}G_n+(rs^2+r^2t+ru-5st)G_{n+1}^2G_{n+3}G_{n+2}+(rt^2-5tu+rsu)G_{n+1}^2G_{n+3}G_n+2(st^2+s^2u+2u^2+2rtu)G_{n+1}^2G_{n+2}G_n+u(-3t+rs)G_n^2G_{n+3}G_{n+2}+2u(-2u+rt)G_n^2G_{n+3}G_{n+1}+u(3ru+4st)G_n^2G_{n+2}G_{n+1}+(3r^2u-4su-3t^2+rst)G_{n+3}G_{n+2}G_{n+1}G_n$

and

$\Lambda_G(0) = -u^2$

and also

$\Lambda_H(n) = -H_{n+3}^4+(u+r^2s+rt)H_{n+2}^4+(st^2-u^2-rtu)H_{n+1}^4+u^3H_n^4+3rH_{n+2}H_{n+3}^3+2sH_{n+1}H_{n+3}^3+tH_nH_{n+3}^3+(-t-2rs+r^3)H_{n+2}^3H_{n+3}+(2rs^2+r^2t+ru+st)H_{n+2}^3H_{n+1}+(r^2u-2su+t^2+rst)H_{n+2}^3H_n+(-r^2u-2su-t^2+rst)H_{n+1}^3H_{n+3}+(rt^2+2s^2t+tu-rsu)H_{n+1}^3H_{n+2}+(-ru^2+t^3+2stu)H_{n+1}^3H_n+ru^2H_n^3H_{n+3}+2su^2H_n^3H_{n+2}+3tu^2H_n^3H_{n+1}+(s-3r^2)H_{n+3}^2H_{n+2}^2+(2u-rt-s^2)H_{n+3}^2H_{n+1}^2-suH_{n+3}^2H_n^2+s(3u+3rt+s^2)H_{n+2}^2H_{n+1}^2+u(-2u+rt+s^2)H_{n+2}^2H_n^2+u(su+3t^2)H_{n+1}^2H_n^2+(3t-4rs)H_{n+3}^2H_{n+2}H_{n+1}+2(2u-rt)H_{n+3}^2H_{n+2}H_n-(3ru+st)H_{n+3}^2H_{n+1}H_n+2(-2u+r^2s-2rt-s^2)H_{n+2}^2H_{n+3}H_{n+1}+(r^2t-5ru-st)H_{n+2}^2H_{n+3}H_n+(rt^2+s^2t+tu+5rsu)H_{n+2}^2H_{n+1}H_n+(rs^2+r^2t+ru-5st)H_{n+1}^2H_{n+3}H_{n+2}+(rt^2-5tu+rsu)H_{n+1}^2H_{n+3}H_n+2(st^2+s^2u+2u^2+2rtu)H_{n+1}^2H_{n+2}H_n+u(-3t+rs)H_n^2H_{n+3}H_{n+2}+2u(-2u+rt)H_n^2H_{n+3}H_{n+1}+u(3ru+4st)H_n^2H_{n+2}H_{n+1}+(3r^2u-4su-3t^2+rst)H_{n+3}H_{n+2}H_{n+1}H_n$

and

$\Lambda_H(0) = 27r^4u^2-18r^3stu+4r^3t^3+4r^2s^3u-r^2s^2t^2+144r^2su^2-6r^2t^2u-80rs^2tu+18rst^3+192rtu^2+16s^4u-4s^3t^2+128s^2u^2-144st^2u+27t^4+256u^3.$

Simson's formulas of W_n, G_n H_n can be given in the following forms.

Lemma 4.1.
For all integers n, we have

(a) $\Lambda_W(n) = (-1)^n u^n \Lambda_W(0).$

(b) $\Lambda_G(n) = (-1)^n u^n \Lambda_G(0) = (-1)^n u^n(-u^2) = (-1)^{n+1}u^{n+2}.$

(c) $\Lambda_H(n) = (-1)^n u^n \Lambda_H(0) = (-1)^n u^n g(r,s,t,u)$

where $g(r,s,t,u) = 27r^4u^2-18r^3stu+4r^3t^3+4r^2s^3u-r^2s^2t^2+144r^2su^2-6r^2t^2u-80rs^2tu+18rst^3+192rtu^2+16s^4u-4s^3t^2+128s^2u^2-144st^2u+27t^4+256u^3.$

Proof.

(a) Note that eq. (25) can be written in the following form:

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}$$

$$\Leftrightarrow$$

$$\frac{1}{u^3}\Lambda_W(n) = (-1)^n u^n \frac{1}{u^3}\Lambda_W(0)$$

$$\Leftrightarrow$$

$$\Lambda_W(n) = (-1)^n u^n \Lambda_W(0)$$

where

$$W_{n-1} = \frac{1}{u}(W_{n+3} - rW_{n+2} - sW_{n+1} - tW_n)$$
$$W_{n-2} = \frac{1}{u^2}(-tW_{n+3} + (u+rt)W_{n+2} + (st-ru)W_{n+1} + (-su+t^2)W_n)$$
$$W_{n-3} = \frac{1}{u^3}((t^2-su)W_{n+3} + (rsu-rt^2-tu)W_{n+2} + (u^2-st^2+rtu+s^2u)W_{n+1} + (2stu-t^3-ru^2)W_n)$$

and

$$W_{-1} = \frac{1}{u}(W_3 - rW_2 - sW_1 - tW_0)$$
$$W_{-2} = \frac{1}{u^2}(-tW_3 + (u+rt)W_2 + (st-ru)W_1 + (-su+t^2)W_0)$$
$$W_{-3} = \frac{1}{u^3}((t^2-su)W_3 + (rsu-rt^2-tu)W_2 + (u^2-st^2+rtu+s^2u)W_1 + (2stu-t^3-ru^2)W_0)$$

So we get the identity in (a).

(b) Take $W_n = G_n$ in (a).

(c) Take $W_n = H_n$ in (a). □

5. Some Identities of Generalized Tetranacci Polynomials

In this section, we obtain some identities of generalized Tetranacci, (r,s,t,u)-Tetranacci and (r,s,t,u)-Tetranacci-Lucas polynomials. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 5.1.
The following equalities are true:

(a) $u^3H_n = (-t^3+3stu-3ru^2)G_{n+4} + (rt^3-2su^2+t^2u+3r^2u^2-3rstu)G_{n+3} + (-3s^2tu+st^3+5rsu^2-rt^2u-tu^2)G_{n+2} + (2s^2u^2+t^4+4u^3+4rtu^2-4st^2u)G_{n+1}$.

(b) $u^2H_n = (-2su+t^2)G_{n+3} - (rt^2+tu-2rsu)G_{n+2} + (-st^2+2s^2u+4u^2+rtu)G_{n+1} - (3ru^2+t^3-3stu)G_n$.

(c) $uH_n = -tG_{n+2} + (4u+rt)G_{n+1} + (-3ru+st)G_n + (-2su+t^2)G_{n-1}$.

(d) $H_n = 4G_{n+1} - 3rG_n - 2sG_{n-1} - tG_{n-2}$.

(e) $(16s^4u+4r^3t^3-4s^3t^2+27r^4u^2+128s^2u^2+27t^4+256u^3-r^2s^2t^2+18rst^3+192rtu^2-144st^2u+144r^2su^2+4r^2s^3u-6r^2t^2u-80rs^2tu-18r^3stu)G_n = (-6st^2+16s^2u-2r^2t^2+64u^2+6r^2su+8rtu)H_{n+3} + (-6r^3su+2r^3t^2-5r^2tu-20rs^2u+7rst^2-16ru^2-32stu+9t^3)H_{n+2} + (-3r^3tu-2r^2s^2u+r^2st^2-12r^2u^2-4rstu-3rt^3-8s^3u+4s^2t^2-32su^2-12t^2u)H_{n+1} + (-9r^3u^2+r^2stu-32rsu^2-3rt^2u+4s^2tu-48tu^2)H_n$.

(f) $(16s^4u+4r^3t^3-4s^3t^2+27r^4u^2+128s^2u^2+27t^4+256u^3-r^2s^2t^2+18rst^3+192rtu^2-144st^2u+144r^2su^2+4r^2s^3u-6r^2t^2u-80rs^2tu-18r^3stu)G_n = (48ru^2+9t^3+rst^2-4rs^2u+3r^2tu-32stu)H_{n+2} + (-3r^3tu+4r^2s^2u-r^2st^2-12r^2u^2+4rstu-3rt^3+8s^3u-2s^2t^2+32su^2-12t^2u)H_{n+1} + (-9r^3u^2+7r^2stu-2r^2t^3-32rsu^2+5rt^2u+20s^2tu-6st^3+16tu^2)H_n + (16s^2u^2+64u^3+8rtu^2-6st^2u+6r^2su^2-2r^2t^2u)H_{n-1}$.

(g) $(16s^4u+4r^3t^3-4s^3t^2+27r^4u^2+128s^2u^2+27t^4+256u^3-r^2s^2t^2+18rst^3+192rtu^2-144st^2u+144r^2su^2+4r^2s^3u-6r^2t^2u-80rs^2tu-18r^3stu)G_n=(6rt^3+32su^2+8s^3u-12t^2u+36r^2u^2-2s^2t^2-28rstu)H_{n+1}+(3st^3+16tu^2-2r^2t^3-9r^3u^2+16rsu^2-4rs^3u+5rt^2u-12s^2tu+rs^2t^2+10r^2stu)H_n+(16s^2u^2+9t^4+64u^3+rst^3+56rtu^2-38st^2u+6r^2su^2+r^2t^2u-4rs^2tu)H_{n-1}+(48ru^3+9t^3u-32stu^2-4rs^2u^2+3r^2tu^2+rst^2u)H_{n-2}.$

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$H_n=a\times G_{n+4}+b\times G_{n+3}+c\times G_{n+2}+d\times G_{n+1}$$

and solving the system of equations

$$H_0=a\times G_4+b\times G_3+c\times G_2+d\times G_1$$
$$H_1=a\times G_5+b\times G_4+c\times G_3+d\times G_2$$
$$H_2=a\times G_6+b\times G_5+c\times G_4+d\times G_3$$
$$H_3=a\times G_7+b\times G_6+c\times G_5+d\times G_4$$

we find that

$$a=\frac{1}{u^3}(-t^3+3stu-3ru^2),$$
$$b=\frac{1}{u^3}(rt^3-2su^2+t^2u+3r^2u^2-3rstu),$$
$$c=\frac{1}{u^3}(-3s^2tu+st^3+5rsu^2-rt^2u-tu^2),$$
$$d=\frac{1}{u^3}(2s^2u^2+t^4+4u^3+4rtu^2-4st^2u).$$

The other equalities can be proved similarly. □

Next, we give a few basic relations between $\{G_n\}$ and $\{W_n\}$.

Lemma 5.2.

The following equalities are true:

(a) $u^2W_n=(-tW_3+(u+rt)W_2+(st-ru)W_1+(t^2-su)W_0)G_{n+3}+((u+rt)W_3-r(2u+rt)W_2+(r^2u-rst-su)W_1+(rsu-tu-rt^2)W_0)G_{n+2}+((st-ru)W_3+(r^2u-su-rst)W_2+s(2ru-st)W_1+(rtu+u^2+s^2u-st^2)W_0)G_{n+1}+((t^2-su)W_3+(rsu-tu-rt^2)W_2+(rtu+u^2-st^2+s^2u)W_1+(2stu-ru^2-t^3)W_0)G_n.$

(b) $uW_n=(W_3-rW_2-sW_1-tW_0)G_{n+2}+(-rW_3+r^2W_2+rsW_1+(u+rt)W_0)G_{n+1}+(-sW_3+rsW_2+(u+s^2)W_1+(st-ru)W_0)G_n+(-tW_3+(u+rt)W_2+(st-ru)W_1+(t^2-su)W_0)G_{n-1}.$

(c) $W_n=W_0G_{n+1}+(W_1-rW_0)G_n+(W_2-rW_1-sW_0)G_{n-1}+(W_3-rW_2-sW_1-tW_0)G_{n-2}.$

Now, we present a basic relation between $\{H_n\}$ and $\{W_n\}$.

Lemma 5.3.

The following equality is true:

$(16s^4u+4r^3t^3-4s^3t^2+27r^4u^2+128s^2u^2+27t^4+256u^3-r^2s^2t^2+18rst^3+192rtu^2-144st^2u+144r^2su^2+4r^2s^3u-6r^2t^2u-80rs^2tu-18r^3stu)W_n=X_1H_{n+3}+X_2H_{n+2}+X_3H_{n+1}+X_4H_n$

where

$X_1=2(3r^3t-6r^2u-r^2s^2-16su-4s^3+18t^2+14rst)W_3+(8rs^3-39rt^2-6r^4t+4s^2t+3r^3u+2r^3s^2-48tu-27r^2st)W_2+(-42st^2+9r^4u+48s^2u+2r^2s^3+r^2t^2+8s^4+64u^2-32rs^2t-7r^3st+50r^2su+56rtu)W_1+(-16ru^2+4s^3t-4r^3t^2-27t^3-18rst^2+12rs^2u+3r^3su+7r^2tu+r^2s^2t+48stu)W_0,$

$X_2=(8rs^3-39rt^2-6r^4t+4s^2t+3r^3u+2r^3s^2-48tu-27r^2st)W_3+2(-3st^2+3r^5t+3r^4u+8s^2u-4r^2s^3+20r^2t^2-r^4s^2+32u^2-4rs^2t+13r^3st+19r^2su+52rtu)W_2+(-8rs^4-80ru^2-4s^3t-9r^5u-2r^3s^3+r^3t^2+9t^3+52rst^2-36rs^2u+7r^4st-47r^3su-61r^2tu+31r^2s^2t+16stu)W_1+(-3r^4su+4r^4t^2-r^3s^2t-r^3tu-14r^2s^2u+18r^2st^2+4r^2u^2-4rs^3t-20rstu+27rt^3-8s^3u-32su^2+36t^2u)W_0,$

$X_3=(-6r^4tu-r^3s^2u+4r^3st^2+3r^3u^2-r^2s^3t-34r^2stu-4rs^3u+18rs^2t^2+16rsu^2-39rt^2u-4s^4t-44s^2tu+27st^3-48tu^2)W_0+2(-6r^4su-r^4t^2+4r^3s^2t+r^3tu-r^2s^4-32r^2s^2u-5r^2st^2+2r^2u^2+18rs^3t-38rstu-6rt^3-4s^5-28s^3u+23s^2t^2-48su^2-6t^2u)W_1+(-8rs^4-80ru^2-4s^3t-9r^5u-2r^3s^3+r^3t^2+9t^3+52rst^2-36rs^2u+7r^4st-47r^3su-61r^2tu+31r^2s^2t+16stu)W_2+(-42st^2+9r^4u+48s^2u+2r^2s^3+r^2t^2+8s^4+64u^2-32rs^2t-7r^3st+50r^2su+56rtu)W_3,$

$X_4=(-16ru^2+4s^3t-4r^3t^2-27t^3-18rst^2+12rs^2u+3r^3su+7r^2tu+r^2s^2t+48stu)W_3+(-3r^4su+4r^4t^2-r^3s^2t-r^3tu-14r^2s^2u+18r^2st^2+4r^2u^2-4rs^3t-20rstu+27rt^3-8s^3u-32su^2+36t^2u)W_2+(-6r^4tu-r^3s^2u+4r^3st^2+3r^3u^2-r^2s^3t-34r^2stu-4rs^3u+18rs^2t^2+16rsu^2-39rt^2u-4s^4t-44s^2tu+27st^3-48tu^2)W_1+(8s^4u+4r^3t^3-4s^3t^2+9r^4u^2+48s^2u^2+27t^4+64u^3-r^2s^2t^2+18rst^3+72rtu^2-90st^2u+50r^2su^2+2r^2s^3u-6r^2t^2u-44rs^2tu-10r^3stu)W_0.$

6. Recurrence Properties of Generalized Tetranacci Polynomials

We can propose an open problem as follows: Whether and how can the generalized Tetranacci polynomial sequence W_n at negative indices be expressed by the sequence itself at positive indices?

We present a result as follows which completely solves the above problem for the generalized Tetranacci polynomial sequence W_n.

Theorem 6.1.
For $n \in \mathbb{Z}$, we have

$$\begin{aligned}W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n}+6H_nW_{2n}-3H_n^2W_n+3H_{2n}W_n+W_0H_n^3+2W_0H_{3n}-3W_0H_nH_{2n})\\&=(-1)^{-n-1}u^{-n}(W_{3n}-H_nW_{2n}+\frac{1}{2}(H_n^2-H_{2n})W_n-\frac{1}{6}(H_n^3+2H_{3n}-3H_{2n}H_n)W_0).\end{aligned}$$

Note that H_n can be written in terms of W_n using remark 6.1 below.

For the proof of theorem 6.1, we need the following lemma.

Lemma 6.1.

(a) *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$). If A_1, A_2, A_3 and A_4 are as in theorem 1.1 (a), then we have*

$$\begin{aligned}(-u)^nW_{-n} &= A_1\beta^n\gamma^n\delta^n+A_2\alpha^n\gamma^n\delta^n+A_3\alpha^n\beta^n\delta^n+A_4\alpha^n\beta^n\gamma^n\\&=(\beta^n\gamma^n\delta^n+\alpha^n\gamma^n\delta^n+\alpha^n\beta^n\delta^n+\alpha^n\beta^n\gamma^n)W_0-\beta^n\gamma^n\alpha^n(A_1+A_2+A_3)\\&\quad-\alpha^n\beta^n\delta^n(A_1+A_2+A_4)-\alpha^n\gamma^n\delta^n(A_1+A_3+A_4)-\beta^n\gamma^n\delta^n(A_2+A_3+A_4).\end{aligned}$$

(b) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = \delta$). If A_1, A_2, A_3 and A_4 are as in theorem 1.1 (b), then we have*

$$\begin{aligned}(-u)^nW_{-n} &= A_1\beta^n\gamma^{2n}+A_2\alpha^n\gamma^{2n}+(A_3-A_4n)\alpha^n\beta^n\gamma^n\\&=\gamma^n(\beta^n\gamma^n+2\alpha^n\beta^n+\alpha^n\gamma^n)W_0-\gamma^{2n}(\alpha^nA_1+\beta^nA_2+(\alpha^n+\beta^n)A_3)\\&\quad-\alpha^n\beta^n\gamma^n(2A_1+2A_2+A_3+nA_4).\end{aligned}$$

(c) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = \delta$). If A_1, A_2, A_3 and A_4 are as in theorem 1.1 (c), then we have*

$$\begin{aligned}(-u)^nW_{-n} &= A_1\beta^{3n}+(A_2-A_3n+A_4n^2)\alpha^n\beta^{2n}\\&=\beta^{2n}\left(3\alpha^n+\beta^n\right)W_0+\beta^{2n}(-\alpha^n(3A_1+2A_2)-\beta^nA_2-n\alpha^nA_3+n^2\alpha^nA_4).\end{aligned}$$

(d) *(Single Root Case: $\alpha = \beta = \gamma = \delta = \frac{r}{4}$). If A_1, A_2, A_3 and A_4 are as in theorem 1.1 (d), then we have*

$$\begin{aligned}(-u)^nW_{-n} &= (A_1-A_2n+A_3n^2-A_4n^3)\alpha^{3n}\\&=(4W_0-3A_1-A_2n+A_3n^2-A_4n^3)\alpha^{3n}.\end{aligned}$$

Proof.

(a) Since $\alpha\beta\gamma\delta = -u$ and using theorem 1.1 (a), i,e., $W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n$, we get

$$\begin{aligned}W_{-n} &= A_1\alpha^{-n}+A_2\beta^{-n}+A_3\gamma^{-n}+A_4\delta^{-n}\\&=A_1\beta^n\gamma^n\delta^n(-u)^{-n}+A_2\alpha^n\gamma^n\delta^n(-u)^{-n}+A_4\alpha^n\beta^n\gamma^n(-u)^{-n}+A_3\alpha^n\beta^n\delta^n(-u)^{-n}\end{aligned}$$

and so

$$(-u)^nW_{-n} = A_1\beta^n\gamma^n\delta^n+A_2\alpha^n\gamma^n\delta^n+A_3\alpha^n\beta^n\delta^n+A_4\alpha^n\beta^n\gamma^n.$$

Note that by using A_1, A_2, A_3 and A_4, we get

$$\begin{aligned}A_1\beta^n\gamma^n\delta^n+A_2\alpha^n\gamma^n\delta^n+A_3\alpha^n\beta^n\delta^n+A_4\alpha^n\beta^n\gamma^n &= (\beta^n\gamma^n\delta^n+\alpha^n\gamma^n\delta^n+\alpha^n\beta^n\delta^n+\alpha^n\beta^n\gamma^n)W_0\\&\quad-\beta^n\gamma^n\alpha^n(W_0-A_4)-\alpha^n\beta^n\delta^n(W_0-A_3)\\&\quad-\alpha^n\gamma^n\delta^n(W_0-A_2)-\beta^n\gamma^n\delta^n(W_0-A_1)\\&=(\beta^n\gamma^n\delta^n+\alpha^n\gamma^n\delta^n+\alpha^n\beta^n\delta^n+\alpha^n\beta^n\gamma^n)W_0\\&\quad-\beta^n\gamma^n\alpha^n(A_1+A_2+A_3)-\alpha^n\beta^n\delta^n(A_1+A_2+A_4)\\&\quad-\alpha^n\gamma^n\delta^n(A_1+A_3+A_4)-\beta^n\gamma^n\delta^n(A_2+A_3+A_4)\end{aligned}$$

where $A_1 + A_2 + A_3 + A_4 = W_0$, (lemma 1.2 (a)).

(b) Since $\alpha\beta\gamma\delta=-u$ (i.e., $\alpha\beta\gamma^2=-u$ in this case) and using theorem 1.1 (b), i,e., $W_n=A_1\alpha^n+A_2\beta^n+(A_3+A_4n)\gamma^n$, we obtain

$$\begin{aligned}W_{-n}&=A_1\alpha^{-n}+A_2\beta^{-n}+(A_3-A_4n)\gamma^{-n}\\&=A_1\beta^n\gamma^{2n}(-u)^{-n}+A_2\alpha^n\gamma^{2n}(-u)^{-n}+(A_3-A_4n)\alpha^n\beta^n\gamma^n(-u)^{-n}\end{aligned}$$

and so

$$(-u)^nW_{-n}=A_1\beta^n\gamma^{2n}+A_2\alpha^n\gamma^{2n}+(A_3-A_4n)\alpha^n\beta^n\gamma^n.$$

Note that by using A_1,A_2,A_3 and A_4, we get

$$\begin{aligned}A_1\beta^n\gamma^{2n}+A_2\alpha^n\gamma^{2n}+(A_3-A_4n)\alpha^n\beta^n\gamma^n&=\gamma^n(\beta^n\gamma^n+2\alpha^n\beta^n+\alpha^n\gamma^n)W_0-\gamma^{2n}(\alpha^nA_1+\beta^nA_2\\&\quad+(\alpha^n+\beta^n)A_3)-\alpha^n\beta^n\gamma^n(2A_1+2A_2+A_3+nA_4)\end{aligned}$$

(c) Since $\alpha\beta\gamma\delta=-u$ (i.e., $\alpha\beta^3=-u$ in this case) and using theorem 1.1 (c), i,e., $W_n=A_1\alpha^n+(A_2+A_3n+A_4n^2)\beta^n$, we get

$$\begin{aligned}W_{-n}&=A_1\alpha^{-n}+(A_2-A_3n+A_4n^2)\beta^{-n}\\&=A_1\beta^{3n}(-u)^{-n}+(A_2-A_3n+A_4n^2)\alpha^n\beta^{2n}(-u)^{-n}\end{aligned}$$

and so

$$(-u)^nW_{-n}=A_1\beta^{3n}+(A_2-A_3n+A_4n^2)\alpha^n\beta^{2n}.$$

Note that by using A_1,A_2,A_3 and A_4, we get

$$A_1\beta^{3n}+(A_2-A_3n+A_4n^2)\alpha^n\beta^{2n}=\beta^{2n}\left(3\alpha^n+\beta^n\right)W_0+\beta^{2n}(-\alpha^n(3A_1+2A_2)-\beta^nA_2-n\alpha^nA_3+n^2\alpha^nA_4)$$

(d) Since $\alpha\beta\gamma\delta=-u$ (i.e., $\alpha^4=-u$ in this case) and using theorem 1.1 (d), i,e., $W_n=(A_1+A_2n+A_3n^2+A_4n^3)\alpha^n$, we have

$$\begin{aligned}W_{-n}&=(A_1-A_2n+A_3n^2-A_4n^3)\alpha^{-n}\\&=(A_1-A_2n+A_3n^2-A_4n^3)\alpha^{3n}(-u)^{-n}\end{aligned}$$

and so

$$(-u)^nW_{-n}=(A_1-A_2n+A_3n^2-A_4n^3)\alpha^{3n}.$$

Note that by using A_1,A_2,A_3 and A_4, we get

$$(A_1-A_2n+A_3n^2-A_4n^3)\alpha^{3n}=(4W_0-3A_1-A_2n+A_3n^2-A_4n^3)\alpha^{3n},$$
$$i.e.,$$
$$A_1-A_2n+A_3n^2-A_4n^3=4W_0-3A_1-A_2n+A_3n^2-A_4n^3.\ \square$$

Now, we shall complete the proof of theorem 6.1.

The Proof of theorem 6.1:

If $\alpha\neq\beta\neq\gamma\neq\delta$ (i.e., if we have the four distinct roots case) then, for $n\in\mathbb{Z}$, we obtain

$$\begin{aligned}W_{2n}H_n&=(A_1\alpha^{2n}+A_2\beta^{2n}+A_3\gamma^{2n}+A_4\delta^{2n})(\alpha^n+\beta^n+\gamma^n+\delta^n)\\&=(A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n)\times\frac{1}{2}((\alpha^n+\beta^n+\gamma^n+\delta^n)^2-(\alpha^{2n}+\beta^{2n}+\gamma^{2n}+\delta^{2n}))\\&\quad+(A_1\alpha^{3n}+A_2\beta^{3n}+A_3\gamma^{3n}+A_4\delta^{3n})\\&\quad-(\beta^n\gamma^n\delta^n+\alpha^n\gamma^n\delta^n+\alpha^n\beta^n\delta^n+\alpha^n\beta^n\gamma^n)W_0\\&\quad+(\beta^n\gamma^n\delta^n+\alpha^n\gamma^n\delta^n+\alpha^n\beta^n\delta^n+\alpha^n\beta^n\gamma^n)W_0-\beta^n\gamma^n\alpha^n(A_1+A_2+A_3)\\&\quad-\alpha^n\beta^n\delta^n(A_1+A_2+A_4)-\alpha^n\gamma^n\delta^n(A_1+A_3+A_4)-\beta^n\gamma^n\delta^n(A_2+A_3+A_4)\\&\Rightarrow\end{aligned}$$

$$W_{2n}H_n=W_n\times\frac{1}{2}(H_n^2-H_{2n})+W_{3n}-S_nW_0+(-u)^nW_{-n}.$$

where we used theorem 1.1 (a), i.e, $W_n=W_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n$, theorem 1.4 (a), i.e., $H_n=\alpha^n+\beta^n+\gamma^n+\delta^n$, lemma 3.2 (a) i.e., $S_n=\beta^n\gamma^n\delta^n+\alpha^n\gamma^n\delta^n+\alpha^n\beta^n\delta^n+\alpha^n\beta^n\gamma^n$ and lemma 6.1 (a).

Note that for the case $\alpha \neq \beta \neq \gamma \neq \delta$ (i.e., if we have the four distinct roots case then) we can also find $W_{2n}H_n$ as follows:

$$\begin{aligned} W_{2n}H_n &= (A_1\alpha^{2n}+A_2\beta^{2n}+A_3\gamma^{2n}+A_4\delta^{2n})(\alpha^n+\beta^n+\gamma^n+\delta^n) \\ &= (A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n)\times(\alpha^n\beta^n+\alpha^n\gamma^n+\alpha^n\delta^n+\beta^n\gamma^n+\beta^n\delta^n+\gamma^n\delta^n) \\ &\quad +(\alpha^{3n}A_1+\beta^{3n}A_2+\gamma^{3n}A_3+\delta^{3n}A_4) \\ &\quad -(\beta^n\gamma^n\delta^n+\alpha^n\gamma^n\delta^n+\alpha^n\beta^n\delta^n+\alpha^n\beta^n\gamma^n)(A_1+A_2+A_3+A_4) \\ &\quad +(A_4\alpha^n\beta^n\gamma^n+A_3\alpha^n\beta^n\delta^n+A_2\alpha^n\gamma^n\delta^n+A_1\beta^n\gamma^n\delta^n) \\ &\Rightarrow \\ W_{2n}H_n &= W_n\times\frac{1}{2}(H_n^2-H_{2n})+W_{3n}-S_nW_0+(-u)^nW_{-n}. \end{aligned}$$

where we used theorem 1.1 (a), i.e., $W_n = A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n$ and

$$A_1+A_2+A_3+A_4 = W_0, \text{ (lemma 1.2 (a))},$$

$$A_4\alpha^n\beta^n\gamma^n+A_3\alpha^n\beta^n\delta^n+A_2\alpha^n\gamma^n\delta^n+A_1\beta^n\gamma^n\delta^n = (-u)^nW_{-n}, \text{ (lemma 6.1 (a))},$$

$$\alpha^n\beta^n+\alpha^n\gamma^n+\alpha^n\delta^n+\beta^n\gamma^n+\beta^n\delta^n+\gamma^n\delta^n = \frac{1}{2}(H_n^2-H_{2n}),$$

$$\beta^n\gamma^n\delta^n+\alpha^n\gamma^n\delta^n+\alpha^n\beta^n\delta^n+\alpha^n\beta^n\gamma^n = S_n \text{ (lemma 3.2 (a))}$$

If $\alpha \neq \beta \neq \gamma = \delta$ (i.e., if we have the three distinct roots case) then, for $n \in \mathbb{Z}$, we get

$$\begin{aligned} W_{2n}H_n &= (A_1\alpha^{2n}+A_2\beta^{2n}+(A_3+2A_4n)\gamma^{2n})(\alpha^n+\beta^n+2\gamma^n) \\ &= (A_1\alpha^n+A_2\beta^n+(A_3+A_4n)\gamma^n)\times\frac{1}{2}((\alpha^n+\beta^n+2\gamma^n)^2-(\alpha^{2n}+\beta^{2n}+2\gamma^{2n})) \\ &\quad +(A_1\alpha^{3n}+A_2\beta^{3n}+(A_3+3A_4n)\gamma^{3n}) \\ &\quad -(\gamma^{2n}(\alpha^n+\beta^n)+2\alpha^n\beta^n\gamma^n)W_0 \\ &\quad +(\gamma^n(\beta^n\gamma^n+2\alpha^n\beta^n+\alpha^n\gamma^n)W_0-\gamma^{2n}(\alpha^nA_1+\beta^nA_2+(\alpha^n+\beta^n)A_3) \\ &\quad -\alpha^n\beta^n\gamma^n(2A_1+2A_2+A_3+nA_4)) \\ &\Rightarrow \\ W_{2n}H_n &= W_n\times\frac{1}{2}(H_n^2-H_{2n})+W_{3n}-S_nW_0+(-u)^nW_{-n} \end{aligned}$$

where we used theorem 1.1 (b), i.e, $W_n = A_1\alpha^n+A_2\beta^n+(A_3+A_4n)\gamma^n$, theorem 1.4 (b), i.e., $H_n = \alpha^n+\beta^n+2\gamma^n$, lemma 3.2 (b) i.e., $S_n = \gamma^{2n}(\alpha^n+\beta^n)+2\alpha^n\beta^n\gamma^n$ and lemma 6.1 (b).

If $\alpha \neq \beta = \gamma = \delta$ (i.e., if we have the two distinct roots case) then, for $n \in \mathbb{Z}$, we have

$$\begin{aligned} W_{2n}H_n &= (A_1\alpha^{2n}+(A_2+2A_3n+4A_4n^2)\beta^{2n})(\alpha^n+3\beta^n) \\ &= \\ &= (A_1\alpha^n+(A_2+A_3n+A_4n^2)\beta^n)\times\frac{1}{2}((\alpha^n+3\beta^n)^2-(\alpha^{2n}+3\beta^{2n})) \\ &\quad +(A_1\alpha^{3n}+(A_2+3A_3n+9A_4n^2)\beta^{3n}) \\ &\quad -\beta^{2n}(3\alpha^n+\beta^n)W_0 \\ &\quad +\beta^{2n}\left(3\alpha^n+\beta^n\right)W_0+\beta^{2n}(-\alpha^n(3A_1+2A_2)-\beta^nA_2-n\alpha^nA_3+n^2\alpha^nA_4) \\ &\Rightarrow \\ W_{2n}H_n &= W_n\times\frac{1}{2}(H_n^2-H_{2n})+W_{3n}-S_nW_0+(-u)^nW_{-n} \end{aligned}$$

where we used theorem 1.1 (c), i.e., $W_n = A_1\alpha^n+(A_2+A_3n+A_4n^2)\beta^n$, theorem 1.4 (c), i.e., $H_n = \alpha^n+3\beta^n$,lemma 3.2 (c) i.e., $S_n = \beta^{2n}(3\alpha^n+\beta^n)$ and lemma 6.1 (c).

If $\alpha = \beta = \gamma = \delta = \frac{r}{4}$ (i.e., if we have single root case) then, for $n \in \mathbb{Z}$, we have

$$\begin{aligned} W_{2n}H_n &= (A_1+2A_2n+4A_3n^2+8A_4n^3)\alpha^{2n}\times 4\alpha^n \\ &= (A_1+A_2n+A_3n^2+A_4n^3)\alpha^n\times\frac{1}{2}(16\alpha^{2n}-4\alpha^{2n}) \\ &\quad +(A_1+3A_2n+9A_3n^2+27A_4n^3)\alpha^{3n} \\ &\quad -4\alpha^{3n}W_0 \\ &\quad +(4W_0-3A_1-A_2n+A_3n^2-A_4n^3)\alpha^{3n} \\ &\Rightarrow \\ W_{2n}H_n &= W_n\times\frac{1}{2}(H_n^2-H_{2n})+W_{3n}-S_nW_0+(-u)^nW_{-n} \end{aligned}$$

where we used theorem 1.1 (d), i.e., $W_n = (A_1 + A_2 n + A_3 n^2 + A_4 n^3)\alpha^n$, theorem 1.4 (d), i.e., $H_n = 4\alpha^n$, lemma 3.2 (d) i.e., $S_n = 4\alpha^{3n}$ and lemma 6.1 (d).

Therefore all the case of the roots of characteristic equation (polynomial) we have the identity

$$W_{2n}H_n = W_n \times \frac{1}{2}(H_n^2 - H_{2n}) + W_{3n} - S_n W_0 + (-u)^n W_{-n}.$$

Now, by lemma 3.3 (c) (using $S_n = \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)$), it follows that

$$\begin{aligned}(-u)^n W_{-n} &= \frac{1}{6}(-6W_{3n} + 6H_n W_{2n} - 3H_n^2 W_n + 3H_{2n}W_n + W_0 H_n^3 + 2W_0 H_{3n} - 3W_0 H_n H_{2n}) \\ &= (-1)^{-n-1}(W_{3n} - H_n W_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \square\end{aligned}$$

Next, we present a remark which presents how H_n can be written in terms of W_n.

Remark 6.1.

To express W_{-n} by the sequence itself at positive indices we need that H_n can be written in terms of W_n. For this, writing

$$H_n = a \times W_{n+3} + b \times W_{n+2} + c \times W_{n+1} + d \times W_n$$

and solving the system of equations

$$\begin{aligned}H_0 &= a \times W_3 + b \times W_2 + c \times W_1 + d \times W_0 \\ H_1 &= a \times W_4 + b \times W_3 + c \times W_2 + d \times W_1 \\ H_2 &= a \times W_5 + b \times W_4 + c \times W_3 + d \times W_2 \\ H_3 &= a \times W_6 + b \times W_5 + c \times W_4 + d \times W_3\end{aligned}$$

or

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} W_3 & W_2 & W_1 & W_0 \\ W_4 & W_3 & W_2 & W_1 \\ W_5 & W_4 & W_3 & W_2 \\ W_6 & W_5 & W_4 & W_3 \end{pmatrix}^{-1} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix}$$

we find a, b, c, d so that H_n can be written in terms of W_n and we can replace this H_n in theorem 6.1. For example, taking $W_n = G_n$ we get

$$\begin{aligned}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &= \begin{pmatrix} G_3 & G_2 & G_1 & G_0 \\ G_4 & G_3 & G_2 & G_1 \\ G_5 & G_4 & G_3 & G_2 \\ G_6 & G_5 & G_4 & G_3 \end{pmatrix}^{-1} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{u^2}(-2su + t^2) \\ \frac{-1}{u^2}(rt^2 + tu - 2rsu) \\ \frac{1}{u^2}(-st^2 + 2s^2u + 4u^2 + rtu) \\ \frac{-1}{u^2}(3ru^2 + t^3 - 3stu) \end{pmatrix}\end{aligned}$$

i.e.,

$$\begin{aligned}a &= \frac{1}{u^2}(-2su + t^2), \\ b &= \frac{-1}{u^2}(rt^2 + tu - 2rsu), \\ c &= \frac{1}{u^2}(-st^2 + 2s^2u + 4u^2 + rtu), \\ d &= \frac{-1}{u^2}(3ru^2 + t^3 - 3stu),\end{aligned}$$

so that

$u^2 H_n = (-2su + t^2)G_{n+3} - (rt^2 + tu - 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1} - (3ru^2 + t^3 - 3stu)G_n.$

In fact, it is the identity given in lemma 5.1 (b).

Using theorem 6.1 and remark 6.1, we have the following corollary by using $G_0 = 0$ and $H_0 = 4$.

Corollary 6.1.
For $n \in \mathbb{Z}$, we have

(a) $G_{-n} = \frac{1}{2}(-u)^{-n}(-2G_{3n} + 2H_n G_{2n} - H_n^2 G_n + H_{2n} G_n) = (-1)^{-n-1} u^{-n} (G_{3n} - H_n G_{2n} + \frac{1}{2}(H_n^2 - H_{2n}) G_n)$

i.e.,

$G_{-n} = -(3ru^2 + t^3 - 3stu)^2 G_n^3 - (2su - t^2)^2 G_{n+3}^2 G_n - (-rt^2 - tu + 2rsu)^2 G_{n+2}^2 G_n - (-st^2 + 2s^2u + 4u^2 + rtu)^2 G_{n+1}^2 G_n + 2(3ru^2 + t^3 - 3stu)((-2su + t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1}) G_n^2 + 2(2su - t^2)(-rt^2 - tu + 2rsu)G_{n+3}G_{n+2}G_n + 2(2su - t^2)(-st^2 + 2s^2u + 4u^2 + rtu)G_{n+3}G_{n+1}G_n - 2(-st^2 + 2s^2u + 4u^2 + rtu)(-rt^2 - tu + 2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su + t^2)G_{2n+3}G_n + u^2(-rt^2 - tu + 2rsu)G_{2n+2}G_n + u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n+1}G_n - 2u^2(2su - t^2)G_{2n}G_{n+3} + 2u^2(-rt^2 - tu + 2rsu)G_{2n}G_{n+2} + 2u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n}G_{n+1} - 3u^2(3ru^2 + t^3 - 3stu)G_{2n}G_n.$

(b) $H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right).$

To express G_{-n} by the sequence itself at positive indices, just replace H_n and H_{2n} in the first identity of the last corollary by using identity given in remark 6.1, i.e., in lemma 5.1 (b).

7. Linear Sum Formulas of Generalized Tetranacci Polynomials with Positive Subscripts: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} z^k W_k$, $\sum_{k=0}^{n} z^k W_{2k}$ and $\sum_{k=0}^{n} z^k W_{2k+1}$

In theorem 7.1 and remark 7.2, we use the following conventions:

- If the roots of $uz^4 + tz^3 + sz^2 + rz - 1 = 0$ are a_1, a_2, a_3, a_4 then we say that $a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact a_1, a_2, a_3, a_4 are real or complex valued function in x),

  for examples:

  - If
  $$uz^4 + tz^3 + sz^2 + rz - 1 = p(z - a_1)^2(z - a_2)(z - a_3) = 0$$
  for some $p, a_1, a_2, a_3 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3$ are real or complex valued function) with $a_1 \neq a_2 \neq a_3$.
  - If
  $$uz^4 + tz^3 + sz^2 + rz - 1 = p(z - a_1)^2(z - a_2)^2 = 0$$
  for some $p, a_1, a_2 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2$ are real or complex valued function) with $a_1 \neq a_2$.

- If the roots of $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = 0$ are a_1, a_2, a_3, a_4 then we say that $a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact a_1, a_2, a_3, a_4 are real or complex valued function in x),

  for examples:

  - If
  $$\begin{aligned} &-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 \\ &= p(z - a_1)(z - a_2)(z - a_3)(z - a_4) = 0 \end{aligned}$$
  for some $p, a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3, a_4$ are real or complex valued function) with $a_1 \neq a_2 \neq a_3 \neq a_4$.
  - If
  $$\begin{aligned} &-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 \\ &= p(z - a_1)^2(z - a_2)^2 = 0 \end{aligned}$$
  for some $p, a_1, a_2 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2$ are real or complex valued function) with $a_1 \neq a_2$.

The following theorem presents some linear summing formulas of generalized Tetranacci polynomials with positive subscripts.

Theorem 7.1.
Let z be a real or complex number (in fact z is a real or complex valued function in x). For $n \geq 0$ we have the following formulas:

(a) (i) *If* $uz^4+tz^3+sz^2+rz-1\neq 0$, *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Theta_{1W}(z)}{uz^4+tz^3+sz^2+rz-1} \tag{26}$$

where

$\Theta_{1W}(z) = z^{n+3}W_{n+3}+(z^{n+2}-rz^{n+3})W_{n+2}+(-sz^{n+3}-rz^{n+2}+z^{n+1})W_{n+1}+uz^{n+4}W_n-z^3W_3+(rz^3-z^2)W_2+(sz^3+rz^2-z)W_1+(tz^3+sz^2+rz-1)W_0.$

(ii) *If* $uz^4+tz^3+sz^2+rz-1=(z-a)f(z)=0$ *for some* $a\in\mathbb{C}$ *(or* $\mathbb{R}$*, in fact* a *is a real or complex valued function) and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\frac{d\Theta_{1W}(z)}{dz}}{\frac{d(uz^4+tz^3+sz^2+rz-1)}{dz}} = \frac{\frac{d\Theta_{1W}(z)}{dz}}{4uz^3+3tz^2+2sz+r}$$

where

$\frac{d\Theta_{1W}(z)}{dz} = (n+3)z^{n+2}W_{n+3}+((n+2)z^{n+1}-r(n+3)z^{n+2})W_{n+2}+(-s(n+3)z^{n+2}-r(n+2)z^{n+1}+(n+1)z^n)W_{n+1}+u(n+4)z^{n+3}W_n-3z^2W_3+(3rz^2-2z)W_2+(3sz^2+2rz-1)W_1+(3tz^2+2sz+r)W_0.$

(iii) *If* $uz^4+tz^3+sz^2+rz-1=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\frac{d^2\Theta_{1W}(z)}{dz^2}}{\frac{d^2}{dz^2}(uz^4+tz^3+sz^2+rz-1)} = \frac{\frac{d^2\Theta_{1W}(z)}{dz^2}}{12uz^2+6tz+2s}$$

where

$\frac{d^2\Theta_{1W}(z)}{dz^2} = (n+2)(n+3)z^{n+1}W_{n+3}+((n+1)(n+2)z^n-r(n+2)(n+3)z^{n+1})W_{n+2}+(-s(n+2)(n+3)z^{n+1}-r(n+1)(n+2)z^n+n(n+1)z^{n-1})W_{n+1}+u(n+3)(n+4)z^{n+2}W_n-6zW_3+(6rz-2)W_2+(6sz+2r)W_1+(6tz+2s)W_0.$

(iv) *If* $uz^4+tz^3+sz^2+rz-1=(z-a)^3f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function* f *(polynomial) in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\frac{d^3\Theta_{1W}(z)}{dz^3}}{\frac{d^3(uz^4+tz^3+sz^2+rz-1)}{dz^3}} = \frac{\frac{d^3\Theta_{1W}(z)}{dz^3}}{24uz+6t}$$

where

$\frac{d^3\Theta_{1W}(z)}{dz^3} = (n+1)(n+2)(n+3)z^nW_{n+3}+(n(n+1)(n+2)z^{n-1}-r(n+1)(n+2)(n+3)z^n)W_{n+2}+(-s(n+1)(n+2)(n+3)z^n-rn(n+1)(n+2)z^{n-1}+(n-1)n(n+1)z^{n-2})W_{n+1}+u(n+2)(n+3)(n+4)z^{n+1}W_n-6W_3+6rW_2+6sW_1+6tW_0$

(v) *If* $uz^4+tz^3+sz^2+rz-1=(z-a)^4f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function* f *(polynomial) in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\frac{d^4\Theta_{1W}(z)}{dz^4}}{\frac{d^4(uz^4+tz^3+sz^2+rz-1)}{dz^4}} = \frac{\frac{d^4\Theta_{1W}(z)}{dz^4}}{24u}$$

where

$\frac{d^4\Theta_{1W}(z)}{dz^4} = n(n+1)(n+2)(n+3)z^{n-1}W_{n+3}+((n-1)n(n+1)(n+2)z^{n-2}-rn(n+1)(n+2)(n+3)z^{n-1})W_{n+2}+(-sn(n+1)(n+2)(n+3)z^{n-1}-r(n-1)n(n+1)(n+2)z^{n-2}+(n-2)(n-1)n(n+1)z^{n-3})W_{n+1}+u(n+1)(n+2)(n+3)(n+4)z^nW_n.$

(b) **(i)** *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Theta_{2W}(z)}{-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1}$$

where
$\Theta_{2W}(z)=(-uz^{n+3}-sz^{n+2}+z^{n+1})W_{2n+2}+(ruz^{n+3}+(t+rs)z^{n+2})W_{2n+1}+(-u^2z^{n+4}+(t^2-su)z^{n+3}+(rt+u)z^{n+2})W_{2n}+u(tz^{n+3}+rz^{n+2})W_{2n-1}-(tz^3+rz^2)W_3+((rt+u)z^3+(r^2+s)z^2-z)W_2+((st-ru)z^3-tz^2)W_1+((t^2-us)z^3+(2rt+u-s^2)z^2+(r^2+2s)z-1)W_0.$

(ii) *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\frac{d\Theta_{2W}(z)}{dz}}{\frac{d(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}{dz}} = \frac{\frac{d\Theta_{2W}(z)}{dz}}{-4u^2z^3+3(t^2-2su)z^2+2(2rt+2u-s^2)z+r^2+2s}$$

where
$\frac{d\Theta_{2W}(z)}{dz}=(-u(n+3)z^{n+2}-s(n+2)z^{n+1}+(n+1)z^n)W_{2n+2}+(ru(n+3)z^{n+2}+(n+2)(t+rs)z^{n+1})W_{2n+1}+(-u^2(n+4)z^{n+3}+(t^2-su)(n+3)z^{n+2}+(rt+u)(n+2)z^{n+1})W_{2n}+u(t(n+3)z^{n+2}+r(n+2)z^{n+1})W_{2n-1}-(3tz^2+2rz)W_3+(3(rt+u)z^2+2(r^2+s)z-1)W_2+(3(st-ru)z^2-2tz)W_1+(3(t^2-us)z^2+2(2rt+u-s^2)z+(r^2+2s))W_0.$

(iii) *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\frac{d^2\Theta_{2W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)} = \frac{\frac{d^2\Theta_{2W}(z)}{dz^2}}{-12u^2z^2+6(t^2-2su)z+4rt+4u-2s^2}$$

where
$\frac{d^2\Theta_{2W}(z)}{dz^2}=(-u(n+2)(n+3)z^{n+1}-s(n+1)(n+2)z^n+n(n+1)z^{n-1})W_{2n+2}+(ru(n+2)(n+3)z^{n+1}+(n+1)(n+2)(t+rs)z^n)W_{2n+1}+(-u^2(n+3)(n+4)z^{n+2}+(t^2-su)(n+2)(n+3)z^{n+1}+(rt+u)(n+1)(n+2)z^n)W_{2n}+u(t(n+2)(n+3)z^{n+1}+r(n+1)(n+2)z^n)W_{2n-1}-(6tz+2r)W_3+(6(rt+u)z+2(r^2+s))W_2+(6(st-ru)z-2t)W_1+(6(t^2-us)z+2(2rt+u-s^2))W_0.$

(iv) *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)^3f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\frac{d^3\Theta_{2W}(z)}{dz^3}}{\frac{d^3(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}{dz^3}} = \frac{\frac{d^3\Theta_{2W}(z)}{dz^3}}{-24u^2z+6t^2-12su}$$

where
$\frac{d^3\Theta_{2W}(z)}{dz^3}=(-u(n+1)(n+2)(n+3)z^n-sn(n+1)(n+2)z^{n-1}+(n-1)n(n+1)z^{n-2})W_{2n+2}+(ru(n+1)(n+2)(n+3)z^n+n(n+1)(n+2)(t+rs)z^{n-1})W_{2n+1}+(-u^2(n+2)(n+3)(n+4)z^{n+1}+(t^2-su)(n+1)(n+2)(n+3)z^n+(rt+u)n(n+1)(n+2)z^{n-1})W_{2n}+u(t(n+1)(n+2)(n+3)z^n+rn(n+1)(n+2)z^{n-1})W_{2n-1}-6tW_3+6(rt+u)W_2+6(st-ru)W_1+6(t^2-us)W_0.$

(v) *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)^4f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function* f *(polynomial) in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\frac{d^4\Theta_{2W}(z)}{dz^4}}{\frac{d^4(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}{dz^4}} = \frac{\Theta_8(z)}{-24u^2}$$

where
$\frac{d^4\Theta_{2W}(z)}{dz^4}=(-un(n+1)(n+2)(n+3)z^{n-1}-s(n-1)n(n+1)(n+2)z^{n-2}+(n-2)(n-1)n(n+1)z^{n-3})W_{2n+2}+(run(n+1)(n+2)(n+3)z^{n-1}+(n-1)n(n+1)(n+2)(t+rs)z^{n-2})W_{2n+1}+(-u^2(n+1)(n+2)(n+3)(n+4)z^n+(t^2-su)n(n+1)(n+2)(n+3)z^{n-1}+(rt+u)(n-1)n(n+1)(n+2)z^{n-2})W_{2n}+u(tn(n+1)(n+2)(n+3)z^{n-1}+r(n-1)n(n+1)(n+2)z^{n-2})W_{2n-1}.$

(c) **(i)** *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1}=\frac{\Theta_{3W}(z)}{-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1}$$

where

$\Theta_{3W}(z)=(tz^{n+2}+rz^{n+1})W_{2n+2}+(-u^2z^{n+4}+(t^2-2su)z^{n+3}+(rt+u-s^2)z^{n+2}+sz^{n+1})W_{2n+1}+((ru-st)z^{n+2}+tz^{n+1})W_{2n}-u(uz^{n+3}+sz^{n+2}-z^{n+1})W_{2n-1}+(uz^3+sz^2-z)W_3+(-ruz^3-(t+rs)z^2)W_2+(-suz^3+(rt+u-s^2)z^2+(r^2+2s)z-1)W_1-u(tz^3+rz^2)W_0.$

(ii) *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function* f *(polynomial) in* z *with* $f(a)\neq 0$ *then, for* $z=a$, *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1}=\frac{\frac{d\Theta_{3W}(z)}{dz}}{\frac{d(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}{dz}}=\frac{\frac{d\Theta_{3W}(z)}{dz}}{-4u^2z^3+3(t^2-2su)z^2+2(2rt+2u-s^2)z+r^2+2s}$$

where

$\frac{d\Theta_{3W}(z)}{dz}=(t(n+2)z^{n+1}+r(n+1)z^n)W_{2n+2}+(-u^2(n+4)z^{n+3}+(n+3)(t^2-2su)z^{n+2}+(n+2)(rt+u-s^2)z^{n+1}+s(n+1)z^n)W_{2n+1}+((n+2)(ru-st)z^{n+1}+t(n+1)z^n)W_{2n}-u(u(n+3)z^{n+2}+s(n+2)z^{n+1}-(n+1)z^n)W_{2n-1}+(3uz^2+2sz-1)W_3+(-3ruz^2-2(t+rs)z)W_2+(-3suz^2+2(rt+u-s^2)z+(r^2+2s))W_1-u(3tz^2+2rz)W_0.$

(iii) *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function* f *(polynomial) in* z *with* $f(a)\neq 0$ *then, for* $z=a$, *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1}=\frac{\frac{d^2\Theta_{3W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}=\frac{\frac{d^2\Theta_{3W}(z)}{dz^2}}{-12u^2z^2+6(t^2-2su)z+4rt+4u-2s^2}$$

where

$\frac{d^2\Theta_{3W}(z)}{dz^2}=(t(n+1)(n+2)z^n+rn(n+1)z^{n-1})W_{2n+2}+(-u^2(n+3)(n+4)z^{n+2}+(n+2)(n+3)(t^2-2su)z^{n+1}+(n+1)(n+2)(rt+u-s^2)z^n+sn(n+1)z^{n-1})W_{2n+1}+((n+1)(n+2)(ru-st)z^n+tn(n+1)z^{n-1})W_{2n}-u(u(n+2)(n+3)z^{n+1}+s(n+1)(n+2)z^n-n(n+1)z^{n-1})W_{2n-1}+(6uz+2s)W_3+(-6ruz-2(t+rs))W_2+(-6suz+2(rt+u-s^2))W_1-u(6tz+2r)W_0.$

(iv) *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)^3f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$, *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1}=\frac{\frac{d^3\Theta_{3W}(z)}{dz^3}}{\frac{d^3(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}{dz^3}}=\frac{\frac{d^3\Theta_{3W}(z)}{dz^3}}{-24u^2z+6t^2-12su}$$

where

$\frac{d^3\Theta_{3W}(z)}{dz^3}=(tn(n+1)(n+2)z^{n-1}+r(n-1)n(n+1)z^{n-2})W_{2n+2}+(-u^2(n+2)(n+3)(n+4)z^{n+1}+(n+1)(n+2)(n+3)(t^2-2su)z^n+n(n+1)(n+2)(rt+u-s^2)z^{n-1}+s(n-1)n(n+1)z^{n-2})W_{2n+1}+(n(n+1)(n+2)(ru-st)z^{n-1}+t(n-1)n(n+1)z^{n-2})W_{2n}-u(u(n+1)(n+2)(n+3)z^n+sn(n+1)(n+2)z^{n-1}-(n-1)n(n+1)z^{n-2})W_{2n-1}+6uW_3-6ruW_2-6suW_1-6utW_0.$

(v) *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)^4f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$, *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1}=\frac{\frac{d^4\Theta_{3W}(z)}{dz^4}}{\frac{d^4(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}{dz^4}}=\frac{\Theta_8(z)}{-24u^2}$$

where

$\frac{d^4\Theta_{3W}(z)}{dz^4}=(t(n-1)n(n+1)(n+2)z^{n-2}+r(n-2)(n-1)n(n+1)z^{n-3})W_{2n+2}+(-u^2(n+1)(n+2)(n+3)(n+4)z^n+n(n+1)(n+2)(n+3)(t^2-2su)z^{n-1}+(n-1)n(n+1)(n+2)(rt+u-s^2)z^{n-2}+s(n-2)(n-1)n(n+1)z^{n-3})W_{2n+1}+((n-1)n(n+1)(n+2)(ru-st)z^{n-2}+t(n-2)(n-1)n(n+1)z^{n-3})W_{2n}-u(un(n+1)(n+2)(n+3)z^{n-1}+s(n-1)n(n+1)(n+2)z^{n-2}-(n-2)(n-1)n(n+1)z^{n-3})W_{2n-1}.$

Proof.

(a)(i) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$uW_{n-4} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3}$$

we obtain

$$\begin{aligned}
uz^0W_0 &= z^0W_4 - rz^0W_3 - sz^0W_2 - tz^0W_1 \\
uz^1W_1 &= z^1W_5 - rz^1W_4 - sz^1W_3 - tz^1W_2 \\
uz^2W_2 &= z^2W_6 - rz^2W_5 - sz^2W_4 - tz^2W_3 \\
uz^3W_3 &= z^3W_7 - rz^3W_6 - sz^3W_5 - tz^3W_4 \\
&\vdots \\
uz^{n-3}W_{n-3} &= z^{n-3}W_{n+1} - rz^{n-3}W_n - sz^{n-3}W_{n-1} - tz^{n-3}W_{n-2} \\
uz^{n-2}W_{n-2} &= z^{n-2}W_{n+2} - rz^{n-2}W_{n+1} - sz^{n-2}W_n - tz^{n-2}W_{n-1} \\
uz^{n-1}W_{n-1} &= z^{n-1}W_{n+3} - rz^{n-1}W_{n+2} - sz^{n-1}W_{n+1} - tz^{n-1}W_n \\
uz^nW_n &= z^nW_{n+4} - rz^nW_{n+3} - sz^nW_{n+2} - tz^nW_{n+1}
\end{aligned}$$

If we add the above equations side by side, we get (a)(i).

(a)(ii) We use eq. (26). For $z = a$, the right hand side of the sum formula eq. (26) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (ii) by using

$$\sum_{k=0}^{n} a^kW_k = \left.\frac{\frac{d\Theta_{1W}(z)}{dz}}{\frac{d(uz^4+tz^3+sz^2+rz-1)}{dz}}\right|_{z=a}.$$

(a)(iii) We use eq. (26). For $z = a$, the right hand side of the sum formula eq. (26) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (iii) by using

$$\sum_{k=0}^{n} a^kW_k = \left.\frac{\frac{d^2\Theta_{1W}(z)}{dz^2}}{\frac{d^2(uz^4+tz^3+sz^2+rz-1)}{dz^2}}\right|_{z=a}.$$

(a)(iv) We use eq. (26). For $z = a$, the right hand side of the sum formula eq. (26) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (iv) by using

$$\sum_{k=0}^{n} a^kW_k = \left.\frac{\frac{d^3\Theta_{1W}(z)}{dz^3}}{\frac{d^3(uz^4+tz^3+sz^2+rz-1)}{dz^3}}\right|_{z=a}.$$

(a)(v) We use eq. (26). For $z = a$, the right hand side of the sum formula eq. (26) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (v) by using

$$\sum_{k=0}^{n} a^kW_k = \left.\frac{\frac{d^4\Theta_{1W}(z)}{dz^4}}{\frac{d^4(uz^4+tz^3+sz^2+rz-1)}{dz^4}}\right|_{z=a}.$$

(b)(i) and (c)(i) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we obtain

$$
\begin{aligned}
rz^1W_3 &= z^1W_4 - sz^1W_2 - tz^1W_1 - uz^1W_0\\
rz^2W_5 &= z^2W_6 - sz^2W_4 - tz^2W_3 - uz^2W_2\\
rz^3W_7 &= z^3W_8 - sz^3W_6 - tz^3W_5 - uz^3W_4\\
rz^4W_9 &= z^4W_{10} - sz^4W_8 - tz^4W_7 - uz^4W_6\\
&\vdots\\
rz^{n-1}W_{2n-1} &= z^{n-1}W_{2n} - sz^{n-1}W_{2n-2} - tz^{n-1}W_{2n-3} - uz^{n-1}W_{2n-4}\\
rz^nW_{2n+1} &= z^nW_{2n+2} - sz^nW_{2n} - tz^nW_{2n-1} - uz^nW_{2n-2}\\
rz^{n+1}W_{2n+3} &= z^{n+1}W_{2n+4} - sz^{n+1}W_{2n+2} - tz^{n+1}W_{2n+1} - uz^{n+1}W_{2n}
\end{aligned}
$$

Now, if we add the above equations side by side, we get

$$
\begin{aligned}
r(-W_1 + \sum_{k=0}^{n} z^kW_{2k+1}) = (z^nW_{2n+2} - W_2 - z^{-1}W_0 + \sum_{k=0}^{n} z^{k-1}W_{2k})\\
-s(-W_0 + \sum_{k=0}^{n} z^kW_{2k}) - t(-z^{n+1}W_{2n+1} + \sum_{k=0}^{n} z^{k+1}W_{2k+1})\\
-u(-z^{n+1}W_{2n} + \sum_{k=0}^{n} z^{k+1}W_{2k}).
\end{aligned}
\tag{27}
$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we write the following obvious equations;

$$
\begin{aligned}
rz^1W_2 &= z^1W_3 - sz^1W_1 - tz^1W_0 - uz^1W_{-1}\\
rz^2W_4 &= z^2W_5 - sz^2W_3 - tz^2W_2 - uz^2W_1\\
rz^3W_6 &= z^3W_7 - sz^3W_5 - tz^3W_4 - uz^3W_3\\
rz^4W_8 &= z^4W_9 - sz^4W_7 - tz^4W_6 - uz^4W_5\\
&\vdots\\
rz^{n-1}W_{2n-2} &= z^{n-1}W_{2n-1} - sz^{n-1}W_{2n-3} - tz^{n-1}W_{2n-4} - uz^{n-1}W_{2n-5}\\
rz^nW_{2n} &= z^nW_{2n+1} - sz^nW_{2n-1} - tz^nW_{2n-2} - uz^nW_{2n-3}\\
rz^{n+1}W_{2n+2} &= z^{n+1}W_{2n+3} - sz^{n+1}W_{2n+1} - tz^{n+1}W_{2n} - uz^{n+1}W_{2n-1}\\
rz^{n+2}W_{2n+4} &= z^{n+2}W_{2n+5} - sz^{n+2}W_{2n+3} - tz^{n+2}W_{2n+2} - uz^{n+2}W_{2n+1}
\end{aligned}
$$

Now, if we add the above equations side by side, we obtain

$$
\begin{aligned}
r(-W_0 + \sum_{k=0}^{n} z^kW_{2k}) = (-W_1 + \sum_{k=0}^{n} z^kW_{2k+1}) - s(-z^{n+1}W_{2n+1} + \sum_{k=0}^{n} z^{k+1}W_{2k+1})\\
-t(-z^{n+1}W_{2n} + \sum_{k=0}^{n} z^{k+1}W_{2k})\\
-u(-z^{n+2}W_{2n+1} - z^{n+1}W_{2n-1} + z^1W_{-1} + \sum_{k=0}^{n} z^{k+2}W_{2k+1})
\end{aligned}
$$

Since

$$W_{-1} = -\frac{t}{u}W_0 - \frac{s}{u}W_1 - \frac{r}{u}W_2 + \frac{1}{u}W_3$$

we have

$$r(-W_0+\sum_{k=0}^{n}z^kW_{2k}) = (-W_1+\sum_{k=0}^{n}z^kW_{2k+1})-s(-z^{n+1}W_{2n+1}+\sum_{k=0}^{n}z^{k+1}W_{2k+1}) \quad (28)$$
$$-t(-z^{n+1}W_{2n}+\sum_{k=0}^{n}z^{k+1}W_{2k})$$
$$-u(-z^{n+2}W_{2n+1}-z^{n+1}W_{2n-1}$$
$$+z^1(-\frac{t}{u}W_0-\frac{s}{u}W_1-\frac{r}{u}W_2+\frac{1}{u}W_3)+\sum_{k=0}^{n}z^{k+2}W_{2k+1}).$$

Then, solving the system eqs. (27) and (28), the required results of (b)(i) and (c)(i) follow.

(b)(ii),(b)(iii),(b)(iv),(b)(v) and (c)(ii),(c)(iii),(c)(iv),(c)(v) Proofs are the same as the proofs of (a)(ii), (a)(iii), (a)(iv), (a)(v). So we omit them. □

Remark 7.1.
Note that (a) (i) of the above theorem can be written as follows:

$$\sum_{k=0}^{n}z^kW_k=\frac{g}{rz+sz^2+tz^3+uz^4-1}$$

where
$g=z^{n+4}W_{n+4}-z^{n+3}(rz-1)W_{n+3}-z^{n+2}(sz^2+rz-1)W_{n+2}-z^{n+1}(tz^3+sz^2+rz-1)W_{n+1}-z^3W_3+z^2(rz-1)W_2+z(sz^2+rz-1)W_1+(tz^3+sz^2+rz-1)W_0.$

To calculate (to evaluate) the sums $\sum_{k=0}^{n}z^kW_k$, $\sum_{k=0}^{n}z^kW_{2k}$ and $\sum_{k=0}^{n}z^kW_{2k+1}$, the following Remark is useful.

Remark 7.2.

(a) Now, we consider theorem 7.1 (a). Some special cases of the roots of $uz^4+tz^3+sz^2+rz-1=0$ can be given as follows and according to the roots of $uz^4+tz^3+sz^2+rz-1=0$, the sum formula $\sum_{k=0}^{n}z^kW_k$ can be evaluated by using theorem 7.1 (a):

- $$uz^4+tz^3+sz^2+rz-1=p(z-a_1)(z-a_2)(z-a_3)(z-a_4)=0$$

  for some $p,a_1,a_2,a_3,a_4\in\mathbb{C}$ (or $\mathbb{R}$, in fact $p\neq0,a_1,a_2,a_3,a_4$ are real or complex valued function) with $a_1\neq a_2\neq a_3\neq a_4$ so that $f(z)=p(z-a_2)(z-a_3)(z-a_4)$.
  In this case,
  - if $z=a_1$ or $z=a_2$ or $z=a_3$ or $z=a_4$ then we use theorem 7.1 (a)(ii) to calculate $\sum_{k=0}^{n}z^kW_k$.

- $$uz^4+tz^3+sz^2+rz-1=p(z-a_1)^2(z-a_2)(z-a_3)=0$$

  for some $p,a_1,a_2,a_3\in\mathbb{C}$ with $a_1\neq a_2\neq a_3$ so that $f(z)=p(z-a_2)(z-a_3)$.
  In this case,
  - if $z=a_1$ then we use theorem 7.1 (a)(iii) to calculate $\sum_{k=0}^{n}z^kW_k$.
  - if $z=a_2$ or $z=a_3$ then we use theorem 7.1 (a)(ii) to calculate $\sum_{k=0}^{n}z^kW_k$.

- $$uz^4+tz^3+sz^2+rz-1=p(z-a_1)^2(z-a_2)^2=0$$

  for some $p,a_1,a_2\in\mathbb{C}$ with $a_1\neq a_2$ so that $f(z)=p(z-a_2)^2$.
  In this case,
  - if $z=a_1$ or $z=a_2$ then we use theorem 7.1 (a)(iii) to calculate $\sum_{k=0}^{n}z^kW_k$.

- $$uz^4+tz^3+sz^2+rz-1=p(z-a_1)^3(z-a_2)=0$$

  for some $p,a_1,a_2\in\mathbb{C}$ with $a_1\neq a_2$ so that $f(z)=p(z-a_2)$.
  In this case,

– if $z = a_1$ then we use theorem 7.1 (a)(iv) to calculate $\sum_{k=0}^{n} z^k W_k$.
– if $z = a_2$ then we use theorem 7.1 (a)(ii) to calculate $\sum_{k=0}^{n} z^k W_k$.

•
$$uz^4 + tz^3 + sz^2 + rz - 1 = p(z - a_1)^4 = 0$$
for some $p, a_1 \in \mathbb{C}$.
In this case,
– if $z = a_1$ then we use theorem 7.1 (a)(v) to calculate $\sum_{k=0}^{n} z^k W_k$.

(b) and (c) Note that to evaluate $\sum_{k=0}^{n} z^k W_{2k}$ and $\sum_{k=0}^{n} z^k W_{2k+1}$ for some special cases of the roots of $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = 0$, theorem 7.1 (b) and theorem 7.1 (c) can be used as in the case of evoluation of $\sum_{k=0}^{n} z^k W_k$ using theorem 7.1 (a) which is given in (a). For example, if

$$\begin{aligned}&-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1\\&= p(z - a_1)(z - a_2)(z - a_3)(z - a_4) = 0\end{aligned}$$

for some $p, a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3, a_4$ are real or complex valued function) with $a_1 \neq a_2 \neq a_3 \neq a_4$ so that $f(z) = p(z - a_2)(z - a_3)(z - a_4)$ then in this case if $z = a_1$ or $z = a_2$ or $z = a_3$ or $z = a_4$ then we use theorem 7.1 (b)(ii) to calculate $\sum_{k=0}^{n} z^k W_{2k}$ and theorem 7.1 (c)(ii) to calculate $\sum_{k=0}^{n} z^k W_{2k+1}$.

8. Linear Sum Formulas of Generalized Tetranacci Polynomials with Negative Subscripts: Closed Forms of the Sum Formulas $\sum_{k=1}^{n} z^k W_{-k}$, $\sum_{k=1}^{n} z^k W_{-2k}$ and $\sum_{k=1}^{n} z^k W_{-2k+1}$

In theorem 8.1 and remark 8.1, we use the following conventions:

• If the roots of $-z^4+rz^3+sz^2+tz+u = 0$ are a_1, a_2, a_3, a_4 then we say that $a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact a_1, a_2, a_3, a_4 are real or complex valued function in x),
for examples:
– If
$$-z^4 + rz^3 + sz^2 + tz + u = p(z - a_1)^2(z - a_2)(z - a_3) = 0$$
for some $p, a_1, a_2, a_3 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3$ are real or complex valued function) with $a_1 \neq a_2 \neq a_3$.
– If
$$-z^4 + rz^3 + sz^2 + tz + u = p(z - a_1)^2(z - a_2)^2 = 0$$
for some $p, a_1, a_2 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2$ are real or complex valued function) with $a_1 \neq a_2$.

• If the roots of $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 = 0$ are a_1, a_2, a_3, a_4 then we say that $a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact a_1, a_2, a_3, a_4 are real or complex valued function in x),
for examples:
– If
$$\begin{aligned}&-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2\\&= p(z - a_1)(z - a_2)(z - a_3)(z - a_4) = 0\end{aligned}$$
for some $p, a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3, a_4$ are real or complex valued function) with $a_1 \neq a_2 \neq a_3 \neq a_4$.
– If
$$\begin{aligned}&-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2\\&= p(z - a_1)^2(z - a_2)^2 = 0\end{aligned}$$
for some $p, a_1, a_2 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2$ are real or complex valued function) with $a_1 \neq a_2$.

The following theorem present some linear summing formulas of generalized Tetranacci polynomials with negative subscripts.

Theorem 8.1.
Let z be a real or complex number (in fact z is a real or complex valued function in x). For $n \geq 1$ we have the following formulas:

(a) (i) *If $-z^4+rz^3+sz^2+tz+u \neq 0$, then*

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{\Theta_{4W}(z)}{-z^4+rz^3+sz^2+tz+u} \tag{29}$$

where
$\Theta_{4W}(z) = -z^{n+1}W_{-n+3}+(-z^{n+2}+rz^{n+1})W_{-n+2}+(-z^{n+3}+rz^{n+2}+sz^{n+1})W_{-n+1}+(-z^{n+4}+rz^{n+3}+sz^{n+2}+tz^{n+1})W_{-n}+zW_3+(z^2-rz)W_2+(z^3-rz^2-sz)W_1+(z^4-rz^3-sz^2-tz)W_0.$

(ii) *If $-z^4+rz^3+sz^2+tz+u=(z-a)f(z)=0$ for some $a \in \mathbb{C}$ (or $\mathbb{R}$, in fact a is a real or complex valued function) and a function (polynomial) f in z with $f(a) \neq 0$ then, for $z=a$, we get*

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{\frac{d\Theta_{4W}(z)}{dz}}{\frac{d(-z^4+rz^3+sz^2+tz+u)}{dz}} = \frac{\frac{d\Theta_{4W}(z)}{dz}}{-4z^3+3rz^2+2sz+t}$$

where
$\frac{d\Theta_{4W}(z)}{dz} = -(n+1)z^nW_{-n+3}+(-(n+2)z^{n+1}+r(n+1)z^n)W_{-n+2}+(-(n+3)z^{n+2}+r(n+2)z^{n+1}+s(n+1)z^n)W_{-n+1}+(-(n+4)z^{n+3}+r(n+3)z^{n+2}+s(n+2)z^{n+1}+t(n+1)z^n)W_{-n}+W_3+(2z-r)W_2+(3z^2-2rz-s)W_1+(4z^3-3rz^2-2sz-t)W_0.$

(iii) *If $-z^4+rz^3+sz^2+tz+u=(z-a)^2f(z)=0$ for some $a \in \mathbb{C}$ and a function f (polynomial) in z with $f(a) \neq 0$ then, for $z=a$, we get*

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{\frac{d^2\Theta_{4W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-z^4+rz^3+sz^2+tz+u)} = \frac{\frac{d^2\Theta_{4W}(z)}{dz^2}}{-12z^2+6rz+2s}$$

where
$\frac{d^2\Theta_{4W}(z)}{dz^2} = -n(n+1)z^{n-1}W_{-n+3}+(-(n+1)(n+2)z^n+rn(n+1)z^{n-1})W_{-n+2}+(-(n+2)(n+3)z^{n+1}+r(n+1)(n+2)z^n+sn(n+1)z^{n-1})W_{-n+1}+(-(n+3)(n+4)z^{n+2}+r(n+2)(n+3)z^{n+1}+s(n+1)(n+2)z^n+tn(n+1)z^{n-1})W_{-n}+2W_2+(6z-2r)W_1+(12z^2-6rz-2s)W_0.$

(iv) *If $-z^4+rz^3+sz^2+tz+u=(z-a)^3f(z)=0$ for some $a \in \mathbb{C}$ and a function f (polynomial) in z with $f(a) \neq 0$ then, for $z=a$, we get*

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{\frac{d^3\Theta_{4W}(z)}{dz^3}}{\frac{d^3(-z^4+rz^3+sz^2+tz+u)}{dz^3}} = \frac{\frac{d^3\Theta_{4W}(z)}{dz^3}}{6r-24z}$$

where
$\frac{d^3\Theta_{4W}(z)}{dz^3} = -(n-1)n(n+1)z^{n-2}W_{-n+3}+(-n(n+1)(n+2)z^{n-1}+r(n-1)n(n+1)z^{n-2})W_{-n+2}+(-(n+1)(n+2)(n+3)z^n+rn(n+1)(n+2)z^{n-1}+s(n-1)n(n+1)z^{n-2})W_{-n+1}+(-(n+2)(n+3)(n+4)z^{n+1}+r(n+1)(n+2)(n+3)z^n+sn(n+1)(n+2)z^{n-1}+t(n-1)n(n+1)z^{n-2})W_{-n}+6W_1+(24z-6r)W_0.$

(v) *If $-z^4+rz^3+sz^2+tz+u=(z-a)^4f(z)=0$ for some $a \in \mathbb{C}$ and a function f (polynomial) in z with $f(a) \neq 0$ then, for $z=a$, we get*

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{\frac{d^4\Theta_{4W}(z)}{dz^4}}{\frac{d^4(-z^4+rz^3+sz^2+tz+u)}{dz^4}} = \frac{\frac{d^4\Theta_{4W}(z)}{dz^4}}{-24}$$

where
$\frac{d^4\Theta_{4W}(z)}{dz^4} = -(n-2)(n-1)n(n+1)z^{n-3}W_{-n+3}+(-(n-1)n(n+1)(n+2)z^{n-2}+r(n-2)(n-1)n(n+1)z^{n-3})W_{-n+2}+(-n(n+1)(n+2)(n+3)z^{n-1}+r(n-1)n(n+1)(n+2)z^{n-2}+s(n-2)(n-1)n(n+1)z^{n-3})W_{-n+1}+(-(n+1)(n+2)(n+3)(n+4)z^n+rn(n+1)(n+2)(n+3)z^{n-1}+s(n-1)n(n+1)(n+2)z^{n-2}+t(n-2)(n-1)n(n+1)z^{n-3})W_{-n}+24W_0.$

(b) (i) *If $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2 \neq 0$ then*

$$\sum_{k=1}^{n} z^k W_{-2k} = \frac{\Theta_{5W}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2}$$

where
$\Theta_{5W}(z) = (-z^{n+3}+sz^{n+2}+uz^{n+1})W_{-2n+2}-((t+rs)z^{n+2}+ruz^{n+1})W_{-2n+1}+(-z^{n+4}+(2s+r^2)z^{n+3}+(rt+u-s^2)z^{n+2}-suz^{n+1})W_{-2n}-u(rz^{n+2}+tz^{n+1})W_{-2n-1}+(rz^2+tz)W_3+(z^3-(s+r^2)z^2-(u+rt)z)W_2+(tz^2+(ru-st)z)W_1+(z^4-(2s+r^2)z^3+(s^2-2rt-u)z^2+(us-t^2)z)W_0.$

(ii) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)f(z)=0$ *for some* $a\in\mathbb{C}$ *(or* $\mathbb{R}$*, in fact* a *is a real or complex valued function) and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=1}^{n} z^k W_{-2k} = \frac{\frac{d\Theta_{5W}(z)}{dz}}{\frac{d(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)}{dz}} = \frac{\frac{d\Theta_{5W}(z)}{dz}}{-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+t^2-2us}$$

$\frac{d\Theta_{5W}(z)}{dz} = (-(n+3)z^{n+2}+s(n+2)z^{n+1}+u(n+1)z^n)W_{-2n+2}-((t+rs)(n+2)z^{n+1}+ru(n+1)z^n)W_{-2n+1}+(-(n+4)z^{n+3}+(2s+r^2)(n+3)z^{n+2}+(rt+u-s^2)(n+2)z^{n+1}-su(n+1)z^n)W_{-2n}-u(r(n+2)z^{n+1}+t(n+1)z^n)W_{-2n-1}+(2rz+t)W_3+(3z^2-2(s+r^2)z-(u+rt))W_2+(2tz+(ru-st))W_1+(4z^3-3(2s+r^2)z^2+2(s^2-2rt-u)z+(us-t^2))W_0$.

(iii) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=1}^{n} z^k W_{-2k} = \frac{\frac{d^2\Theta_{5W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)} = \frac{\frac{d^2\Theta_{5W}(z)}{dz^2}}{-12z^2+6(r^2+2s)z-2s^2+4tr+4u}$$

where
$\frac{d^2\Theta_{5W}(z)}{dz^2} = (-(n+2)(n+3)z^{n+1}+s(n+1)(n+2)z^n+un(n+1)z^{n-1})W_{-2n+2}-((t+rs)(n+1)(n+2)z^n+run(n+1)z^{n-1})W_{-2n+1}+(-(n+3)(n+4)z^{n+2}+(2s+r^2)(n+2)(n+3)z^{n+1}+(rt+u-s^2)(n+1)(n+2)z^n-sun(n+1)z^{n-1})W_{-2n}-u(r(n+1)(n+2)z^n+tn(n+1)z^{n-1})W_{-2n-1}+2rW_3+(6z-2(s+r^2))W_2+2tW_1+(12z^2-6(2s+r^2)z+2(s^2-2rt-u))W_0$.

(iv) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)^3f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=1}^{n} z^k W_{-2k} = \frac{\frac{d^3\Theta_{5W}(z)}{dz^3}}{\frac{d^3(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)}{dz^3}} = \frac{\frac{d^3\Theta_{5W}(z)}{dz^3}}{-24z+6r^2+12s}$$

where
$\frac{d^3\Theta_{5W}(z)}{dz^3} = (-(n+1)(n+2)(n+3)z^n+sn(n+1)(n+2)z^{n-1}+u(n-1)n(n+1)z^{n-2})W_{-2n+2}-((t+rs)n(n+1)(n+2)z^{n-1}+ru(n-1)n(n+1)z^{n-2})W_{-2n+1}+(-(n+2)(n+3)(n+4)z^{n+1}+(2s+r^2)(n+1)(n+2)(n+3)z^n+(rt+u-s^2)n(n+1)(n+2)z^{n-1}-su(n-1)n(n+1)z^{n-2})W_{-2n}-u(rn(n+1)(n+2)z^{n-1}+t(n-1)n(n+1)z^{n-2})W_{-2n-1}+6W_2+(24z-6(2s+r^2))W_0$.

(v) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)^4f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* f *in* z *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=1}^{n} z^k W_{2k} = \frac{\frac{d^4\Theta_{5W}(z)}{dz^4}}{\frac{d^4(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)}{dz^4}} = \frac{\frac{d^4\Theta_{5W}(z)}{dz^4}}{-24}$$

where
$\frac{d^4\Theta_{5W}(z)}{dz^4} = (-n(n+1)(n+2)(n+3)z^{n-1}+s(n-1)n(n+1)(n+2)z^{n-2}+u(n-2)(n-1)n(n+1)z^{n-3})W_{-2n+2}-((t+rs)(n-1)n(n+1)(n+2)z^{n-2}+ru(n-2)(n-1)n(n+1)z^{n-3})W_{-2n+1}+(-(n+1)(n+2)(n+3)(n+4)z^n+(2s+r^2)n(n+1)(n+2)(n+3)z^{n-1}+(rt+u-s^2)(n-1)n(n+1)(n+2)z^{n-2}-su(n-2)(n-1)n(n+1)z^{n-3})W_{-2n}-u(r(n-1)n(n+1)(n+2)z^{n-2}+t(n-2)(n-1)n(n+1)z^{n-3})W_{-2n-1}+24W_0$.

(c) **(i)** *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2\neq 0$ *then*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\Theta_{6W}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2}$$

where
$\Theta_{6W}(z) = -(rz^{n+3}+tz^{n+2})W_{-2n+2}+(-z^{n+4}+(r^2+s)z^{n+3}+(u+rt)z^{n+2})W_{-2n+1}-(tz^{n+3}+(ru-st)z^{n+2})W_{-2n}+u(-z^{n+3}+sz^{n+2}+uz^{n+1})W_{-2n-1}+(z^3-sz^2-uz)W_3+((t+rs)z^2+ruz)W_2+(z^4-(r^2+2s)z^3+(s^2-tr-u)z^2+usz)W_1+u(rz^2+tz)W_0$.

(ii) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)f(z)=0$ *for some* $a\in\mathbb{C}$ *(or* $\mathbb{R}$*, in fact a is a real or complex valued function) and a function (polynomial) f in z with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\frac{d\Theta_{6W}(z)}{dz}}{\frac{d(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)}{dz}} = \frac{\frac{d\Theta_{6W}(z)}{dz}}{-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+t^2-2us}$$

$\frac{d\Theta_{6W}(z)}{dz} = -(r(n+3)z^{n+2}+t(n+2)z^{n+1})W_{-2n+2}+(-(n+4)z^{n+3}+(r^2+s)(n+3)z^{n+2}+(u+rt)(n+2)z^{n+1})W_{-2n+1}-(t(n+3)z^{n+2}+(ru-st)(n+2)z^{n+1})W_{-2n}+u(-(n+3)z^{n+2}+s(n+2)z^{n+1}+u(n+1)z^{n})W_{-2n-1}+(3z^2-2sz-u)W_3+(2(t+rs)z+ru)W_2+(4z^3-3(r^2+2s)z^2+2(s^2-tr-u)z+us)W_1+u(2rz+t)W_0.$

(iii) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial) f in z with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\frac{d^2\Theta_{6W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)} = \frac{\frac{d^2\Theta_{6W}(z)}{dz^2}}{-12z^2+6(r^2+2s)z-2s^2+4tr+4u}$$

where
$\frac{d^2\Theta_{6W}(z)}{dz^2} = -(r(n+2)(n+3)z^{n+1}+t(n+1)(n+2)z^{n})W_{-2n+2}+(-(n+3)(n+4)z^{n+2}+(r^2+s)(n+2)(n+3)z^{n+1}+(u+rt)(n+1)(n+2)z^{n})W_{-2n+1}-(t(n+2)(n+3)z^{n+1}+(ru-st)(n+1)(n+2)z^{n})W_{-2n}+u(-(n+2)(n+3)z^{n+1}+s(n+1)(n+2)z^{n}+un(n+1)z^{n-1})W_{-2n-1}+(6z-2s)W_3+2(t+rs)W_2+(12z^2-6(r^2+2s)z+2(s^2-tr-u))W_1+2ruW_0.$

(iv) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)^3f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial) f in z with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\frac{d^3\Theta_{6W}(z)}{dz^3}}{\frac{d^3(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)}{dz^3}} = \frac{\frac{d^3\Theta_{6W}(z)}{dz^3}}{-24z+6r^2+12s}$$

where
$\frac{d^3\Theta_{6W}(z)}{dz^3} = -(r(n+1)(n+2)(n+3)z^{n}+tn(n+1)(n+2)z^{n-1})W_{-2n+2}+(-(n+2)(n+3)(n+4)z^{n+1}+(r^2+s)(n+1)(n+2)(n+3)z^{n}+(u+rt)n(n+1)(n+2)z^{n-1})W_{-2n+1}-(t(n+1)(n+2)(n+3)z^{n}+(ru-st)n(n+1)(n+2)z^{n-1})W_{-2n}+u(-(n+1)(n+2)(n+3)z^{n}+sn(n+1)(n+2)z^{n-1}+u(n-1)n(n+1)z^{n-2})W_{-2n-1}+6W_3+(24z-6(r^2+2s))W_1.$

(v) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)^4f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial) f in z with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\frac{d^4\Theta_{6W}(z)}{dz^4}}{\frac{d^4(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)}{dz^4}} = \frac{\frac{d^4\Theta_{6W}(z)}{dz^4}}{-24}$$

where
$\frac{d^4\Theta_{6W}(z)}{dz^4} = -(rn(n+1)(n+2)(n+3)z^{n-1}+t(n-1)n(n+1)(n+2)z^{n-2})W_{-2n+2}+(-(n+1)(n+2)(n+3)(n+4)z^{n}+(r^2+s)n(n+1)(n+2)(n+3)z^{n-1}+(u+rt)(n-1)n(n+1)(n+2)z^{n-2})W_{-2n+1}-(tn(n+1)(n+2)(n+3)z^{n-1}+(ru-st)(n-1)n(n+1)(n+2)z^{n-2})W_{-2n}+u(-n(n+1)(n+2)(n+3)z^{n-1}+s(n-1)n(n+1)(n+2)z^{n-2}+u(n-2)(n-1)n(n+1)z^{n-3})W_{-2n-1}+24W_1.$

Proof.

(a)(i) Using the recurrence relation

$$W_{-n+4}=rW_{-n+3}+sW_{-n+2}+tW_{-n+1}+uW_{-n}$$

i.e.

$$uW_{-n}=W_{-n+4}-rW_{-n+3}-sW_{-n+2}-tW_{-n+1}$$

we obtain

$$
\begin{aligned}
uz^{n}W_{-n} &= z^{n}W_{-n+4}-rz^{n}W_{-n+3}-sz^{n}W_{-n+2}-tz^{n}W_{-n+1}\\
uz^{n-1}W_{-n+1} &= z^{n-1}W_{-n+5}-rz^{n-1}W_{-n+4}-sz^{n-1}W_{-n+3}-tz^{n-1}W_{-n+2}\\
uz^{n-2}W_{-n+2} &= z^{n-2}W_{-n+6}-rz^{n-2}W_{-n+5}-sz^{n-2}W_{-n+4}-tz^{n-2}W_{-n+3}\\
&\vdots\\
uz^{4}W_{-4} &= z^{4}W_{0}-rz^{4}W_{-1}-sz^{4}W_{-2}-tz^{4}W_{-3}\\
uz^{3}W_{-3} &= z^{3}W_{1}-rz^{3}W_{0}-sz^{3}W_{-1}-tz^{3}W_{-2}\\
uz^{2}W_{-2} &= z^{2}W_{2}-rz^{2}W_{1}-sz^{2}W_{0}-tz^{2}W_{-1}\\
uz^{1}W_{-1} &= z^{1}W_{3}-rz^{1}W_{2}-sz^{1}W_{1}-tz^{1}W_{0}
\end{aligned}
$$

If we add the above equations side by side, we obtain

$$
\begin{aligned}
u(\sum_{k=1}^{n}z^{k}W_{-k}) = (&-z^{n+1}W_{-n+3}-z^{n+2}W_{-n+2}-z^{n+3}W_{-n+1}-z^{n+4}W_{-n}\\
&+z^{1}W_{3}+z^{2}W_{2}+z^{3}W_{1}+z^{4}W_{0}+z^{4}\sum_{k=1}^{n}z^{k}W_{-k})\\
&-r(-z^{n+1}W_{-n+2}-z^{n+2}W_{-n+1}-z^{n+3}W_{-n}+z^{1}W_{2}+z^{2}W_{1}+z^{3}W_{0}+z^{3}\sum_{k=1}^{n}z^{k}W_{-k})\\
&-s(-z^{n+1}W_{-n+1}-z^{n+2}W_{-n}+z^{1}W_{1}+z^{2}W_{0}+z^{2}\sum_{k=1}^{n}z^{k}W_{-k})\\
&-t(-z^{n+1}W_{-n}+z^{1}W_{0}+z^{1}\sum_{k=1}^{n}z^{k}W_{-k})
\end{aligned}
$$

From the last equation we get (a)(i).

(a)(ii) We use eq. (29)). For $z=a$, the right hand side of the sum formula eq. (29) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (ii) by using

$$\sum_{k=1}^{n}a^{k}W_{-k}=\left.\frac{\frac{d\Theta_{4W}(z)}{dz}}{\frac{d(-z^{4}+rz^{3}+sz^{2}+tz+u)}{dz}}\right|_{z=a}.$$

(a)(iii) We use eq. (29). For $z=a$, the right hand side of the sum formula eq. (29) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (iii) by using

$$\sum_{k=1}^{n}a^{k}W_{-k}=\left.\frac{\frac{d^{2}\Theta_{4W}(z)}{dz^{2}}}{\frac{d^{2}(-z^{4}+rz^{3}+sz^{2}+tz+u)}{dz^{2}}}\right|_{z=a}.$$

(a)(iv) We use eq. (29). For $z=a$, the right hand side of the sum formula eq. (29) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (iv) by using

$$\sum_{k=1}^{n}a^{k}W_{-k}=\left.\frac{\frac{d^{3}\Theta_{4W}(z)}{dz^{3}}}{\frac{d^{3}(-z^{4}+rz^{3}+sz^{2}+tz+u)}{dz^{3}}}\right|_{z=a}.$$

(a)(v) We use eq. (29). For $z=a$, the right hand side of the sum formula eq. (29) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (v) by using

$$\sum_{k=1}^{n}a^{k}W_{-k}=\left.\frac{\frac{d^{4}\Theta_{4W}(z)}{dz^{4}}}{\frac{d^{4}(-z^{4}+rz^{3}+sz^{2}+tz+u)}{dz^{4}}}\right|_{z=a}.$$

(b)(i) and (c)(i) Using the recurrence relation

$$W_{-n+4}=rW_{-n+3}+sW_{-n+2}+tW_{-n+1}+uW_{-n}$$

i.e.

$$tW_{-n+1}=W_{-n+4}-rW_{-n+3}-sW_{-n+2}-uW_{-n}$$

we obtain

$$
\begin{aligned}
tz^{n}W_{-2n+1} &= z^{n}W_{-2n+4}-rz^{n}W_{-2n+3}-sz^{n}W_{-2n+2}-uz^{n}W_{-2n}\\
tz^{n-1}W_{-2n+3} &= z^{n-1}W_{-2n+6}-rz^{n-1}W_{-2n+5}-sz^{n-1}W_{-2n+4}-uz^{n-1}W_{-2n+2}\\
tz^{n-2}W_{-2n+5} &= z^{n-2}W_{-2n+8}-rz^{n-2}W_{-2n+7}-sz^{n-2}W_{-2n+6}-uz^{n-2}W_{-2n+4}\\
tz^{n-3}W_{-2n+7} &= z^{n-3}W_{-2n+10}-rz^{n-3}W_{-2n+9}-sz^{n-3}W_{-2n+8}-uz^{n-3}W_{-2n+6}\\
&\vdots\\
tz^{3}W_{-5} &= z^{3}W_{-2}-rz^{3}W_{-3}-sz^{3}W_{-4}-uz^{3}W_{-6}\\
tz^{2}W_{-3} &= z^{2}W_{0}-rz^{2}W_{-1}-sz^{2}W_{-2}-uz^{2}W_{-4}\\
tz^{1}W_{-1} &= z^{1}W_{2}-rz^{1}W_{1}-sz^{1}W_{0}-uz^{1}W_{-2}.
\end{aligned}
$$

If we add the above equations side by side, we get

$$
\begin{aligned}
t\sum_{k=1}^{n}z^{k}W_{-2k+1} &= (-z^{n+1}W_{-2n+2}-z^{n+2}W_{-2n}+z^{2}W_{0}+z^{1}W_{2}+\sum_{k=1}^{n}z^{k+2}W_{-2k})\\
&\quad -r(-z^{n+1}W_{-2n+1}+z^{1}W_{1}+\sum_{k=1}^{n}z^{k+1}W_{-2k+1})\\
&\quad -s(-z^{n+1}W_{-2n}+z^{1}W_{0}+\sum_{k=1}^{n}z^{k+1}W_{-2k})-u(\sum_{k=1}^{n}z^{k}W_{-2k}).
\end{aligned}
\tag{30}
$$

Similarly, using the recurrence relation

$$W_{-n+4}=rW_{-n+3}+sW_{-n+2}+tW_{-n+1}+uW_{-n}$$

i.e.

$$tW_{-n}=W_{-n+3}-rW_{-n+2}-sW_{-n+1}-uW_{-n-1}$$

we obtain

$$
\begin{aligned}
tz^{n}W_{-2n} &= z^{n}W_{-2n+3}-rz^{n}W_{-2n+2}-sz^{n}W_{-2n+1}-uz^{n}W_{-2n-1}\\
tz^{n-1}W_{-2n+2} &= z^{n-1}W_{-2n+5}-rz^{n-1}W_{-2n+4}-sz^{n-1}W_{-2n+3}-uz^{n-1}W_{-2n+1}\\
tz^{n-2}W_{-2n+4} &= z^{n-2}W_{-2n+7}-rz^{n-2}W_{-2n+6}-sz^{n-2}W_{-2n+5}-uz^{n-2}W_{-2n+3}\\
tz^{n-3}W_{-2n+6} &= z^{n-3}W_{-2n+9}-rz^{n-3}W_{-2n+8}-sz^{n-3}W_{-2n+7}-uz^{n-3}W_{-2n+5}\\
&\vdots\\
tz^{4}W_{-8} &= z^{4}W_{-5}-rz^{4}W_{-6}-sz^{4}W_{-7}-uz^{4}W_{-9}\\
tz^{3}W_{-6} &= z^{3}W_{-3}-rz^{3}W_{-4}-sz^{3}W_{-5}-uz^{3}W_{-7}\\
tz^{2}W_{-4} &= z^{2}W_{-1}-rz^{2}W_{-2}-sz^{2}W_{-3}-uz^{2}W_{-5}\\
tz^{1}W_{-2} &= z^{1}W_{1}-rz^{1}W_{0}-sz^{1}W_{-1}-uz^{1}W_{-3}.
\end{aligned}
$$

If we add the above equations side by side, we get

$$
\begin{aligned}
t\sum_{k=1}^{n}z^{k}W_{-2k} &= (-z^{n+1}W_{-2n+1}+z^{1}W_{1}+\sum_{k=1}^{n}z^{k+1}W_{-2k+1})\\
&\quad -r(-z^{n+1}W_{-2n}+z^{1}W_{0}+\sum_{k=1}^{n}z^{k+1}W_{-2k})\\
&\quad -s(\sum_{k=1}^{n}z^{k}W_{-2k+1})-u(z^{n}W_{-2n-1}-z^{0}W_{-1}+\sum_{k=1}^{n}z^{k-1}W_{-2k+1}).
\end{aligned}
$$

Since

$$W_{-1}=-\frac{t}{u}W_{0}-\frac{s}{u}W_{1}-\frac{r}{u}W_{2}+\frac{1}{u}W_{3}$$

it follows that

$$
\begin{aligned}
t\sum_{k=1}^{n}z^{k}W_{-2k} &= (-z^{n+1}W_{-2n+1}+z^{1}W_{1}+\sum_{k=1}^{n}z^{k+1}W_{-2k+1})\\
&\quad -r(-z^{n+1}W_{-2n}+z^{1}W_{0}+\sum_{k=1}^{n}z^{k+1}W_{-2k})-s(\sum_{k=1}^{n}z^{k}W_{-2k+1})\\
&\quad -u(z^{n}W_{-2n-1}-z^{0}(-\frac{t}{u}W_{0}-\frac{s}{u}W_{1}-\frac{r}{u}W_{2}+\frac{1}{u}W_{3})+\sum_{k=1}^{n}z^{k-1}W_{-2k+1}).
\end{aligned}
\tag{31}
$$

Then, solving system eqs. (30) and (31) the required results of (b)(i) and (c)(i) follow.

(b)(ii),(b)(iii),(b)(iv),(b)(v) and (c)(ii),(c)(iii),(c)(iv),(c)(v) Proofs are the same as the proofs of (a)(ii), (a)(iii), (a)(iv), (a)(v). So we omit them. □

To calculate (to evaluate) the sums $\sum_{k=0}^{n} z^k W_{-k}$, $\sum_{k=1}^{n} z^k W_{-2k}$ and $\sum_{k=1}^{n} z^k W_{-2k+1}$, the following Remark is useful.

Remark 8.1.

(a) Now, we consider theorem 8.1 (a). Some special cases of the roots of $-z^4+rz^3+sz^2+tz+u=0$ can be given as follows and according to the roots of $-z^4+rz^3+sz^2+tz+u=0$, the sum formula $\sum_{k=1}^{n} z^k W_{-k}$ can be evaluated by using theorem 8.1 (a):

- $$-z^4+rz^3+sz^2+tz+u=p(z-a_1)(z-a_2)(z-a_3)(z-a_4)=0$$

  for some $p, a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3, a_4$ are real or complex valued function) with $a_1 \neq a_2 \neq a_3 \neq a_4$ so that $f(z)=p(z-a_2)(z-a_3)(z-a_4)$.
  In this case,
  - if $z=a_1$ or $z=a_2$ or $z=a_3$ or $z=a_4$ then we use theorem 8.1 (a)(ii) to calculate $\sum_{k=1}^{n} z^k W_{-k}$.

- $$-z^4+rz^3+sz^2+tz+u=p(z-a_1)^2(z-a_2)(z-a_3)=0$$

  for some $p, a_1, a_2, a_3 \in \mathbb{C}$ with $a_1 \neq a_2 \neq a_3$ so that $f(z)=p(z-a_2)(z-a_3)$.
  In this case,
  - if $z=a_1$ then we use theorem 8.1 (a)(iii) to calculate $\sum_{k=1}^{n} z^k W_{-k}$.
  - if $z=a_2$ or $z=a_3$ then we use theorem 8.1 (a)(ii) to calculate $\sum_{k=1}^{n} z^k W_{-k}$.

- $$-z^4+rz^3+sz^2+tz+u=p(z-a_1)^2(z-a_2)^2=0$$

  for some $p, a_1, a_2 \in \mathbb{C}$ with $a_1 \neq a_2$ so that $f(z)=p(z-a_2)^2$.
  In this case,
  - if $z=a_1$ or $z=a_2$ then we use theorem 8.1 (a)(iii) to calculate $\sum_{k=1}^{n} z^k W_{-k}$.

- $$-z^4+rz^3+sz^2+tz+u=p(z-a_1)^3(z-a_2)=0$$

  for some $p, a_1, a_2 \in \mathbb{C}$ with $a_1 \neq a_2$ so that $f(z)=p(z-a_2)$.
  In this case,
  - if $z=a_1$ then we use theorem 8.1 (a)(iv) to calculate $\sum_{k=1}^{n} z^k W_{-k}$.
  - if $z=a_2$ then we use theorem 8.1 (a)(ii) to calculate $\sum_{k=1}^{n} z^k W_{-k}$.

- $$-z^4+rz^3+sz^2+tz+u=p(z-a_1)^4=0$$

  for some $p, a_1 \in \mathbb{C}$.
  In this case,
  - if $z=a_1$ then we use theorem 8.1 (a)(v) to calculate $\sum_{k=1}^{n} z^k W_{-k}$.

(b) and (c) Note that to evaluate $\sum_{k=1}^{n} z^k W_{-2k}$ and $\sum_{k=1}^{n} z^k W_{-2k+1}$ for some special cases of the roots of $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=0$, theorem 8.1 (b) and theorem 8.1 (c) can be used as in the case of evoluation of $\sum_{k=1}^{n} z^k W_{-k}$ using theorem 8.1 (a) which is given in (a). For example, if

$$\begin{aligned}&-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2\\&=p(z-a_1)(z-a_2)(z-a_3)(z-a_4)=0\end{aligned}$$

for some $p, a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3, a_4$ are real or complex valued function) with $a_1 \neq a_2 \neq a_3 \neq a_4$ so that $f(z)=p(z-a_2)(z-a_3)(z-a_4)$ then in this case if $z=a_1$ or $z=a_2$ or $z=a_3$ or $z=a_4$ then we use theorem 8.1 (b)(ii) to calculate $\sum_{k=1}^{n} z^k W_{-2k}$ and theorem 8.1 (c)(ii) to calculate $\sum_{k=1}^{n} z^k W_{-2k+1}$.

9. Generating Function of Generalized Tetranacci Polynomials: Closed Formulas of $\sum_{n=0}^{\infty} W_n z^n, \sum_{n=0}^{\infty} W_{2n} z^n, \sum_{n=0}^{\infty} W_{2n+1} z^n, \sum_{n=1}^{\infty} W_{-n} z^n, \sum_{n=1}^{\infty} W_{-2n} z^n, \sum_{n=1}^{\infty} W_{-2n+1} z^n$

Next, we give the ordinary generating functions of some special cases of generalized Tetranacci polynomials.

Lemma 9.1.
The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:

(a) $(|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\})$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{\Gamma_{1W}(z)}{uz^4 + tz^3 + sz^2 + rz - 1}$$

where

$\Gamma_{1W}(z) = -z^3 W_3 + (rz^3 - z^2) W_2 + (sz^3 + rz^2 - z) W_1 + (tz^3 + sz^2 + rz - 1) W_0.$

(b) $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{\Gamma_{2W}(z)}{-u^2 z^4 + (t^2 - 2su) z^3 + (2u + 2rt - s^2) z^2 + (2s + r^2) z - 1}$$

where

$\Gamma_{2W}(z) = -(tz^3 + rz^2) W_3 + ((rt + u) z^3 + (r^2 + s) z^2 - z) W_2 + ((st - ru) z^3 - tz^2) W_1 + ((t^2 - us) z^3 + (2rt + u - s^2) z^2 + (r^2 + 2s) z - 1) W_0.$

(c) $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{\Gamma_{3W}(z)}{-u^2 z^4 + (t^2 - 2su) z^3 + (2u + 2rt - s^2) z^2 + (2s + r^2) z - 1}$$

where

$\Gamma_{3W}(z) = (uz^3 + sz^2 - z) W_3 + (-ruz^3 - (t + rs) z^2) W_2 + (-suz^3 + (rt + u - s^2) z^2 + (r^2 + 2s) z - 1) W_1 - u(tz^3 + rz^2) W_0.$

(d) $(|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\})$.

$$\sum_{n=1}^{\infty} W_{-n} z^n = \frac{\Gamma_{4W}(z)}{-z^4 + rz^3 + sz^2 + tz + u}$$

where

$\Gamma_{4W}(z) = zW_3 + (z^2 - rz) W_2 + (z^3 - rz^2 - sz) W_1 + (z^4 - rz^3 - sz^2 - tz) W_0.$

(e) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=1}^{\infty} W_{-2n} z^n = \frac{\Gamma_{5W}(z)}{-z^4 + (r^2 + 2s) z^3 + (2u + 2rt - s^2) z^2 + (t^2 - 2su) z - u^2}$$

where

$\Gamma_{5W}(z) = (rz^2 + tz) W_3 + (z^3 - (s + r^2) z^2 - (u + rt) z) W_2 + (tz^2 + (ru - st) z) W_1 + (z^4 - (2s + r^2) z^3 + (s^2 - 2rt - u) z^2 + (us - t^2) z) W_0$

(f) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=1}^{\infty} W_{-2n+1} z^n = \frac{\Gamma_{6W}(z)}{-z^4 + (r^2 + 2s) z^3 + (2u + 2rt - s^2) z^2 + (t^2 - 2su) z - u^2}$$

where

$\Gamma_{6W}(z) = (z^3 - sz^2 - uz) W_3 + ((t + rs) z^2 + ruz) W_2 + (z^4 - (r^2 + 2s) z^3 + (s^2 - tr - u) z^2 + usz) W_1 + u(rz^2 + tz) W_0.$

Proof. Use theorem 7.1 for the ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1} and theorem 8.1 for the ordinary generating functions of the sequences W_{-n}, W_{-2n}, W_{-2n+1}. □

(d), (e) and (f) of lemma 9.1 can be given in the standart form as the following Lemma shows.

Lemma 9.2.
The ordinary generating functions of the sequences W_{-n}, W_{-2n}, W_{-2n+1} can be given as follows:

(i) $(|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\})$.

$$\sum_{n=0}^{\infty} W_{-n}z^n = \frac{zW_3 + (z^2 - rz)W_2 + (z^3 - rz^2 - sz)W_1 + uW_0}{-z^4 + rz^3 + sz^2 + tz + u}.$$

(ii) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=0}^{\infty} W_{-2n}z^n = \frac{\Gamma_{5aW}(z)}{-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2}$$

where

$$\Gamma_{5aW}(z) = (rz^2 + tz)W_3 + (z^3 - (s + r^2)z^2 - (u + rt)z)W_2 + (tz^2 + (ru - st)z)W_1 + (uz^2 - suz - u^2)W_0.$$

(iii) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=0}^{\infty} W_{-2n+1}z^n = \frac{\Gamma_{6aW}(z)}{-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2}$$

where

$$\Gamma_{6aW}(z) = (z^3 - sz^2 - uz)W_3 + ((t + rs)z^2 + ruz)W_2 + ((rt + u)z^2 + (t^2 - su)z - u^2)W_1 + u(rz^2 + tz)W_0.$$

Now, we consider special cases of lemma 9.1.

Corollary 9.1.
The ordinary generating functions of the sequences G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} are given as follows:

(a) $(|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\})$.

$$\sum_{n=0}^{\infty} G_n z^n = \frac{-z}{uz^4 + tz^3 + sz^2 + rz - 1},$$

$$\sum_{n=0}^{\infty} H_n z^n = \frac{tz^3 + 2sz^2 + 3rz - 4}{uz^4 + tz^3 + sz^2 + rz - 1}.$$

(b) $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty} G_{2n} z^n = \frac{-tz^2 - rz}{-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1},$$

$$\sum_{n=0}^{\infty} H_{2n} z^n = \frac{(t^2 - 2su)z^3 + 2(2u + 2rt - s^2)z^2 + 3(r^2 + 2s)z - 4}{-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1}.$$

(c) $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty} G_{2n+1} z^n = \frac{uz^2 + sz - 1}{-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1},$$

$$\sum_{n=0}^{\infty} H_{2n+1} z^n = \frac{-tuz^3 + (st - 3ru)z^2 - (rs + 3t)z - r}{-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1}.$$

(d) $(|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\})$.

$$\sum_{n=1}^{\infty} G_{-n} z^n = \frac{z^3}{-z^4 + rz^3 + sz^2 + tz + u},$$

$$\sum_{n=1}^{\infty} H_{-n} z^n = \frac{4z^4 - 3rz^3 - 2sz^2 - tz}{-z^4 + rz^3 + sz^2 + tz + u}.$$

(e) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=1}^{\infty} G_{-2n} z^n = \frac{rz^3 + tz^2}{-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2},$$
$$\sum_{n=1}^{\infty} H_{-2n} z^n = \frac{4z^4 - 3(r^2+2s)z^3 + 2(s^2-2rt-2u)z^2 + (2us-t^2)z}{-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2}.$$

(f) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=1}^{\infty} G_{-2n+1} z^n = \frac{z^4 - sz^3 - uz^2}{-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2},$$
$$\sum_{n=1}^{\infty} H_{-2n+1} z^n = \frac{rz^4 + (3t+rs)z^3 + (3ru-st)z^2 + tuz}{-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2}.$$

(d), (e) and (f) of corollary 9.1 can be given in the standart form as the following Corollary shows.

Lemma 9.3.
The ordinary generating functions of the sequences G_{-n}, G_{-2n}, G_{-2n+1} and H_{-n}, H_{-2n}, H_{-2n+1} can be given as follows:

(i) $(|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\})$.

$$\sum_{n=0}^{\infty} G_{-n} z^n = \frac{z^3}{-z^4 + rz^3 + sz^2 + tz + u},$$
$$\sum_{n=0}^{\infty} H_{-n} z^n = \frac{rz^3 + 2sz^2 + 3tz + 4u}{-z^4 + rz^3 + sz^2 + tz + u}.$$

(ii) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=0}^{\infty} G_{-2n} z^n = \frac{rz^3 + tz^2}{-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2},$$
$$\sum_{n=0}^{\infty} H_{-2n} z^n = \frac{(r^2+2s)z^3 + 2(2rt+2u-s^2)z^2 + 3(t^2-2su)z - 4u^2}{-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2}.$$

(iii) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=0}^{\infty} G_{-2n+1} z^n = \frac{(r^2+s)z^3 + (2rt+u-s^2)z^2 + (t^2-2su)z - u^2}{-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2},$$
$$\sum_{n=0}^{\infty} H_{-2n+1} z^n = \frac{(r^3+3rs+3t)z^3 + (2r^2t+5ru-st-rs^2)z^2 + (rt^2+tu-2rsu)z - ru^2}{-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2}.$$

10. Closed Formulas of the Sum $\sum_{k=0}^{n} W_{mk+j}$

The following theorem presents sum formulas of generalized Tetranacci polynomials.

Theorem 10.1.
For all integers m and j, if $(H_m^2 - H_{2m} - 2H_m + 2(1-H_{-m})(-u)^m + 2) \neq 0$ then we have

$$\sum_{k=0}^{n} W_{mk+j} = \frac{\Theta_{7W}}{\frac{1}{2}(H_m^2 - H_{2m} - 2H_m + 2(1-H_{-m})(-u)^m + 2)} \tag{32}$$

where
$\Theta_{7W} = -W_{mn+2m+j} + (H_m - 1)W_{mn+m+j} + (-u)^m(1-H_{-m})W_{mn+j} + (-u)^m W_{mn-m+j} - (-u)^m W_{-m+j} + W_{2m+j} - (H_m - 1)W_{m+j} - (H_m + \frac{1}{2}(H_{2m} - H_m^2) - 1)W_j$.

Proof. Note that

$$\sum_{k=0}^{n} W_{mk+j} = W_{mn+j} + \sum_{k=0}^{n-1} W_{mk+j} = W_{mn+j} + \sum_{k=0}^{n-1} (A_1\alpha^{mk+j} + A_2\beta^{mk+j} + A_3\gamma^{mk+j} + A_4\delta^{mk+j})$$
$$= W_{mn+j} + A_1\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right) + A_2\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right) + A_3\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right) + A_4\delta^j\left(\frac{\delta^{mn}-1}{\delta^m-1}\right).$$

Simplifying the last equalities in the last two expression imply eq. (32) as required. □

Note that eq. (32) can be written in the following form:

$$\sum_{k=1}^{n} W_{mk+j} = \frac{\Theta_{8W}}{\frac{1}{2}(H_m^2 - H_{2m} - 2H_m + 2(1-H_{-m})(-u)^m + 2)}$$

where

$\Theta_{8W} = -W_{mn+2m+j} + (H_m - 1)W_{mn+m+j} + (H_m + \frac{1}{2}(H_{2m} - H_m^2) - 1)W_{mn+j} + (-u)^m W_{mn-m+j} - (-u)^m W_{-m+j} + W_{2m+j} - (H_m - 1)W_{m+j} + (-u)^m (H_{-m} - 1)W_j.$

11. Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kz^k W_k$, $\sum_{k=0}^{n} kz^k W_{2k}$ and $\sum_{k=0}^{n} kz^k W_{2k+1}$

The following Theorem present some sum formulas of generalized Tetranacci polynomials with positive subscripts.

Theorem 11.1.
Let z be a real or complex number (in fact z is a real or complex valued function in x). For $n \geq 0$ we have the following formulas:

(a) *If $uz^4 + tz^3 + sz^2 + rz - 1 \neq 0$ then*

$$\sum_{k=0}^{n} kz^k W_k = \frac{\Theta_{9W}}{(uz^4 + tz^3 + sz^2 + rz - 1)^2} \tag{33}$$

where

$\Theta_{9W} = z^{n+3}(n(sz^2+tz^3+uz^4+rz-1)+sz^2+2rz-uz^4-3)W_{n+3}+z^{n+2}(n(1-rz)(sz^2+tz^3+uz^4+rz-1)-2+4rz-tz^3-2uz^4-2r^2z^2-rsz^3+ruz^5)W_{n+2}+z^{n+1}(-n(sz^2+rz-1)(sz^2+tz^3+uz^4+rz-1)-1+2sz^2-2tz^3-3uz^4-r^2z^2-s^2z^4+2rz-2rsz^3+rtz^4+2ruz^5+suz^6)W_{n+1}+uz^{n+4}(n(sz^2+tz^3+uz^4+rz-1)-4+2sz^2+tz^3+3rz)W_n+z^3(-sz^2+uz^4-2rz+3)W_3+z^2(tz^3+2uz^4+2r^2z^2-4rz+rsz^3-ruz^5+2)W_2+z(-2sz^2+2tz^3+3uz^4+r^2z^2+s^2z^4-2rz+2rsz^3-rtz^4-2ruz^5-suz^6+1)W_1-uz^4(2sz^2+tz^3+3rz-4)W_0.$

(b) *If $-u^2z^4 + (t^2-2su)z^3 + (2u+2rt-s^2)z^2 + (2s+r^2)z - 1 \neq 0$ then*

$$\sum_{k=0}^{n} kz^k W_{2k} = \frac{\Theta_{10W}}{(-u^2z^4 + (t^2-2su)z^3 + (2u+2rt-s^2)z^2 + (2s+r^2)z - 1)^2} \tag{34}$$

where

$\Theta_{10W} = z^{n+1}(-n(uz^2+sz-1)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2-2suz^3-1)-1-s^2z^2-2t^2z^3+u^2z^4-u^3z^6+2sz-2rtz^2-r^2sz^2-2r^2uz^3+st^2z^4-s^2uz^4-2su^2z^5-2rtuz^4+uz^2)W_{2n+2}+z^{n+2}(n(t+rs+ruz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2-2suz^3-1)+2rs^2z-t^3z^3-2rs-2t+r^3sz+r^2tz+2ru^2z^3+2r^3uz^2+ru^3z^5+2tu^2z^4-3ruz+2stz+4rsuz^2+2stuz^3-rst^2z^3+rs^2uz^3+2rsu^2z^4+2r^2tuz^3)W_{2n+1}+uz^{n+2}(n(r+tz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2-2suz^3-1)+r^3z-2r-3tz+4stz^2+2tuz^3+2r^2tz^2+rt^2z^3-s^2tz^3+2ru^2z^4+tu^2z^5+2rsz+2rsuz^3)W_{2n-1}+z^{n+2}(n(u+t^2z-u^2z^2+rt-suz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2-2suz^3-1)+4u^2z^2-3t^2z-2u-2u^3z^4-2rt+2r^2t^2z^2-3r^2u^2z^3-s^2t^2z^3+2s^2u^2z^4+r^3tz+r^2uz+rt^3z^3+4st^2z^2-4s^2uz^2-6su^2z^3+s^3uz^3+t^2uz^3+su^3z^5+5suz-2r^2suz^2-2rtu^2z^4+2rstz)W_{2n}+z^2(2r-r^3z+3tz-4stz^2-2tuz^3-2r^2tz^2-rt^2z^3+s^2tz^3-2ru^2z^4-tu^2z^5-2rsz-2rsuz^3)W_3+z(-2r^2z-uz^2+r^4z^2+s^2z^2+2t^2z^3-u^2z^4+u^3z^6-2sz+r^2t^2z^4+2r^2u^2z^5-rtz^2+3r^2sz^2+2r^3tz^3+2r^2uz^3-st^2z^4+s^2uz^4+2su^2z^5+4rstz^3+4rtuz^4-rs^2tz^4+2r^2suz^4+rtu^2z^6+1)W_2+z^2(2t+t^3z^3-r^2tz+4s^2tz^2-2ru^2z^3-2r^3uz^2-s^3tz^3-ru^3z^5-2tu^2z^4+3ruz-5stz-4rsuz^2+2r^2stz^2+2rst^2z^3+rs^2uz^3-2r^2tuz^3+stu^2z^5)W_1+uz^2(-r^2z-4uz^2+4s^2z^2-s^3z^3+t^2z^3+2u^2z^4-5sz+6suz^3+2r^2sz^2+3r^2uz^3-2s^2uz^4-su^2z^5+t^2uz^5+2rstz^3+4rtuz^4+2)W_0.$

(c) *If $-u^2z^4 + (t^2-2su)z^3 + (2u+2rt-s^2)z^2 + (2s+r^2)z - 1 \neq 0$ then*

$$\sum_{k=0}^{n} kz^k W_{2k+1} = \frac{\Theta_{11W}}{(-u^2z^4 + (t^2-2su)z^3 + (2u+2rt-s^2)z^2 + (2s+r^2)z - 1)^2} \tag{35}$$

where

$\Theta_{11W} = z^{n+1}(n(r+tz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2-2suz^3-1)-t^3z^4-2tz-r-2ruz^2+2stz^2+rs^2z^2-r^2tz^2-2rt^2z^3+3ru^2z^4+2tu^2z^5+4rsuz^3+2stuz^4)W_{2n+2}+z^{n+1}(n(s-s^2z+t^2z^2-u^2z^3+uz-2suz^2+rtz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2-2suz^3-1)+2s^2z-s-s^3z^2-3t^2z^2+4u^2z^3-2u^3z^5-r^2s^2z^2+2r^2t^2z^3-3r^2u^2z^4+6suz^2+r^3tz^2+r^2uz^2+rt^3z^4+2st^2z^3-4s^2uz^3-5su^2z^4+t^2uz^4-2rtz-4r^2suz^3-2rtu^2z^5-2uz-2rstuz^4)W_{2n+1}+z^{n+1}(n(t+ruz-stz)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2-2suz^3-1)-2t^3z^3-2tuz^2-t-2rt^2z^2-s^2tz^2+r^3uz^2+st^3z^4+2ru^3z^5+3tu^2z^4-2ruz+2stz+2rsuz^2+4stuz^3-r^2stz^2+2rsu^2z^4-rt^2uz^4-2s^2tuz^4-2stu^2z^5)W_{2n}+uz^{n+1}(-n(uz^2+sz-1)(r^2z+2uz^2-s^2z^2+t^2z^3-u^2z^4+2sz+2rtz^2-2suz^3-1)-1-s^2z^2-2t^2z^3+u^2z^4-u^3z^6+2sz-r^2sz^2-2r^2uz^3+st^2z^4-s^2uz^4-2su^2z^5-2rtuz^4+uz^2-2rtz^2)W_{2n-1}+$

$z(-uz^2+s^2z^2+2t^2z^3-u^2z^4+u^3z^6-2sz+2rtz^2+r^2sz^2+2r^2uz^3-st^2z^4+s^2uz^4+2su^2z^5+2rtuz^4+1)W_3+z^2(2t+t^3z^3+2rs-2rs^2z-r^3sz-r^2tz-2ru^2z^3-2r^3uz^2-ru^3z^5-2tu^2z^4+3ruz-2stz-4rsuz^2-2stuz^3+rst^2z^3-rs^2uz^3-2rsu^2z^4-2r^2tuz^3)W_2+z^2(2u+3t^2z-4u^2z^2+2u^3z^4+2rt-2r^2t^2z^2+3r^2u^2z^3+s^2t^2z^3-2s^2u^2z^4-r^3tz-r^2uz-rt^3z^3-4st^2z^2+4s^2uz^2+6su^2z^3-s^3uz^3-t^2uz^3-su^3z^5-5suz+2r^2suz^2+2rtu^2z^4-2rstz)W_1+uz^2(2r-r^3z+3tz-4stz^2-2tuz^3-2r^2tz^2-rt^2z^3+s^2tz^3-2ru^2z^4-tu^2z^5-2rsz-2rsuz^3)W_0.$

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$uW_{n-4} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3}$$

we obtain

$$\begin{aligned}
u\times 0\times z^0W_0 &= 0\times z^0W_4 - r\times 0\times z^0W_3 - s\times 0\times z^0W_2 - t\times 0\times z^0W_1\\
u\times 1\times z^1W_1 &= 1\times z^1W_5 - r\times 1\times z^1W_4 - s\times 1\times z^1W_3 - t\times 1\times z^1W_2\\
u\times 2\times z^2W_2 &= 2\times z^2W_6 - r\times 2\times z^2W_5 - s\times 2\times z^2W_4 - t\times 2\times z^2W_3\\
u\times 3\times z^3W_3 &= 3\times z^3W_7 - r\times 3\times z^3W_6 - s\times 3\times z^3W_5 - t\times 3\times z^3W_4\\
&\vdots\\
u(n-4)z^{n-4}W_{n-4} &= (n-4)z^{n-4}W_n - r(n-4)z^{n-4}W_{n-1} - s(n-4)z^{n-4}W_{n-2} - t(n-4)z^{n-4}W_{n-3}\\
u(n-3)z^{n-3}W_{n-3} &= (n-3)z^{n-3}W_{n+1} - r(n-3)z^{n-3}W_n - s(n-3)z^{n-3}W_{n-1} - t(n-3)z^{n-3}W_{n-2}\\
u(n-2)z^{n-2}W_{n-2} &= (n-2)z^{n-2}W_{n+2} - r(n-2)z^{n-2}W_{n+1} - s(n-2)z^{n-2}W_n - t(n-2)z^{n-2}W_{n-1}\\
u(n-1)z^{n-1}W_{n-1} &= (n-1)z^{n-1}W_{n+3} - r(n-1)z^{n-1}W_{n+2} - s(n-1)z^{n-1}W_{n+1} - t(n-1)z^{n-1}W_n\\
u\times n\times z^nW_n &= u\times n\times z^nW_{n+4} - ru\times n\times z^nW_{n+3} - su\times n\times z^nW_{n+2} - tu\times n\times z^nW_{n+1}
\end{aligned}$$

If we add the equations side by side we get

$u\sum_{k=0}^{n}kz^kW_k = (nz^nW_{n+4}+(n-1)z^{n-1}W_{n+3}+(n-2)z^{n-2}W_{n+2}+(n-3)z^{n-3}W_{n+1}-(-1)z^{-1}W_3-(-2)z^{-2}W_2-(-3)z^{-3}W_1-(-4)z^{-4}W_0+\sum_{k=0}^{n}kz^{k-4}W_k-4\sum_{k=0}^{n}z^{k-4}W_k)$
$-r(nz^nW_{n+3}+(n-1)z^{n-1}W_{n+2}+(n-2)z^{n-2}W_{n+1}-(-1)z^{-1}W_2-(-2)z^{-2}W_1-(-3)z^{-3}W_0+\sum_{k=0}^{n}kz^{k-3}W_k-3\sum_{k=0}^{n}z^{k-3}W_k)$
$-s(nz^nW_{n+2}+(n-1)z^{n-1}W_{n+1}-(-1)z^{-1}W_1-(-2)z^{-2}W_0+\sum_{k=0}^{n}kz^{k-2}W_k-2\sum_{k=0}^{n}z^{k-2}W_k)$
$-t(nz^nW_{n+1}-(-1)z^{-1}W_0+\sum_{k=0}^{n}kz^{k-1}W_k-\sum_{k=0}^{n}z^{k-1}W_k).$

Then if we denote $\sum_{k=0}^{n}z^kW_k$ and $\sum_{k=0}^{n}kz^kW_k$ as

$$A = \sum_{k=0}^{n}z^kW_k,$$
$$a = \sum_{k=0}^{n}kz^kW_k,$$

and use

$$W_{n+4} = rW_{n+3} + sW_{n+2} + tW_{n+1} + uW_n,$$

we obtain

$$ua = (nz^n(rW_{n+3}+sW_{n+2}+tW_{n+1}+uW_n)+(n-1)z^{n-1}W_{n+3}+(n-2)z^{n-2}W_{n+2}+(n-3)z^{n-3}W_{n+1}-(-1)z^{-1}W_3-(-2)z^{-2}W_2-(-3)z^{-3}W_1-(-4)z^{-4}W_0+z^{-4}a-4z^{-4}A)$$
$$-r(nz^nW_{n+3}+(n-1)z^{n-1}W_{n+2}+(n-2)z^{n-2}W_{n+1}-(-1)z^{-1}W_2-(-2)z^{-2}W_1-(-3)z^{-3}W_0+z^{-3}a-3z^{-3}A)$$
$$-s(nz^nW_{n+2}+(n-1)z^{n-1}W_{n+1}-(-1)z^{-1}W_1-(-2)z^{-2}W_0+z^{-2}a-2z^{-2}A)$$
$$-t(nz^nW_{n+1}-(-1)z^{-1}W_0+z^{-1}a-z^{-1}A).$$

Using theorem 7.1 (a)(i) and solving the last equation for a, we get (a).

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we obtain

$$\begin{aligned}
r\times 1\times z^1W_3 &= 1\times z^1W_4 - s\times 1\times z^1W_2 - t\times 1\times z^1W_1 - u\times 1\times z^1W_0\\
r\times 2\times z^2W_5 &= 2\times z^2W_6 - s\times 2\times z^2W_4 - t\times 2\times z^2W_3 - u\times 2\times z^2W_2\\
r\times 3\times z^3W_7 &= 3\times z^3W_8 - s\times 3\times z^3W_6 - t\times 3\times z^3W_5 - u\times 3\times z^3W_4\\
r\times 4\times rz^4W_9 &= 4\times rz^4W_{10} - s\times 4\times rz^4W_8 - t\times 4\times rz^4W_7 - u\times 4\times rz^4W_6\\
&\vdots\\
r(n-1)z^{n-1}W_{2n-1} &= (n-1)z^{n-1}W_{2n} - s(n-1)z^{n-1}W_{2n-2}\\
&\quad -t(n-1)z^{n-1}W_{2n-3} - u(n-1)z^{n-1}W_{2n-4}\\
rnz^nW_{2n+1} &= nz^nW_{2n+2} - snz^nW_{2n} - tnz^nW_{2n-1} - unz^nW_{2n-2}
\end{aligned}$$

Now, if we add the above equations side by side, we get

$$\begin{aligned}
r(-0\times z^0W_1 + \sum_{k=0}^{n} kz^kW_{2k+1}) = {} & (nz^nW_{2n+2} - 0\times z^0W_2 - (-1)z^{-1}W_0 + \sum_{k=0}^{n}(k-1)z^{k-1}W_{2k})\\
& -s(-0\times z^0W_0 + \sum_{k=0}^{n} kz^kW_{2k})\\
& -t(-(n+1)z^{n+1}W_{2n+1} + \sum_{k=0}^{n}(k+1)z^{k+1}W_{2k+1})\\
& -u(-(n+1)z^{n+1}W_{2n} + \sum_{k=0}^{n}(k+1)z^{k+1}W_{2k})
\end{aligned}$$

and so

$$\begin{aligned}
r(-0\times z^0W_1 + \sum_{k=0}^{n} kz^kW_{2k+1}) = {} & (nz^nW_{2n+2} - 0\times z^0W_2 - (-1)z^{-1}W_0 \qquad (36)\\
& +z^{-1}\sum_{k=0}^{n} kz^kW_{2k} - z^{-1}\sum_{k=0}^{n} z^kW_{2k})\\
& -s(-0\times z^0W_0 + \sum_{k=0}^{n} kz^kW_{2k})\\
& -t(-(n+1)z^{n+1}W_{2n+1} + z^1\sum_{k=0}^{n} kz^kW_{2k+1} + z^1\sum_{k=0}^{n} z^kW_{2k+1})\\
& -u(-(n+1)z^{n+1}W_{2n} + z^1\sum_{k=0}^{n} kz^kW_{2k} + z^1\sum_{k=0}^{n} z^kW_{2k}).
\end{aligned}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we write the following obvious equations;

$$\begin{aligned}
r\times 1\times z^1W_2 &= 1\times z^1W_3 - s\times 1\times z^1W_1 - t\times 1\times z^1W_0 - u\times 1\times z^1W_{-1}\\
r\times 2\times z^2W_4 &= 2\times z^2W_5 - s\times 2\times z^2W_3 - t\times 2\times z^2W_2 - u\times 2\times z^2W_1\\
r\times 3\times z^3W_6 &= 3\times z^3W_7 - s\times 3\times z^3W_5 - t\times 3\times z^3W_4 - u\times 3\times z^3W_3\\
r\times 8\times z^4W_8 &= 4\times z^4W_9 - s\times 8\times z^4W_7 - t\times 8\times z^4W_6 - u\times 8\times z^4W_5\\
&\vdots\\
r(n-1)z^{n-1}W_{2n-2} &= (n-1)z^{n-1}W_{2n-1} - s(n-1)z^{n-1}W_{2n-3}\\
&\quad -t(n-1)z^{n-1}W_{2n-4} - u(n-1)z^{n-1}W_{2n-5}\\
rnz^nW_{2n} &= nz^nW_{2n+1} - snz^nW_{2n-1} - tnz^nW_{2n-2} - unz^nW_{2n-3}\\
r(n+1)z^{n+1}W_{2n+2} &= (n+1)z^{n+1}W_{2n+3} - s(n+1)z^{n+1}W_{2n+1}\\
&\quad -t(n+1)z^{n+1}W_{2n} - u(n+1)z^{n+1}W_{2n-1}.
\end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$\begin{aligned}
r(-0\times z^0W_0 + \sum_{k=0}^{n} kz^kW_{2k}) &= (-0\times z^0W_1 + \sum_{k=0}^{n} kz^kW_{2k+1})\\
&\quad -s(-(n+1)z^{n+1}W_{2n+1} + \sum_{k=0}^{n}(k+1)z^{k+1}W_{2k+1})\\
&\quad -t(-(n+1)z^{n+1}W_{2n} + \sum_{k=0}^{n}(k+1)z^{k+1}W_{2k})\\
&\quad -u(-(n+2)z^{n+2}W_{2n+1} - (n+1)z^{n+1}W_{2n-1}\\
&\quad +1\times z^1W_{-1} + \sum_{k=0}^{n}(k+2)z^{k+2}W_{2k+1}).
\end{aligned}$$

Since

$$W_{-1} = -\frac{t}{u}W_0 - \frac{s}{u}W_1 - \frac{r}{u}W_2 + \frac{1}{u}W_3$$

we have

$$\begin{aligned}
r(-0\times z^0W_0 + \sum_{k=0}^{n} kz^kW_{2k}) &= (-0\times z^0W_1 + \sum_{k=0}^{n} kz^kW_{2k+1}) \qquad (37)\\
&\quad -s(-(n+1)z^{n+1}W_{2n+1} + z^1\sum_{k=0}^{n} kz^kW_{2k+1} + z^1\sum_{k=0}^{n} z^kW_{2k+1})\\
&\quad -t(-(n+1)z^{n+1}W_{2n} + z^1\sum_{k=0}^{n} kz^kW_{2k} + z^1\sum_{k=0}^{n} z^kW_{2k})\\
&\quad -u(-(n+2)z^{n+2}W_{2n+1} - (n+1)z^{n+1}W_{2n-1}\\
&\quad +1\times z^1(-\frac{t}{u}W_0 - \frac{s}{u}W_1 - \frac{r}{u}W_2 + \frac{1}{u}W_3)\\
&\quad +z^2\sum_{k=0}^{n} kz^kW_{2k+1} + 2z^2\sum_{k=0}^{n} z^kW_{2k+1}).
\end{aligned}$$

Then, solving the system eqs. (36) and (37) (using theorem 7.1 (b)(i) and (c)(i)), the required result of (b) and (c) follow.

In fact, if we denote

$$a=\sum_{k=0}^{n}kz^{k}W_{2k},$$
$$b=\sum_{k=0}^{n}kz^{k}W_{2k+1},$$
$$f=\sum_{k=0}^{n}z^{k}W_{2k},$$
$$g=\sum_{k=0}^{n}z^{k}W_{2k+1},$$

Equations (36) and (37) can be written as follows:

$$\begin{aligned}r(-0\times z^{0}W_{1}+b) &= (nz^{n}W_{2n+2}-0\times z^{0}W_{2}-(-1)z^{-1}W_{0}+z^{-1}a-z^{-1}f)\\ &\quad -s(-0\times z^{0}W_{0}+a)-t(-(n+1)z^{n+1}W_{2n+1}+z^{1}b+z^{1}g)\\ &\quad -u(-(n+1)z^{n+1}W_{2n}+z^{1}a+z^{1}f)\\ r(-0\times z^{0}W_{0}+a) &= (-0\times z^{0}W_{1}+b)\\ &\quad -s(-(n+1)z^{n+1}W_{2n+1}+z^{1}b+z^{1}g)-t(-(n+1)z^{n+1}W_{2n}+z^{1}a+z^{1}f)\\ &\quad -u(-(n+2)z^{n+2}W_{2n+1}-(n+1)z^{n+1}W_{2n-1}\\ &\quad +1\times z^{1}(-\frac{t}{u}W_{0}-\frac{s}{u}W_{1}-\frac{r}{u}W_{2}+\frac{1}{u}W_{3})+z^{2}b+2z^{2}g)\end{aligned}$$

Using theorem 7.1 (b)(i) and (c)(i) and solving the last two simultaneous equations with respect to a and b, we get (b) and (c). □

Note that the proof of theorem 11.1 can be done by taking the derivative of the formulas in theorem 7.1 so that theorem 11.1 can be given in the following form. Here, $\Theta_{1W}^{'}(z)$, $\Theta_{2W}^{'}(z)$ and $\Theta_{3W}^{'}(z)$ denotes the derivatives of $\Theta_{1W}(z)$, $\Theta_{2W}(z)$ and $\Theta_{3W}(z)$ with respect to z, respectively.

Theorem 11.2.
Let z be a real or complex number (in fact z is a real or complex valued function in x). For $n\geq 0$ we have the following formulas:

(a) *If $uz^{4}+tz^{3}+sz^{2}+rz-1\neq 0$ then*

$$\sum_{k=0}^{n}kz^{k}W_{k}=\frac{\Theta_{9W}}{(uz^{4}+tz^{3}+sz^{2}+rz-1)^{2}} \tag{38}$$

where

$\Theta_{9W}=z((uz^{4}+tz^{3}+sz^{2}+rz-1)\Theta_{1W}^{'}(z)-(4uz^{3}+3tz^{2}+2sz+r)\Theta_{1W}(z)).$

(b) *If $-u^{2}z^{4}+(t^{2}-2su)z^{3}+(2u+2rt-s^{2})z^{2}+(2s+r^{2})z-1\neq 0$ then*

$$\sum_{k=0}^{n}kz^{k}W_{2k}=\frac{\Theta_{10W}}{(-u^{2}z^{4}+(t^{2}-2su)z^{3}+(2u+2rt-s^{2})z^{2}+(2s+r^{2})z-1)^{2}}$$

where

$\Theta_{10W}=z((-u^{2}z^{4}+(t^{2}-2su)z^{3}+(2u+2rt-s^{2})z^{2}+(2s+r^{2})z-1)\Theta_{2W}^{'}(z)-(-4u^{2}z^{3}+3(t^{2}-2su)z^{2}+2(2u+2rt-s^{2})z+(2s+r^{2}))\Theta_{2W}(z)).$

(c) *If $-u^{2}z^{4}+(t^{2}-2su)z^{3}+(2u+2rt-s^{2})z^{2}+(2s+r^{2})z-1\neq 0$ then*

$$\sum_{k=0}^{n}kz^{k}W_{2k+1}=\frac{\Theta_{11W}}{(-u^{2}z^{4}+(t^{2}-2su)z^{3}+(2u+2rt-s^{2})z^{2}+(2s+r^{2})z-1)^{2}}$$

where

$\Theta_{11W}=z((-u^{2}z^{4}+(t^{2}-2su)z^{3}+(2u+2rt-s^{2})z^{2}+(2s+r^{2})z-1)\Theta_{3W}^{'}(z)-(-4u^{2}z^{3}+3(t^{2}-2su)z^{2}+2(2u+2rt-s^{2})z+(2s+r^{2}))\Theta_{3W}(z)).$

Proof. From theorem 7.1, we have

$$\sum_{k=0}^{n} z^k W_k = \frac{\Theta_{1W}(z)}{uz^4+tz^3+sz^2+rz-1},$$
$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Theta_{2W}(z)}{-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1},$$
$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Theta_{3W}(z)}{-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1}.$$

By taking the derivative of the both sides of the above formulas with respect to z, we get

$$\sum_{k=0}^{n} kz^{k-1} W_k = \frac{C_1(z)}{(uz^4+tz^3+sz^2+rz-1)^2},$$
$$\sum_{k=0}^{n} kz^{k-1} W_{2k} = \frac{C_2(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2},$$
$$\sum_{k=0}^{n} kz^{k-1} W_{2k+1} = \frac{C_3(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2},$$

where

$C_1(z) = (uz^4+tz^3+sz^2+rz-1)\Theta_{1W}'(z) - (4uz^3+3tz^2+2sz+r)\Theta_{1W}(z)$,

$C_2(z) = (-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)\Theta_{2W}'(z) - (-4u^2z^3+3(t^2-2su)z^2+2(2u+2rt-s^2)z+(2s+r^2))\Theta_{2W}(z)$,

$C_3(z) = (-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)\Theta_{3W}'(z) - (-4u^2z^3+3(t^2-2su)z^2+2(2u+2rt-s^2)z+(2s+r^2))\Theta_{3W}(z)$.

Now, it follows that

$$\sum_{k=0}^{n} kz^{k} W_k = \frac{z\times C_1(z)}{(uz^4+tz^3+sz^2+rz-1)^2} = \frac{\Theta_{9W}}{(uz^4+tz^3+sz^2+rz-1)^2},$$
$$\sum_{k=0}^{n} kz^{k} W_{2k} = \frac{z\times C_2(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2} = \frac{\Theta_{10W}}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2},$$
$$\sum_{k=0}^{n} kz^{k} W_{2k+1} = \frac{z\times C_3(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2} = \frac{\Theta_{11W}}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2}.$$

where

$\Theta_{9W} = z\times C_1(z) = z((uz^4+tz^3+sz^2+rz-1)\Theta_{1W}'(z) - (4uz^3+3tz^2+2sz+r)\Theta_{1W}(z))$,

$\Theta_{10W} = z\times C_2(z) = z((-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)\Theta_{2W}'(z) - (-4u^2z^3+3(t^2-2su)z^2+2(2u+2rt-s^2)z+(2s+r^2))\Theta_{2W}(z))$,

$\Theta_{11W} = z\times C_3(z) = z((-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)\Theta_{3W}'(z) - (-4u^2z^3+3(t^2-2su)z^2+2(2u+2rt-s^2)z+(2s+r^2))\Theta_{3W}(z))$. □

To calculate (to evaluate) the sums $\sum_{k=0}^{n} kz^kW_k$, $\sum_{k=0}^{n} kz^kW_{2k}$ and $\sum_{k=0}^{n} kz^kW_{2k+1}$, the following Remark is useful.

Remark 11.1.

(a) Now, we consider theorem 11.1 (a) (and so Theorem theorem 11.2 (a)). Some special cases of the roots of $uz^4+tz^3+sz^2+rz-1=0$ can be given as follows and according to the roots of $uz^4+tz^3+sz^2+rz-1=0$, the sum formula $\sum_{k=0}^{n} kz^kW_k$ can be evaluated by using theorem 11.1 (a) (and so theorem 11.2 (a)):

- $$(uz^4+tz^3+sz^2+rz-1)^2 = p(z-a_1)^2(z-a_2)^2(z-a_3)^2(z-a_4)^2 = 0$$

  for some $p, a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3, a_4$ are real or complex valued function in x) with $a_1 \neq a_2 \neq a_3 \neq a_4$.

  In this case,

– if $z = a_1$ or $z = a_2$ or $z = a_3$ or $z = a_4$ then (we use theorem 11.1 (a), (and so theorem 11.2 (a) since both sum formulas are the same), to calculate $\sum_{k=0}^{n} kz^k W_k$), we get

$$\sum_{k=0}^{n} kz^k W_k = \frac{\frac{d^2\Theta_{9W}(z)}{dz^2}}{\frac{d^2(uz^4+tz^3+sz^2+rz-1)^2}{dz^2}}$$

because for $z = a_1$ or $z = a_2$ or $z = a_3$ or $z = a_4$, the right hand side of the sum formula eq. (33) in theorem 11.1 (a), (and so eq. (38) in theorem 11.2 (a) since both sum formulas are the same), is an indeterminate form so we can use L'Hospital rule (twice).

•
$$(uz^4 + tz^3 + sz^2 + rz - 1)^2 = p(z - a_1)^4 (z - a_2)^2 (z - a_3)^2 = 0$$

for some $p, a_1, a_2, a_3 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3$ are real or complex valued function in x) with $a_1 \neq a_2 \neq a_3$.
In this case,

– if $z = a_1$ then we get

$$\sum_{k=0}^{n} kz^k W_k = \frac{\frac{d^4\Theta_{9W}(z)}{dz^4}}{\frac{d^4(uz^4+tz^3+sz^2+rz-1)^2}{dz^4}}$$

because for $z = a_1$, the right hand side of the sum formula eq. (33) in theorem 11.1 (a), (and so eq. (38)in theorem 11.2 (a)), is an indeterminate form so we can use L'Hospital rule (four times).

– if $z = a_2$ or $z = a_3$ then we get

$$\sum_{k=0}^{n} kz^k W_k = \frac{\frac{d^2\Theta_{9W}(z)}{dz^2}}{\frac{d^2(uz^4+tz^3+sz^2+rz-1)^2}{dz^2}}$$

because for $z = a_2$ or $z = a_3$, the right hand side of the sum formula (eq. (33) in theorem 11.1 (a), (and so eq. (38) in theorem 11.2 (a)), is an indeterminate form so we can use L'Hospital rule (twice).

•
$$(uz^4 + tz^3 + sz^2 + rz - 1)^2 = p(z - a_1)^4 (z - a_2)^4 = 0$$

for some $p, a_1, a_2 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2$ are real or complex valued function in x) with $a_1 \neq a_2$.
In this case,

– if $z = a_1$ or $z = a_2$ then we get

$$\sum_{k=0}^{n} kz^k W_k = \frac{\frac{d^4\Theta_{9W}(z)}{dz^4}}{\frac{d^4(uz^4+tz^3+sz^2+rz-1)^2}{dz^4}}$$

because for $z = a_1$ or $z = a_2$, the right hand side of the sum formula (eq. (33)) in theorem 11.1 (a), (and so eq. (38) in theorem 11.2 (a)), is an indeterminate form so we can use L'Hospital rule (four times).

•
$$(uz^4 + tz^3 + sz^2 + rz - 1)^2 = p(z - a_1)^6 (z - a_2)^2 = 0$$

for some $p, a_1, a_2 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2$ are real or complex valued function in x) with $a_1 \neq a_2$.
In this case,

– if $z = a_1$ then we get

$$\sum_{k=0}^{n} kz^k W_k = \frac{\frac{d^6\Theta_{9W}(z)}{dz^6}}{\frac{d^6(uz^4+tz^3+sz^2+rz-1)^2}{dz^6}}$$

because for $z = a_1$, the right hand side of the sum formula eq. (33) in theorem 11.1 (a), (and so eq. (38)in theorem 11.2 (a)), is an indeterminate form so we can use L'Hospital rule (six times).

– if $z = a_2$ then we get

$$\sum_{k=0}^{n} kz^k W_k = \frac{\frac{d^2\Theta_{9W}(z)}{dz^2}}{\frac{d^2(uz^4+tz^3+sz^2+rz-1)^2}{dz^2}}$$

because for $z = a_2$, the right hand side of the sum formula eq. (33) in theorem 11.1 (a), (and so eq. (38)in theorem 11.2 (a)), is an indeterminate form so we can use L'Hospital rule (twice).

•

$$(uz^4 + tz^3 + sz^2 + rz - 1)^2 = p(z - a_1)^8 = 0$$

for some $p, a_1 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1$ are real or complex valued function in x).
In this case,

– if $z = a_1$ then we get

$$\sum_{k=0}^{n} kz^k W_k = \frac{\frac{d^8\Theta_{9W}(z)}{dz^8}}{\frac{d^8(uz^4+tz^3+sz^2+rz-1)^2}{dz^8}} = \frac{\frac{d^8\Theta_{9W}(z)}{dz^8}}{40320u^2}$$

because for $z = a_1$, the right hand side of the sum formula eq. (33) in theorem 11.1 (a), (and so eq. (38)in theorem 11.2 (a)), is an indeterminate form so we can use L'Hospital rule (eight times).

(b) and (c) Note that to evaluate $\sum_{k=0}^{n} kz^k W_{2k}$ and $\sum_{k=0}^{n} kz^k W_{2k+1}$ for some special cases of the roots of $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = 0$, Theorem theorem 11.1 (b) and theorem 11.1 (c), (and so theorem 11.2 (a) and (b)), can be used as in the case of evolution of $\sum_{k=0}^{n} kz^k W_k$ using theorem 11.1 (a) which is given in (a). For example, if

$$\begin{aligned}&(-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1)^2 \\ &= p(z - a_1)^2(z - a_2)^2(z - a_3)^2(z - a_4)^2 = 0\end{aligned}$$

for some $p, a_1, a_2, a_3, a_4 \in \mathbb{C}$ (or $\mathbb{R}$, in fact $p \neq 0, a_1, a_2, a_3, a_4$ are real or complex valued function in x) with $a_1 \neq a_2 \neq a_3 \neq a_4$ then in this case if $z = a_1$ or $z = a_2$ or $z = a_3$ or $z = a_4$ then (we use theorem 11.1 (b) to calculate $\sum_{k=0}^{n} kz^k W_{2k}$ and Theorem theorem 11.1 (c) to calculate $\sum_{k=0}^{n} kz^k W_{2k+1}$) :

• – if $z = a_1$ or $z = a_2$ or $z = a_3$ or $z = a_4$ then (we use theorem 11.1 (b), (and so theorem 11.2 (b)), to calculate $\sum_{k=0}^{n} kz^k W_{2k}$), we get

$$\sum_{k=0}^{n} kz^k W_{2k} = \frac{\frac{d^2\Theta_{10W}(z)}{dz^2}}{\frac{d^2(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2}{dz^2}}$$

because for $z = a_1$ or $z = a_2$ or $z = a_3$ or $z = a_4$, the right hand side of the sum formula eq. (34) in theorem 11.1 (b) is an indeterminate form so we can use L'Hospital rule (twice).

– if $z = a_1$ or $z = a_2$ or $z = a_3$ or $z = a_4$ then (we use theorem 11.1 (c), (and so theorem 11.2 (c)), to calculate $\sum_{k=0}^{n} kz^k W_{2k+1}$), we get

$$\sum_{k=0}^{n} kz^k W_{2k+1} = \frac{\frac{d^2\Theta_{11W}(z)}{dz^2}}{\frac{d^2(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2}{dz^2}}$$

because for $z = a_1$ or $z = a_2$ or $z = a_3$ or $z = a_4$, the right hand side of the sum formula eq. (35) in theorem 11.1 (c) is an indeterminate form so we can use L'Hospital rule (twice).

12. Closed Forms of the Sum Formulas $\sum_{k=1}^{n} kz^k W_{-k}$, $\sum_{k=1}^{n} kz^k W_{-2k}$ and $\sum_{k=1}^{n} kz^k W_{-2k+1}$

The following Theorem present somesum formulas of generalized Tetranacci polynomials with negative subscripts.

Theorem 12.1.
Let z be a real or complex number (in fact z is a real or complex valued function in x). For $n \geq 1$ we have the following formulas:

(a) *If* $-z^4+rz^3+sz^2+tz+u\neq 0$, *then*

$$\sum_{k=1}^{n} kz^kW_{-k}=\sum_{k=0}^{n} kz^kW_{-k}=\frac{\Theta_{12W}}{(-z^4+rz^3+sz^2+tz+u)^2}$$

where

$\Theta_{12W}=z^{n+1}(n(-u-rz^3-sz^2-tz+z^4)-u+2rz^3+sz^2-3z^4)W_{-n+3}+z^{n+1}(n(r-z)(u+rz^3+sz^2+tz-z^4)+4rz^4-tz^2-2r^2z^3+ru-2uz-2z^5-rsz^2)W_{-n+2}+z^{n+1}(n(s+rz-z^2)(u+rz^3+sz^2+tz-z^4)+2rz^5+2sz^4-2tz^3-3uz^2-r^2z^4-s^2z^2+su-z^6-2rsz^3+rtz^2+2ruz)W_{-n+1}+z^{n+1}(n(t+rz^2+sz-z^3)(u+rz^3+sz^2+tz-z^4)-4uz^3+tu+3ruz^2+2suz)W_{-n}+z(u-2rz^3-sz^2+3z^4)W_3+z(-4rz^4+tz^2+2r^2z^3-ru+2uz+2z^5+rsz^2)W_2+z(-2rz^5-2sz^4+2tz^3+3uz^2+r^2z^4+s^2z^2-su+z^6+2rsz^3-rtz^2-2ruz)W_1+uz(-t-3rz^2-2sz+4z^3)W_0.$

(b) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2\neq 0$ *then*

$$\sum_{k=1}^{n} kz^kW_{-2k}=\sum_{k=0}^{n} kz^kW_{-2k}=\frac{\Theta_{13W}}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2}$$

where

$\Theta_{13W}=z^{n+1}(n(u+sz-z^2)(2sz^3+t^2z+2uz^2+r^2z^3-s^2z^2-u^2-z^4+2rtz^2-2suz)+2sz^5+uz^4-s^2z^4-2t^2z^3+u^2z^2-u^3-z^6-2rtz^4-2su^2z-r^2sz^4+st^2z^2-2r^2uz^3-s^2uz^2-2rtuz^2)W_{-2n+2}+z^{n+1}(n(ru+tz+rsz)(-2sz^3-t^2z-2uz^2-r^2z^3+s^2z^2+u^2+z^4-2rtz^2+2suz)+ru^3-2tz^5-t^3z^2-2rsz^5-3ruz^4+2stz^4+2tu^2z+2rs^2z^4+r^3sz^4+2ru^2z^2+r^2tz^4+2r^3uz^3+2rsu^2z+4rsuz^3+2stuz^2-rst^2z^2+rs^2uz^2+2r^2tuz^2)W_{-2n+1}+z^{n+1}(n(2sz^2-s^2z+r^2z^2-su+uz-z^3+rtz)(2sz^3+t^2z+2uz^2+r^2z^3-s^2z^2-u^2-z^4+2rtz^2-2suz)+su^3-2u^3z-2uz^5-3t^2z^4+4u^2z^3+2r^2t^2z^3-3r^2u^2z^2-s^2t^2z^2-2rtz^5+5suz^4+rt^3z^2+4st^2z^3+r^3tz^4-6su^2z^2+r^2uz^4+2s^2u^2z-4s^2uz^3+s^3uz^2+t^2uz^2+2rstz^4-2rtu^2z-2r^2suz^3)W_{-2n}+uz^{n+1}(n(t+rz)(-2sz^3-t^2z-2uz^2-r^2z^3+s^2z^2+u^2+z^4-2rtz^2+2suz)+tu^2-2rz^5-3tz^4+r^3z^4+2rsz^4+2ru^2z+4stz^3+2tuz^2+rt^2z^2+2r^2tz^3-s^2tz^2+2rsuz^2)W_{-2n-1}-z(tu^2-2rz^5-3tz^4+r^3z^4+2rsz^4+2ru^2z+4stz^3+2tuz^2+rt^2z^2+2r^2tz^3-s^2tz^2+2rsuz^2)W_3+z(-2sz^5-uz^4-2r^2z^5+r^4z^4+s^2z^4+2t^2z^3-u^2z^2+u^3+z^6+r^2t^2z^2+rtu^2-rtz^4+2su^2z+3r^2sz^4-st^2z^2+2r^2u^2z+2r^3tz^3+2r^2uz^3+s^2uz^2+4rstz^3+4rtuz^2-rs^2tz^2+2r^2suz^2)W_2-z(ru^3-2tz^5-t^3z^2-stu^2-3ruz^4+5stz^4+2tu^2z+2ru^2z^2+r^2tz^4-4s^2tz^3+s^3tz^2+2r^3uz^3+4rsuz^3-2rst^2z^2-2r^2stz^3-rs^2uz^2+2r^2tuz^2)W_1+uz(-su^2+t^2u-5sz^4+2u^2z-4uz^3-r^2z^4+4s^2z^3-s^3z^2+t^2z^2+2z^5+6suz^2-2s^2uz+2r^2sz^3+3r^2uz^2+2rstz^2+4rtuz)W_0.$

(c) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2\neq 0$ *then*

$$\sum_{k=1}^{n} kz^kW_{-2k+1}=\sum_{k=0}^{n} kz^kW_{-2k+1}=\frac{\Theta_{14W}}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2}$$

where

$\Theta_{14W}=z^{n+2}(n(t+rz)(-2sz^3-t^2z-2uz^2-r^2z^3+s^2z^2+u^2+z^4-2rtz^2+2suz)+2tu^2-rz^5-t^3z-2tz^4+3ru^2z-2ruz^3+2stz^3+rs^2z^3-2rt^2z^2-r^2tz^3+4rsuz^2+2stuz)W_{-2n+2}+z^{n+2}(n(u+r^2z+rt+sz-z^2)(2sz^3+t^2z+2uz^2+r^2z^3-s^2z^2-u^2-z^4+2rtz^2-2suz)+2s^2z^4-s^3z^3-3t^2z^3+4u^2z^2-2u^3-r^2s^2z^3+2r^2t^2z^2-2rtu^2+rt^3z-2rtz^4-5su^2z+6suz^3+t^2uz-sz^5-2uz^4+2st^2z^2-3r^2u^2z+r^3tz^3+r^2uz^3-4s^2uz^2-2rstuz-4r^2suz^2)W_{-2n+1}+z^{n+2}(n(ru-st+tz)(-2sz^3-t^2z-2uz^2-r^2z^3+s^2z^2+u^2+z^4-2rtz^2+2suz)+2ru^3-tz^5-2t^3z^2-2stu^2+st^3z-2ruz^4+2stz^4+3tu^2z-2tuz^3-2rt^2z^3-s^2tz^3+r^3uz^3+2rsu^2z+2rsuz^3-rt^2uz+4stuz^2-2s^2tuz-r^2stz^3)W_{-2n}+uz^{n+1}(n(u+sz-z^2)(2sz^3+t^2z+2uz^2+r^2z^3-s^2z^2-u^2-z^4+2rtz^2-2suz)+2sz^5+uz^4-s^2z^4-2t^2z^3+u^2z^2-u^3-z^6-2rtz^4-2su^2z-r^2sz^4+st^2z^2-2r^2uz^3-s^2uz^2-2rtuz^2)W_{-2n-1}+z(-2sz^5-uz^4+s^2z^4+2t^2z^3-u^2z^2+u^3+z^6+2rtz^4+2su^2z+r^2sz^4-st^2z^2+2r^2uz^3+s^2uz^2+2rtuz^2)W_3-z(ru^3-2tz^5-t^3z^2-2rsz^5-3ruz^4+2stz^4+2tu^2z+2rs^2z^4+r^3sz^4+2ru^2z^2+r^2tz^4+2r^3uz^3+2rsu^2z+4rsuz^3+2stuz^2-rst^2z^2+rs^2uz^2+2r^2tuz^2)W_2-z(su^3-2u^3z-2uz^5-3t^2z^4+4u^2z^3+2r^2t^2z^3-3r^2u^2z^2-s^2t^2z^2-2rtz^5+5suz^4+rt^3z^2+4st^2z^3+r^3tz^4-6su^2z^2+r^2uz^4+2s^2u^2z-4s^2uz^3+s^3uz^2+t^2uz^2+2rstz^4-2rtu^2z-2r^2suz^3)W_1-uz(tu^2-2rz^5-3tz^4+r^3z^4+2rsz^4+2ru^2z+4stz^3+2tuz^2+rt^2z^2+2r^2tz^3-s^2tz^2+2rsuz^2)W_0.$

Proof.

(a) Using the recurrence relation

$$W_{-n+4} = r \times W_{-n+3} + s \times W_{-n+2} + t \times W_{-n+1} + u \times W_{-n}$$

i.e.

$$uW_{-n} = W_{-n+4} - rW_{-n+3} - sW_{-n+2} - tW_{-n+1}$$

we obtain

$$\begin{aligned}
unz^n W_{-n} &= nz^n W_{-n+4} - rnz^n W_{-n+3} - snz^n W_{-n+2} - tnz^n W_{-n+1} \\
u(n-1)z^{n-1} W_{-n+1} &= (n-1)z^{n-1} W_{-n+5} - r(n-1)z^{n-1} W_{-n+4} \\
&\quad - s(n-1)z^{n-1} W_{-n+3} - t(n-1)z^{n-1} W_{-n+2} \\
u(n-2)z^{n-2} W_{-n+2} &= (n-2)z^{n-2} W_{-n+6} - r(n-2)z^{n-2} W_{-n+5} \\
&\quad - s(n-2)z^{n-2} W_{-n+4} - t(n-2)z^{n-2} W_{-n+3} \\
&\vdots \\
u \times 5 \times W_{-5} &= 5 \times W_{-1} - r \times 5 \times W_{-2} - s \times 5 \times W_{-3} - t \times 5 \times W_{-4} \\
u \times 4 \times z^4 W_{-4} &= 4 \times z^4 W_0 - r \times 4 \times z^4 W_{-1} - s \times 4 \times z^4 W_{-2} - t \times 4 \times z^4 W_{-3} \\
u \times 3 \times z^3 W_{-3} &= 3 \times z^3 W_1 - r \times 3 \times z^3 W_0 - s \times 3 \times z^3 W_{-1} - t \times 3 \times z^3 W_{-2} \\
u \times 2 \times z^2 W_{-2} &= 2 \times z^2 W_2 - r \times 2 \times z^2 W_1 - s \times 2 \times z^2 W_0 - t \times 2 \times z^2 W_{-1} \\
u \times 1 \times z^1 W_{-1} &= 1 \times z^1 W_3 - r \times 1 \times z^1 W_2 - s \times 1 \times z^1 W_1 - t \times 1 \times z^1 W_0.
\end{aligned}$$

If we add the above equations side by side (and using theorem 8.1 (a)(i)), we get (a)

(b) and (c) Using the recurrence relation

$$W_{-n+4} = rW_{-n+3} + sW_{-n+2} + tW_{-n+1} + uW_{-n}$$

i.e.

$$tW_{-n+1} = W_{-n+4} - rW_{-n+3} - sW_{-n+2} - uW_{-n}$$

we obtain

$$\begin{aligned}
tnz^n W_{-2n+1} &= nz^n W_{-2n+4} - rnz^n W_{-2n+3} - snz^n W_{-2n+2} - unz^n W_{-2n} \\
t(n-1)z^{n-1} W_{-2n+3} &= (n-1)z^{n-1} W_{-2n+6} - r(n-1)z^{n-1} W_{-2n+5} \\
&\quad - s(n-1)z^{n-1} W_{-2n+4} - u(n-1)z^{n-1} W_{-2n+2} \\
t(n-2)z^{n-2} W_{-2n+5} &= (n-2)z^{n-2} W_{-2n+8} - r(n-2)z^{n-2} W_{-2n+7} \\
&\quad - s(n-2)z^{n-2} W_{-2n+6} - u(n-2)z^{n-2} W_{-2n+4} \\
&\vdots \\
t \times 3 \times z^3 W_{-5} &= 3 \times z^3 W_{-2} - r \times 3 \times z^3 W_{-3} - s \times 3 \times z^3 W_{-4} - u \times 3 \times z^3 W_{-6} \\
t \times 2 \times z^2 W_{-3} &= 2 \times z^2 W_0 - r \times 2 \times z^2 W_{-1} - s \times 2 \times z^2 W_{-2} - u \times 2 \times z^2 W_{-4} \\
t \times 1 \times z^1 W_{-1} &= 1 \times z^1 W_2 - r \times 1 \times z^1 W_1 - s \times 1 \times z^1 W_0 - u \times 1 \times z^1 W_{-2}.
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
t\sum_{k=1}^{n} kz^k W_{-2k+1} &= (-(n+1)z^{n+1} W_{-2n+2} - (n+2)z^{n+2} W_{-2n} + 2 \times z^2 W_0 \\
&\quad + 1 \times z^1 W_2 + z^2 \sum_{k=1}^{n} kz^k W_{-2k} + 2z^2 \sum_{k=1}^{n} z^k W_{-2k}) \\
&\quad - r(-(n+1)z^{n+1} W_{-2n+1} + 1 \times z^1 W_1 + z^1 \sum_{k=1}^{n} kz^k W_{-2k+1} + z^1 \sum_{k=1}^{n} z^k W_{-2k+1}) \\
&\quad - s(-(n+1)z^{n+1} W_{-2n} + 1 \times z^1 W_0 + z^1 \sum_{k=1}^{n} kz^k W_{-2k} + z^1 \sum_{k=1}^{n} z^k W_{-2k}) \\
&\quad - u(\sum_{k=1}^{n} kz^k W_{-2k}).
\end{aligned} \tag{39}$$

Similarly, using the recurrence relation

$$W_{-n+4} = rW_{-n+3} + sW_{-n+2} + tW_{-n+1} + uW_{-n}$$

i.e.

$$tW_{-n} = W_{-n+3} - rW_{-n+2} - sW_{-n+1} - uW_{-n-1}$$

we obtain

$$\begin{aligned}
tnz^nW_{-2n} &= nz^nW_{-2n+3} - rnz^nW_{-2n+2} - snz^nW_{-2n+1} - unz^nW_{-2n-1} \\
t(n-1)z^{n-1}W_{-2n+2} &= (n-1)\times z^{n-1}W_{-2n+5} - r(n-1)z^{n-1}W_{-2n+4} \\
&\quad - s(n-1)z^{n-1}W_{-2n+3} - u(n-1)z^{n-1}W_{-2n+1} \\
t(n-2)z^{n-2}W_{-2n+4} &= (n-2)\times z^{n-2}W_{-2n+7} - r(n-2)z^{n-2}W_{-2n+6} \\
&\quad - s(n-2)z^{n-2}W_{-2n+5} - u(n-2)z^{n-2}W_{-2n+3} \\
&\vdots \\
t\times 3\times z^3W_{-6} &= 3\times z^3W_{-3} - r\times 3\times z^3W_{-4} - s\times 3\times z^3W_{-5} - u\times 3\times z^3W_{-7} \\
t\times 2\times z^2W_{-4} &= 2\times z^2W_{-1} - r\times 2\times z^2W_{-2} - s\times 2\times z^2W_{-3} - u\times 2\times z^2W_{-5} \\
t\times 1\times z^1W_{-2} &= 1\times z^1W_{1} - r\times 1\times z^1W_{0} - s\times 1\times z^1W_{-1} - u\times 1\times z^1W_{-3}.
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
t\sum_{k=1}^{n} kz^kW_{-2k} &= (-(n+1)z^{n+1}W_{-2n+1} + 1\times z^1W_1 + z^1\sum_{k=1}^{n} kz^kW_{-2k+1} + z^1\sum_{k=1}^{n} z^kW_{-2k+1}) \\
&\quad - r(-(n+1)z^{n+1}W_{-2n} + 1\times z^1W_0 + z^1\sum_{k=1}^{n} kz^kW_{-2k} + z^1\sum_{k=1}^{n} z^kW_{-2k}) \\
&\quad - s(\sum_{k=1}^{n} kz^kW_{-2k+1}) - u(nz^nW_{-2n-1} + z^{-1}\sum_{k=1}^{n} kz^kW_{-2k+1} - z^{-1}\sum_{k=1}^{n} z^kW_{-2k+1}).
\end{aligned} \tag{40}$$

Then, solving system eqs. (39) and (40) (using theorem 8.1 (b)(i) and (c)(i)), the required result of (b) and (c) follow. □

Note that

$$\begin{aligned}
\sum_{k=1}^{n} kz^kW_{-k} &= \sum_{k=0}^{n} kz^kW_{-k}, \\
\sum_{k=1}^{n} kz^kW_{-2k} &= \sum_{k=0}^{n} kz^kW_{-2k}, \\
\sum_{k=1}^{n} kz^kW_{-2k+1} &= \sum_{k=0}^{n} kz^kW_{-2k+1}.
\end{aligned}$$

Note that the proof of theorem 12.1 can be done by taking the derivative of the formulas in theorem 8.1 so that theorem 12.1 can be given in the following form. Here, $\Theta_{4W}'(z)$, $\Theta_{5W}'(z)$ and $\Theta_{6W}'(z)$ denotes the derivatives of $\Theta_{4W}(z)$, $\Theta_{5W}(z)$ and $\Theta_{6W}(z)$ with respect to z, respectively.

Theorem 12.2.
Let z be a real or complex number (in fact z is a real or complex valued function in x). For $n \geq 0$ we have the following formulas:

(a) *If $-z^4 + rz^3 + sz^2 + tz + u \neq 0$ then*

$$\sum_{k=1}^{n} kz^kW_{-k} = \sum_{k=0}^{n} kz^kW_{-k} = \frac{\Theta_{12W}}{(-z^4 + rz^3 + sz^2 + tz + u)^2}$$

where

$$\Theta_{12W} = z((-z^4 + rz^3 + sz^2 + tz + u)\Theta_{4W}'(z) - (-4z^3 + 3rz^2 + 2sz + t)\Theta_{4W}(z)).$$

(b) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2\neq 0$ *then*

$$\sum_{k=1}^{n} kz^kW_{-2k}=\sum_{k=0}^{n} kz^kW_{-2k}$$
$$=\frac{\Theta_{13W}}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2}$$

where

$\Theta_{13W}=z((-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)\Theta_{5W}^{'}(z)-(-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+(t^2-2su))\Theta_{5W}(z))$.

(c) *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2\neq 0$ *then*

$$\sum_{k=1}^{n} kz^kW_{-2k+1}=\sum_{k=0}^{n} kz^kW_{-2k+1}$$
$$=\frac{\Theta_{14W}}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2}$$

where

$\Theta_{14W}=z((-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)\Theta_{6W}^{'}(z)-(-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+(t^2-2su))\Theta_{6W}(z))$.

Proof. From theorem 8.1, we have

$$\sum_{k=1}^{n} z^kW_{-k}=\frac{\Theta_{4W}(z)}{-z^4+rz^3+sz^2+tz+u},$$
$$\sum_{k=1}^{n} z^kW_{-2k}=\frac{\Theta_{5W}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2},$$
$$\sum_{k=1}^{n} z^kW_{-2k+1}=\frac{\Theta_{6W}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2}.$$

By taking the derivative of the both sides of the above formulas with respect to z, we get

$$\sum_{k=1}^{n} kz^{k-1}W_{-k}=\frac{C_4(z)}{(-z^4+rz^3+sz^2+tz+u)^2},$$
$$\sum_{k=1}^{n} kz^{k-1}W_{-2k}=\frac{C_5(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2},$$
$$\sum_{k=1}^{n} kz^{k-1}W_{-2k+1}=\frac{C_6(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2},$$

where

$C_4(z)=(-z^4+rz^3+sz^2+tz+u)\Theta_{4W}^{'}(z)-(-4z^3+3rz^2+2sz+t)\Theta_{4W}(z)$,

$C_5(z)=(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)\Theta_{5W}^{'}(z)-(-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+(t^2-2su))\Theta_{5W}(z)$,

$C_6(z)=(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)\Theta_{6W}^{'}(z)-(-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+(t^2-2su))\Theta_{6W}(z)$.

Now, it follows that

$$\sum_{k=1}^{n} kz^kW_{-k}=\frac{z\times C_4(z)}{(-z^4+rz^3+sz^2+tz+u)^2}=\frac{\Theta_{12W}}{(-z^4+rz^3+sz^2+tz+u)^2},$$
$$\sum_{k=1}^{n} kz^kW_{-2k}=\frac{z\times C_5(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2}$$
$$=\frac{\Theta_{13W}}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2},$$
$$\sum_{k=1}^{n} kz^kW_{-2k+1}=\frac{z\times C_6(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2}$$
$$=\frac{\Theta_{14W}}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2},$$

where

$\Theta_{12W}=z\times C_4(z)=z((-z^4+rz^3+sz^2+tz+u)\Theta_{4W}^{'}(z)-(-4z^3+3rz^2+2sz+t)\Theta_{4W}(z)),$

$\Theta_{13W}=z\times C_5(z)=z((-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)\Theta_{5W}^{'}(z)-(-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+(t^2-2su))\Theta_{5W}(z)),$

$\Theta_{14W}=z\times C_6(z)=z((-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)\Theta_{6W}^{'}(z)-(-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+(t^2-2su))\Theta_{6W}(z)).$ □

To calculate (to evaluate) the sums $\sum_{k=1}^{n}kz^kW_{-k}$, $\sum_{k=1}^{n}kz^kW_{-2k}$ and $\sum_{k=1}^{n}kz^kW_{-2k+1}$, the following Remark is useful.

Remark 12.1.
To calculate the sums $\sum_{k=1}^{n}kz^kW_{-k}$, $\sum_{k=1}^{n}kz^kW_{-2k}$ and $\sum_{k=1}^{n}kz^kW_{-2k+1}$ we use theorem 12.1 (and so theorem 12.2). If there is indeterminate form in the right sides of the sum formulas which is given in theorem 12.1 (and so in theorem 12.2) then we can use L'Hospital rule as theorem 8.1 and remark 8.1.

13. Generating Function of Generalized Tetranacci Polynomials: Closed Formulas of $\sum_{n=0}^{\infty}nW_nz^n$, $\sum_{n=0}^{\infty}nW_{2n}z^n$, $\sum_{n=0}^{\infty}nW_{2n+1}z^n$, $\sum_{n=0}^{\infty}nW_{-n}z^n$, $\sum_{n=0}^{\infty}nW_{-2n}z^n$, $\sum_{n=0}^{\infty}nW_{-2n+1}z^n$

Next, we give the ordinary generating functions of some special cases of generalized Tetranacci polynomials.

Lemma 13.1.
The ordinary generating functions of the sequences nW_n, nW_{2n}, nW_{2n+1}, nW_{-n}, nW_{-2n}, nW_{-2n+1} are given as follows:

(a) $(|z|<\min\{|\alpha|^{-1},|\beta|^{-1},|\gamma|^{-1},|\delta|^{-1}\})$.

$$\sum_{n=0}^{\infty}nW_nz^n=\frac{\Gamma_{7W}(z)}{(uz^4+tz^3+sz^2+rz-1)^2}$$

where

$\Gamma_{7W}(z)=z^3(-sz^2+uz^4-2rz+3)W_3+z^2(tz^3+2uz^4+2r^2z^2-4rz+rsz^3-ruz^5+2)W_2+z(-2sz^2+2tz^3+3uz^4+r^2z^2+s^2z^4-2rz+2rsz^3-rtz^4-2ruz^5-suz^6+1)W_1-uz^4(2sz^2+tz^3+3rz-4)W_0.$

(b) $(|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2},|\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty}nW_{2n}z^n=\frac{\Gamma_{8W}(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2}$$

where

$\Gamma_{8W}(z)=z^2(2r-r^3z+3tz-4stz^2-2tuz^3-2r^2tz^2-rt^2z^3+s^2tz^3-2ru^2z^4-tu^2z^5-2rsz-2rsuz^3)W_3+z(-2r^2z-uz^2+r^4z^2+s^2z^2+2t^2z^3-u^2z^4+u^3z^6-2sz+r^2t^2z^4+2r^2u^2z^5-rtz^2+3r^2sz^2+2r^3tz^3+2r^2uz^3-st^2z^4+s^2uz^4+2su^2z^5+4rstz^3+4rtuz^4-rs^2tz^4+2r^2suz^4+rtu^2z^6+1)W_2+z^2(2t+t^3z^3-r^2tz+4s^2tz^2-2ru^2z^3-2r^3uz^2-s^3tz^3-ru^3z^5-2tu^2z^4+3ruz-5stz-4rsuz^2+2r^2stz^2+2rst^2z^3+rs^2uz^3-2r^2tuz^3+stu^2z^5)W_1+uz^2(-r^2z-4uz^2+4s^2z^2-s^3z^3+t^2z^3+2u^2z^4-5sz+6suz^3+2r^2sz^2+3r^2uz^3-2s^2uz^4-su^2z^5+t^2uz^5+2rstz^3+4rtuz^4+2)W_0.$

(c) $(|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2},|\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty}nW_{2n+1}z^n=\frac{\Gamma_{9W}(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2}$$

where

$\Gamma_{9W}(z)=z(-uz^2+s^2z^2+2t^2z^3-u^2z^4+u^3z^6-2sz+2rtz^2+r^2sz^2+2r^2uz^3-st^2z^4+s^2uz^4+2su^2z^5+2rtuz^4+1)W_3+z^2(2t+t^3z^3+2rs-2rs^2z-r^3sz-r^2tz-2ru^2z^3-2r^3uz^2-ru^3z^5-2tu^2z^4+3ruz-2stz-4rsuz^2-2stuz^3+rst^2z^3-rs^2uz^3-2rsu^2z^4-2r^2tuz^3)W_2+z^2(2u+3t^2z-4u^2z^2+2u^3z^4+2rt-2r^2t^2z^2+3r^2u^2z^3+s^2t^2z^3-2s^2u^2z^4-r^3tz-r^2uz-rt^3z^3-4st^2z^2+4s^2uz^2+6su^2z^3-s^3uz^3-t^2uz^3-su^3z^5-5suz+2r^2suz^2+2rtu^2z^4-2rstz)W_1+uz^2(2r-r^3z+3tz-4stz^2-2tuz^3-2r^2tz^2-rt^2z^3+s^2tz^3-2ru^2z^4-tu^2z^5-2rsz-2rsuz^3)W_0.$

(d) $(|z|<\min\{|\alpha|,|\beta|,|\gamma|,|\delta|\})$.

$$\sum_{n=0}^{\infty}nW_{-n}z^n=\sum_{n=1}^{\infty}nW_{-n}z^n$$
$$=\frac{\Gamma_{10W}(z)}{(-z^4+rz^3+sz^2+tz+u)^2}$$

where

$\Gamma_{10W}(z) = z(u-2rz^3-sz^2+3z^4)W_3 + z(-4rz^4+tz^2+2r^2z^3-ru+2uz+2z^5+rsz^2)W_2 + z(-2rz^5-2sz^4+2tz^3+3uz^2+r^2z^4+s^2z^2-su+z^6+2rsz^3-rtz^2-2ruz)W_1 + uz(-t-3rz^2-2sz+4z^3)W_0.$

(e) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=0}^{\infty} nW_{-2n}z^n = \sum_{n=1}^{\infty} nW_{-2n}z^n = \frac{\Gamma_{11W}(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2}$$

where

$\Gamma_{11W}(z) = -z(tu^2-2rz^5-3tz^4+r^3z^4+2rsz^4+2ru^2z+4stz^3+2tuz^2+rt^2z^2+2r^2tz^3-s^2tz^2+2rsuz^2)W_3 + z(-2sz^5-uz^4-2r^2z^5+r^4z^4+s^2z^4+2t^2z^3-u^2z^2+u^3+z^6+r^2t^2z^2+rtu^2-rtz^4+2su^2z+3r^2sz^4-st^2z^2+2r^2u^2z+2r^3tz^3+2r^2uz^3+s^2uz^2+4rstz^3+4rtuz^2-rs^2tz^2+2r^2suz^2)W_2 - z(ru^3-2tz^5-t^3z^2-stu^2-3ruz^4+5stz^4+2tu^2z+2ru^2z^2+r^2tz^4-4s^2tz^3+s^3tz^2+2r^3uz^3+4rsuz^3-2rst^2z^2-2r^2stz^3-rs^2uz^2+2r^2tuz^2)W_1 + uz(-su^2+t^2u-5sz^4+2u^2z-4uz^3-r^2z^4+4s^2z^3-s^3z^2+t^2z^2+2z^5+6suz^2-2s^2uz+2r^2sz^3+3r^2uz^2+2rstz^2+4rtuz)W_0.$

(f) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=0}^{\infty} nW_{-2n+1}z^n = \sum_{n=1}^{\infty} nW_{-2n+1}z^n = \frac{\Gamma_{12W}(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2}$$

where

$\Gamma_{12W}(z) = z(-2sz^5-uz^4+s^2z^4+2t^2z^3-u^2z^2+u^3+z^6+2rtz^4+2su^2z+r^2sz^4-st^2z^2+2r^2uz^3+s^2uz^2+2rtuz^2)W_3 - z(ru^3-2tz^5-t^3z^2-2rsz^5-3ruz^4+2stz^4+2tu^2z+2rs^2z^4+r^3sz^4+2ru^2z^2+r^2tz^4+2r^3uz^3+2rsu^2z+4rsuz^3+2stuz^2-rst^2z^2+rs^2uz^2+2r^2tuz^2)W_2 - z(su^3-2u^3z-2uz^5-3t^2z^4+4u^2z^3+2r^2t^2z^3-3r^2u^2z^2-s^2t^2z^2-2rtz^5+5suz^4+rt^3z^2+4st^2z^3+r^3tz^4-6su^2z^2+r^2uz^4+2s^2u^2z-4s^2uz^3+s^3uz^2+t^2uz^2+2rstz^4-2rtu^2z-2r^2suz^3)W_1 - uz(tu^2-2rz^5-3tz^4+r^3z^4+2rsz^4+2ru^2z+4stz^3+2tuz^2+rt^2z^2+2r^2tz^3-s^2tz^2+2rsuz^2)W_0.$

Proof. Use theorem 11.1 and theorem 11.2 for the ordinary generating functions of the sequences nW_n, nW_{2n}, nW_{2n+1} and use theorem 12.1 and theorem 12.2 for the ordinary generating functions of the sequences nW_{-n}, nW_{-2n}, nW_{-2n+1}. □

Now, we consider special cases of lemma 13.1.

Corollary 13.1.
The ordinary generating functions of the sequences nG_n, nG_{2n}, nG_{2n+1}, nG_{-n}, nG_{-2n}, nG_{-2n+1} and nH_n, nH_{2n}, nH_{2n+1}, nH_{-n}, nH_{-2n}, nH_{-2n+1} are given as follows:

(a) $(|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\})$.

$$\sum_{n=0}^{\infty} nG_nz^n = \frac{3uz^5+2tz^4+sz^3+z}{(uz^4+tz^3+sz^2+rz-1)^2},$$

$$\sum_{n=0}^{\infty} nH_nz^n = \frac{-tuz^7-4suz^6-(9ru+st)z^5+4(4u-rt)z^4+(9t-rs)z^3+4sz^2+rz}{(uz^4+tz^3+sz^2+rz-1)^2}.$$

(b) $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty} nG_{2n}z^n = \frac{\Gamma_{8aG}(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2},$$

$$\sum_{n=0}^{\infty} nH_{2n}z^n = \frac{\Gamma_{8aH}(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2},$$

where

$\Gamma_{8aG}(z) = -2tu^2z^6+(t^3-3ru^2-2stu)z^5+2(rt^2-2rsu)z^4+(r^2t+2ru-rs^2-2st)z^3+2tz^2+rz$

and

$\Gamma_{8aH}(z) = (t^2u^2-2su^3)z^7+(8rtu^2+8u^3-4s^2u^2)z^6+(9r^2u^2+s^2t^2+4rstu+22su^2-2t^2u-2rt^3-2s^3u)z^5+(8r^2su+16s^2u-16u^2-8st^2-4r^2t^2)z^4+(r^2s^2+9t^2-4rst-2r^3t-2r^2u-22su+2s^3)z^3+(8rt+8u-4s^2)z^2+(r^2+2s)z.$

(c) $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty} nG_{2n+1}z^n = \frac{\Gamma_{9aG}(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2},$$
$$\sum_{n=0}^{\infty} nH_{2n+1}z^n = \frac{\Gamma_{9aH}(z)}{(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)^2},$$

where

$\Gamma_{9aG}(z) = 2u^3z^6+(5su^2-t^2u)z^5+(4s^2u-2st^2-4u^2)z^4+(r^2u+s^3+3t^2-2rst-6su)z^3+2(2rt+u-s^2)z^2+(r^2+s)z$

and

$\Gamma_{9aH}(z) = -tu^3z^7+(2stu^2-6ru^3)z^6+(rt^2u+3s^2tu-11tu^2-st^3-9rsu^2)z^5+(2rst^2+6t^3-16stu-4ru^2-2r^2tu-4rs^2u)z^4+(3r^2st+9tu+9rt^2-s^2t-10rsu-rs^3-3r^3u)z^3+(4r^2t-2rs^2+10ru-2st)z^2+(r^3+3rs+3t)z.$

(d) $(|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\})$.

$$\sum_{n=0}^{\infty} nG_{-n}z^n = \sum_{n=1}^{\infty} nG_{-n}z^n = \frac{z^7+sz^5+2tz^4+3uz^3}{(-z^4+rz^3+sz^2+tz+u)^2},$$
$$\sum_{n=0}^{\infty} nH_{-n}z^n = \sum_{n=1}^{\infty} nH_{-n}z^n = \frac{rz^7+4sz^6+(9t-rs)z^5+(16u-4rt)z^4-(9ru+st)z^3-4suz^2-tuz}{(-z^4+rz^3+sz^2+tz+u)^2}.$$

(e) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=0}^{\infty} nG_{-2n}z^n = \sum_{n=1}^{\infty} nG_{-2n}z^n = \frac{\Gamma_{11aG}(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2},$$
$$\sum_{n=0}^{\infty} nH_{-2n}z^n = \sum_{n=1}^{\infty} nH_{-2n}z^n = \frac{\Gamma_{11aH}(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2},$$

where

$\Gamma_{11aG}(z) = rz^7+2tz^6+(2ru+r^2t-rs^2-2st)z^5+2(rt^2-2rsu)z^4+(t^3-3ru^2-2stu)z^3-2tu^2z^2$

and

$\Gamma_{11aH}(z) = (r^2+2s)z^7+(8rt+8u-4s^2)z^6+(r^2s^2+9t^2+2s^3-22su-2r^3t-2r^2u-4rst)z^5+(16s^2u-16u^2+8r^2su-4r^2t^2-8st^2)z^4+(9r^2u^2+4rstu+s^2t^2+22su^2-2t^2u-2s^3u-2rt^3)z^3+(8rtu^2-4s^2u^2+8u^3)z^2+(t^2u^2-2su^3)z.$

(f) $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\})$.

$$\sum_{n=0}^{\infty} nG_{-2n+1}z^n = \sum_{n=1}^{\infty} nG_{-2n+1}z^n = \frac{\Gamma_{12aG}(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2},$$
$$\sum_{n=0}^{\infty} nH_{-2n+1}z^n = \sum_{n=1}^{\infty} nH_{-2n+1}z^n = \frac{\Gamma_{12aH}(z)}{(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)^2},$$

where

$\Gamma_{12aG}(z) = (r^2+s)z^7+2(2rt+u-s^2)z^6+(r^2u+s^3+3t^2-6su-2rst)z^5+2(2s^2u-st^2-2u^2)z^4+(5su^2-t^2u)z^3+2u^3z^2$

and

$\Gamma_{12aH}(z) = (r^3+3rs+3t)z^7+(4r^2t+10ru-2st-2rs^2)z^6+(3r^2st+9tu+9rt^2-s^2t-10rsu-3r^3u-rs^3)z^5+(2rst^2+6t^3-16stu-4ru^2-2r^2tu-4rs^2u)z^4+(3s^2tu+rt^2u-11tu^2-st^3-9rsu^2)z^3+(2stu^2-6ru^3)z^2-tu^3z.$

14. Closed Forms of the Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1} W_k$, $\sum_{k=0}^{n} z^k W_{k+2} W_k$ and $\sum_{k=0}^{n} z^k W_{k+3} W_k$ for the Generalized Tetranacci Numbers

The following theorem presents some summing formulas of generalized Tetranacci polynomials with positive subscripts.

Theorem 14.1.
Let z be a real or complex number (in fact z is a real or complex valued function in x). If $\Delta = (-u^3z^6 + su^2z^5 - u(u+rt)z^4 + (2su + r^2u - t^2)z^3 + (rt+u)z^2 + sz + 1)(-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (r^2 + 2s)z - 1) \neq 0$ then For $n \geq 0$ we have the following formulas:

(a)
$$\sum_{k=0}^{n} z^k W_k^2 = \frac{\Theta_{15W}}{\Delta},$$

(b)
$$\sum_{k=0}^{n} z^k W_{k+1} W_k = \frac{\Theta_{16W}}{\Delta},$$

(c)
$$\sum_{k=0}^{n} z^k W_{k+2} W_k = \frac{\Theta_{17W}}{\Delta},$$

(d)
$$\sum_{k=0}^{n} z^k W_{k+3} W_k = \frac{\Theta_{18W}}{\Delta},$$

where

$$\Theta_{15W} = \sum_{k=1}^{20} \Omega_k,\ \Theta_{16W} = \sum_{k=1}^{20} \Lambda_k,\ \Theta_{17W} = \sum_{k=1}^{20} \Phi_k,\ \Theta_{18W} = \sum_{k=1}^{20} \Psi_k$$

with

$\Omega_1 = -z^{n+4}(uz^2 + t^2z^3 + u^2z^4 - u^3z^6 + sz + rtz^2 + r^2uz^3 - su^2z^5 + rtuz^4 - 1)W_{n+4}^2,$

$\Omega_2 = -z^{n+3}(r^2z + uz^2 + t^2z^3 + u^2z^4 - u^3z^6 + sz + r^2t^2z^4 + r^2u^2z^5 - r^2u^3z^7 + rtz^2 + r^2sz^2 + r^3tz^3 + r^4uz^4 - su^2z^5 - r^2su^2z^6 + 2rstz^3 + 3rtuz^4 + 2r^2suz^4 - 2rtu^2z^6 + r^3tuz^5 - 1)W_{n+3}^2,$

$\Omega_3 = -z^{n+2}(r^2z + uz^2 + s^2z^2 - s^3z^3 + t^2z^3 + u^2z^4 - u^3z^6 + sz + r^2t^2z^4 + r^2u^2z^5 + s^2t^2z^5 - r^2u^3z^7 - s^2u^2z^6 - s^3u^2z^7 - s^2u^3z^8 + rtz^2 + r^2sz^2 + r^3tz^3 + r^4uz^4 + s^2uz^4 - su^2z^5 - 2s^3uz^5 - r^2s^2uz^5 - r^2su^2z^6 + 4rstz^3 + 3rtuz^4 - rs^2tz^4 + 4r^2suz^4 - 2rtu^2z^6 + r^3tuz^5 + 2st^2uz^6 + rs^2tuz^6 + 2rstu^2z^7 - 1)W_{n+2}^2,$

$\Omega_4 = -z^{n+1}(r^2z + uz^2 + s^2z^2 - s^3z^3 + 2t^2z^3 + u^2z^4 - t^4z^6 - u^3z^6 + sz + r^2t^2z^4 + r^2u^2z^5 + s^2t^2z^5 - r^2u^3z^7 - s^2u^2z^6 - s^3u^2z^7 - s^2u^3z^8 + t^2u^2z^7 - t^2u^3z^9 + rtz^2 + r^2sz^2 + r^3tz^3 - rt^3z^5 - st^2z^4 + r^4uz^4 + s^2uz^4 - su^2z^5 - 2s^3uz^5 - t^2uz^5 - r^2s^2uz^5 - r^2su^2z^6 - r^2t^2uz^6 + st^2u^2z^8 + 4rstz^3 + 5rtuz^4 - rs^2tz^4 + 4r^2suz^4 - 4rtu^2z^6 + r^3tuz^5 - rt^3uz^7 + 4st^2uz^6 + rs^2tuz^6 + 4rstu^2z^7 - 1)W_{n+1}^2,$

$\Omega_5 = 2z^{n+5}(rs + r^2tz + tuz^2 + rt^2z^2 + ru^2z^3 + r^3uz^2 - ru^3z^5 - tu^2z^4 + stz + rsuz^2 - rsu^2z^4 + r^2tuz^3)W_{n+4}W_{n+3},$

$\Omega_6 = 2z^{n+5}(rt - s^2u^2z^4 + r^2uz + st^2z^2 - s^2uz^2 + t^2uz^3 - su^3z^5 + suz + rtu^2z^4 + rstuz^3)W_{n+4}W_{n+2},$

$\Omega_7 = 2uz^{n+5}(r - ruz^2 + stz^2 + tuz^3 - tu^2z^5 + rsuz^3)W_{n+4}W_{n+1},$

$\Omega_8 = -2z^{n+5}(-st - rt^2z + s^2tz - ru^2z^2 + ru^3z^4 + tu^2z^3 - tuz - rs^2u^2z^4 + r^2tu^2z^4 + stuz^2 + rst^2z^2 + rsu^2z^3 - r^2tuz^2 + rt^2uz^3 - rsu^3z^5 - stu^2z^4 - rsuz + r^2stuz^3)W_{n+3}W_{n+2},$

$\Omega_9 = -2uz^{n+5}(-s + s^2z - r^2uz^2 + s^2uz^3 + su^2z^4 - t^2uz^4 - rtz + rstz^2 + r^2suz^3 - rtu^2z^5)W_{n+3}W_{n+1},$

$\Omega_{10} = -2uz^{n+5}(-t + t^3z^3 + tuz^2 + rt^2z^2 + ru^2z^3 - ruz + stz - stuz^3 - rsu^2z^4 + r^2tuz^3 + rt^2uz^4 - stu^2z^5)W_{n+2}W_{n+1},$

$\Omega_{11} = z^3(uz^2 + t^2z^3 + u^2z^4 - u^3z^6 + sz + rtz^2 + r^2uz^3 - su^2z^5 + rtuz^4 - 1)W_3^2,$

$\Omega_{12} = z^2(r^2z + uz^2 + t^2z^3 + u^2z^4 - u^3z^6 + sz + r^2t^2z^4 + r^2u^2z^5 - r^2u^3z^7 + rtz^2 + r^2sz^2 + r^3tz^3 + r^4uz^4 - su^2z^5 - r^2su^2z^6 + 2rstz^3 + 3rtuz^4 + 2r^2suz^4 - 2rtu^2z^6 + r^3tuz^5 - 1)W_2^2,$

$\Omega_{13} = z(r^2z + uz^2 + s^2z^2 - s^3z^3 + t^2z^3 + u^2z^4 - u^3z^6 + sz + r^2t^2z^4 + r^2u^2z^5 + s^2t^2z^5 - r^2u^3z^7 - s^2u^2z^6 - s^3u^2z^7 - s^2u^3z^8 + rtz^2 + r^2sz^2 + r^3tz^3 + r^4uz^4 + s^2uz^4 - su^2z^5 - 2s^3uz^5 - r^2s^2uz^5 - r^2su^2z^6 + 4rstz^3 + 3rtuz^4 - rs^2tz^4 + 4r^2suz^4 - 2rtu^2z^6 + r^3tuz^5 + 2st^2uz^6 + rs^2tuz^6 + 2rstu^2z^7 - 1)W_1^2,$

$\Omega_{14}=(r^2z+uz^2+s^2z^2-s^3z^3+2t^2z^3+u^2z^4-t^4z^6-u^3z^6+sz+r^2t^2z^4+r^2u^2z^5+s^2t^2z^5-r^2u^3z^7-s^2u^2z^6-s^3u^2z^7-s^2u^3z^8+t^2u^2z^7-t^2u^3z^9+rtz^2+r^2sz^2+r^3tz^3-rt^3z^5-st^2z^4+r^4uz^4+s^2uz^4-su^2z^5-2s^3uz^5-t^2uz^5-r^2s^2uz^5-r^2su^2z^6-r^2t^2uz^6+st^2u^2z^8+4rstz^3+5rtuz^4-rs^2tz^4+4r^2suz^4-4rtu^2z^6+r^3tuz^5-rt^3uz^7+4st^2uz^6+rs^2tuz^6+4rstu^2z^7-1)W_0^2,$

$\Omega_{15}=-2z^4(rs+r^2tz+tuz^2+rt^2z^2+ru^2z^3+r^3uz^2-ru^3z^5-tu^2z^4+stz+rsuz^2-rsu^2z^4+r^2tuz^3)W_3W_2,$

$\Omega_{16}=-2z^4(rt-s^2u^2z^4+r^2uz+st^2z^2-s^2uz^2+t^2uz^3-su^3z^5+suz+rtu^2z^4+rstuz^3)W_3W_1,$

$\Omega_{17}=2z^4(-st-rt^2z+s^2tz-ru^2z^2+ru^3z^4+tu^2z^3-tuz-rs^2u^2z^4+r^2tu^2z^4+stuz^2+rst^2z^2+rsu^2z^3-r^2tuz^2+rt^2uz^3-rsu^3z^5-stu^2z^4-rsuz+r^2stuz^3)W_2W_1,$

$\Omega_{18}=-2uz^4(r-ruz^2+stz^2+tuz^3-tu^2z^5+rsuz^3)W_3W_0,$

$\Omega_{19}=2uz^4(-s+s^2z-r^2uz^2+s^2uz^3+su^2z^4-t^2uz^4-rtz+rstz^2+r^2suz^3-rtu^2z^5)W_2W_0,$

$\Omega_{20}=2uz^4(-t+t^3z^3+tuz^2+rt^2z^2+ru^2z^3-ruz+stz-stuz^3-rsu^2z^4+r^2tuz^3+rt^2uz^4-stu^2z^5)W_1W_0,$

and

$\Lambda_1=z^{n+4}(r-ruz^2+stz^2+tuz^3-tu^2z^5+rsuz^3)W_{n+4}^2,$

$\Lambda_2=z^{n+5}(rs^2+st+rt^2z+ru^2z^2-ru^3z^4-tu^2z^3+tuz-r^2tu^2z^4+r^2stz+rs^2uz^2+r^3suz^2+r^2tuz^2-rt^2uz^3+rsuz)W_{n+3}^2,$

$\Lambda_3=z^{n+5}(rt^2+tu+ru^2z+st^3z^2-ru^3z^3-tu^2z^2-r^2tu^2z^3-s^2tu^2z^4+r^2tuz-rt^2uz^2+rsu^3z^4-s^2tuz^2+stu^2z^3+rst^2uz^3)W_{n+2}^2,$

$\Lambda_4=u^2z^{n+5}(r-ruz^2+stz^2+tuz^3-tu^2z^5+rsuz^3)W_{n+1}^2,$

$\Lambda_5=-z^{n+3}(r^2z+uz^2+s^2z^2+t^2z^3+u^2z^4-u^3z^6-r^2uz^3+s^2uz^4-t^2uz^5+2rstz^3+2rtuz^4+2r^2suz^4-2rtu^2z^6-1)W_{n+4}W_{n+3},$

$\Lambda_6=z^{n+4}(t-t^3z^3+r^2tz-s^2tz^2+r^3uz^2-ru^3z^5-tu^2z^4+ruz+2rsuz^2-rs^2uz^3-rt^2uz^4+2stu^2z^5)W_{n+4}W_{n+2},$

$\Lambda_7=uz^{n+4}(r^2z-uz^2-s^2z^2-t^2z^3-u^2z^4+u^3z^6-r^2uz^3-s^2uz^4+t^2uz^5+1)W_{n+4}W_{n+1},$

$\Lambda_8=-z^{n+2}(r^2z+uz^2+s^2z^2-s^3z^3+t^2z^3+u^2z^4-u^3z^6+sz-r^2u^3z^7+rtz^2-suz^3+r^2sz^2+r^3tz^3-rt^3z^5-st^2z^4+r^4uz^4+s^2uz^4-su^2z^5-s^3uz^5-t^2uz^5+su^3z^7-r^2s^2uz^5-r^2t^2uz^6+2rstz^3+2rtuz^4-rs^2tz^4+3r^2suz^4-3rtu^2z^6+st^2uz^6+2rstu^2z^7-1)W_{n+3}W_{n+2},$

$\Lambda_9=uz^{n+5}(t-t^3z^3+2rs+r^2tz+s^2tz^2+r^3uz^2-ru^3z^5-tu^2z^4+ruz+2stuz^3+rs^2uz^3-rt^2uz^4)W_{n+3}W_{n+1},$

$\Lambda_{10}=-z^{n+1}(r^2z+uz^2+s^2z^2-s^3z^3+2t^2z^3+u^2z^4-t^4z^6-u^3z^6+sz+r^2t^2z^4+s^2t^2z^5-r^2u^3z^7-s^3u^2z^7-t^2u^2z^7+rtz^2+r^2sz^2+r^3tz^3-rt^3z^5-st^2z^4+r^4uz^4+s^2uz^4-2su^2z^5-2s^3uz^5-t^2uz^5+su^4z^9-r^2s^2uz^5-r^2su^2z^6-r^2t^2uz^6+st^2u^2z^8+4rstz^3+3rtuz^4-rs^2tz^4+4r^2suz^4-3rtu^2z^6+r^3tuz^5-rt^3uz^7+2st^2uz^6-rtu^3z^8+rs^2tuz^6+2rstu^2z^7-1)W_{n+2}W_{n+1},$

$\Lambda_{11}=-z^3(r-ruz^2+stz^2+tuz^3-tu^2z^5+rsuz^3)W_3^2,$

$\Lambda_{12}=-z^4(rs^2+st+rt^2z+ru^2z^2-ru^3z^4-tu^2z^3+tuz-r^2tu^2z^4+r^2stz+rs^2uz^2+r^3suz^2+r^2tuz^2-rt^2uz^3+rsuz)W_2^2,$

$\Lambda_{13}=z^4(-rt^2-tu-ru^2z-st^3z^2+ru^3z^3+tu^2z^2+r^2tu^2z^3+s^2tu^2z^4-r^2tuz+rt^2uz^2-rsu^3z^4+s^2tuz^2-stu^2z^3-rst^2uz^3)W_1^2,$

$\Lambda_{14}=-u^2z^4(r-ruz^2+stz^2+tuz^3-tu^2z^5+rsuz^3)W_0^2,$

$\Lambda_{15}=z^2(r^2z+uz^2+s^2z^2+t^2z^3+u^2z^4-u^3z^6-r^2uz^3+s^2uz^4-t^2uz^5+2rstz^3+2rtuz^4+2r^2suz^4-2rtu^2z^6-1)W_3W_2,$

$\Lambda_{16}=-z^3(t-t^3z^3+r^2tz-s^2tz^2+r^3uz^2-ru^3z^5-tu^2z^4+ruz+2rsuz^2-rs^2uz^3-rt^2uz^4+2stu^2z^5)W_3W_1,$

$\Lambda_{17}=z(r^2z+uz^2+s^2z^2-s^3z^3+t^2z^3+u^2z^4-u^3z^6+sz-r^2u^3z^7+rtz^2-suz^3+r^2sz^2+r^3tz^3-rt^3z^5-st^2z^4+r^4uz^4+s^2uz^4-su^2z^5-s^3uz^5-t^2uz^5+su^3z^7-r^2s^2uz^5-r^2t^2uz^6+2rstz^3+2rtuz^4-rs^2tz^4+3r^2suz^4-3rtu^2z^6+st^2uz^6+2rstu^2z^7-1)W_2W_1,$

$\Lambda_{18}=uz^3(-r^2z+uz^2+s^2z^2+t^2z^3+u^2z^4-u^3z^6+r^2uz^3+s^2uz^4-t^2uz^5-1)W_3W_0,$

$\Lambda_{19}=-uz^4(t-t^3z^3+2rs+r^2tz+s^2tz^2+r^3uz^2-ru^3z^5-tu^2z^4+ruz+2stuz^3+rs^2uz^3-rt^2uz^4)W_2W_0,$

$\Lambda_{20}=(r^2z+uz^2+s^2z^2-s^3z^3+2t^2z^3+u^2z^4-t^4z^6-u^3z^6+sz+r^2t^2z^4+s^2t^2z^5-r^2u^3z^7-s^3u^2z^7-t^2u^2z^7+rtz^2+r^2sz^2+r^3tz^3-rt^3z^5-st^2z^4+r^4uz^4+s^2uz^4-2su^2z^5-2s^3uz^5-t^2uz^5+su^4z^9-r^2s^2uz^5-r^2su^2z^6-r^2t^2uz^6+st^2u^2z^8+4rstz^3+3rtuz^4-rs^2tz^4+4r^2suz^4-3rtu^2z^6+r^3tuz^5-rt^3uz^7+2st^2uz^6-rtu^3z^8+rs^2tuz^6+2rstu^2z^7-1)W_1W_0,$

and

$\Phi_1=z^{n+4}(s-s^2z+r^2-s^2uz^3-su^2z^4+t^2uz^4+rtz+rtuz^3)W_{n+4}^2,$

$\Phi_2 = z^{n+3}(s - s^2z + r^2t^2z^3 + r^2u^2z^4 - r^2sz + rt^3z^4 - s^2uz^3 - su^2z^4 + t^2uz^4 - r^2s^2uz^4 - r^2su^2z^5 + r^2t^2uz^5 + rtuz^3 - rs^2tz^3 + rtu^2z^5 + r^3tuz^4 - 2rstuz^4)W_{n+3}^2,$

$\Phi_3 = z^{n+5}(st^2 + r^2t^2 - s^2u^2z^2 - s^2u^3z^4 + rt^3z + su^2z + t^2uz + r^2u^2z - s^2t^2z - su^4z^5 + rtu + r^2t^2uz^2 + st^2u^2z^4 + r^3tuz + rtu^2z^2 - rs^2tuz^2 + 2rstuz)W_{n+2}^2,$

$\Phi_4 = u^2z^{n+5}(s - s^2z + r^2 - s^2uz^3 - su^2z^4 + t^2uz^4 + rtz + rtuz^3)W_{n+1}^2,$

$\Phi_5 = z^{n+3}(r - r^3z - t^3z^4 + tz + rs^2z^2 - r^2tz^2 - rt^2z^3 + s^2tz^3 - ru^2z^4 - tu^2z^5 + 2stuz^4 + 2rs^2uz^4 + 2rsu^2z^5 - 2r^2tuz^4 - 2rt^2uz^5)W_{n+4}W_{n+3},$

$\Phi_6 = -z^{n+2}(r^2z + s^2z^2 - s^3z^3 + t^2z^3 + 2u^2z^4 - u^4z^8 + sz + r^2u^2z^5 - s^2u^2z^6 + t^2u^2z^7 + suz^3 + r^2sz^2 - st^2z^4 - su^2z^5 - s^3uz^5 - su^3z^7 + 2rstz^3 + 4rtuz^4 + r^2suz^4 + st^2uz^6 - 1)W_{n+4}W_{n+2},$

$\Phi_7 = uz^{n+4}(r + r^3z - t^3z^4 + tz - rs^2z^2 + r^2tz^2 - rt^2z^3 + s^2tz^3 - ru^2z^4 - tu^2z^5 + 2rsz + 2stuz^4)W_{n+4}W_{n+1},$

$\Phi_8 = z^{n+3}(t - t^3z^3 + r^3u^2z^4 - r^2tz + s^2tz^2 + ru^2z^3 - s^3tz^3 + st^3z^4 - tu^2z^4 - ru^4z^7 - stz - rs^2u^2z^5 + rt^2u^2z^6 + rsuz^2 + 2stuz^3 + r^2stz^2 + 2rs^2uz^3 + 2rsu^2z^4 + r^3suz^3 - rs^3uz^4 + 2r^2tuz^3 - 2rt^2uz^4 - rsu^3z^6 - 2s^2tuz^4 + stu^2z^5 + rst^2uz^5) W_{n+3}W_{n+2},$

$\Phi_9 = -z^{n+1}(r^2z + s^2z^2 - s^3z^3 + 2t^2z^3 + 2u^2z^4 - t^4z^6 - u^4z^8 + sz + r^2t^2z^4 + r^2u^2z^5 + s^2t^2z^5 - r^2u^3z^7 - s^2u^2z^6 + rtz^2 + suz^3 + r^2sz^2 + r^3tz^3 + r^2uz^3 - rt^3z^5 - st^2z^4 + r^4uz^4 - su^2z^5 - s^3uz^5 - su^3z^7 - r^2s^2uz^5 - r^2t^2uz^6 + 4rstz^3 + 5rtuz^4 - rs^2tz^4 + 3r^2suz^4 - rtu^2z^6 + r^3tuz^5 - rt^3uz^7 + 3st^2uz^6 - rtu^3z^8 + rs^2tuz^6 + 2rstu^2z^7 - 1)W_{n+3}W_{n+1},$

$\Phi_{10} = uz^{n+5}(2r^2t + ru + 2st + 2rt^2z - 2s^2tz + r^3uz - ru^3z^4 + t^3uz^4 - tu^3z^5 + tuz - rs^2uz^2 + r^2tuz^2 + rt^2uz^3 - s^2tuz^3 + 2rsuz)W_{n+2}W_{n+1},$

$\Phi_{11} = -z^3(s - s^2z + r^2 - s^2uz^3 - su^2z^4 + t^2uz^4 + rtz + rtuz^3)W_3^2,$

$\Phi_{12} = -z^2(s - s^2z + r^2t^2z^3 + r^2u^2z^4 - r^2sz + rt^3z^4 - s^2uz^3 - su^2z^4 + t^2uz^4 - r^2s^2uz^4 - r^2su^2z^5 + r^2t^2uz^5 + rtuz^3 - rs^2tz^3 + rtu^2z^5 + r^3tuz^4 - 2rstuz^4)W_2^2,$

$\Phi_{13} = -z^4(st^2 + r^2t^2 - s^2u^2z^2 - s^2u^3z^4 + rt^3z + su^2z + t^2uz + r^2u^2z - s^2t^2z - su^4z^5 + rtu + r^2t^2uz^2 + st^2u^2z^4 + r^3tuz + rtu^2z^2 - rs^2tuz^2 + 2rstuz)W_1^2,$

$\Phi_{14} = -u^2z^4(s - s^2z + r^2 - s^2uz^3 - su^2z^4 + t^2uz^4 + rtz + rtuz^3)W_0^2,$

$\Phi_{15} = z^2(-r + r^3z + t^3z^4 - tz - rs^2z^2 + r^2tz^2 + rt^2z^3 - s^2tz^3 + ru^2z^4 + tu^2z^5 - 2stuz^4 - 2rs^2uz^4 - 2rsu^2z^5 + 2r^2tuz^4 + 2rt^2uz^5)W_3W_2,$

$\Phi_{16} = z(r^2z + s^2z^2 - s^3z^3 + t^2z^3 + 2u^2z^4 - u^4z^8 + sz + r^2u^2z^5 - s^2u^2z^6 + t^2u^2z^7 + suz^3 + r^2sz^2 - st^2z^4 - su^2z^5 - s^3uz^5 - su^3z^7 + 2rstz^3 + 4rtuz^4 + r^2suz^4 + st^2uz^6 - 1)W_3W_1,$

$\Phi_{17} = -z^2(t - t^3z^3 + r^3u^2z^4 - r^2tz + s^2tz^2 + ru^2z^3 - s^3tz^3 + st^3z^4 - tu^2z^4 - ru^4z^7 - stz - rs^2u^2z^5 + rt^2u^2z^6 + rsuz^2 + 2stuz^3 + r^2stz^2 + 2rs^2uz^3 + 2rsu^2z^4 + r^3suz^3 - rs^3uz^4 + 2r^2tuz^3 - 2rt^2uz^4 - rsu^3z^6 - 2s^2tuz^4 + stu^2z^5 + rst^2uz^5)W_2W_1,$

$\Phi_{18} = -uz^3(r + r^3z - t^3z^4 + tz - rs^2z^2 + r^2tz^2 - rt^2z^3 + s^2tz^3 - ru^2z^4 - tu^2z^5 + 2rsz + 2stuz^4)W_3W_0,$

$\Phi_{19} = (r^2z + s^2z^2 - s^3z^3 + 2t^2z^3 + 2u^2z^4 - t^4z^6 - u^4z^8 + sz + r^2t^2z^4 + r^2u^2z^5 + s^2t^2z^5 - r^2u^3z^7 - s^2u^2z^6 + rtz^2 + suz^3 + r^2sz^2 + r^3tz^3 + r^2uz^3 - rt^3z^5 - st^2z^4 + r^4uz^4 - su^2z^5 - s^3uz^5 - su^3z^7 - r^2s^2uz^5 - r^2t^2uz^6 + 4rstz^3 + 5rtuz^4 - rs^2tz^4 + 3r^2suz^4 - rtu^2z^6 + r^3tuz^5 - rt^3uz^7 + 3st^2uz^6 - rtu^3z^8 + rs^2tuz^6 + 2rstu^2z^7 - 1)W_2W_0,$

$\Phi_{20} = -uz^4(2r^2t + ru + 2st + 2rt^2z - 2s^2tz + r^3uz - ru^3z^4 + t^3uz^4 - tu^3z^5 + tuz - rs^2uz^2 + r^2tuz^2 + rt^2uz^3 - s^2tuz^3 + 2rsuz)W_1W_0,$

and

$\Psi_1 = z^{n+4}(t - t^3z^3 + 2rs + r^3 - rs^2z + r^2tz - tuz^2 - rt^2z^2 + s^2tz^2 - ru^2z^3 + ruz - stz - rsuz^2 + 2stuz^3)W_{n+4}^2,$

$\Psi_2 = z^{n+3}(t - t^3z^3 + rs + r^3u^2z^4 - rs^2z - r^3sz - r^2tz - tuz^2 + rs^3z^2 - rt^2z^2 + s^2tz^2 + ru^3z^5 - ru^4z^7 - stz + rs^2u^2z^5 + r^2tu^2z^5 + rt^2u^2z^6 + 2stuz^3 - r^2stz^2 - rst^2z^3 + rs^2uz^3 + rsu^2z^4 + rs^3uz^4 + r^2tuz^3 - rt^2uz^4 - rsu^3z^6 - 2r^2stuz^4 - rst^2uz^5)W_{n+3}^2,$

$\Psi_3 = z^{n+2}(t - t^3z^3 + r^3u^2z^4 - r^2tz - tuz^2 - rt^2z^2 - s^2tz^2 + s^3tz^3 + ru^3z^5 - ru^4z^7 - stz - rs^2u^2z^5 + r^2tu^2z^5 + rt^2u^2z^6 + s^2tu^2z^6 + stuz^3 - r^2stz^2 - 2rst^2z^3 + 2rsu^2z^4 + r^2tuz^3 - rt^2uz^4 - rsu^3z^6 - s^2tuz^4 + s^3tuz^5 - st^3uz^6 + stu^3z^7 - r^2stuz^4 - rst^2uz^5)W_{n+2}^2,$

$\Psi_4 = u^2z^{n+5}(t - t^3z^3 + 2rs + r^3 - rs^2z + r^2tz - tuz^2 - rt^2z^2 + s^2tz^2 - ru^2z^3 + ruz - stz - rsuz^2 + 2stuz^3)W_{n+1}^2,$

$\Psi_5 = z^{n+3}(s - r^4z - s^3z^2 - u^2z^3 - u^3z^5 + u^4z^7 + uz + r^2 + r^2s^2z^2 + r^2t^2z^3 - s^2u^2z^5 - t^2u^2z^6 - r^2sz - suz^2 - r^3tz^2 - r^2uz^2 + rt^3z^4 + st^2z^3 - s^2uz^3 - su^2z^4 - s^3uz^4 + t^2uz^4 + su^3z^6 + rtz + 2rstz^2 - rs^2tz^3 + r^2suz^3 - rtu^2z^5 + st^2uz^5) W_{n+4}W_{n+3},$

$\Psi_6 = z^{n+2}(r - r^3z + stz^2 + tuz^3 - rs^2z^2 - r^3sz^2 + rs^3z^3 + rt^2z^3 - ru^2z^4 - s^3tz^4 + st^3z^5 + t^3uz^6 - tu^3z^7 - rsz + 2stuz^4 - r^2stz^3 + rst^2z^4 + 2rs^2uz^4 + 3rsu^2z^5 - r^2tuz^4 - 3s^2tuz^5 - stu^2z^6)W_{n+4}W_{n+2},$

$\Psi_7 = -z^{n+1}(r^2z+uz^2+s^2z^2-s^3z^3+2t^2z^3+u^2z^4-t^4z^6-u^3z^6+sz+r^2t^2z^4+s^2t^2z^5-t^2u^2z^7+rtz^2-suz^3+r^2sz^2+r^3tz^3-r^2uz^3-rt^3z^5-st^2z^4+s^2uz^4-su^2z^5-s^3uz^5-t^2uz^5+su^3z^7+4rstz^3+2rtuz^4-rs^2tz^4+r^2suz^4-3rtu^2z^6+3st^2uz^6-1)W_{n+4}W_{n+1},$

$\Psi_8 = z^{n+2}(s-s^2z-s^3z^2+s^4z^3-u^2z^3-u^3z^5+u^4z^7+uz-r^2s^2z^2+r^2u^2z^4-s^2t^2z^4+s^3u^2z^6-s^2u^3z^7-t^2u^2z^6-r^2sz-2suz^2-r^3tz^2-r^2uz^2+rt^3z^4+st^2z^3+t^2uz^4+2su^3z^6+s^4uz^5-su^4z^8+rtz-r^2s^2uz^4-r^2su^2z^5-s^2t^2uz^6+st^2u^2z^7-2rstz^2-rtuz^3-rs^2tz^3-rtu^2z^5+r^3tuz^4-rt^3uz^6+rtu^3z^7-rs^2tuz^5+2rstu^2z^6)W_{n+3}W_{n+2},$

$\Psi_9 = uz^{n+3}(r-r^3z+3stz^2+tuz^3+3rs^2z^2+r^3sz^2-rs^3z^3+rt^2z^3-2s^2tz^3-ru^2z^4+s^3tz^4-st^3z^5+t^3uz^6-tu^3z^7-rsz+2rsuz^3+r^2stz^3-rst^2z^4+rsu^2z^5-r^2tuz^4+s^2tuz^5-stu^2z^6)W_{n+3}W_{n+1},$

$\Psi_{10} = uz^{n+2}(-r^2z-uz^2-s^2z^2+s^3z^3-u^2z^4-t^4z^6+u^3z^6-sz+r^2t^2z^4+s^2t^2z^5+t^2u^2z^7-rtz^2+suz^3-r^2sz^2+r^3tz^3+r^2uz^3-rt^3z^5-st^2z^4-s^2uz^4+su^2z^5+s^3uz^5-t^2uz^5-su^3z^7-rs^2tz^4-r^2suz^4+rtu^2z^6+st^2uz^6-2rstuz^5+1)W_{n+2}W_{n+1},$

$\Psi_{11} = -z^3(t-t^3z^3+2rs+r^3-rs^2z+r^2tz-tuz^2-rt^2z^2+s^2tz^2-ru^2z^3+ruz-stz-rsuz^2+2stuz^3)W_3^2,$

$\Psi_{12} = z^2(-t+t^3z^3-rs-r^3u^2z^4+rs^2z+r^3sz+r^2tz+tuz^2-rs^3z^2+rt^2z^2-s^2tz^2-ru^3z^5+ru^4z^7+stz-rs^2u^2z^5-r^2tu^2z^5-rt^2u^2z^6-2stuz^3+r^2stz^2+rst^2z^3-rs^2uz^3-rsu^2z^4-rs^3uz^4-r^2tuz^3+rt^2uz^4+rsu^3z^6+2r^2stuz^4+rst^2uz^5)W_2^2,$

$\Psi_{13} = -z(t-t^3z^3+r^3u^2z^4-r^2tz-tuz^2-rt^2z^2-s^2tz^2+s^3tz^3+ru^3z^5-ru^4z^7-stz-rs^2u^2z^5+r^2tu^2z^5+rt^2u^2z^6+s^2tu^2z^6+stuz^3-r^2stz^2-2rst^2z^3+2rsu^2z^4+r^2tuz^3-rt^2uz^4-rsu^3z^6-s^2tuz^4+s^3tuz^5-st^3uz^6+stu^3z^7-r^2stuz^4-rst^2uz^5)W_1^2,$

$\Psi_{14} = -u^2z^4(t-t^3z^3+2rs+r^3-rs^2z+r^2tz-tuz^2-rt^2z^2+s^2tz^2-ru^2z^3+ruz-stz-rsuz^2+2stuz^3)W_0^2,$

$\Psi_{15} = z^2(-s+r^4z+s^3z^2+u^2z^3+u^3z^5-u^4z^7-uz-r^2-r^2s^2z^2-r^2t^2z^3+s^2u^2z^5+t^2u^2z^6+r^2sz+suz^2+r^3tz^2+r^2uz^2-rt^3z^4-st^2z^3+s^2uz^3+su^2z^4+s^3uz^4-t^2uz^4-su^3z^6-rtz-2rstz^2+rs^2tz^3-r^2suz^3+rtu^2z^5-st^2uz^5) W_3W_2,$

$\Psi_{16} = z(-r+r^3z-stz^2-tuz^3+rs^2z^2+r^3sz^2-rs^3z^3-rt^2z^3+ru^2z^4+s^3tz^4-st^3z^5-t^3uz^6+tu^3z^7+rsz-2stuz^4+r^2stz^3-rst^2z^4-2rs^2uz^4-3rsu^2z^5+r^2tuz^4+3s^2tuz^5+stu^2z^6)W_3W_1,$

$\Psi_{17} = -z(s-s^2z-s^3z^2+s^4z^3-u^2z^3-u^3z^5+u^4z^7+uz-r^2s^2z^2+r^2u^2z^4-s^2t^2z^4+s^3u^2z^6-s^2u^3z^7-t^2u^2z^6-r^2sz-2suz^2-r^3tz^2-r^2uz^2+rt^3z^4+st^2z^3+t^2uz^4+2su^3z^6+s^4uz^5-su^4z^8+rtz-r^2s^2uz^4-r^2su^2z^5-s^2t^2uz^6+st^2u^2z^7-2rstz^2-rtuz^3-rs^2tz^3-rtu^2z^5+r^3tuz^4-rt^3uz^6+rtu^3z^7-rs^2tuz^5+2rstu^2z^6)W_2W_1,$

$\Psi_{18} = (r^2z+uz^2+s^2z^2-s^3z^3+2t^2z^3+u^2z^4-t^4z^6-u^3z^6+sz+r^2t^2z^4+s^2t^2z^5-t^2u^2z^7+rtz^2-suz^3+r^2sz^2+r^3tz^3-r^2uz^3-rt^3z^5-st^2z^4+s^2uz^4-su^2z^5-s^3uz^5-t^2uz^5+su^3z^7+4rstz^3+2rtuz^4-rs^2tz^4+r^2suz^4-3rtu^2z^6+3st^2uz^6-1)W_3W_0,$

$\Psi_{19} = -uz^2(r-r^3z+3stz^2+tuz^3+3rs^2z^2+r^3sz^2-rs^3z^3+rt^2z^3-2s^2tz^3-ru^2z^4+s^3tz^4-st^3z^5+t^3uz^6-tu^3z^7-rsz+2rsuz^3+r^2stz^3-rst^2z^4+rsu^2z^5-r^2tuz^4+s^2tuz^5-stu^2z^6)W_2W_0,$

$\Psi_{20} = uz(r^2z+uz^2+s^2z^2-s^3z^3+u^2z^4+t^4z^6-u^3z^6+sz-r^2t^2z^4-s^2t^2z^5-t^2u^2z^7+rtz^2-suz^3+r^2sz^2-r^3tz^3-r^2uz^3+rt^3z^5+st^2z^4+s^2uz^4-su^2z^5-s^3uz^5+t^2uz^5+su^3z^7+rs^2tz^4+r^2suz^4-rtu^2z^6-st^2uz^6+2rstuz^5-1) W_1W_0.$

Proof. First, we obtain $\sum_{k=0}^{n} z^kW_k^2$. Using the recurrence relation

$$W_{n+4} = rW_{n+3} + sW_{n+2} + tW_{n+1} + uW_n$$

or

$$uW_n = W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1}$$

i.e.

$$\begin{aligned} u^2W_n^2 = {} & W_{n+4}^2 + r^2W_{n+3}^2 + s^2W_{n+2}^2 + t^2W_{n+1}^2 - 2rW_{n+4}W_{n+3} - 2sW_{n+4}W_{n+2} \\ & -2tW_{n+4}W_{n+1} + 2rsW_{n+3}W_{n+2} + 2rtW_{n+3}W_{n+1} + 2stW_{n+2}W_{n+1} \end{aligned}$$

we obtain

$$
\begin{aligned}
u^2z^nW_n^2 &= z^nW_{n+4}^2+r^2z^nW_{n+3}^2+s^2z^nW_{n+2}^2+t^2z^nW_{n+1}^2-2rz^nW_{n+4}W_{n+3}-2sz^nW_{n+4}W_{n+2}\\
&\quad -2tz^nW_{n+4}W_{n+1}+2rsz^nW_{n+3}W_{n+2}+2rtz^nW_{n+3}W_{n+1}+2stz^nW_{n+2}W_{n+1}\\
u^2z^{n-1}W_{n-1}^2 &= z^{n-1}W_{n+3}^2+r^2z^{n-1}W_{n+2}^2+s^2z^{n-1}W_{n+1}^2+t^2z^{n-1}W_n^2-2rz^{n-1}W_{n+3}W_{n+2}-2sz^{n-1}W_{n+3}W_{n+1}\\
&\quad -2tz^{n-1}W_{n+3}W_n+2rsz^{n-1}W_{n+2}W_{n+1}+2rtz^{n-1}W_{n+2}W_n+2stz^{n-1}W_{n+1}W_n\\
u^2z^{n-2}W_{n-2}^2 &= z^{n-2}W_{n+2}^2+r^2z^{n-2}W_{n+1}^2+s^2z^{n-2}W_n^2+t^2z^{n-2}W_{n-1}^2-2rz^{n-2}W_{n+2}W_{n+1}-2sz^{n-2}W_{n+2}W_n\\
&\quad -2tz^{n-2}W_{n+2}W_{n-1}+2rsz^{n-2}W_{n+1}W_n+2rtz^{n-2}W_{n+1}W_{n-1}+2stz^{n-2}W_nW_{n-1}\\
&\quad \vdots\\
u^2z^2W_2^2 &= z^2W_6^2+r^2z^2W_5^2+s^2z^2W_4^2+t^2z^2W_3^2-2rz^2W_6W_5-2sz^2W_6W_4\\
&\quad -2tz^2W_6W_3+2rsz^2W_5W_4+2rtz^2W_5W_3+2stz^2W_4W_3\\
u^2z^1W_1^2 &= z^1W_5^2+r^2z^1W_4^2+s^2z^1W_3^2+t^2z^1W_2^2-2rz^1W_5W_4-2sz^1W_5W_3\\
&\quad -2tz^1W_5W_2+2rsz^1W_4W_3+2rtz^1W_4W_2+2stz^1W_3W_2\\
u^2z^0W_0^2 &= z^0W_4^2+r^2z^0W_3^2+s^2z^0W_2^2+t^2z^0W_1^2-2rz^0W_4W_3-2sz^0W_4W_2\\
&\quad -2tz^0W_4W_1+2rsz^0W_3W_2+2rtz^0W_3W_1+2stz^0W_2W_1
\end{aligned}
$$

If we add the equations side by side, we get

$$
\begin{aligned}
u^2\sum_{k=0}^{n}z^kW_k^2 &= (r^2z+s^2z^2+t^2z^3+1)z^{-4}\sum_{k=0}^{n}z^kW_k^2-2tz^{-1}\sum_{k=0}^{n}z^kW_{k+3}W_k \qquad (41)\\
&\quad +2(-s+rtz)z^{-2}\sum_{k=0}^{n}z^kW_{k+2}W_k+2(-r+stz^2+rsz)z^{-3}\sum_{k=0}^{n}z^kW_{k+1}W_k\\
&\quad +z^nW_{n+4}^2+(r^2z+1)z^{n-1}W_{n+3}^2+(r^2z+s^2z^2+1)z^{n-2}W_{n+2}^2\\
&\quad +(r^2z+s^2z^2+t^2z^3+1)z^{n-3}W_{n+1}^2-2rz^nW_{n+4}W_{n+3}-2sz^nW_{n+4}W_{n+2}\\
&\quad -2tz^nW_{n+4}W_{n+1}+2r(sz-1)z^{n-1}W_{n+3}W_{n+2}+2(-s+rtz)z^{n-1}W_{n+3}W_{n+1}\\
&\quad +2(-r+stz^2+rsz)z^{n-2}W_{n+2}W_{n+1}-z^{-1}W_3^2-(r^2z+1)z^{-2}W_2^2\\
&\quad -(r^2z+s^2z^2+1)z^{-3}W_1^2-(r^2z+s^2z^2+t^2z^3+1)z^{-4}W_0^2\\
&\quad +2rz^{-1}W_3W_2+2sz^{-1}W_3W_1-2r(sz-1)z^{-2}W_2W_1+2tz^{-1}W_3W_0\\
&\quad -2(-s+rtz)z^{-2}W_2W_0-2(-r+stz^2+rsz)z^{-3}W_1W_0.
\end{aligned}
$$

Next we obtain $\sum_{k=0}^{n}z^kW_{k+1}W_k$. Multiplying the both side of the recurrence relation

$$uW_n=W_{n+4}-rW_{n+3}-sW_{n+2}-tW_{n+1}$$

by W_{n+1} we get

$$uW_{n+1}W_n=W_{n+4}W_{n+1}-rW_{n+3}W_{n+1}-sW_{n+2}W_{n+1}-tW_{n+1}^2$$

Then using last recurrence relation, we obtain

$$
\begin{aligned}
uz^nW_{n+1}W_n &= z^nW_{n+4}W_{n+1}-rz^nW_{n+3}W_{n+1}-sz^nW_{n+2}W_{n+1}-tz^nW_{n+1}^2\\
uz^{n-1}W_nW_{n-1} &= z^{n-1}W_{n+3}W_n-rz^{n-1}W_{n+2}W_n-sz^{n-1}W_{n+1}W_n-tz^{n-1}W_n^2\\
uz^{n-2}W_{n-1}W_{n-2} &= z^{n-2}W_{n+2}W_{n-1}-rz^{n-2}W_{n+1}W_{n-1}-sz^{n-2}W_nW_{n-1}-tz^{n-2}W_{n-1}^2\\
&\quad \vdots\\
uz^2W_3W_2 &= z^2W_6W_3-rz^2W_5W_3-sz^2W_4W_3-tz^2W_3^2\\
uz^1W_2W_1 &= z^1W_5W_2-rz^1W_4W_2-sz^1W_3W_2-tz^1W_2^2\\
uz^0W_1W_0 &= z^0W_4W_1-rz^0W_3W_1-sz^0W_2W_1-tz^0W_1^2
\end{aligned}
$$

If we add the equations side by side, we get

$$u\sum_{k=0}^{n} z^k W_{k+1}W_k = (z^n W_{n+4}W_{n+1} - z^{-1}W_3W_0 + z^{-1}\sum_{k=0}^{n} z^k W_{k+3}W_k) \tag{42}$$
$$-r(z^n W_{n+3}W_{n+1} - z^{-1}W_2W_0 + z^{-1}\sum_{k=0}^{n} z^k W_{k+2}W_k)$$
$$-s(z^n W_{n+2}W_{n+1} - z^{-1}W_1W_0 + z^{-1}\sum_{k=0}^{n} z^k W_{k+1}W_k)$$
$$-t(z^n W_{n+1}^2 - z^{-1}W_0^2 + z^{-1}\sum_{k=0}^{n} z^k W_k^2)$$

Next we obtain $\sum_{k=0}^{n} z^k W_{k+2}W_k$. Multiplying the both side of the recurrence relation

$$uW_n = W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1}$$

by W_{n+2} we get

$$uW_{n+2}W_n = W_{n+4}W_{n+2} - rW_{n+3}W_{n+2} - sW_{n+2}^2 - tW_{n+2}W_{n+1}$$

Then using last recurrence relation, we obtain

$$uz^n W_{n+2}W_n = z^n W_{n+4}W_{n+2} - rz^n W_{n+3}W_{n+2} - sz^n W_{n+2}^2 - tz^n W_{n+2}W_{n+1}$$
$$uz^{n-1}W_{n+1}W_{n-1} = z^{n-1}W_{n+3}W_{n+1} - rz^{n-1}W_{n+2}W_{n+1} - sz^{n-1}W_{n+1}^2 - tz^{n-1}W_{n+1}W_n$$
$$uz^{n-2}W_nW_{n-2} = z^{n-2}W_{n+2}W_n - rz^{n-2}W_{n+1}W_n - sz^{n-2}W_n^2 - tz^{n-2}W_nW_{n-1}$$
$$\vdots$$
$$uz^2W_4W_2 = z^2W_6W_4 - rz^2W_5W_4 - sz^2W_4^2 - tz^2W_4W_3$$
$$uz^1W_3W_1 = z^1W_5W_3 - rz^1W_4W_3 - sz^1W_3^2 - tz^1W_3W_2$$
$$uz^0W_2W_0 = z^0W_4W_2 - rz^0W_3W_2 - sz^0W_2^2 - tz^0W_2W_1$$

If we add the equations side by side, we get

$$u\sum_{k=0}^{n} z^k W_{k+2}W_k = (z^n W_{n+4}W_{n+2} + z^{n-1}W_{n+3}W_{n+1} - z^{-1}W_3W_1 - z^{-2}W_2W_0 + z^{-2}\sum_{k=0}^{n} z^k W_{k+2}W_k) \tag{43}$$
$$-r(z^n W_{n+3}W_{n+2} + z^{n-1}W_{n+2}W_{n+1} - z^{-1}W_2W_1 - z^{-2}W_1W_0 + z^{-2}\sum_{k=0}^{n} z^k W_{k+1}W_k)$$
$$-s(z^n W_{n+2}^2 + z^{n-1}W_{n+1}^2 - z^{-1}W_1^2 - z^{-2}W_0^2 + z^{-2}\sum_{k=0}^{n} z^k W_k^2)$$
$$-t(z^n W_{n+2}W_{n+1} - z^{-1}W_1W_0 + z^{-1}\sum_{k=0}^{n} z^k W_{k+1}W_k)$$

Next we obtain $\sum_{k=0}^{n} z^k W_{k+3}W_k$. Multiplying the both side of the recurrence relation

$$uW_n = W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1}$$

by W_{n+3} we get

$$uW_{n+3}W_n = W_{n+4}W_{n+3} - rW_{n+3}^2 - sW_{n+3}W_{n+2} - tW_{n+3}W_{n+1}$$

Then using last recurrence relation, we obtain

$$uz^n W_{n+3}W_n = z^n W_{n+4}W_{n+3} - rz^n W_{n+3}^2 - sz^n W_{n+3}W_{n+2} - tz^n W_{n+3}W_{n+1}$$
$$uz^{n-1}W_{n+2}W_{n-1} = z^{n-1}W_{n+3}W_{n+2} - rz^{n-1}W_{n+2}^2 - sz^{n-1}W_{n+2}W_{n+1} - tz^{n-1}W_{n+2}W_n$$
$$uz^{n-2}W_{n+1}W_{n-2} = z^{n-2}W_{n+2}W_{n+1} - rz^{n-2}W_{n+1}^2 - sz^{n-2}W_{n+1}W_n - tz^{n-2}W_{n+1}W_{n-1}$$
$$\vdots$$
$$uz^2W_5W_2 = z^2W_6W_5 - rz^2W_5^2 - sz^2W_5W_4 - tz^2W_5W_3$$
$$uz^1W_4W_1 = z^1W_5W_4 - rz^1W_4^2 - sz^1W_4W_3 - tz^1W_4W_2$$
$$uz^0W_3W_0 = z^0W_4W_3 - rz^0W_3^2 - sz^0W_3W_2 - tz^0W_3W_1$$

If we add the equations side by side, we get

$$
\begin{aligned}
u\sum_{k=0}^{n} z^k W_{k+3}W_k = {} & (z^n W_{n+4}W_{n+3} + z^{n-1}W_{n+3}W_{n+2} + z^{n-2}W_{n+2}W_{n+1} \\
& - z^{-1}W_3W_2 - z^{-2}W_2W_1 - z^{-3}W_1W_0 + z^{-3}\sum_{k=0}^{n} z^k W_{k+1}W_k) \\
& - r(z^n W_{n+3}^2 + z^{n-1}W_{n+2}^2 + z^{n-2}W_{n+1}^2 - z^{-1}W_2^2 - z^{-2}W_1^2 - z^{-3}W_0^2 + z^{-3}\sum_{k=0}^{n} z^k W_k^2) \\
& - s(z^n W_{n+3}W_{n+2} + z^{n-1}W_{n+2}W_{n+1} - z^{-1}W_2W_1 - z^{-2}W_1W_0 + z^{-2}\sum_{k=0}^{n} z^k W_{k+1}W_k) \\
& - t(z^n W_{n+3}W_{n+1} - z^{-1}W_2W_0 + z^{-1}\sum_{k=0}^{n} z^k W_{k+2}W_k)
\end{aligned} \tag{44}
$$

Solving the system eqs. (41)–(44), the results in (a), (b), (c) and (d) follow.

To calculate (to evaluate) the sums $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and $\sum_{k=0}^{n} z^k W_{k+3}W_k$, the following Remark is useful.

Remark 14.1.
To calculate the sums $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and $\sum_{k=0}^{n} z^k W_{k+3}W_k$ we use theorem 14.1. If there is indeterminate form in the right sides of the sum formulas which is given in theorem 14.1 then we can use L'Hospital rule as theorem 8.1 and remark 8.1.

15. Generating Function of Generalized Tetranacci Polynomials: Closed Formulas of $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ and $\sum_{n=0}^{\infty} W_{n+3}W_n z^n$

Next, we give the ordinary generating functions of some special cases of generalized Tetranacci polynomials.

Lemma 15.1.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\alpha\delta|^{-1}, |\beta\gamma|^{-1}, |\beta\delta|^{-1}, |\gamma\delta|^{-1}\}$. The ordinary generating functions of the sequences W_n^2, $W_{n+1}W_n$, $W_{n+2}W_n$ and $W_{n+3}W_n$ are given as follows:

(a)
$$\sum_{n=0}^{\infty} W_n^2 z^n = \frac{\Gamma_{13W}(z)}{\Delta}$$

(b)
$$\sum_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{\Gamma_{14W}(z)}{\Delta}$$

(c)
$$\sum_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{\Gamma_{15W}(z)}{\Delta}$$

(d)
$$\sum_{n=0}^{\infty} W_{n+3}W_n z^n = \frac{\Gamma_{16W}(z)}{\Delta}$$

where

$\Delta = (-u^3z^6 + su^2z^5 - u(u+rt)z^4 + (2su + r^2u - t^2)z^3 + (rt+u)z^2 + sz + 1)(-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (r^2 + 2s)z - 1)$

and

$\Gamma_{13W}(z) = \sum_{k=11}^{20} \Omega_k = \Omega_{11} + \Omega_{12} + \Omega_{13} + \Omega_{14} + \Omega_{15} + \Omega_{16} + \Omega_{17} + \Omega_{18} + \Omega_{19} + \Omega_{20} = z^3(uz^2 + t^2z^3 + u^2z^4 - u^3z^6 + sz + rtz^2 + r^2uz^3 - su^2z^5 + rtuz^4 - 1)W_3^2 + z^2(r^2z + uz^2 + t^2z^3 + u^2z^4 - u^3z^6 + sz + r^2t^2z^4 + r^2u^2z^5 - r^2u^3z^7 + rtz^2 + r^2sz^2 + r^3tz^3 + r^4uz^4 - su^2z^5 - r^2su^2z^6 + 2rstz^3 + 3rtuz^4 + 2r^2suz^4 - 2rtu^2z^6 + r^3tuz^5 - 1)W_2^2 + z(r^2z + uz^2 + s^2z^2 - s^3z^3 + t^2z^3 + u^2z^4 - u^3z^6 + sz + r^2t^2z^4 + r^2u^2z^5 + s^2t^2z^5 - r^2u^3z^7 - s^2u^2z^6 - s^3u^2z^7 - s^2u^3z^8 + rtz^2 + r^2sz^2 + r^3tz^3 + r^4uz^4 + s^2uz^4 - su^2z^5 - 2s^3uz^5 - r^2s^2uz^5 - r^2su^2z^6 + 4rstz^3 + 3rtuz^4 - rs^2tz^4 + 4r^2suz^4 - 2rtu^2z^6 + r^3tuz^5 + 2st^2uz^6 + rs^2tuz^6 + 2rstu^2z^7 - 1)W_1^2 + (r^2z + uz^2 + s^2z^2 - s^3z^3 + 2t^2z^3 + u^2z^4 - t^4z^6 - u^3z^6 + sz + r^2t^2z^4 +$

$$r^2u^2z^5+s^2t^2z^5-r^2u^3z^7-s^2u^2z^6-s^3u^2z^7-s^2u^3z^8+t^2u^2z^7-t^2u^3z^9+rtz^2+r^2sz^2+r^3tz^3-rt^3z^5-st^2z^4+r^4uz^4+s^2uz^4-su^2z^5-2s^3uz^5-t^2uz^5-r^2s^2uz^5-r^2su^2z^6-r^2t^2uz^6+st^2u^2z^8+4rstz^3+5rtuz^4-rs^2tz^4+4r^2suz^4-4rtu^2z^6+r^3tuz^5-rt^3uz^7+4st^2uz^6+rs^2tuz^6+4rstu^2z^7-1)W_0^2-2z^4(rs+r^2tz+tuz^2+rt^2z^2+ru^2z^3+r^3uz^2-ru^3z^5-tu^2z^4+stz+rsuz^2-rsu^2z^4+r^2tuz^3)W_3W_2-2z^4(rt-s^2u^2z^4+r^2uz+st^2z^2-s^2uz^2+t^2uz^3-su^3z^5+suz+rtu^2z^4+rstuz^3)W_3W_1+2z^4(-st-rt^2z+s^2tz-ru^2z^2+ru^3z^4+tu^2z^3-tuz-rs^2u^2z^4+r^2tu^2z^4+stuz^2+rst^2z^2+rsu^2z^3-r^2tuz^2+rt^2uz^3-rsu^3z^5-stu^2z^4-rsuz+r^2stuz^3)W_2W_1-2uz^4(r-ruz^2+stz^2+tuz^3-tu^2z^5+rsuz^3)W_3W_0+2uz^4(-s+s^2z-r^2uz^2+s^2uz^3+su^2z^4-t^2uz^4-rtz+rstz^2+r^2suz^3-rtu^2z^5)W_2W_0+2uz^4(-t+t^3z^3+tuz^2+rt^2z^2+ru^2z^3-ruz+stz-stuz^3-rsu^2z^4+r^2tuz^3+rt^2uz^4-stu^2z^5)W_1W_0$$

and

$$\Gamma_{14W}(z)=\sum_{k=11}^{20}\Lambda_k=\Lambda_{11}+\Lambda_{12}+\Lambda_{13}+\Lambda_{14}+\Lambda_{15}+\Lambda_{16}+\Lambda_{17}+\Lambda_{18}+\Lambda_{19}+\Lambda_{20}=-z^3(r-ruz^2+stz^2+tuz^3-tu^2z^5+rsuz^3)W_3^2-z^4(rs^2+st+rt^2z+ru^2z^2-ru^3z^4-tu^2z^3+tuz-r^2tu^2z^4+r^2stz+rs^2uz^2+r^3suz^2+r^2tuz^2-rt^2uz^3+rsuz)W_2^2+z^4(-rt^2-tu-ru^2z-st^3z^2+ru^3z^3+tu^2z^2+r^2tu^2z^3+s^2tu^2z^4-r^2tuz+rt^2uz^2-rsu^3z^4+s^2tuz^2-stu^2z^3-rst^2uz^3)W_1^2-u^2z^4(r-ruz^2+stz^2+tuz^3-tu^2z^5+rsuz^3)W_0^2+z^2(r^2z+uz^2+s^2z^2+t^2z^3+u^2z^4-u^3z^6-r^2uz^3+s^2uz^4-t^2uz^5+2rstz^3+2rtuz^4+2r^2suz^4-2rtu^2z^6-1)W_3W_2-z^3(t-t^3z^3+r^2tz-s^2tz^2+r^3uz^2-ru^3z^5-tu^2z^4+ruz+2rsuz^2-rs^2uz^3-rt^2uz^4+2stu^2z^5)W_3W_1+z(r^2z+uz^2+s^2z^2-s^3z^3+t^2z^3+u^2z^4-u^3z^6+sz-r^2u^3z^7+rtz^2-suz^3+r^2sz^2+r^3tz^3-rt^3z^5-st^2z^4+r^4uz^4+s^2uz^4-su^2z^5-s^3uz^5-t^2uz^5+su^3z^7-r^2s^2uz^5-r^2t^2uz^6+2rstz^3+2rtuz^4-rs^2tz^4+3r^2suz^4-3rtu^2z^6+st^2uz^6+2rstu^2z^7-1)W_2W_1+uz^3(-r^2z+uz^2+s^2z^2+t^2z^3+u^2z^4-u^3z^6+r^2uz^3+s^2uz^4-t^2uz^5-1)W_3W_0-uz^4(t-t^3z^3+2rs+r^2tz+s^2tz^2+r^3uz^2-ru^3z^5-tu^2z^4+ruz+2stuz^3+rs^2uz^3-rt^2uz^4)W_2W_0+(r^2z+uz^2+s^2z^2-s^3z^3+2t^2z^3+u^2z^4-t^4z^6-u^3z^6+sz+r^2t^2z^4+s^2t^2z^5-r^2u^3z^7-s^3u^2z^7-t^2u^2z^7+rtz^2+r^2sz^2+r^3tz^3-rt^3z^5-st^2z^4+r^4uz^4+s^2uz^4-2su^2z^5-2s^3uz^5-t^2uz^5+su^4z^9-r^2s^2uz^5-r^2su^2z^6-r^2t^2uz^6+st^2u^2z^8+4rstz^3+3rtuz^4-rs^2tz^4+4r^2suz^4-3rtu^2z^6+r^3tuz^5-rt^3uz^7+2st^2uz^6-rtu^3z^8+rs^2tuz^6+2rstu^2z^7-1)W_1W_0$$

and

$$\Gamma_{15W}(z)=\sum_{k=11}^{20}\Phi_k=\Phi_{11}+\Phi_{12}+\Phi_{13}+\Phi_{14}+\Phi_{15}+\Phi_{16}+\Phi_{17}+\Phi_{18}+\Phi_{19}+\Phi_{20}=-z^3(s-s^2z+r^2-s^2uz^3-su^2z^4+t^2uz^4+rtz+rtuz^3)W_3^2-z^2(s-s^2z+r^2t^2z^3+r^2u^2z^4-r^2sz+rt^3z^4-s^2uz^3-su^2z^4+t^2uz^4-r^2s^2uz^4-r^2su^2z^5+r^2t^2uz^5+rtuz^3-rs^2tz^3+rtu^2z^5+r^3tuz^4-2rstuz^4)W_2^2-z^4(st^2+r^2t^2-s^2u^2z^2-s^2u^3z^4+rt^3z+su^2z+t^2uz+r^2u^2z-s^2t^2z-su^4z^5+rtu+r^2t^2uz^2+st^2u^2z^4+r^3tuz+rtu^2z^2-rs^2tuz^2+2rstuz)W_1^2-u^2z^4(s-s^2z+r^2-s^2uz^3-su^2z^4+t^2uz^4+rtz+rtuz^3)W_0^2+z^2(-r+r^3z+t^3z^4-tz-rs^2z^2+r^2tz^2+rt^2z^3-s^2tz^3+ru^2z^4+tu^2z^5-2stuz^4-2rs^2uz^4-2rsu^2z^5+2r^2tuz^4+2rt^2uz^5)W_3W_2+z(r^2z+s^2z^2-s^3z^3+t^2z^3+2u^2z^4-u^4z^8+sz+r^2u^2z^5-s^2u^2z^6+t^2u^2z^7+suz^3+r^2sz^2-st^2z^4-su^2z^5-s^3uz^5-su^3z^7+2rstz^3+4rtuz^4+r^2suz^4+st^2uz^6-1)W_3W_1-z^2(t-t^3z^3+r^3u^2z^4-r^2tz+s^2tz^2+ru^2z^3-s^3tz^3+st^3z^4-tu^2z^4-ru^4z^7-stz-rs^2u^2z^5+rt^2u^2z^6+rsuz^2+2stuz^3+r^2stz^2+2rs^2uz^3+2rsu^2z^4+r^3suz^3-rs^3uz^4+2r^2tuz^3-2rt^2uz^4-rsu^3z^6-2s^2tuz^4+stu^2z^5+rst^2uz^5)W_2W_1-uz^3(r+r^3z-t^3z^4+tz-rs^2z^2+r^2tz^2-rt^2z^3+s^2tz^3-ru^2z^4-tu^2z^5+2rsz+2stuz^4)W_3W_0+(r^2z+s^2z^2-s^3z^3+2t^2z^3+2u^2z^4-t^4z^6-u^4z^8+sz+r^2t^2z^4+r^2u^2z^5+s^2t^2z^5-r^2u^3z^7-s^2u^2z^6+rtz^2+suz^3+r^2sz^2+r^3tz^3+r^2uz^3-rt^3z^5-st^2z^4+r^4uz^4-su^2z^5-s^3uz^5-su^3z^7-r^2s^2uz^5-r^2t^2uz^6+4rstz^3+5rtuz^4-rs^2tz^4+3r^2suz^4-rtu^2z^6+r^3tuz^5-rt^3uz^7+3st^2uz^6-rtu^3z^8+rs^2tuz^6+2rstu^2z^7-1)W_2W_0-uz^4(2r^2t+ru+2st+2rt^2z-2s^2tz+r^3uz-ru^3z^4+t^3uz^4-tu^3z^5+tuz-rs^2uz^2+r^2tuz^2+rt^2uz^3-s^2tuz^3+2rsuz)W_1W_0$$

and

$$\Gamma_{16W}(z)=\sum_{k=11}^{20}\Psi_k=\Psi_{11}+\Psi_{12}+\Psi_{13}+\Psi_{14}+\Psi_{15}+\Psi_{16}+\Psi_{17}+\Psi_{18}+\Psi_{19}+\Psi_{20}=-z^3(t-t^3z^3+2rs+r^3-rs^2z+r^2tz-tuz^2-rt^2z^2+s^2tz^2-ru^2z^3+ruz-stz-rsuz^2+2stuz^3)W_3^2+z^2(-t+t^3z^3-rs-r^3u^2z^4+rs^2z+r^3sz+r^2tz+tuz^2-rs^3z^2+rt^2z^2-s^2tz^2-ru^3z^5+ru^4z^7+stz-rs^2u^2z^5-r^2tu^2z^5-rt^2u^2z^6-2stuz^3+r^2stz^2+rst^2z^3-rs^2uz^3-rsu^2z^4-rs^3uz^4-r^2tuz^3+rt^2uz^4+rsu^3z^6+2r^2stuz^4+rst^2uz^5)W_2^2-z(t-t^3z^3+r^3u^2z^4-r^2tz-tuz^2-rt^2z^2-s^2tz^2+s^3tz^3+ru^3z^5-ru^4z^7-stz-rs^2u^2z^5+r^2tu^2z^5+rt^2u^2z^6+s^2tu^2z^6+stuz^3-r^2stz^2-2rst^2z^3+2rsu^2z^4+r^2tuz^3-rt^2uz^4-rsu^3z^6-s^2tuz^4+s^3tuz^5-st^3uz^6+stu^3z^7-r^2stuz^4-rst^2uz^5)W_1^2-u^2z^4(t-t^3z^3+2rs+r^3-rs^2z+r^2tz-tuz^2-rt^2z^2+s^2tz^2-ru^2z^3+ruz-stz-rsuz^2+2stuz^3)W_0^2+z^2(-s+r^4z+s^3z^2+u^2z^3+u^3z^5-u^4z^7-uz-r^2-r^2s^2z^2-r^2t^2z^3+s^2u^2z^5+t^2u^2z^6+r^2sz+suz^2+r^3tz^2+r^2uz^2-rt^3z^4-st^2z^3+s^2uz^3+su^2z^4+s^3uz^4-t^2uz^4-su^3z^6-rtz-2rstz^2+rs^2tz^3-r^2suz^3+rtu^2z^5-st^2uz^5)W_3W_2+z(-r+r^3z-stz^2-tuz^3+rs^2z^2+r^3sz^2-rs^3z^3-rt^2z^3+ru^2z^4+s^3tz^4-st^3z^5-t^3uz^6+tu^3z^7+rsz-2stuz^4+r^2stz^3-rst^2z^4-2rs^2uz^4-3rsu^2z^5+r^2tuz^4+3s^2tuz^5+stu^2z^6)W_3W_1-z(s-s^2z-s^3z^2+s^4z^3-u^2z^3-u^3z^5+u^4z^7+uz-r^2s^2z^2+r^2u^2z^4-s^2t^2z^4+s^3u^2z^6-s^2u^3z^7-t^2u^2z^6-r^2sz-2suz^2-r^3tz^2-r^2uz^2+rt^3z^4+st^2z^3+t^2uz^4+2su^3z^6+s^4uz^5-su^4z^8+rtz-r^2s^2uz^4-r^2su^2z^5-s^2t^2uz^6+st^2u^2z^7-2rstz^2-rtuz^3-rs^2tz^3-rtu^2z^5+r^3tuz^4-rt^3uz^6+rtu^3z^7-rs^2tuz^5+2rstu^2z^6)W_2W_1+(r^2z+uz^2+s^2z^2-s^3z^3+2t^2z^3+u^2z^4-t^4z^6-u^3z^6+sz+r^2t^2z^4+s^2t^2z^5-t^2u^2z^7+rtz^2-suz^3+r^2sz^2+r^3tz^3-r^2uz^3-rt^3z^5-st^2z^4+s^2uz^4-su^2z^5-s^3uz^5-t^2uz^5+su^3z^7+4rstz^3+2rtuz^4-rs^2tz^4+r^2suz^4-3rtu^2z^6+3st^2uz^6-1)W_3W_0-uz^2(r-r^3z+3stz^2+tuz^3+3rs^2z^2+r^3sz^2-rs^3z^3+rt^2z^3-2s^2tz^3-ru^2z^4+s^3tz^4-st^3z^5+t^3uz^6-tu^3z^7-rsz+2rsuz^3+r^2stz^3-rst^2z^4+rsu^2z^5-r^2tuz^4+s^2tuz^5-stu^2z^6)W_2W_0+uz(r^2z+uz^2+s^2z^2-s^3z^3+u^2z^4+t^4z^6-u^3z^6+sz-r^2t^2z^4-s^2t^2z^5-t^2u^2z^7+rtz^2-suz^3+r^2sz^2-r^3tz^3-r^2uz^3+rt^3z^5+st^2z^4+s^2uz^4-su^2z^5-s^3uz^5+t^2uz^5+su^3z^7+rs^2tz^4+r^2suz^4-rtu^2z^6-st^2uz^6+2rstuz^5-1)W_1W_0$$

Proof. Use theorem 14.1. Here,
$\Omega_{11},\Omega_{12},\Omega_{13},\Omega_{14},\Omega_{15},\Omega_{16},\Omega_{17},\Omega_{18},\Omega_{19},\Omega_{20}$ and
$\Lambda_{11},\Lambda_{12},\Lambda_{13},\Lambda_{14},\Lambda_{15},\Lambda_{16},\Lambda_{17},\Lambda_{18},\Lambda_{19},\Lambda_{20}$ and
$\Phi_{11},\Phi_{12},\Phi_{13},\Phi_{14},\Phi_{15},\Phi_{16},\Phi_{17},\Phi_{18},\Phi_{19},\Phi_{20}$ and
$\Psi_{11},\Psi_{12},\Psi_{13},\Psi_{14},\Psi_{15},\Psi_{16},\Psi_{17},\Psi_{18},\Psi_{19},\Psi_{20}$ are as in theorem 14.1. □
Now, we consider special cases of lemma 15.1.

Corollary 15.1.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\alpha\delta|^{-1}, |\beta\gamma|^{-1}, |\beta\delta|^{-1}, |\gamma\delta|^{-1}\}$. *The ordinary generating functions of the sequences* G_n^2, $G_{n+1}G_n$, $G_{n+2}G_n$, $G_{n+3}G_n$ *and* H_n^2, $H_{n+1}H_n$, $H_{n+2}H_n$, $H_{n+3}H_n$ *are given as follows:*

(a)
$$\sum_{n=0}^{\infty} G_n^2 z^n = \frac{-u^3z^7 - su^2z^6 + (u^2+rtu)z^5 + (r^2u+t^2)z^4 + (rt+u)z^3 + sz^2 - z}{\Delta},$$
$$\sum_{n=0}^{\infty} H_n^2 z^n = \frac{\Gamma_{13aH}(z)}{\Delta},$$

where

$\Delta = (-u^3z^6 + su^2z^5 - u(u+rt)z^4 + (2su+r^2u-t^2)z^3 + (rt+u)z^2 + sz + 1)(-u^2z^4 + (t^2-2su)z^3 + (2u+2rt-s^2)z^2 + (r^2+2s)z - 1)$

and

$\Gamma_{13aH}(z) = -t^2u^3z^9 + su^2(3t^2-4su)z^8 + u(t^2u+18rstu-5rt^3-4s^3u-9r^2u^2)z^7 + (6rs^2tu+28st^2u-12s^2u^2-7t^4-16u^3-32rtu^2-9r^2su^2-5r^2t^2u)z^6 + (9r^3tu+9r^2u^2+8s^2t^2-8r^2s^2u-7rt^3-16s^3u-16su^2-7t^2u)z^5 + (9r^4u+4s^2u+11r^2t^2+16u^2+36r^2su+48rtu-10rs^2t-7st^2)z^4 + (11r^3t+r^2u+23t^2+46rst-12s^3)z^3 + (13r^2s+12s^2+16rt+16u)z^2 + (15r^2+16s)z - 16.$

(b)
$$\sum_{n=0}^{\infty} G_{n+1}G_n z^n = \frac{tu^2z^6 - u(t+rs)z^4 + (ru-st)z^3 - rz}{\Delta},$$
$$\sum_{n=0}^{\infty} H_{n+1}H_n z^n = \frac{\Gamma_{14aH}(z)}{\Delta},$$

where

$\Gamma_{14aH}(z) = 4tu^4z^9 + tu^2(2su-3t^2)z^8 + u(2st^3-4tu^2-6rsu^2-rt^2u)z^7 + (4r^2tu^2+12ru^3+3t^3u+3rst^2u-rt^4-10stu^2-2s^3tu)z^6 + (rs^2t^2+5rt^2u+8s^2tu+12tu^2-2r^2stu+2rs^3u-r^2t^3-2rsu^2-3st^3)z^5 + (2s^3t+r^3t^2-12ru^2-2rst^2-8rs^2u-3r^3su-5r^2tu-r^2s^2t-2stu)z^4 + (3r^2st+r^4t-2rs^3-4rt^2-3r^3u-12tu-10rsu)z^3 + (2r^3s+r^2t+4ru-6st)z^2 + r(3r^2+2s)z - 4r.$

(c)
$$\sum_{n=0}^{\infty} G_{n+2}G_n z^n = \frac{su^2z^5 - t^2uz^5 + u(s^2-rt)z^4 + (s^2-rt)z^2 - (r^2+s)z}{\Delta},$$
$$\sum_{n=0}^{\infty} H_{n+2}H_n z^n = \frac{\Gamma_{15aH}(z)}{\Delta},$$

where

$\Gamma_{15aH}(z) = rtu^4z^9 + u^2(8su^2+rstu-rt^3-4t^2u)z^8 + u(8s^2u^2+3t^4+rst^3-4rtu^2-9st^2u-r^2t^2u-rs^2tu)z^7 + (4r^2st^2u+5s^2t^2u+3rt^3u-r^2t^4-2r^3tu^2-rs^3tu-3rstu^2-4s^3u^2-2st^4)z^6 + (r^2s^2t^2+8s^2u^2+2s^3t^2+12rs^2tu-3rst^3-9r^2t^2u-r^3stu-r^3t^3-r^2su^2-14rtu^2-4s^4u)z^5 + (r^4t^2+r^2s^2u+s^2t^2-12t^2u-5r^3tu-r^3s^2t-12r^2u^2-rs^3t-rstu-6rt^3)z^4 + (r^5t+7r^3st+13rs^2t+8s^2u+st^2-2r^2s^3-4r^2t^2-12rtu-4s^4)z^3 + (2r^4s+3r^3t+5r^2s^2+3rst+4s^3)z^2 + (3r^4+9r^2s+8s^2-3rt)z - 4(r^2+2s).$

(d)
$$\sum_{n=0}^{\infty} G_{n+3}G_n z^n = \frac{\Gamma_{16aG}(z)}{\Delta},$$
$$\sum_{n=0}^{\infty} H_{n+3}H_n z^n = \frac{\Gamma_{16aH}(z)}{\Delta},$$

where

$\Gamma_{16aG}(z) = (ru^2 + t^3 - 2stu)z^4 + (rt^2 + tu + rsu - s^2t)z^3 + (rs^2 + st - r^2t - ru)z^2 - (r^3 + t + 2rs)z$

and

$\Gamma_{16aH}(z) = tu^4(r^2+2s)z^9 + u^2(2s^2tu + r^2stu + 2rsu^2 - 2st^3 - r^2t^3 - rt^2u)z^8 + u(rt^4 + 12ru^3 + r^2st^3 + 2rs^2u^2 + 4t^3u + 2s^2t^3 - 14stu^2 - 2s^3tu - r^2tu^2 - r^3t^2u - 5rst^2u - r^2s^2tu)z^7 + (12st^3u + 4tu^3 + 11rs^2t^2u + r^2t^3u + 10rsu^3 + 4r^3st^2u - r^3t^4 - 2r^4tu^2 - 3rst^4 - 3t^5 - 7rt^2u^2 - 2s^4tu - r^2s^3tu - 6r^2stu^2 - 2rs^3u^2 - 10s^2tu^2)z^6 + (r^3s^2t^2 + s^2t^3 + 5r^2s^2tu + 3rst^2u + 2s^3tu + 3rs^3t^2 + 2stu^2 - r^4stu - r^4t^3 - 4r^3su^2 - 3r^3t^2u - 4r^2st^3 - 14r^2tu^2 - 2rs^4u - 2rs^2u^2 - 3t^3u - 3rt^4 - 12ru^3)z^5 + (2s^4t + r^2t^3 + r^5t^2 + r^3st^2 + 17r^2stu + 12rt^2u + r^4tu - 3st^3 - 4tu^2 - 15r^3u^2 - 22rsu^2 - 10rs^3u - 2s^2tu - 7rs^2t^2 - 2r^2s^3t - r^4s^2t - 2r^3s^2u)z^4 + (3r^3t^2 + 15t^3 + 27rst^2 + 8r^4st + r^6t + 16r^2s^2t + 4ru^2 - 6rs^4 - 8s^3t - 2r^3s^3 - 2rs^2u - r^3su - 10r^2tu - 18stu)z^3 + (4rs^3 + 12rt^2 + 2r^5s + 3r^4t + 2s^2t + 3r^3u + 7r^3s^2 + 12tu + 13r^2st + 6rsu) z^2 + (10rs^2 + 12r^3s + 8r^2t + 12st + 3r^5 - 4ru)z - 4(r^3 + 3t + 3rs).$

16. Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kz^kW_k^2$, $\sum_{k=0}^{n} kz^kW_{k+1}W_k$, $\sum_{k=0}^{n} kz^kW_{k+2}W_k$ and $\sum_{k=0}^{n} kz^kW_{k+3}W_k$ for the Generalized Tetranacci Numbers

The following theorem presents some summing formulas of generalized Tetranacci polynomials with positive subscripts. In the following Theorem, $\Theta_{15W}^{'}(z)$, $\Theta_{16W}^{'}(z)$,, $\Theta_{17W}^{'}(z)$ and $\Theta_{18W}^{'}(z)$ and $\frac{d\Delta}{dz} = \Delta'$ denotes the derivatives of $\Theta_{15W}(z)$, $\Theta_{16W}(z)$, $\Theta_{17W}(z)$ and $\Theta_{18W}(z)$ and Δ with respect to z, respectively, where

$\Delta = (-u^3z^6 + su^2z^5 - u(u+rt)z^4 + (2su + r^2u - t^2)z^3 + (rt+u)z^2 + sz + 1)(-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (r^2 + 2s)z - 1)$.

Theorem 16.1.
Let z be a real or complex number (in fact z is a real or complex valued function in x). If $\Delta^2 = (-u^3z^6 + su^2z^5 - u(u+rt)z^4 + (2su + r^2u - t^2)z^3 + (rt+u)z^2 + sz + 1)^2(-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (r^2 + 2s)z - 1)^2 \neq 0$ then, for $n \geq 0$, we have the following formulas:

(a)
$$\sum_{k=0}^{n} kz^kW_k^2 = \frac{\Theta_{19W}}{\Delta^2}$$

where

$$\Theta_{19W} = z(\Theta_{15W}^{'} \times \Delta - \frac{d\Delta}{dz} \times \Theta_{15W}).$$

(b)
$$\sum_{k=0}^{n} kz^kW_{k+1}W_k = \frac{\Theta_{20W}}{\Delta^2}$$

where

$$\Theta_{20W} = z(\Theta_{16W}^{'} \times \Delta - \frac{d\Delta}{dz} \times \Theta_{16W}).$$

(c)
$$\sum_{k=0}^{n} kz^kW_{k+2}W_k = \frac{\Theta_{21W}}{\Delta^2}$$

where

$$\Theta_{21W} = z(\Theta_{17W}^{'} \times \Delta - \frac{d\Delta}{dz} \times \Theta_{17W}).$$

(d)
$$\sum_{k=0}^{n} kz^kW_{k+3}W_k = \frac{\Theta_{22W}}{\Delta^2}$$

where

$$\Theta_{22W} = z(\Theta_{18W}^{'} \times \Delta - \frac{d\Delta}{dz} \times \Theta_{18W}).$$

Proof. From theorem 14.1, we have

$$\sum_{k=0}^{n} z^k W_k^2 = \frac{\Theta_{15W}}{\Delta},$$
$$\sum_{k=0}^{n} z^k W_{k+1} W_k = \frac{\Theta_{16W}}{\Delta},$$
$$\sum_{k=0}^{n} z^k W_{k+2} W_k = \frac{\Theta_{17W}}{\Delta},$$
$$\sum_{k=0}^{n} z^k W_{k+3} W_k = \frac{\Theta_{18W}}{\Delta}.$$

By taking the derivative of the both sides of the above formulas with respect to z, we get

$$\sum_{k=0}^{n} kz^{k-1} W_k^2 = \frac{C_7(z)}{\Delta^2},$$
$$\sum_{k=0}^{n} kz^{k-1} W_{k+1} W_k = \frac{C_8(z)}{\Delta^2},$$
$$\sum_{k=0}^{n} kz^{k-1} W_{k+2} W_k = \frac{C_9(z)}{\Delta^2},$$
$$\sum_{k=0}^{n} kz^{k-1} W_{k+3} W_k = \frac{C_{10}(z)}{\Delta^2},$$

where

$$C_7(z) = \Theta_{15W}' \times \Delta - \frac{d\Delta}{dz} \times \Theta_{15W} = \frac{d\Theta_{15W}}{dz} \times \Delta - \frac{d\Delta}{dz} \times \Theta_{15W},$$
$$C_8(z) = \Theta_{16W}' \times \Delta - \frac{d\Delta}{dz} \times \Theta_{16W} = \frac{d\Theta_{16W}}{dz} \times \Delta - \frac{d\Delta}{dz} \times \Theta_{16W},$$
$$C_9(z) = \Theta_{17W}' \times \Delta - \frac{d\Delta}{dz} \times \Theta_{17W} = \frac{d\Theta_{17W}}{dz} \times \Delta - \frac{d\Delta}{dz} \times \Theta_{17W},$$
$$C_{10}(z) = \Theta_{18W}' \times \Delta - \frac{d\Delta}{dz} \times \Theta_{18W} = \frac{d\Theta_{18W}}{dz} \times \Delta - \frac{d\Delta}{dz} \times \Theta_{18W}.$$

Now, it follows that

$$\sum_{k=0}^{n} kz^{k} W_k^2 = \frac{z \times C_7(z)}{\Delta^2} = \frac{\Theta_{19W}}{\Delta^2},$$
$$\sum_{k=0}^{n} kz^{k} W_{k+1} W_k = \frac{z \times C_8(z)}{\Delta^2} = \frac{\Theta_{20W}}{\Delta^2},$$
$$\sum_{k=0}^{n} kz^{k} W_{k+2} W_k = \frac{z \times C_9(z)}{\Delta^2} = \frac{\Theta_{21W}}{\Delta^2},$$
$$\sum_{k=0}^{n} kz^{k} W_{k+3} W_k = \frac{z \times C_{10}(z)}{\Delta^2} = \frac{\Theta_{22W}}{\Delta^2},$$

where

$$\Theta_{19W} = z \times C_7(z) = z(\Theta_{15W}' \times \Delta - \frac{d\Delta}{dz} \times \Theta_{15W}),$$
$$\Theta_{20W} = z \times C_8(z) = z(\Theta_{16W}' \times \Delta - \frac{d\Delta}{dz} \times \Theta_{16W}),$$
$$\Theta_{21W} = z \times C_9(z) = z(\Theta_{17W}' \times \Delta - \frac{d\Delta}{dz} \times \Theta_{17W}),$$
$$\Theta_{22W} = z \times C_{10}(z) = z(\Theta_{18W}' \times \Delta - \frac{d\Delta}{dz} \times \Theta_{18W}). \square$$

To calculate (to evaluate) the sums $\sum_{k=0}^{n} kz^k W_k^2$, $\sum_{k=0}^{n} kz^k W_{k+1} W_k$, $\sum_{k=0}^{n} kz^k W_{k+2} W_k$ and $\sum_{k=0}^{n} kz^k W_{k+3} W_k$, the following Remark is useful.

Remark 16.1.
To calculate the sums $\sum_{k=0}^{n} kz^k W_k^2$, $\sum_{k=0}^{n} kz^k W_{k+1} W_k$, $\sum_{k=0}^{n} kz^k W_{k+2} W_k$ and $\sum_{k=0}^{n} kz^k W_{k+3} W_k$ we use theorem 16.1. If there is indeterminate form in the right sides of the sum formulas which is given in theorem 16.1 then we can use L'Hospital rule as theorem 8.1 and remark 8.1.

17. Generating Function of Generalized Tetranacci Polynomials: Closed Formulas of $\sum_{n=0}^{\infty} nW_n^2 z^n$, $\sum_{n=0}^{\infty} nW_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} nW_{n+2}W_n z^n$ and $\sum_{n=0}^{\infty} nW_{n+3}W_n z^n$

Next, we give the ordinary generating functions of some special cases of generalized Tetranacci polynomials.

Lemma 17.1.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\alpha\delta|^{-1}, |\beta\gamma|^{-1}, |\beta\delta|^{-1}, |\gamma\delta|^{-1}\}$. *The ordinary generating functions of the sequences* nW_n^2, $nW_{n+1}W_n$, $nW_{n+2}W_n$ *and* $nW_{n+3}W_n$ *are given as follows:*

(a)
$$\sum_{n=0}^{\infty} nW_n^2 z^n = \frac{\Gamma_{17W}(z)}{\Delta^2},$$

(b)
$$\sum_{n=0}^{\infty} nW_{n+1}W_n z^n = \frac{\Gamma_{18W}(z)}{\Delta^2},$$

(c)
$$\sum_{n=0}^{\infty} nW_{n+2}W_n z^n = \frac{\Gamma_{19W}(z)}{\Delta^2},$$

(d)
$$\sum_{n=0}^{\infty} nW_{n+3}W_n z^n = \frac{\Gamma_{20W}(z)}{\Delta^2},$$

where

$\Delta = (-u^3z^6 + su^2z^5 - u(u+rt)z^4 + (2su + r^2u - t^2)z^3 + (rt+u)z^2 + sz + 1)(-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (r^2 + 2s)z - 1)$

and

$$\Gamma_{17W}(z) = z(\frac{d(\Gamma_{13W}(z))}{dz} \times \Delta - \frac{d\Delta}{dz} \times \Gamma_{13W}(z))$$

and

$$\Gamma_{18W}(z) = z(\frac{d(\Gamma_{14W}(z))}{dz} \times \Delta - \frac{d\Delta}{dz} \times \Gamma_{14W}(z))$$

and

$$\Gamma_{19W}(z) = z(\frac{d(\Gamma_{15W}(z))}{dz} \times \Delta - \frac{d\Delta}{dz} \times \Gamma_{15W}(z))$$

and

$$\Gamma_{20W}(z) = z(\frac{d(\Gamma_{16W}(z))}{dz} \times \Delta - \frac{d\Delta}{dz} \times \Gamma_{16W}(z)).$$

Proof. Use theorem 16.1. Here,
$\Omega_{11}, \Omega_{12}, \Omega_{13}, \Omega_{14}, \Omega_{15}, \Omega_{16}, \Omega_{17}, \Omega_{18}, \Omega_{19}, \Omega_{20}$ and
$\Lambda_{11}, \Lambda_{12}, \Lambda_{13}, \Lambda_{14}, \Lambda_{15}, \Lambda_{16}, \Lambda_{17}, \Lambda_{18}, \Lambda_{19}, \Lambda_{20}$ and
$\Phi_{11}, \Phi_{12}, \Phi_{13}, \Phi_{14}, \Phi_{15}, \Phi_{16}, \Phi_{17}, \Phi_{18}, \Phi_{19}, \Phi_{20}$ and
$\Psi_{11}, \Psi_{12}, \Psi_{13}, \Psi_{14}, \Psi_{15}, \Psi_{16}, \Psi_{17}, \Psi_{18}, \Psi_{19}, \Psi_{20}$ are as in theorem 14.1
and $\Gamma_{13W}(z) = \sum_{k=11}^{20} \Omega_k$, $\Gamma_{14W}(z) = \sum_{k=11}^{20} \Lambda_k$, $\Gamma_{15W}(z) = \sum_{k=11}^{20} \Phi_k$, $\Gamma_{16W}(z) = \sum_{k=11}^{20} \Psi_k$ are as in lemma 15.1. □
Now, we consider special cases of lemma 17.1.

Corollary 17.1.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\alpha\delta|^{-1}, |\beta\gamma|^{-1}, |\beta\delta|^{-1}, |\gamma\delta|^{-1}\}$. *The ordinary generating functions of the sequences* nG_n^2, $nG_{n+1}G_n$, $nG_{n+2}G_n$, $nG_{n+3}G_n$ *are given as follows:*

(a)
$$\sum_{n=0}^{\infty} nG_n^2 z^n = \frac{\Gamma_{17aG}(z)}{\Delta^2},$$

where

$\Delta=(-u^3z^6+su^2z^5-u(u+rt)z^4+(2su+r^2u-t^2)z^3+(rt+u)z^2+sz+1)(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(r^2+2s)z-1)$

and

$\Gamma_{17aG}(z)=3u^8z^{17}+2u^6(3su-t^2)z^{16}+2u^5(s^2u-st^2-3rtu-3u^2)z^{15}+2u^4(-3r^2u^2-3rstu+2rt^3-s^3u+s^2t^2-3su^2-t^2u)z^{14}+u^3(-6r^2su^2+9r^2t^2u+6rs^2tu-4rst^3+4rtu^2-s^4u+4s^2u^2-12st^2u+6t^4-2u^3)z^{13}+2u^2(-2st^4-5su^3+r^2t^4+3r^2u^3+3s^3u^2+6t^2u^2+3r^2s^2u^2+7rt^3u+3r^3tu^2+s^2t^2u-7rstu^2+rs^3tu-6r^2st^2u)z^{12}+u(4rt^5+t^4u+3r^4u^3-2s^2u^3+2s^4u^2+10u^4+4r^2s^3u^2+14r^2t^2u^2+2rtu^3-12r^2su^3+2st^2u^2+4r^3t^3u+2s^3t^2u-r^2s^2t^2u-20rst^3u+8rs^2tu^2-12r^3stu^2)z^{11}+2(5su^4+6r^2u^4+6s^3u^3-6t^2u^3+t^6-3r^2s^2u^3+r^4t^2u^2-s^2t^2u^2-4st^4u+7rt^3u^2+4r^2t^4u+7r^3tu^3-12r^2st^2u^2-r^3s^2tu^2-18rstu^3-rs^2t^3u+3rs^3tu^2)z^{10}+(4rt^5+4t^4u-s^2t^4-8s^2u^3+8s^4u^2-4u^4+2r^2s^3u^2+24r^2t^2u^2+r^4s^2u^2+16rtu^3+14r^2su^3-10st^2u^2+4r^3t^3u+2s^3t^2u-4r^2s^2t^2u-8rst^3u-32rs^2tu^2+4r^3stu^2)z^9+2(2st^4+su^3+r^2t^4-10r^2u^3-2s^3u^2+3t^2u^2+r^2s^2u^2+rt^3u-rs^2t^3-r^3tu^2-9s^2t^2u+26rstu^2-rs^3tu+6r^2st^2u+r^3s^2tu)z^8+(-r^4u^2+2r^2s^3u+r^2s^2t^2-14r^2su^2-6r^2t^2u+8rs^2tu+8rst^3-26rtu^2+6s^4u-4s^3t^2-2s^2u^2+24st^2u-3t^4-2u^3)z^7+2(-rt^3-su^2-5s^3u-t^2u+5r^2u^2+4s^2t^2+rs^3t+5r^3tu-5r^2s^2u-3rstu)z^6+(-2st^2+6r^4u+4s^2u+7r^2t^2+s^4+2u^2-6rs^2t+16r^2su+12rtu)z^5+2(2r^3t-r^2u+su-s^3+5rst)z^4+2(-u+r^2s-rt+s^2)z^3-2sz^2+z.$

(b)

$$\sum_{n=0}^{\infty}nG_{n+1}G_nz^n=\frac{\Gamma_{18aG}(z)}{\Delta^2},$$

where

$\Gamma_{18aG}(z)=-4tu^7z^{16}+3tu^5(t^2-su)z^{15}+2u^4(-st^3+4tu^2+3rsu^2+rt^2u+s^2tu)z^{14}+u^3(rt^4-7ru^3-5t^3u+12stu^2+s^3tu+5rs^2u^2+2r^2tu^2-9rst^2u)z^{13}-2u^3(st^3+2tu^2+5rsu^2+2rs^3u-rt^2u-s^2tu-2rs^2t^2+2r^2stu)z^{12}-u^2(4rt^4-14ru^3+t^3u-6s^2t^3+3rs^4u+7stu^2+10s^3tu+3r^2st^3-5rs^2u^2+6r^3su^2-r^3t^2u-2rst^2u-11r^2s^2tu)z^{11}+2u(4r^3u^3-t^5-r^3s^2u^2-3rst^4+2rsu^3+3st^3u-2s^4tu+rs^3u^2-4rt^2u^2+2r^2t^3u+r^4tu^2+12rs^2t^2u-14r^2stu^2+r^2s^3tu-r^3st^2u)z^{10}+(-ru^4-3st^5+5t^3u^2-r^3s^3u^2+7r^3t^2u^2+2rt^4u-8stu^3-6rs^2u^3-2rs^4u^2+4r^3su^3+9r^2tu^3+13s^2t^3u-8s^3tu^2-7r^2s^2tu^2-9rst^2u^2+4rs^3t^2u-4r^2st^3u+r^4stu^2)z^9-2(-2tu^3+7r^3u^3-s^3t^3-r^3s^2u^2+rst^4-2rsu^3+st^3u+2s^4tu+rs^3u^2-3rt^2u^2+2r^2t^3u+r^4tu^2+rs^2t^2u-15r^2stu^2+r^2s^3tu-r^3st^2u)z^8+(-r^5u^2-9r^3su^2-7r^3t^2u+6r^2s^2tu+r^2st^3-25r^2tu^2+rs^4u-rs^3t^2-6rs^2u^2+20rst^2u-5rt^4-12ru^3+2s^3tu-s^2t^3+7stu^2-2t^3u)z^7+2(-4tu^2-2r^2t^3+2r^3u^2-5rsu^2-5rs^3u-3rt^2u+2r^4tu-s^2tu+2rs^2t^2-3r^3s^2u-2r^2stu)z^6+(7ru^2-s^3t+3r^5u+3r^3t^2-4rst^2+rs^2u+10r^3su+13r^2tu-4r^2s^2t-4stu)z^5+2(-rs^3+2rt^2+r^4t-s^2t+r^3u+2tu+3r^2st+3rsu)z^4+(rs^2+r^3s+r^2t-2ru+3st)z^3+rz.$

(c)

$$\sum_{n=0}^{\infty}nG_{n+2}G_nz^n=\frac{\Gamma_{19aG}(z)}{\Delta^2},$$

where

$\Gamma_{19aG}(z)=5u^6(t^2-su)z^{15}+2u^4(-5s^2u^2-2t^4+3rtu^2+4st^2u)z^{14}+u^3(su-t^2)(-3st^2-2s^2u+3u^2+8rtu)z^{13}+2u^2(-rt^5-2s^2u^3+3s^4u^2-4r^2t^2u^2+2rtu^3+2r^2su^3-3s^3t^2u+7rst^3u-4rs^2tu^2)z^{12}+u(-6r^3tu^3+7r^2s^2u^3+12r^2st^2u^2-4r^2t^4u+9r^2u^4-16rs^3tu^2+4rs^2t^3u+12rstu^3-12rt^3u^2+3s^5u^2-6s^3u^3+2s^2t^2u^2+5st^4u+11su^4-t^6-2t^2u^3)z^{11}+2u(-r^3stu^2-r^3t^3u+r^2s^3u^2+2r^2s^2t^2u+4r^2su^3-12r^2t^2u^2-rs^4tu+4rs^2tu^2+7rst^3u-rt^5-5rtu^3+4s^4u^2-7s^3t^2u+s^2t^4+9s^2u^3-4st^2u^2)z^{10}+u(st^4-5su^3+7s^5u-7r^2t^4-7r^2u^3-s^4t^2-4s^3u^2-2t^2u^2+6r^2s^2u^2-6rt^3u+8rs^2t^3+r^2s^4u+r^4su^2-16r^3tu^2+6s^2t^2u-4rstu^2-28rs^3tu+24r^2st^2u-2r^3s^2tu)z^9+2(-6r^4u^3+11r^3stu^2-6r^3t^3u-r^2s^3u^2+4r^2s^2t^2u-6r^2su^3-10r^2t^2u^2-2rs^4tu+24rs^2tu^2+rst^3u-2rt^5-4rtu^3-2s^4u^2-7s^3t^2u+2s^2t^4+4s^2u^3+2st^2u^2-2t^4u)z^8+(-5r^4su^2-3r^4t^2u-22r^3tu^2+3r^2s^4u-10r^2s^2u^2+8r^2st^2u-8r^2t^4-10r^2u^3+4rs^3tu+6rs^2t^3-28rstu^2-8rt^3u+5s^5u-3s^4t^2+4s^3u^2+22s^2t^2u-5st^4-7su^3-3t^2u^2)z^7+2(-4s^4u-r^3t^3+3s^3t^2+2r^4u^2-3s^2u^2-3rst^3+rs^4t+rtu^2-4st^2u+3r^5tu-9r^2s^3u-3r^4s^2u-4rs^2tu+5r^3stu)z^6+(2rt^3+su^2+6s^3u+3r^6u+5t^2u+6r^2u^2+4r^4t^2-5s^2t^2+s^5-8rs^3t+15r^4su+12r^3tu+4r^2st^2-4r^3s^2t+18r^2s^2u+12rstu)z^5+2(2st^2+r^5t-2s^2u-r^2s^3+2r^2t^2-s^4+4rs^2t+5r^3st+2rtu)z^4+(s+r^2)(u+r^2s+2s^2)z^3+2(rt-s^2)z^2+(s+r^2)z.$

(d)

$$\sum_{n=0}^{\infty}nG_{n+3}G_nz^n=\frac{\Gamma_{20aG}(z)}{\Delta^2},$$

where

$\Gamma_{20aG}(z)=6u^5(-t^3+2stu-ru^2)z^{14}+u^3(-7tu^3+5t^5-12rsu^3-15st^3u-2rt^2u^2+17s^2tu^2)z^{13}+2u^2(6ru^4-2st^5+5t^3u^2+5rt^4u-11stu^3-5rs^2u^3+6r^2tu^3+3s^2t^3u-s^3tu^2-6rst^2u^2)z^{12}+u(3rt^6+14tu^4+15r^3u^4+7r^2t^3u^2+30rsu^4+rs^3u^3+3rt^2u^3+2st^3u^2-7s^2tu^3+8s^3t^3u-11s^4tu^2+26rs^2t^2u^2-23rst^4u-12r^2stu^3)$

$z^{11}+2(-ru^5-2t^3u^3+t^7-2r^3t^2u^3+6s^2t^3u^2+3stu^4-6st^5u+9rs^2u^4+5rs^4u^3+9r^3su^4+8rt^4u^2+3r^2tu^4+3r^2t^5u+3s^3tu^3-2s^5tu^2+9rs^3t^2u^2-4r^2st^3u^2-15r^2s^2tu^3-24rst^2u^3-3rs^2t^4u)z^{10}+(4rt^6-tu^4+4t^5u-18r^3u^4-4s^2t^5+7r^3s^2u^3+13r^2t^3u^2-6rsu^4-14rs^3u^3+15rt^2u^3+5rs^5u^2-5st^3u^2-3r^3t^4u-18r^4tu^3-14s^2tu^3+16s^3t^3u-2s^4tu^2-34rs^2t^2u^2-33r^2s^3tu^2+35r^3st^2u^2-6rst^4u+36r^2stu^3+3rs^4t^2u)z^9+2(-6r^5u^3+9r^4stu^2-6r^4t^3u+6r^3s^2t^2u-15r^3su^3-14r^3t^2u^2-3r^2s^4tu+38r^2s^2tu^2+5r^2st^3u-r^2t^5-21r^2tu^3-2rs^4u^2-9rs^3t^2u+4rs^2u^3+33rst^2u^2-3rt^4u-4ru^4-2s^5tu+s^4t^3-s^3tu^2-10s^2t^3u+2st^5+6stu^3+t^3u^2)z^8+(-6r^5su^2-3r^5t^2u-19r^4tu^2+3r^3s^4u-25r^3s^2u^2-8r^3t^4-7r^3u^3+12r^2s^3tu+8r^2s^2t^3-69r^2stu^2-17r^2t^3u+6rs^5u-4rs^4t^2-12rs^3u^2+49rs^2t^2u-2rst^4-28rsu^3-33rt^2u^2+9s^4tu-5s^3t^3+2s^2tu^2+20st^3u-3t^5-12tu^3)z^7+2(-rt^4+3ru^3-t^3u-r^4t^3+4s^2t^3+3r^5u^2-9rs^4u-8stu^2-7s^3tu+3r^6tu+6rs^3t^2-5r^2st^3-8rs^2u^2+r^2s^4t+5r^3su^2+10r^2tu^2-12r^3s^3u+6r^3t^2u-3r^5s^2u-12rst^2u+7r^4stu-13r^2s^2tu)z^6+(rs^5-2st^3+7tu^2+3r^7u+9r^2t^3+9r^3u^2+4r^5t^2+16rsu^2+6rs^3u+19rt^2u+18r^5su+19r^4tu-3s^2tu-15rs^2t^2-12r^2s^3t+7r^3st^2-4r^4s^2t+28r^3s^2u+42r^2stu)z^5+2(-2rs^4-2ru^2-2s^3t+r^6t-r^3s^3+4r^3t^2+9rst^2+rs^2u+6r^4st+r^3su+r^2tu+7r^2s^2t+5stu)z^4+(3rs^3-2rt^2+r^5s+5s^2t+4r^3s^2-2tu+3r^2st-2rsu)z^3+2(-rs^2+r^2t+ru-st)z^2+(t+2rs+r^3)z.$

18. Generalized Tetranacci Polynomials by Matrix Methods

In this section, we present matrix representations of the sequences W_n and G_n. We also introduce Simson matrix and investigate its properties.

18.1. Matrix Representations of the Sequences W_n and G_n

We define the square matrix A of order 4 as:

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -u$. Some properties of matrix A^n can be given as

$$\begin{aligned} A^n &= rA^{n-1} + sA^{n-2} + tA^{n-3} + uA^{n-4}, \\ A^{n+m} &= A^n A^m = A^m A^n, \\ \det(A^n) &= (-u)^n, \end{aligned}$$

for all integers m and n.

From eq. (1), we have

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \tag{45}$$

and using eq. (45) and induction, we have the matrix formulation of W_n as

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{46}$$

If we take $W_n = G_n$ in eq. (45) we have

$$\begin{pmatrix} G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{47}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix}.$$

Theorem 18.1.
For all integers m, n, we have the following properties:

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

(b) $C_1 A^n = A^n C_1$.

(c) $C_{n+m} = C_n B_m = B_m C_n$.

(d)

$$A^n = G_{n-2}A^3 + (sG_{n-3} + tG_{n-4} + uG_{n-5})A^2 + (tG_{n-3} + uG_{n-4})A + uG_{n-3}I,$$

i.e.,

$A^n = \frac{1}{u^2}((-tG_{n+3} + (u+rt)G_{n+2} + (st-ru)G_{n+1} + (-su+t^2)G_n)A^3 + ((u+rt)G_{n+3} - r(2u+rt)G_{n+2} + (r^2u - su - rst)G_{n+1} + (rsu - tu - rt^2)G_n)A^2 + ((st-ru)G_{n+3} + (r^2u - su - rst)G_{n+2} + (2rsu - s^2t)G_{n+1} + (u^2 - st^2 + s^2u + rtu)G_n)A$
$+((t^2 - su)G_{n+3} + (rsu - rt^2 - tu)G_{n+2} + (u^2 - st^2 + rtu + s^2u)G_{n+1} + (2stu - t^3 - ru^2)G_n)I)$

i.e.,

$A^n = \frac{1}{u^2}(G_{n+3}(-tA^3 + (u+rt)A^2 + (st-ru)A + (t^2-su)I) + G_{n+2}((u+rt)A^3 - r(2u+rt)A^2 + (r^2u - rst - su)A + (-rt^2 + rsu - tu)I) + G_{n+1}((st-ru)A^3 + (r^2u - su - rst)A^2 + (2rsu - s^2t)A + (s^2u + u^2 - st^2 + rtu)I) + G_n((t^2 - su)A^3 + (rsu - rt^2 - tu)A^2 + (s^2u + rtu + u^2 - st^2)A + (2stu - t^3 - ru^2)I)$

where

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(e)

$$A^{n+m} = G_{n-2}A^{m+3} + (sG_{n-3} + tG_{n-4} + uG_{n-5})A^{m+2} + (tG_{n-3} + uG_{n-4})A^{m+1} + uG_{n-3}A^m$$

i.e.,

$A^{n+m} = \frac{1}{u^2}((-tG_{n+3} + (u+rt)G_{n+2} + (st-ru)G_{n+1} + (-su+t^2)G_n)A^{m+3} + ((u+rt)G_{n+3} - r(2u+rt)G_{n+2} + (r^2u - su - rst)G_{n+1} + (rsu - tu - rt^2)G_n)A^{m+2} + ((st-ru)G_{n+3} + (r^2u - su - rst)G_{n+2} + (2rsu - s^2t)G_{n+1} + (u^2 - st^2 + s^2u + rtu)G_n)A^{m+1} + ((t^2 - su)G_{n+3} + (rsu - rt^2 - tu)G_{n+2} + (u^2 - st^2 + rtu + s^2u)G_{n+1} + (2stu - t^3 - ru^2)G_n)A^m)$

i.e.,

$A^{n+m} = \frac{1}{u^2}(G_{n+3}(-tA^{m+3} + (u+rt)A^{m+2} + (st-ru)A^{m+1} + (t^2-su)A^m) + G_{n+2}((u+rt)A^{m+3} - r(2u+rt)A^{m+2} + (r^2u - rst - su)A^{m+1} + (-rt^2 + rsu - tu)A^m) + G_{n+1}((st-ru)A^{m+3} + (r^2u - su - rst)A^{m+2} + (2rsu - s^2t)A^{m+1} + (s^2u + u^2 - st^2 + rtu)A^m) + G_n((t^2 - su)A^{m+3} + (rsu - rt^2 - tu)A^{m+2} + (s^2u + rtu + u^2 - st^2)A^{m+1} + (2stu - t^3 - ru^2)$
$A^m)$.

Proof.

(a) We use induction on n. First we assume that $n \geq 0$. For $n = 0$, we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} G_1 & sG_0 + tG_{-1} + uG_{-2} & tG_0 + uG_{-1} & uG_0 \\ G_0 & sG_{-1} + tG_{-2} + uG_{-3} & tG_{-1} + uG_{-2} & uG_{-1} \\ G_{-1} & sG_{-2} + tG_{-3} + uG_{-4} & tG_{-2} + uG_{-3} & uG_{-2} \\ G_{-2} & sG_{-3} + tG_{-4} + uG_{-5} & tG_{-3} + uG_{-4} & uG_{-3} \end{pmatrix}$$

which is true because $G_{-4} = -\frac{t}{u^2}, G_{-3} = \frac{1}{u}, G_{-2} = 0, G_{-1} = 0, G_0 = 0, G_1 = 1$. Suppose that the relation holds for $n = k$.

Then, we get

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix} \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and so

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n+1}$$

$$= \begin{pmatrix} rG_{n+1} + sG_n + tG_{n-1} + uG_{n-2} & sG_{n+1} + tG_n + uG_{n-1} & tG_{n+1} + uG_n & uG_{n+1} \\ rG_n + sG_{n-1} + tG_{n-2} + uG_{n-3} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ rG_{n-1} + sG_{n-2} + tG_{n-3} + uG_{n-4} & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ rG_{n-2} + sG_{n-3} + tG_{n-4} + uG_{n-5} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} G_{n+2} & sG_{n+1} + tG_n + uG_{n-1} & tG_{n+1} + uG_n & uG_{n+1} \\ G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \end{pmatrix}.$$

So the relation is true for $n = k+1$ which completes the case $n \geq 0$.

For the case $n \leq 0$, the proof can be done induction as well. So for all integers n we have $B_n = A^n$.

Note that proof of the case $n \geq 0$ can also be given as follows.

By expanding the vectors on the both sides of eq. (47) to 4-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

(d) Use induction.

(e) Multiply both side of the identity given in (d) with the matrix A^m. □

Now, we present an identity for W_{n+m}.

Theorem 18.2.
(Honsberger's Identity) For all integers m and n, we have

$$W_{n+m} = W_nG_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m,$$

i.e.,

$$W_{m+n} = W_{m+3}G_{n-2} + W_{m+2}(sG_{n-3} + tG_{n-4} + uG_{n-5}) + W_{m+1}(tG_{n-3} + uG_{n-4}) + uW_mG_{n-3}.$$

Proof. From the equation $C_{n+m}=C_nB_m=B_mC_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof of first identitiy. For the second identity, replace m with $n-3$ and n with $m+3$ in the first identity.

Proof can also be given by using induction. □

Corollary 18.1.
For all integers m,n, we have

$$G_{n+m}=G_nG_{m+1}+G_{n-1}(sG_m+tG_{m-1}+uG_{m-2})+G_{n-2}(tG_m+uG_{m-1})+uG_{n-3}G_m,$$
$$H_{n+m}=H_nG_{m+1}+H_{n-1}(sG_m+tG_{m-1}+uG_{m-2})+H_{n-2}(tG_m+uG_{m-1})+uH_{n-3}G_m,$$

i.e.,

$$G_{m+n}=G_{m+3}G_{n-2}+G_{m+2}(sG_{n-3}+tG_{n-4}+uG_{n-5})+G_{m+1}(tG_{n-3}+uG_{n-4})+uG_mG_{n-3},$$
$$H_{m+n}=H_{m+3}G_{n-2}+H_{m+2}(sG_{n-3}+tG_{n-4}+uG_{n-5})+H_{m+1}(tG_{n-3}+uG_{n-4})+uH_mG_{n-3}.$$

Proof. Set $W_n=G_n$ with $G_0=0,G_1=1,G_2=r,G_3=r^2+s$ and $W_n=H_n$ with $H_0=4,H_1=r,H_2=2s+r^2,H_3=r^3+3sr+3t$ in theorem 18.2, respectively. □

Corollary 18.2.
For all integers m,n,j, we have the following properties:

$$W_{mn+j}=W_{j+2}G_{mn-1}+W_{j+1}(sG_{mn-2}+tG_{mn-3}+uG_{mn-4})+W_j(tG_{mn-2}+uG_{mn-3})+uW_{j-1}G_{mn-2},$$
$$W_{mn+j}=W_{j+2}G_{mn-1}+W_{j+1}(sG_{mn-2}+tG_{mn-3}+uG_{mn-4})+W_j(tG_{mn-2}+uG_{mn-3})+uW_{j-1}G_{mn-2},$$
$$W_{mn+j}=W_{j+2}G_{mn-1}+W_{j+1}(sG_{mn-2}+tG_{mn-3}+uG_{mn-4})+W_j(tG_{mn-2}+uG_{mn-3})+uW_{j-1}G_{mn-2}.$$

Proof. If we make the following changes

$$n\Leftrightarrow a$$
$$m\Leftrightarrow b$$

in the first identity of theorem 18.2, i.e.,

$$W_{n+m}=W_nG_{m+1}+W_{n-1}(sG_m+tG_{m-1}+uG_{m-2})+W_{n-2}(tG_m+uG_{m-1})+uW_{n-3}G_m,$$

we get

$$W_{a+b}=W_aG_{b+1}+W_{a-1}(sG_b+tG_{b-1}+uG_{b-2})+W_{a-2}(tG_b+uG_{b-1})+uW_{a-3}G_b \tag{48}$$

Now, if we make the following changes

$$a\Leftrightarrow j+2$$
$$b\Leftrightarrow mn-2$$

in eq. (48) we obtain

$$W_{mn+j}=W_{j+2}G_{mn-1}+W_{j+1}(sG_{mn-2}+tG_{mn-3}+uG_{mn-4})+W_j(tG_{mn-2}+uG_{mn-3})+uW_{j-1}G_{mn-2}.$$

To complete the proof, set $W_n=G_n$ and $W_n=H_n$ in the last identity, respectively. □

18.2. Simson Matrix and its Properties

For $n\in\mathbb{Z}$, we define

$$f_W(n)=\begin{pmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence W_n. So, Simson determinant of the sequence W_n can be defined as

$$\det(f_W(n)) = \begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix}.$$

Similarly, as special cases of W_n, Simson matrices of the sequences G_n and H_n are

$$f_G(n) = \begin{pmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{pmatrix},$$

$$f_H(n) = \begin{pmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{pmatrix},$$

respectively and Simson determinants of the sequences G_n and H_n are

$$\det(f_G(n)) = \begin{vmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{vmatrix},$$

$$\det(f_H(n)) = \begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix},$$

respectively.

Note that from theorem 4.1 we have

$$\det(f_W(n)) = (-1)^n u^n \det(f_W(0))$$

i.e.,

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}.$$

Note also that from the proof of lemma 4.1, we have

$$\det(f_W(n)) = \frac{1}{u^3}\Lambda_W(n),$$

i.e.,

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = \frac{1}{u^3}(-W_{n+3}^4 + (u + r^2 s + rt)W_{n+2}^4 + (st^2 - u^2 - rtu)W_{n+1}^4 + u^3 W_n^4 + 3rW_{n+2}W_{n+3}^3 +$$

$2sW_{n+1}W_{n+3}^3 + tW_nW_{n+3}^3 + (-t-2rs+r^3)W_{n+2}^3W_{n+3} + (2rs^2+r^2t+ru+st)W_{n+2}^3W_{n+1} + (r^2u-2su+t^2+rst)W_{n+2}^3W_n + (-r^2u-2su-t^2+rst)W_{n+1}^3W_{n+3} + (rt^2+2s^2t+tu-rsu)W_{n+1}^3W_{n+2} + (-ru^2+t^3+2stu)W_{n+1}^3W_n + ru^2W_n^3W_{n+3} + 2su^2W_n^3W_{n+2} + 3tu^2W_n^3W_{n+1} + (s-3r^2)W_{n+3}^2W_{n+2}^2 + (2u-rt-s^2)W_{n+3}^2W_{n+1}^2 - suW_{n+3}^2W_n^2 + s(3u+3rt+s^2)W_{n+2}^2W_{n+1}^2 + u(-2u+rt+s^2)W_{n+2}^2W_n^2 + u(su+3t^2)W_{n+1}^2W_n^2 + (3t-4rs)W_{n+3}^2W_{n+2}W_{n+1} + 2(2u-rt)W_{n+3}^2W_{n+2}W_n - (3ru + st)W_{n+3}^2W_{n+1}W_n + 2(-2u+r^2s-2rt-s^2)W_{n+2}^2W_{n+3}W_{n+1} + (r^2t-5ru-st)W_{n+2}^2W_{n+3}W_n + (rt^2+s^2t+tu+5rsu)W_{n+2}^2W_{n+1}W_n + (rs^2+r^2t+ru-5st)W_{n+1}^2W_{n+3}W_{n+2} + (rt^2-5tu+rsu)W_{n+1}^2W_{n+3}W_n + 2(st^2+s^2u+2u^2+2rtu) W_{n+1}^2W_{n+2}W_n + u(-3t+rs)W_n^2W_{n+3}W_{n+2} + 2u(-2u+rt)W_n^2W_{n+3}W_{n+1} + u(3ru+4st)W_n^2W_{n+2}W_{n+1} + (3r^2u-4su-3t^2+rst)W_{n+3}W_{n+2}W_{n+1}W_n)$.

So

$$\det(f_G(n)) = (-1)^n u^n \det(f_G(0)),$$

$$\det(f_G(n)) = \frac{1}{u^3}\Lambda_G(n),$$

i.e.,

$$\begin{vmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{vmatrix} = (-1)^{n+1}u^{n-1},$$

and

$$\begin{vmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{vmatrix} = \frac{1}{u^3}(-G_{n+3}^4 + (u + r^2s + rt)G_{n+2}^4 + (st^2 - u^2 - rtu)G_{n+1}^4 + u^3G_n^4 + 3rG_{n+2}G_{n+3}^3 + 2sG_{n+1}G_{n+3}^3 + tG_nG_{n+3}^3 + (-t-2rs+r^3)G_{n+2}^3G_{n+3} + (2rs^2+r^2t+ru+st)G_{n+2}^3G_{n+1} + (r^2u-2su+t^2+rst)G_{n+2}^3G_n + (-r^2u-2su-t^2+rst)G_{n+1}^3G_{n+3} + (rt^2+2s^2t+tu-rsu)G_{n+1}^3G_{n+2} + (-ru^2+t^3+2stu)G_{n+1}^3G_n + ru^2G_n^3G_{n+3} + 2su^2G_n^3G_{n+2} + 3tu^2G_n^3G_{n+1} + (s-3r^2)G_{n+3}^2G_{n+2}^2 + (2u-rt-s^2)G_{n+3}^2G_{n+1}^2 - suG_{n+3}^2G_n^2 + s(3u+3rt+s^2)G_{n+2}^2G_{n+1}^2 + u(-2u+rt+s^2)G_{n+2}^2G_n^2 + u(su+3t^2)G_{n+1}^2G_n^2 + (3t-4rs)G_{n+3}^2G_{n+2}G_{n+1} + 2(2u-rt)G_{n+3}^2G_{n+2}G_n - (3ru+st)G_{n+3}^2G_{n+1}G_n + 2(-2u+r^2s-2rt-s^2)G_{n+2}^2G_{n+3}G_{n+1} + (r^2t-5ru-st)G_{n+2}^2G_{n+3}G_n + (rt^2+s^2t+tu+5rsu)G_{n+2}^2G_{n+1}G_n + (rs^2+r^2t+ru-5st)G_{n+1}^2G_{n+3}G_{n+2} + (rt^2-5tu+rsu)G_{n+1}^2G_{n+3}G_n + 2(st^2+s^2u+2u^2+2rtu)G_{n+1}^2G_{n+2}G_n + u(-3t+rs)G_n^2G_{n+3}G_{n+2} + 2u(-2u+rt)G_n^2G_{n+3}G_{n+1} + u(3ru+4st)G_n^2G_{n+2}G_{n+1} + (3r^2u-4su-3t^2+rst)G_{n+3}G_{n+2}G_{n+1}G_n)$$

and

$$\det(f_H(n)) = (-1)^n u^n \det(f_H(0)),$$

$$\det(f_H(n)) = \frac{1}{u^3}\Lambda_H(n),$$

i.e.,

$$\begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix} = (-1)^n u^{n-3} g(r,s,t,u)$$

and

$$\begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix} = \frac{1}{u^3}(-H_{n+3}^4 + (u + r^2s + rt)H_{n+2}^4 + (st^2 - u^2 - rtu)H_{n+1}^4 + u^3H_n^4 + 3rH_{n+2}H_{n+3}^3 + 2sH_{n+1}H_{n+3}^3 + tH_nH_{n+3}^3 + (-t-2rs+r^3)H_{n+2}^3H_{n+3} + (2rs^2+r^2t+ru+st)H_{n+2}^3H_{n+1} + (r^2u-2su+t^2+rst)H_{n+2}^3H_n + (-r^2u-2su-t^2+rst)H_{n+1}^3H_{n+3} + (rt^2+2s^2t+tu-rsu)H_{n+1}^3H_{n+2} + (-ru^2+t^3+2stu)H_{n+1}^3H_n + ru^2H_n^3H_{n+3} + 2su^2H_n^3H_{n+2} + 3tu^2H_n^3H_{n+1} + (s-3r^2)H_{n+3}^2H_{n+2}^2 + (2u-rt-s^2)H_{n+3}^2H_{n+1}^2 - suH_{n+3}^2H_n^2 + s(3u+3rt+s^2)H_{n+2}^2H_{n+1}^2 + u(-2u+rt+s^2)H_{n+2}^2H_n^2 + u(su+3t^2)H_{n+1}^2H_n^2 + (3t-4rs)H_{n+3}^2H_{n+2}H_{n+1} + 2(2u-rt)H_{n+3}^2H_{n+2}H_n - (3ru+st)H_{n+3}^2H_{n+1}H_n + 2(-2u+r^2s-2rt-s^2)H_{n+2}^2H_{n+3}H_{n+1} + (r^2t-5ru-st)H_{n+2}^2H_{n+3}H_n + (rt^2+s^2t+tu+5rsu)H_{n+2}^2H_{n+1}H_n + (rs^2+r^2t+ru-5st)H_{n+1}^2H_{n+3}H_{n+2} + (rt^2-5tu+rsu)H_{n+1}^2H_{n+3}H_n + 2(st^2+s^2u+2u^2+2rtu)H_{n+1}^2H_{n+2}H_n + u(-3t+rs)H_n^2H_{n+3}H_{n+2} + 2u(-2u+rt)H_n^2H_{n+3}H_{n+1} + u(3ru+4st)H_n^2H_{n+2}H_{n+1} + (3r^2u-4su-3t^2+rst)H_{n+3}H_{n+2}H_{n+1}H_n)$$

where

$g(r,s,t,u) = 27r^4u^2 - 18r^3stu + 4r^3t^3 + 4r^2s^3u - r^2s^2t^2 + 144r^2su^2 - 6r^2t^2u - 80rs^2tu + 18rst^3 + 192rtu^2 + 16s^4u - 4s^3t^2 + 128s^2u^2 - 144st^2u + 27t^4 + 256u^3.$

Note that, as in the proof of lemma 4.1, the following identities

$$W_{n-1} = \frac{1}{u}(W_{n+3} - rW_{n+2} - sW_{n+1} - tW_n),$$

$$W_{n-2} = \frac{1}{u^2}(-tW_{n+3} + (u+rt)W_{n+2} + (st-ru)W_{n+1} + (-su+t^2)W_n),$$

$$W_{n-3} = \frac{1}{u^3}((t^2-su)W_{n+3} + (rsu-rt^2-tu)W_{n+2} + (u^2-st^2+rtu+s^2u)W_{n+1} + (2stu-t^3-ru^2)W_n),$$

can be used in Simson matrix and the following identities (by taking $n=0$)

$$W_{-1} = \frac{1}{u}(W_3 - rW_2 - sW_1 - tW_0),$$

$$W_{-2} = \frac{1}{u^2}(-tW_3 + (u+rt)W_2 + (st-ru)W_1 + (-su+t^2)W_0),$$

$$W_{-3} = \frac{1}{u^3}((t^2-su)W_3 + (rsu-rt^2-tu)W_2 + (u^2-st^2+rtu+s^2u)W_1 + (2stu-t^3-ru^2)W_0),$$

can be used in the matrix

$$\begin{pmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{pmatrix}.$$

Lemma 18.1.
For all integers n, m and j, the followings hold.

(a) $f_W(n) = r f_W(n-1) + s f_W(n-2) + t f_W(n-3) + u f_W(n-4)$.

(b) $f_W(n) = A f_W(n-1)$ *and* $f_W(n) = A^n f_W(0)$, *i.e.,*

$$\begin{pmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{pmatrix}.$$

.

(c) $f_W(n+m) = A^n f_W(m)$ *and* $f_W(n+m) = A^m f_W(n)$ *i.e.,*

$$\begin{pmatrix} W_{n+m+3} & W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+2} & W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} & W_{n+m-2} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} & W_{n+m-3} \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_{m+3} & W_{m+2} & W_{m+1} & W_m \\ W_{m+2} & W_{m+1} & W_m & W_{m-1} \\ W_{m+1} & W_m & W_{m-1} & W_{m-2} \\ W_m & W_{m-1} & W_{m-2} & W_{m-3} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+m+3} & W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+2} & W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} & W_{n+m-2} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} & W_{n+m-3} \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix}$$

and $f_W(n) = A^m f_W(n-m)$, *i.e.,*

$$\begin{pmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n-m+3} & W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+2} & W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} & W_{n-m-2} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} & W_{n-m-3} \end{pmatrix}.$$

(d)

$$f_W(mn+j) = A^{mn} f_W(j)$$

and

$$f_W(mn+j) = (G_{n-2}A^3 + (sG_{n-3} + tG_{n-4} + uG_{n-5})A^2 + (tG_{n-3} + uG_{n-4})A + uG_{n-3}I)^m f_W(j).$$

Proof.

(a) Use $W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$.

(b) By using the definition of W_n, i.e., $W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$, we get

$$
\begin{aligned}
Af_W(n-1) &= \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \end{pmatrix} \\
&= \begin{pmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix} \\
&= f_W(n).
\end{aligned}
$$

Now it follows that $f_W(n) = A^n f_W(0)$.

(c) By using (b) we get

$$f_W(n+m) = A^{n+m} f_W(0) = A^n A^m f_W(0) = A^n f_W(m).$$

By interchanging m and n in $f_W(n+m) = A^n f_W(m)$, we get $f_W(n+m) = A^m f_W(n)$.
Then it follows that

$$
\begin{aligned}
f_W(n+m) &= A^n f_W(m) \Leftrightarrow A^{-n} f_W(n+m) = f_W(m) \\
&\Leftrightarrow A^m f_W(-m+n) = f_W(n) \Leftrightarrow f_W(n) = A^m f_W(n-m).
\end{aligned}
$$

(d) By using theorem 18.1 (d), i.e.,

$$A^n = G_{n-2}A^3 + (sG_{n-3} + tG_{n-4} + uG_{n-5})A^2 + (tG_{n-3} + uG_{n-4})A + uG_{n-3}I$$

and the first identity of corollary 18.2, i.e.,

$$W_{mn+j} = W_{j+2}G_{mn-1} + W_{j+1}(sG_{mn-2} + tG_{mn-3} + uG_{mn-4}) + W_j(tG_{mn-2} + uG_{mn-3}) + uW_{j-1}G_{mn-2},$$

we get

$$
\begin{aligned}
&(G_{n-2}A^3 + (sG_{n-3} + tG_{n-4} + uG_{n-5})A^2 + (tG_{n-3} + uG_{n-4})A + uG_{n-3}I)^m f_W(j) \\
&= A^{mn} f_W(j) \\
&= \begin{pmatrix} G_{mn+1} & sG_{mn} + tG_{mn-1} + uG_{mn-2} & tG_{mn} + uG_{mn-1} & uG_{mn} \\ G_{mn} & sG_{mn-1} + tG_{mn-2} + uG_{mn-3} & tG_{mn-1} + uG_{mn-2} & uG_{mn-1} \\ G_{mn-1} & sG_{mn-2} + tG_{mn-3} + uG_{mn-4} & tG_{mn-2} + uG_{mn-3} & uG_{mn-2} \\ G_{mn-2} & sG_{mn-3} + tG_{mn-4} + uG_{mn-5} & tG_{mn-3} + uG_{mn-4} & uG_{mn-3} \end{pmatrix} \begin{pmatrix} W_{j+3} & W_{j+2} & W_{j+1} & W_j \\ W_{j+2} & W_{j+1} & W_j & W_{j-1} \\ W_{j+1} & W_j & W_{j-1} & W_{j-2} \\ W_j & W_{j-1} & W_{j-2} & W_{j-3} \end{pmatrix} \\
&= \begin{pmatrix} W_{mn+j+3} & W_{mn+j+2} & W_{mn+j+1} & W_{mn+j} \\ W_{mn+j+2} & W_{mn+j+1} & W_{mn+j} & W_{mn+j-1} \\ W_{mn+j+1} & W_{mn+j} & W_{mn+j-1} & W_{mn+j-2} \\ W_{mn+j} & W_{mn+j-1} & W_{mn+j-2} & W_{mn+j-3} \end{pmatrix} \\
&= f_W(mn+j).
\end{aligned}
$$

Note that $A^{mn} f_W(j) = f_W(mn+j)$ also follows from the identity $f_W(n+m) = A^n f_W(m)$ which is given in (c), by replacing n and m by mn and j respectively in $f_W(n+m) = A^n f_W(m)$. □

Taking the determinant of both sides of the identities given in lemma 18.1, we obtain the following Theorem.

Theorem 18.3.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_W(n+m)) = (-u)^n \det(f_W(m)),$$

and

$$\det(f_W(n)) = (-u)^m \det(f_W(n-m)),$$

i.e.,

$$\begin{vmatrix} W_{n+m+3} & W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+2} & W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} & W_{n+m-2} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} & W_{n+m-3} \end{vmatrix} = (-u)^n \begin{vmatrix} W_{m+3} & W_{m+2} & W_{m+1} & W_m \\ W_{m+2} & W_{m+1} & W_m & W_{m-1} \\ W_{m+1} & W_m & W_{m-1} & W_{m-2} \\ W_m & W_{m-1} & W_{m-2} & W_{m-3} \end{vmatrix},$$

and

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-u)^m \begin{vmatrix} W_{n-m+3} & W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+2} & W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} & W_{n-m-2} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} & W_{n-m-3} \end{vmatrix}.$$

(b) *(see theorem 4.1) Simson's (or Cassini's) Identity:*

$$\det(f_W(n)) = (-u)^n \det(f_W(0)),$$

i.e.,

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-u)^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}.$$

Proof.

(a) Taking the determinant of both sides of the identities

$$f_W(n+m) = A^n f_W(m)$$

and

$$f_W(n) = A^m f_W(n-m)$$

which are given in lemma 18.1 (c), we get the required results.

(b) Take $m = 0$ in $\det(f_W(n+m)) = (-u)^n \det(f_W(m))$ in (a) or take the determinant of both sides of the identity $f_W(n) = A^n f_W(0)$ which is given in lemma 18.1 (b). □

Remark 18.1.
To prove the second matrix identity in lemma 18.1 (d), we used a consequence corollary 18.2 of Honsberger's Identity theorem 18.2. However, firstly, the second matrix identity in lemma 18.1 (d) can be proved by induction and then Honsberger's Identity, i.e.,

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m,$$

can be obtained just comparing the linear combination of the 3rd row and 1st column entries of the matrices.

From the last Theorem, we have the following Corollary which gives determinantal formulas of (r,s,t,u)-Tetranacci polynomials (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r, G_3 = r^2 + s$).

Corollary 18.3.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_G(n+m)) = (-u)^n \det(f_G(m)),$$

and

$$\det(f_G(n)) = (-u)^m \det(f_G(n-m)),$$

i.e.,

$$\begin{vmatrix} G_{n+m+3} & G_{n+m+2} & G_{n+m+1} & G_{n+m} \\ G_{n+m+2} & G_{n+m+1} & G_{n+m} & G_{n+m-1} \\ G_{n+m+1} & G_{n+m} & G_{n+m-1} & G_{n+m-2} \\ G_{n+m} & G_{n+m-1} & G_{n+m-2} & G_{n+m-3} \end{vmatrix} = (-u)^n \begin{vmatrix} G_{m+3} & G_{m+2} & G_{m+1} & G_m \\ G_{m+2} & G_{m+1} & G_m & G_{m-1} \\ G_{m+1} & G_m & G_{m-1} & G_{m-2} \\ G_m & G_{m-1} & G_{m-2} & G_{m-3} \end{vmatrix},$$

and

$$\begin{vmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{vmatrix} = (-u)^m \begin{vmatrix} G_{n-m+3} & G_{n-m+2} & G_{n-m+1} & G_{n-m} \\ G_{n-m+2} & G_{n-m+1} & G_{n-m} & G_{n-m-1} \\ G_{n-m+1} & G_{n-m} & G_{n-m-1} & G_{n-m-2} \\ G_{n-m} & G_{n-m-1} & G_{n-m-2} & G_{n-m-3} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_G(n)) = (-u)^n \det(f_G(0)),$$

i.e.,

$$\begin{vmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{vmatrix} = (-1)^{n+1} u^{n-1}.$$

Taking $W_n = H_n$ with $H_0 = 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t$ in the last Theorem, we have the following Corollary which gives determinantal formulas of (r, s, t, u)-Tetranacci-Lucas polynomials.

Corollary 18.4.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_H(n+m)) = (-u)^n \det(f_H(m)),$$

and

$$\det(f_H(n)) = (-u)^m \det(f_H(n-m)),$$

i.e.,

$$\begin{vmatrix} H_{n+m+3} & H_{n+m+2} & H_{n+m+1} & H_{n+m} \\ H_{n+m+2} & H_{n+m+1} & H_{n+m} & H_{n+m-1} \\ H_{n+m+1} & H_{n+m} & H_{n+m-1} & H_{n+m-2} \\ H_{n+m} & H_{n+m-1} & H_{n+m-2} & H_{n+m-3} \end{vmatrix} = (-u)^n \begin{vmatrix} H_{m+3} & H_{m+2} & H_{m+1} & H_m \\ H_{m+2} & H_{m+1} & H_m & H_{m-1} \\ H_{m+1} & H_m & H_{m-1} & H_{m-2} \\ H_m & H_{m-1} & H_{m-2} & H_{m-3} \end{vmatrix},$$

and

$$\begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix} = (-u)^m \begin{vmatrix} H_{n-m+3} & H_{n-m+2} & H_{n-m+1} & H_{n-m} \\ H_{n-m+2} & H_{n-m+1} & H_{n-m} & H_{n-m-1} \\ H_{n-m+1} & H_{n-m} & H_{n-m-1} & H_{n-m-2} \\ H_{n-m} & H_{n-m-1} & H_{n-m-2} & H_{n-m-3} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_H(n)) = (-u)^n \det(f_H(0)),$$

i.e.,

$$\begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix} = (-1)^n u^{n-3} g(r,s,t,u),$$

where

$g(r,s,t,u) = 27r^4u^2 - 18r^3stu + 4r^3t^3 + 4r^2s^3u - r^2s^2t^2 + 144r^2su^2 - 6r^2t^2u - 80rs^2tu + 18rst^3 + 192rtu^2 + 16s^4u - 4s^3t^2 + 128s^2u^2 - 144st^2u + 27t^4 + 256u^3.$

References

[1] Andrica, D., Bagdasar, O., Recurrent Sequences Key Results, Applications, and Problems, Springer, 2020.

[2] Djordjević, G.B., Milovanović, G.V., Special Classes of Plynomials, University of Niš, Faculty of Technology, Leskovac, 2014. http://www.mi.sanu.ac.rs/~gvm/Teze/Special%20Classes%20of%20Polynomials.pdf

[3] Frei, G., Binary Lucas and Fibonacci Polynomials, I, Math. Nadir. 96, 83-112, 1980.

[4] Flórez, R., McAnally, N., Mukherjee, A., Identities for the Generalized Fibonacci Polynomial, Integers, 18B, 2018.

[5] He,T.X., Peter J.-S. Shiue, P.J.S., On Sequences of Numbers and Polynomials Defined by Linear Recurrence Relations of Order 2, International Journal of Mathematics and Mathematical Sciences, Volume 2009, Article ID 709386, 21 pages, doi:10.1155/2009/709386.

[6] Howard F.T., Saidak, F., Zhou's Theory of Constructing Identities, Congress Numer. 200 (2010), 225-237.

[7] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[8] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 1 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), Second Edition, John Wiley&Sons, New York, 2018.

[9] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 2 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), John Wiley&Sons, New York, 2019.

[10] Soykan, Y., On Generalized Fibonacci Polynomials: Horadam Polynomials, Earthline Journal of Mathematical Sciences, 11(1), 23-114, 2023. https://doi.org/10.34198/ejms.11123.23114

[11] Soykan, Y., Generalized Fibonacci Numbers: Sum Formulas, Minel Yayın, 2022. https://www.minelyayin.com/generalized-fibonacci-numbers-sum-formulas-51

[12] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 7(2), 45-56, 2019.

[13] Soykan, Y., Properties of Generalized (r,s,t,u)-Numbers, Earthline Journal of Mathematical Sciences, 5(2), 297-327, 2021. https://doi.org/10.34198/ejms.5221.297327

[14] Soykan, Y., Polatlı, E. E., A Study on Generalized Fourth-Order Jacobsthal Sequences, Int. J. Adv. Appl. Math. and Mech. 9(4), 34-50, 2022. ISSN: 2347-2529.

[15] Soykan, Y., A Study of Generalized Fourth-Order Pell Sequences, Journal of Scientific Research and Reports, 25(1-2), 1-18, 2019.

[16] Soykan, Y., Generalized Richard Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 10(3), 38-51, 2023.

[17] Soykan, Y., Generalized Olivier Numbers, Asian Research Journal of Mathematics, 19(1), Page 1-22, 2023. DOI: 10.9734/ARJOM/2023/v19i1634

[18] Soykan, Y., A Study on Generalized Blaise Numbers, Asian Journal of Advanced Research and Reports, 17(1), 32-53, 2023. DOI: 10.9734/AJARR/2023/v17i1463

[19] Soykan, Y., Generalized Friedrich Numbers, Journal of Advances in Mathematics and Computer Science, 38(3), 12-31, 2023. DOI: 10.9734/JAMCS/2023/v38i31748

[20] Soykan, Y., Generalized Pierre Numbers, Journal of Progressive Research in Mathematics, 20(1), 16-38, 2023.

[21] Soykan, Y., Generalized Pandita Numbers, International Journal of Mathematics, Statistics and Operations Research, 3(1), 107-123, 2023. https://doi.org/10.47509/IJMSOR.2023.v03i01.06

[22] Soykan, Y., Generalized Adrien Numbers, Applied Mathematics and Computer Science, 7(1), 37-51, 2023.

[23] Vajda, S,. Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications, Dover Publications Inc.; 2008.

[24] Wang, J., Some New Results for the (p,q)-Fibonacci and Lucas Polynomials, Advances in Difference Equations, 2014. http://www.advancesindifferenceequations.com/content/2014/1/64

[25] Wang, W., Wang, H., Generalized-Humbert Polynomials via Generalized Fibonacci Polynomials, Applied Mathematics and Computation 307, 204–216, 2017. https://doi.org/10.1016/j.amc.2017.02.050.