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Application of the SBA method to fractional Kolmogorov-Fisher-Piskunov-Petrovsky models

Research Article

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Abstract: In this paper we determine the analytical solution by the Somé Blaise Abbo(SBA) method of some models of non-linear partial differential equations(PDE) of fractional order in the sense of Caputo. The models studied are of the Kolmogorov-Fisher-Piskunov-Petrovsky(KFPP) type with fractional-order time derivatives. decisive role in facilitating the search for solutions to the various models.

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1. Introduction

In our work we are interested in solving equations modelling quite varied phenomena with applications in combustion theory, biology, ecology, physics, chemistry and many others. Models of the Kolmogorov-Fisher-Piskunov-Petrovsky type have been the subject of several works, in 2022 Maged Z. Youssef et al in [5] proved the generalized solution of these models using a coupled finite difference method and a variant of the spectral method known as the collocation algorithm. In 2019 Maud El-Hachem et al in [6] numerically interpreted by a diffusive extension of the dichotomy method. In 2019 K. M. Saad and al in [8] obtained by the spectral collocation method the numerical solutions of fractional Fisher's equations in the Atangana-Baleanu sense. In 2017 M. M. Al Qurashi et al in [11] used a new iterative algorithm to determine the residual series of fractional-order Fisher equations in time. On the basis of this, we interested in applying the SBA method to fractional-order edp models of the type Kolmogorov-Fisher-Piskunov-Petrovsky . This project is structured in four points: the second is entitled point is entitled preliminaries, the third point is a description of the SBA method in Caputo's of Caputo, the fourth is the application results of a few examples and finally a conclusion.

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2. Preliminaries

In this section we recall some definitions of fractional operators:

2.1. Derivatives in Caputo's sense

Definition 2.1.
the following formulas define the fractional right-hand and left-hand derivatives respectively and left derivatives respectively:

$$c_{D^{\alpha}_{d^+}} H(x) = \frac{1}{\Gamma(j-\alpha)} \int_d^x (x-u)^{j-\alpha-1} H^{(j)}(u) du,\ \alpha > 0 \tag{1}$$

$$c_{D^{\alpha}_{s^-}} H(x) = \frac{1}{\Gamma(j-\alpha)} \int_x^s (u-x)^{n-\alpha-1} H^{(j)}(u) du,\ \alpha > 0 \tag{2}$$

Where $j = [\alpha]+1$, $[\alpha]$ defines the integer part of α

2.2. Integral in the Riemann-Liouville sense

Definition 2.2.
If H is defined in $C([0,+\infty])$ then the integral of $\alpha > 0$ of the function p is defined by :

$$I^{\alpha}(H)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} H(u) du,\ t > 0 \tag{3}$$

$$I^0(H)(x) = H(x) \tag{4}$$

The above expression defines the left-hand integral in the Riemmann-Liouville sense.

3. Description of the SBA method in the Caputo sense

We describe this approach along the lines of the one described in [1, 3, 4] , and the solution is defined in a suitable space denoted J.

$$\begin{cases} D_t^{\alpha} u = R(u) + N(u),\ t \geq 0,\ \alpha > 0 \\ u^{(i)}(0) = w_i,\ i = 0,1,...,j-1 \end{cases} \tag{5}$$

With:
R and N are linear and non-linear operators respectively of J in J ;
$D_t^{\alpha} u$ defined as the fractional derivative of order α in the sense of Caputo;
$u \in B$ solution of the problem;
$n \in \mathbb{N}$.
Posing :

$$\begin{cases} L(.) = D_t^{\alpha} u \\ L^{-1}(.) = I^{\alpha}(.) \end{cases} \tag{6}$$

Where L^{-1} is invertible in the Adomian sense. .

So the canonical form associated with eq. (5) is:

$$L^{-1}Lu = L^{-1}R(u) + L^{-1}N(u) \tag{7}$$

$L^{-1}Lu(t) = u(t) - \sum_{i=0}^{j-1} \frac{u^{(i)}(0)}{i!} t^i,\ h = \sum_{i=0}^{j-1} \frac{u^{(i)}(0)}{i!} t^i$
$j = [\alpha]+1$, $[\alpha]$ designates the entire part of α
$L^{-1}Lu(t)$ is contracting.
The method of successive approximations applied to eq. (7) gives us :

$$u^k = h + L^{-1}R(u^k) + L^{-1}N(u^{k-1}),\ k \geq 1 \tag{8}$$

Applying the SBA algorithm to eq. (8) we have :

$$\begin{cases} u_0^k = h + L^{-1}N(u^{k-1}),\ k \geq 1 \\ u_n^k = L^{-1}R(u_{n-1}^k),\ n \geq 1 \end{cases} \tag{9}$$

Picard's principle:
let u^0 be any solution such that $N(u^0) = 0$
For k=1, calculate the approximate solution at the first iteration, u^1 defined by:

$$u^1 = \sum_{n=0}^{+\infty} u_n^1$$

Where u_n^1 is the generalization of $u_0^1, u_1^1, u_2^1 ... u_{n-1}^1$
For $k \geq 2$ we will first examine $N(u^1)$ as follows:

- if $N(u^1)$=0, then we can say that the analytical solution in step 1 is the solution to the given problem.
- if not, we are forced to improve the given problem or resort to other optimal theories for convergence.

In conclusion the exact solution u, if it exists we will have :

$$u = \lim_{k \longmapsto +\infty} u^k = \lim_{k \longmapsto +\infty} \left(\sum_{n=0}^{+\infty} u_n^1 \right)$$

4. Application results

Example 4.1.
Consider the Fisher equation of fractional order:

$$\begin{cases} D_t^\alpha g(x,t) - \dfrac{\partial^2 g(x,t)}{\partial x^2} - 6g(x,t)(1-g(x,t)) = f(x,t), t \geq 0, 0 \prec \alpha \leq 1 \\ g(x,0) = \dfrac{1}{(1+e^x)^2}, 0 \leq x \leq 1 \end{cases} \tag{10}$$

Where

$$f(x,t) = \frac{-\sqrt{2}b\left(18\sqrt{2}b - \sqrt{2}e^x(e^x-2)\right)}{(e^x+1)^4},\ b = E_\alpha(6t^\alpha)$$

Equation eq. (23) can be contracted as follows:

$$\left\{ L_t(g(x,t)) = R_1(g(x,t)) + N_1(g(x,t)) \right. \tag{11}$$

With:

$$\begin{cases} L_t(g(x,t)) = D_t^\alpha(g(x,t)) \\ L_t^{-1}(g(x,t)) = I^\alpha(g(x,t)) \\ R_1(g(x,t)) = 6g(x,t) \\ N_1(g(x,t)) = (-6g^2(x,t) + \dfrac{\partial^2 g(x,t)}{\partial x^2} + f(x,t)) \end{cases} \tag{12}$$

Composing eq. (12) by I^α we obtain :

$$g(x,t) = g(x,0) + L_t^{-1}(R_1(g)) + L_t^{-1}(N_1(g)) \tag{13}$$

The successive approximation applied to eq. (13) gives us :

$$g^k(x,t) = g^k(x,0) + L_t^{-1}(R_1(g^k)) + L_t^{-1}(N_1(g^{k-1})) \tag{14}$$

We have the following SBA algorithm:

$$\begin{cases} g_0^k(x,t) = g^k(x,0) + L_t^{-1}(N_1(g^{k-1})) \\ g_n^k(x,t) = L_t^{-1}(R_1(g_{n-1}^k)) \end{cases} \tag{15}$$

For k=1, SBA algorithm becomes :

$$\begin{cases} g_0^1(x,t) = g^1(x,0) = \dfrac{1}{(1+e^x)^2} \\ g_1^1(x,t) = L^{-1}(R_1(g_0^1)) \end{cases} \tag{16}$$

Determining the solution in step 1

$$g_1^1(x,t) = L_t^{-1}(R_1(g_0^1) = I^{\alpha}(6g_0^1) = \frac{6}{(1+e^x)^2\Gamma(\alpha+1)}t^{\alpha}$$

$$g_2^1(x,t) = I^{\alpha}(6g_1^1) = \frac{6^2}{(1+e^x)^2\Gamma(2\alpha+1)}t^{2\alpha}$$

$$g_3^1(x,t) = I^{\alpha}(6g_0^1) = \frac{6^3}{(1+e^x)^2\Gamma(3\alpha+1)}t^{3\alpha}$$

$$g_4^1(x,t) = \frac{6^4}{(1+e^x)^2\Gamma(4\alpha+1)}t^{4\alpha}$$

Step by step, we deduce:

$$\begin{cases} g_0^1(x,t) = v^1(x,0) = \dfrac{1}{(1+e^x)^2} \\ g_1^1(x,t) = \dfrac{6}{(1+e^x)^2\Gamma(\alpha+1)}t^{\alpha} \\ g_2^1(x,t) = \dfrac{6^2}{(1+e^x)^2\Gamma(2\alpha+1)}t^{2\alpha} \\ . \\ . \\ . \\ g_n^1(x,t) = \dfrac{6^n}{(1+e^x)^2\Gamma(n\alpha+1)}t^{n\alpha} \end{cases}$$

Then the solution of the first iteration is:

$$g^1 = \sum_{n=0}^{+\infty} \frac{6^n}{(1+e^x)^2\Gamma(n\alpha+1)}t^{\alpha}$$

$$g^1 = \frac{1}{(1+e^x)^2}\sum_{n=0}^{+\infty}\frac{[6^n t^{\alpha}]^n}{\Gamma(n\alpha+1)}$$

$$g^1 = \frac{1}{(1+e^x)^2}E_{\alpha}(6t^{\alpha})$$

For k=2, let's calculate $N(_1g^1)(b = E_{\alpha}(6t^{\alpha}))$

$$\begin{cases} N_1(g^1) = (-(6g^1)^2 + \dfrac{\partial^2 g^1}{\partial x^2} + f(x,t)) \\ = -\left[\dfrac{6b}{(1+e^x)^2}\right]^2 - \dfrac{-2be^x(e^x-2)}{(e^x+1)^4} - \dfrac{\sqrt{2}b\left(\sqrt{2}b - 2\sqrt{2}e^x(e^x-2)\right)}{2(e^x+1)^4} \\ = \dfrac{36b^2 - 2be^x(e^x-2)}{(e^x+1)^4} - \dfrac{\sqrt{2}b\left(36\sqrt{2}b - 2\sqrt{2}e^x(e^x-2)\right)}{2(e^x+1)^4} \\ = \dfrac{\sqrt{2}}{\sqrt{2}}\left[\dfrac{36b^2 - 2be^x(e^x-2)}{(e^x+1)^4}\right] - \dfrac{\sqrt{2}b\left(36\sqrt{2}b - 2\sqrt{2}e^x(e^x-2)\right)}{2(e^x+1)^4} \\ = \dfrac{\sqrt{2}b\left(18\sqrt{2}b - \sqrt{2}e^x(e^x-2)\right)}{(e^x+1)^4} - \dfrac{\sqrt{2}b\left(18\sqrt{2}b - \sqrt{2}e^x(e^x-2)\right)}{(e^x+1)^4} \\ = 0 \end{cases}$$

$N_1(g^1) = 0$, so the analytical solution of eq. (23) is :

$$g(x,t) = \frac{E_{\alpha}(6t^{\alpha})}{(1+e^x)^2}$$

Example 4.2.
Consider the nonlinear fractional equation of Kolmogorov-Piskunov-Petrovsky(KPP) :

$$\begin{cases} D_t^\alpha g(x,t) + \dfrac{\partial^2 g(x,t)}{\partial x^2} + 2e^g + 9c = f(x,t), t \geq 0, 0 < \alpha \leq 1 \\ g(x,0) = 4x(1-x), 0 \leq x \leq 1 \end{cases} \tag{17}$$

Where

$$f(x,t) = 4\sinh(4x(1-x)c) + 2(2x-1)\left[\frac{e^{-(4x(1-x)c)}}{2(2x-1)} + 2c2(x+1)\right],\ c = E_\alpha(t^\alpha)$$

Equation eq. (17) can be written as follows:

$$\begin{cases} D_t^\alpha g(x,t) = g(x,t) - \dfrac{\partial^2 g(x,t)}{\partial x^2} - 2e^g - 9c - g(x,t) + f(x,t) \end{cases} \tag{18}$$

Composing eq. (18) by I^α gives:

$$g(x,t) = g(x,0) + L_t^{-1}(R_2(g)) + L_t^{-1}(N_2(g)) \tag{19}$$

The successive approximation applied to eq. (19) gives us:

$$g^k(x,t) = g^k(x,0) + L_t^{-1}(R_2(g^k)) + L_t^{-1}(N_2(g^{k-1})) \tag{20}$$

Then the SBA algorithm is deduced:

$$\begin{cases} g_0^k(x,t) = g^k(x,0) + L_t^{-1}(N_2(g^{k-1})) \\ g_n^k(x,t) = L_t^{-1}(R_2(g_{n-1}^k)) \end{cases} \tag{21}$$

For k=1, the SBA algorithm SBA becomes:

$$\begin{cases} g_0^1(x,t) = g^1(x,0) = 4x(1-x) \\ g_1^1(x,t) = L^{-1}(R_1(g_0^1)) \end{cases} \tag{22}$$

Determining the solution in step 1

$$g_1^1(x,t) = L^{-1}(R_1(g_0^1) = I^\alpha(g_0^1) = \frac{4x(1-x)t^\alpha}{\Gamma(\alpha+1)}$$

$$g_2^1(x,t) = I^\alpha(g_1^1) = \frac{4x(1-x)t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$g_3^1(x,t) = I^\alpha(g_2^1) = \frac{4x(1-x)t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$g_4^1(x,t) = \frac{4x(1-x)t^{4\alpha}}{\Gamma(4\alpha+1)}$$

So we'll describe these different approximate solutions from step 1 as follows:

$$\begin{cases} g_0^1(x,t) = g^1(x,0) = 4x(1-x) \\ g_1^1(x,t) = \dfrac{4x(1-x)t^\alpha}{\Gamma(\alpha+1)} \\ g_2^1(x,t) = \dfrac{4x(1-x)t^{2\alpha}}{\Gamma(2\alpha+1)} \\ . \\ . \\ . \\ g_n^1(x,t) = \dfrac{4x(1-x)t^{n\alpha}}{n\Gamma(\alpha+1)} \end{cases}$$

So the solution to the first iteration is:

$$g^1 = \sum_{n=0}^{+\infty} \frac{4x(1-x)t^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$g^1 = 4x(1-x)E_\alpha(t^\alpha)$$
$$g^1 = 4x(1-x)c$$

Let's calulate $N_2(g^1)$

$$\begin{cases} N_2(g^1) = -\dfrac{\partial^2 g^1}{\partial x^2} - 2e^{g^1} - g^1 - 9c + f(x,t) \\ = 8c - 2e^{(4x(1-x)c)} - 4x(1-x)c - 9c + f(x,t) \\ = -2e^{(4x(1-x)c)} - 4c(-x^2 + x + 2) - c + f(x,t) \\ = -2e^{(4x(1-x)c)} + 2e^{-(4x(1-x)c)} - 4c(-x^2 + x + 2) - c - 2e^{-(4x(1-x)c)} + f(x,t) \\ = -4\sinh(4x(1-x)c) - 2(2x-1)\left[\dfrac{e^{-(4x(1-x)c)}}{2(2x-1)} + 2c(x+1)\right] + f(x,t) \\ = f(x,t) - f(x,t) \\ = 0 \end{cases}$$

Sence $N_2(g^1) = 0$, then the analytical solution of problem (eq. (17)) is:

$$g(x,t) = 4x(1-x)E_\alpha(t^\alpha)$$

Example 4.3.
Consider the nonlinear KPP equation of fractional order in time with Neperaen logarithme :

$$\begin{cases} D_t^\alpha g(x,t) - \dfrac{\partial^2 g(x,t)}{\partial x^2} + 5\ln(g+1) = f(x,t), t \geq 0, 0 < \alpha \leq 1 \\ g(x,0) = x(1-x), 0 \leq x \leq 1 \end{cases} \tag{23}$$

Where

$$f(x,t) = -\ln\left(\frac{e^{(-2-x+x^2)b}}{(x(1-x)b+1)^5}\right), b = E_\alpha(t^\alpha)$$

Following the process described in example 4.1, the solution in step 1 is obtained as follows:

$$g_1^1(x,t) = L^{-1}(R_1(g_0^1) = I^\alpha(\frac{\partial^2 g_0^1}{\partial x^2}) = \frac{x(1-x)t^\alpha}{\Gamma(\alpha+1)}$$

$$g_2^1(x,t) = I^\alpha(\frac{\partial^2 g_0^1}{\partial x^2}) = \frac{x(1-x)t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$g_3^1(x,t) = I^\alpha(\frac{\partial^2 g_0^1}{\partial x^2}) = \frac{x(1-x)t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$g_4^1(x,t) = \frac{x(1-x)t^{4\alpha}}{\Gamma(4\alpha+1)}$$

We have these different approximate solutions from stepe 1 Nous avons ces différentes solutions approchées de l'étape 1 comme suit:

$$\begin{cases} g_0^1(x,t) = g^1(x,0) = x(1-x) \\ g_1^1(x,t) = \dfrac{x(1-x)t^\alpha}{\Gamma(\alpha+1)} \\ g_2^1(x,t) = \dfrac{x(1-x)t^{2\alpha}}{\Gamma(2\alpha+1)} \\ . \\ . \\ . \\ g_n^1(x,t) = \dfrac{x(1-x)t^{n\alpha}}{\Gamma(n\alpha+1)} \end{cases}$$

The solution of the first iterationis :

$$g^1 = \sum_{n=0}^{+\infty} \frac{x(1-x)t^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$g^1 = x(1-x)E_\alpha(t^\alpha)$$

For k=2, let's calculate $N(g^1)(b = E_\alpha(t^\alpha))$

$$\begin{cases} N_3(g^1) = \dfrac{\partial^2 g^1}{\partial x^2} - 5\ln(g^1+1) - g^1 + f(x,t)) \\ = -2b - 5\ln(x(1-x)b+1) - x(1-x)b - \ln\left(\dfrac{e^{(-2-x+x^2)b}}{(x(1-x)b+1)^5}\right) \\ = (-2-x+x^2)b - 5\ln(x(1-x)b+1) - \ln\left(\dfrac{e^{(-2-x+x^2)b}}{(x(1-x)b+1)^5}\right) \\ = \ln\left(\dfrac{e^{(-2-x+x^2)b}}{(x(1-x)b+1)^5}\right) - \ln\left(\dfrac{e^{(-2-x+x^2)b}}{(x(1-x)b+1)^5}\right) \\ = 0 \end{cases}$$

The analytical solution of eq. (23) is :

$$g(x,t) = \frac{E_\alpha(6t^\alpha)}{(1+e^x)^2}$$

5. Conclusion

The resolution of strongly nonlinear fractional-order Kolmogorov-Fisher-Piskunov-Petrovsky models linear fractional order models has been a success, thanks to the approach used with the SBA algorithm. This approach enabled us to avoid tedious computations and obtain algorithms converging towards the analytical solution.

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