Int. J. Adv. Appl. Math. and Mech. 12(3) (2025) pages 1–72 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Sums and Generating Functions of Squares of Special Cases of Generalized Tribonacci Polynomials: Closed Formulas of $\sum_{k=0}^{n} z^k W_k^2$ and $\sum_{n=0}^{\infty} W_n^2 z^n$

Research Article

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Volume 12, Issue 03
Received: 06 November 2024
Accepted (in revised version): 04 December 2024

Abstract: In this paper, the closed forms of the sum formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1} W_k$ and $\sum_{k=0}^{n} z^k W_{k+2} W_k$ for the special cases of generalized Tribonacci polynomials are presented. We also present the closed forms of formulas of generating functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1} W_n z^n$ and $\sum_{n=0}^{\infty} W_{n+2} W_n z^n$.

MSC: 11B37 • 11B39 • 11B83

Keywords: Sums of squares • third order recurrence • Tribonacci polynomials • Padovan polynomials • Perrin polynomials • Narayana polynomials • Tribonacci numbers • Padovan numbers • Perrin numbers • Narayana numbers

1. Introduction

In this paper, we present sum formulas and generating functions for special cases of generalized Tribonacci polynomials, namely, generalized Tribonacci numbers, generalized third-order Pell numbers, generalized Padovan numbers, generalized Pell-Padovan numbers, generalized Jacobsthal-Padovan numbers, generalized Narayana numbers, generalized third order Jacobsthal numbers. Moreover, we evaluate the infinite sums of special cases of generalized Tribonacci numbers. First, we recall the definition of generalized Tribonacci polynomials.

The generalized Tribonacci polynomials (or generalized $(r(x), s(x), t(x))-$Tribonacci polynomials or x-Tribonacci numbers or generalized $(r(x), s(x), t(x))$-polynomials or 3-step Fibonacci polynomials)

$$\{W_n(W_0(x), W_1(x), W_2(x); r(x), s(x), t(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n(x) = r(x)W_{n-1}(x) + s(x)W_{n-2}(x) + t(x)W_{n-3}(x), \quad W_0(x) = a(x), W_1(x) = b(x), W_2(x) = c(x), \quad n \geq 3 \tag{1}$$

where $W_0(x), W_1(x), W_2(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$, $s(x)$ and $t(x)$ are polynomials with real coefficients and $t(x) \neq 0$.

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Special cases of this sequence has been studied by many authors. For some references on special cases of generalized Tribonacci polynomials, see for example [2, 4–6, 17].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = -\frac{s(x)}{t(x)}W_{-(n-1)}(x) - \frac{r(x)}{t(x)}W_{-(n-2)}(x) + \frac{1}{t(x)}W_{-(n-3)}(x)$$

for $n = 1,2,3,...$ when $t(x) \neq 0$. Therefore, recurrence eq. (1) holds for all integers n. Note that for $n \geq 1$, $W_{-n}(x)$ need not to be a polynomial in the ordinary sense.

If $r(x) = r$, $s(x) = s$, $t(x) = t$ are real or complex numbers then this polynomials are called as the generalized Tribonacci numbers (or generalized $(r,s,t)-$Tribonacci numbers or 3-step Fibonacci numbers).

Binet's formula of generalized Tribonacci polynomials, as $\{W_n\}$ is a third-order recurrence sequence (difference equation), can be calculated using its characteristic equation which is given as

$$z^3 - r(x)z^2 - s(x)z - t(x) = 0. \tag{2}$$

The roots of characteristic equation of $\{W_n\}$ will be denoted as $\alpha(x) = \alpha(x,r,s,t), \beta(x) = \beta(x,r,s,t), \gamma(x) = \gamma(x,r,s,t)$.

Remark 1.1.

For the sake of simplicity throughout the rest of the paper, we use

$$W_n, r, s, t, W_0, W_1, W_2, \alpha, \beta, \gamma,$$

instead of

$$W_n(x), r(x), s(x), t(x), W_0(x), W_1(x), W_2(x), \alpha(x), \beta(x), \gamma(x),$$

respectively, unless otherwise stated. For example, we write

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3$$

for the equation eq. (1).

2. Sum Formulas

In this paper, we will use the following sum formulas extensively which is given in [25].

Theorem 2.1.

[25, Theorem 2.1] Let z be a real or complex number (in fact z is a real or complex valued function in x). . Then

(a)

(i) *If* $\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) = -z^6t^4 + z^5t^2(s^2 - rt) + z^4t(r^2t - rs^2 - st) + z^3(r^3t - s^3 + 2t^2 + 4rst) + z^2(r^2s + s^2 + rt) + z(s + r^2) - 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k^2 = \frac{\Theta_{1W}(z)}{\Gamma(z)} \tag{3}$$

where

$\Theta_{1W}(z) = -z^{n+6}\Theta_1 - z^{n+5}\Theta_2 - z^{n+4}\Theta_3 + z^{n+3}\Theta_4 + z^{n+2}\Theta_5 + z^{n+1}\Theta_6 + z^5\Theta_7 + z^4\Theta_8 + z^3\Theta_9 + z^2\Theta_{10} + z\Theta_{11} + \Theta_{12}$
$= -z^{n+6}t^2(W_{n+3}^2 + r^2W_{n+2}^2 + s^2W_{n+1}^2 + 2(-rW_{n+2}W_{n+3} - sW_{n+1}W_{n+3} + rsW_{n+1}W_{n+2})) - z^{n+5}t(rW_{n+3}^2 + (r^3 + 2rs + t)W_{n+2}^2 + r(rt - s^2)W_{n+1}^2 + 2(-(s + r^2)W_{n+3} + (s^2 - tr)W_{n+1})W_{n+2}) - z^{n+4}(sW_{n+3}^2 + r(t + rs)W_{n+2}^2 + (r^3t - s^3 + t^2 + 4rst)W_{n+1}^2 + 2(-rsW_{n+2}W_{n+3} - stW_{n+1}W_{n+2} - rtW_{n+1}W_{n+3}))$
$+z^{n+3}(W_{n+3}^2 - (s + r^2)W_{n+2}^2 - (r^2s + rt + s^2)W_{n+1}^2) + z^{n+2}(W_{n+2}^2 - (s + r^2)W_{n+1}^2) + z^{n+1}W_{n+1}^2 + z^5t^2(-W_2 + rW_1 + sW_0)^2 + z^4t(rW_2^2 + (t + 2rs + r^3)W_1^2 + r(rt - s^2)W_0^2 - 2(s + r^2)W_1W_2 - 2(rt - s^2)W_0W_1) + z^3(sW_2^2 + r(t + rs)W_1^2 + (r^3t - s^3 + t^2 + 4rst)W_0^2 - 2rsW_1W_2 - 2rtW_0W_2 - 2stW_0W_1)$
$+z^2(-W_2^2 + (r^2 + s)W_1^2 + s(s + r^2)W_0^2 + rtW_0^2) + z(-W_1^2 + (r^2 + s)W_0^2) - W_0^2$

(ii) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^kW_k^2=\frac{\frac{d}{dz}\Theta_{1W}(z)}{\frac{d}{dz}\Gamma(z)} \tag{4}$$

where

$\frac{d}{dz}\Theta_{1W}(z)=-(n+6)z^{n+5}t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-(n+5)z^{n+4}t(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})-(n+4)z^{n+3}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))$
$+(n+3)z^{n+2}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)+(n+2)z^{n+1}(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n+1)z^nW_{n+1}^2+5z^4t^2(-W_2+rW_1+sW_0)^2+4z^3t(rW_2^2+(t+2rs+r^3)W_1^2+r(rt-s^2)W_0^2-2(s+r^2)W_1W_2-2(rt-s^2)W_0W_1)+3z^2(sW_2^2+r(t+rs)W_1^2+(r^3t-s^3+t^2+4rst)W_0^2-2rsW_1W_2-2rtW_0W_2-2stW_0W_1)$
$+2z(-W_2^2+(r^2+s)W_1^2+s(s+r^2)W_0^2+rtW_0^2)+(-W_1^2+(r^2+s)W_0^2)$

and

$\frac{d}{dz}\Gamma(z)=-6z^5t^4+5z^4t^2(s^2-rt)+4z^3t(r^2t-rs^2-st)+3z^2(r^3t-s^3+2t^2+4rst)+2z(r^2s+s^2+rt)+(s+r^2)$

(iii) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^2f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^kW_k^2=\frac{\frac{d^2}{dz^2}\Theta_{1W}(z)}{\frac{d^2}{dz^2}\Gamma(z)} \tag{5}$$

where

$\frac{d^2}{dz^2}\Theta_{1W}(z)=-(n+6)(n+5)z^{n+4}t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-(n+5)(n+4)z^{n+3}t(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})$
$-(n+4)(n+3)z^{n+2}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))+(n+3)(n+2)z^{n+1}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)+(n+2)(n+1)z^n(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n+1)nz^{n-1}W_{n+1}^2$
$+20z^3t^2(-W_2+rW_1+sW_0)^2+12z^2t(rW_2^2+(t+2rs+r^3)W_1^2+r(rt-s^2)W_0^2-2(s+r^2)W_1W_2-2(rt-s^2)W_0W_1)+6z(sW_2^2+r(t+rs)W_1^2+(r^3t-s^3+t^2+4rst)W_0^2-2rsW_1W_2-2rtW_0W_2-2stW_0W_1)+2(-W_2^2+(r^2+s)W_1^2+s(s+r^2)W_0^2+rtW_0^2)$

and

$\frac{d^2}{dz^2}\Gamma(z)=-30z^4t^4+20z^3t^2(s^2-rt)+12z^2t(r^2t-rs^2-st)+6z(r^3t-s^3+2t^2+4rst)+2(r^2s+s^2+rt)$

(iv) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^3f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^kW_k^2=\frac{\frac{d^3}{dz^3}\Theta_{1W}(z)}{\frac{d^3}{dz^3}\Gamma(z)} \tag{6}$$

$$=\frac{\frac{d^3}{dz^3}\Theta_{1W}(z)}{-120z^3t^4+60z^2t^2(s^2-rt)+24zt(r^2t-rs^2-st)+6(r^3t-s^3+2t^2+4rst)}$$

where

$\frac{d^3}{dz^3}\Theta_{1W}(z)=-(n+4)(n+5)(n+6)z^{n+3}t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-(n+3)(n+4)(n+5)z^{n+2}t(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})$
$-(n+2)(n+3)(n+4)z^{n+1}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))+(n+1)(n+2)(n+3)z^n(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)+n(n+1)(n+2)z^{n-1}(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n-1)n(n+1)z^{n-2}W_{n+1}^2$

$+60z^2t^2(-W_2+rW_1+sW_0)^2+24zt(rW_2^2+(t+2rs+r^3)W_1^2+r(rt-s^2)W_0^2-2(s+r^2)W_1W_2-2(rt-s^2)W_0W_1)+6(sW_2^2+r(t+rs)W_1^2+(r^3t-s^3+t^2+4rst)W_0^2-2rsW_1W_2-2rtW_0W_2-2stW_0W_1)$

and

$$\frac{d^3}{dz^3}\Gamma(z)=-120z^3t^4+60z^2t^2(s^2-rt)+24zt(r^2t-rs^2-st)+6(r^3t-s^3+2t^2+4rst)$$

(v) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^4f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n}z^kW_k^2=\frac{\frac{d^4}{dz^4}\Theta_{1W}(z)}{\frac{d^4}{dz^4}\Gamma(z)}=\frac{\frac{d^4}{dz^4}\Theta_{1W}(z)}{-360z^2t^4+120zt^2(s^2-rt)+24t(r^2t-rs^2-st)} \tag{7}$$

where

$\frac{d^4}{dz^4}\Theta_{1W}(z)=-(n+3)(n+4)(n+5)(n+6)z^{n+2}t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-(n+2)(n+3)(n+4)(n+5)z^{n+1}t(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})$
$-(n+1)(n+2)(n+3)(n+4)z^n(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))+n(n+1)(n+2)(n+3)z^{n-1}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)+(n-1)n(n+1)(n+2)z^{n-2}(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n-2)(n-1)n(n+1)z^{n-3}W_{n+1}^2$
$+120zt^2(-W_2+rW_1+sW_0)^2+24t(rW_2^2+(t+2rs+r^3)W_1^2+r(rt-s^2)W_0^2-2(s+r^2)W_1W_2-2(rt-s^2)W_0W_1)$

and

$$\frac{d^4}{dz^4}\Gamma(z)=-360z^2t^4+120zt^2(s^2-rt)+24t(r^2t-rs^2-st)$$

(vi) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^5f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n}z^kW_k^2=\frac{\frac{d^5}{dz^5}\Theta_{1W}(z)}{\frac{d^5}{dz^5}\Gamma(z)}=\frac{\frac{d^5}{dz^5}\Theta_{1W}(z)}{-720zt^4+120t^2(s^2-rt)} \tag{8}$$

where

$\frac{d^5}{dz^5}\Theta_{1W}(z)=-(n+2)(n+3)(n+4)(n+5)(n+6)z^{n+1}t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-(n+1)(n+2)(n+3)(n+4)(n+5)z^nt(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})$
$-n(n+1)(n+2)(n+3)(n+4)z^{n-1}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))+(n-1)n(n+1)(n+2)(n+3)z^{n-2}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)+(n-2)(n-1)n(n+1)(n+2)z^{n-3}(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n-3)(n-2)(n-1)n(n+1)z^{n-4}W_{n+1}^2$
$+120t^2(-W_2+rW_1+sW_0)^2$

and

$$\frac{d^5}{dz^5}\Gamma(z)=-720zt^4+120t^2(s^2-rt)$$

(vii) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^6=0$ *for some* $a_1\in\mathbb{C}$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n}z^kW_k^2=\frac{\frac{d^6}{dz^6}\Theta_{1W}(z)}{\frac{d^6}{dz^6}\Gamma(z)}=\frac{\frac{d^6}{dz^6}\Theta_{1W}(z)}{-720t^4} \tag{9}$$

where

$\frac{d^6}{dz^6}\Theta_{1W}(z)=-(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)z^n t^2(W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+2(-rW_{n+2}W_{n+3}-sW_{n+1}W_{n+3}+rsW_{n+1}W_{n+2}))-n(n+1)(n+2)(n+3)(n+4)(n+5)z^{n-1}t(rW_{n+3}^2+(r^3+2rs+t)W_{n+2}^2+r(rt-s^2)W_{n+1}^2+2(-(s+r^2)W_{n+3}+(s^2-tr)W_{n+1})W_{n+2})$
$-(n-1)n(n+1)(n+2)(n+3)(n+4)z^{n-2}(sW_{n+3}^2+r(t+rs)W_{n+2}^2+(r^3t-s^3+t^2+4rst)W_{n+1}^2+2(-rsW_{n+2}W_{n+3}-stW_{n+1}W_{n+2}-rtW_{n+1}W_{n+3}))+(n-2)(n-1)n(n+1)(n+2)(n+3)z^{n-3}(W_{n+3}^2-(s+r^2)W_{n+2}^2-(r^2s+rt+s^2)W_{n+1}^2)$
$+(n-3)(n-2)(n-1)n(n+1)(n+2)z^{n-4}(W_{n+2}^2-(s+r^2)W_{n+1}^2)+(n-4)(n-3)(n-2)(n-1)n(n+1)z^{n-5}W_{n+1}^2$

and

$\frac{d^6}{dz^6}\Gamma(z)=-720t^4$

(b)

(i) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)\neq 0$ *then*

$$\sum_{k=0}^{n} z^kW_{k+1}W_k=\frac{\Theta_{2W}(z)}{\Gamma(z)} \tag{10}$$

where

$\Theta_{2W}(z)=z^{n+6}\Theta_{13}+z^{n+5}\Theta_{14}+z^{n+4}\Theta_{15}+z^{n+3}\Theta_{16}+z^{n+2}\Theta_{17}+z^{n+1}\Theta_{18}+z^5\Theta_{19}+z^4\Theta_{20}+z^3\Theta_{21}+z^2\Theta_{22}+z\Theta_{23}+\Theta_{24}$
$=z^{n+6}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+z^{n+5}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+z^{n+4}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})+z^{n+3}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})$
$+z^{n+2}(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+z^{n+1}W_{n+1}W_{n+2}+z^5t^3(W_2-rW_1-sW_0)W_0+z^4t(W_2-rW_1-sW_0)(-sW_2+(rs+t)W_1)+z^3(-s(t+rs)W_1^2-rt^2W_0^2+s^2W_1W_2-r^2tW_0W_2+(r^3t-s^3+t^2+2rst)W_0W_1)+z^2(-rW_2^2+r^2W_1W_2-tW_0W_2+(r^2s+rt+s^2)W_0W_1)+z(-W_2+(r^2+s)W_0)W_1-W_0W_1$

(ii) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with and* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^kW_{k+1}W_k=\frac{\frac{d}{dz}\Theta_{2W}(z)}{\frac{d}{dz}\Gamma(z)} \tag{11}$$

where

$\frac{d}{dz}\Theta_{2W}(z)=(n+6)z^{n+5}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+(n+5)z^{n+4}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n+4)z^{n+3}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})+(n+3)z^{n+2}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})$
$+(n+2)z^{n+1}(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+(n+1)z^nW_{n+1}W_{n+2}+5z^4t^3(W_2-rW_1-sW_0)W_0+4z^3t(W_2-rW_1-sW_0)(-sW_2+(rs+t)W_1)+3z^2(-s(t+rs)W_1^2-rt^2W_0^2+s^2W_1W_2-r^2tW_0W_2+(r^3t-s^3+t^2+2rst)W_0W_1)+2z(-rW_2^2+r^2W_1W_2-tW_0W_2+(r^2s+rt+s^2)W_0W_1)+(-W_2+(r^2+s)W_0)W_1$

and

$\frac{d}{dz}\Gamma(z)=-6z^5t^4+5z^4t^2(s^2-rt)+4z^3t(r^2t-rs^2-st)+3z^2(r^3t-s^3+2t^2+4rst)+2z(r^2s+s^2+rt)+(s+r^2)$

(iii) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^2f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^kW_{k+1}W_k=\frac{\frac{d^2}{dz^2}\Theta_{2W}(z)}{\frac{d^2}{dz^2}\Gamma(z)} \tag{12}$$

where

$\frac{d^2}{dz^2}\Theta_{2W}(z)=(n+5)(n+6)z^{n+4}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+(n+4)(n+5)z^{n+3}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n+3)(n+4)z^{n+2}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})+(n+2)(n+3)z^{n+1}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})$

$+(n+1)(n+2)z^{n}(W_{n+3}-(s+r^{2})W_{n+1})W_{n+2}+n(n+1)z^{n-1}W_{n+1}W_{n+2}+20z^{3}t^{3}(W_{2}-rW_{1}-sW_{0})W_{0}+12z^{2}t(W_{2}-rW_{1}-sW_{0})(-sW_{2}+(rs+t)W_{1})+6z(-s(t+rs)W_{1}^{2}-rt^{2}W_{0}^{2}+s^{2}W_{1}W_{2}-r^{2}tW_{0}W_{2}+(r^{3}t-s^{3}+t^{2}+2rst)W_{0}W_{1})+2(-rW_{2}^{2}+r^{2}W_{1}W_{2}-tW_{0}W_{2}+(r^{2}s+rt+s^{2})W_{0}W_{1})$

and

$$\frac{d^{2}}{dz^{2}}\Gamma(z)=-30z^{4}t^{4}+20z^{3}t^{2}(s^{2}-rt)+12z^{2}t(r^{2}t-rs^{2}-st)+6z(r^{3}t-s^{3}+2t^{2}+4rst)+2(r^{2}s+s^{2}+rt)$$

(iv) *If* $\Gamma(z)=(-t^{2}z^{3}+sz+rtz^{2}+1)(r^{2}z-s^{2}z^{2}+t^{2}z^{3}+2sz+2rtz^{2}-1)=(z-a_{1})^{3}f(z)=0$ *for some* $a_{1}\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_{1})\neq 0$ *then, for* $z=a_{1}$*, we get*

$$\sum_{k=0}^{n}z^{k}W_{k+1}W_{k}=\frac{\frac{d^{3}}{dz^{3}}\Theta_{2W}(z)}{\frac{d^{3}}{dz^{3}}\Gamma(z)}=\frac{\frac{d^{3}}{dz^{3}}\Theta_{2W}(z)}{-120z^{3}t^{4}+60z^{2}t^{2}(s^{2}-rt)+24zt(r^{2}t-rs^{2}-st)+6(r^{3}t-s^{3}+2t^{2}+4rst)} \tag{13}$$

where

$\frac{d^{3}}{dz^{3}}\Theta_{2W}(z)=(n+4)(n+5)(n+6)z^{n+3}t^{3}(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+(n+3)(n+4)(n+5)z^{n+2}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n+2)(n+3)(n+4)z^{n+1}(s(t+rs)W_{n+2}^{2}+rt^{2}W_{n+1}^{2}-s^{2}W_{n+2}W_{n+3}+r^{2}tW_{n+1}W_{n+3}+(-r^{3}t+s^{3}-t^{2}-2rst)W_{n+2}W_{n+1})$
$+(n+1)(n+2)(n+3)z^{n}(rW_{n+3}^{2}-r^{2}W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^{2}s+rt+s^{2})W_{n+2}W_{n+1})+n(n+1)(n+2)z^{n-1}(W_{n+3}-(s+r^{2})W_{n+1})W_{n+2}+(n-1)n(n+1)z^{n-2}W_{n+1}W_{n+2}+60z^{2}t^{3}(W_{2}-rW_{1}-sW_{0})W_{0}+24zt(W_{2}-rW_{1}-sW_{0})(-sW_{2}+(rs+t)W_{1})+6(-s(t+rs)W_{1}^{2}-rt^{2}W_{0}^{2}+s^{2}W_{1}W_{2}-r^{2}tW_{0}W_{2}+(r^{3}t-s^{3}+t^{2}+2rst)W_{0}W_{1})$

and

$$\frac{d^{3}}{dz^{3}}\Gamma(z)=-120z^{3}t^{4}+60z^{2}t^{2}(s^{2}-rt)+24zt(r^{2}t-rs^{2}-st)+6(r^{3}t-s^{3}+2t^{2}+4rst)$$

(v) *If* $\Gamma(z)=(-t^{2}z^{3}+sz+rtz^{2}+1)(r^{2}z-s^{2}z^{2}+t^{2}z^{3}+2sz+2rtz^{2}-1)=(z-a_{1})^{4}f(z)=0$ *for some* $a_{1}\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_{1})\neq 0$ *then, for* $z=a_{1}$*, we get*

$$\sum_{k=0}^{n}z^{k}W_{k+1}W_{k}=\frac{\frac{d^{4}}{dz^{4}}\Theta_{2W}(z)}{\frac{d^{4}}{dz^{4}}\Gamma(z)}=\frac{\frac{d^{4}}{dz^{4}}\Theta_{2W}(z)}{-360z^{2}t^{4}+120zt^{2}(s^{2}-rt)+24t(r^{2}t-rs^{2}-st)} \tag{14}$$

where

$\frac{d^{4}}{dz^{4}}\Theta_{2W}(z)=(n+3)(n+4)(n+5)(n+6)z^{n+2}t^{3}(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+(n+2)(n+3)(n+4)(n+5)z^{n+1}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n+1)(n+2)(n+3)(n+4)z^{n}(s(t+rs)W_{n+2}^{2}+rt^{2}W_{n+1}^{2}-s^{2}W_{n+2}W_{n+3}+r^{2}tW_{n+1}W_{n+3}+(-r^{3}t+s^{3}-t^{2}-2rst)W_{n+2}W_{n+1})$
$+n(n+1)(n+2)(n+3)z^{n-1}(rW_{n+3}^{2}-r^{2}W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^{2}s+rt+s^{2})W_{n+2}W_{n+1})+(n-1)n(n+1)(n+2)z^{n-2}(W_{n+3}-(s+r^{2})W_{n+1})W_{n+2}+(n-2)(n-1)n(n+1)z^{n-3}W_{n+1}W_{n+2}+120zt^{3}(W_{2}-rW_{1}-sW_{0})W_{0}+24t(W_{2}-rW_{1}-sW_{0})(-sW_{2}+(rs+t)W_{1})$

and

$$\frac{d^{4}}{dz^{4}}\Gamma(z)=-360z^{2}t^{4}+120zt^{2}(s^{2}-rt)+24t(r^{2}t-rs^{2}-st)$$

(vi) *If* $\Gamma(z)=(-t^{2}z^{3}+sz+rtz^{2}+1)(r^{2}z-s^{2}z^{2}+t^{2}z^{3}+2sz+2rtz^{2}-1)=(z-a_{1})^{5}f(z)=0$ *for some* $a_{1}\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_{1})\neq 0$ *then, for* $z=a_{1}$*, we get*

$$\sum_{k=0}^{n}z^{k}W_{k+1}W_{k}=\frac{\frac{d^{5}}{dz^{5}}\Theta_{2W}(z)}{\frac{d^{5}}{dz^{5}}\Gamma(z)}=\frac{\frac{d^{5}}{dz^{5}}\Theta_{2W}(z)}{-720zt^{4}+120t^{2}(s^{2}-rt)} \tag{15}$$

where
$\frac{d^5}{dz^5}\Theta_{2W}(z) = (n+2)(n+3)(n+4)(n+5)(n+6)z^{n+1}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+(n+1)(n+2)(n+3)(n+4)(n+5)z^n t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+n(n+1)(n+2)(n+3)(n+4)z^{n-1}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})$
$+(n-1)n(n+1)(n+2)(n+3)z^{n-2}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})+(n-2)(n-1)n(n+1)(n+2)z^{n-3}(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+(n-3)(n-2)(n-1)n(n+1)z^{n-4}W_{n+1}W_{n+2}+120t^3(W_2-rW_1-sW_0)W_0$
and
$\frac{d^5}{dz^5}\Gamma(z) = -720zt^4+120t^2(s^2-rt)$

(vii) *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = (z-a_1)^6 = 0$ *for some* $a_1 \in \mathbb{C}$ *then, for* $z = a_1$, *we get*

$$\sum_{k=0}^{n} z^k W_{k+1}W_k = \frac{\frac{d^6}{dz^6}\Theta_{2W}(z)}{\frac{d^6}{dz^6}\Gamma(z)} = \frac{\frac{d^6}{dz^6}\Theta_{2W}(z)}{-720t^4} \tag{16}$$

where
$\frac{d^6}{dz^6}\Theta_{2W}(z) = (n+1)(n+2)(n+3)(n+4)(n+5)(n+6)z^n t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+1}+n(n+1)(n+2)(n+3)(n+4)(n+5)z^{n-1}t(-W_{n+3}+rW_{n+2}+sW_{n+1})(-sW_{n+3}+tW_{n+2}+rsW_{n+2})+(n-1)n(n+1)(n+2)(n+3)(n+4)z^{n-2}(s(t+rs)W_{n+2}^2+rt^2W_{n+1}^2-s^2W_{n+2}W_{n+3}+r^2tW_{n+1}W_{n+3}+(-r^3t+s^3-t^2-2rst)W_{n+2}W_{n+1})$
$+(n-2)(n-1)n(n+1)(n+2)(n+3)z^{n-3}(rW_{n+3}^2-r^2W_{n+2}W_{n+3}+tW_{n+1}W_{n+3}-(r^2s+rt+s^2)W_{n+2}W_{n+1})+(n-3)(n-2)(n-1)n(n+1)(n+2)z^{n-4}(W_{n+3}-(s+r^2)W_{n+1})W_{n+2}+(n-4)(n-3)(n-2)(n-1)n(n+1)z^{n-5}W_{n+1}W_{n+2}$
and
$\frac{d^6}{dz^6}\Gamma(z) = -720t^4$

(c)

(i) *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{k+2}W_k = \frac{\Theta_{3W}(z)}{\Gamma(z)} \tag{17}$$

where
$\Theta_{3W}(z) = z^{n+6}\Theta_{25}+z^{n+5}\Theta_{26}+z^{n+4}\Theta_{27}+z^{n+3}\Theta_{28}+z^{n+2}\Theta_{29}+z^{n+1}\Theta_{30}+z^5\Theta_{31}+z^4\Theta_{32}+z^3\Theta_{33}+z^2\Theta_{34}+z\Theta_{35}+\Theta_{36} = z^{n+6}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+z^{n+5}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})+z^{n+4}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})$
$+z^{n+3}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})+z^{n+2}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+z^{n+1}W_{n+1}W_{n+3}+z^5t^3(W_2-rW_1-sW_0)W_1+z^4t(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)$
$+z^3((s^2-rt)W_2^2-t^2(r^2+s)W_0^2+r\left(rt-s^2\right)W_1W_2+(t^2-s^3+2rst)W_0W_2-st(r^2+s)W_0W_1)+z^2(-(r^2+s)W_2^2+s(r^2+s)W_1^2+(r^3-t)W_1W_2+s(r^2+s)W_0W_2+t(r^2+s)W_0W_1)+z(-sW_1^2-rW_1W_2+(r^2+s)W_0W_2-tW_0W_1)-W_0W_2$

(ii) *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = (z-a_1)f(z) = 0$ *for some* $a_1 \in \mathbb{C}$ *and a function* f *in* z *with* $f(a_1) \neq 0$ *then, for* $z = a_1$, *we get*

$$\sum_{k=0}^{n} z^k W_{k+2}W_k = \frac{\frac{d}{dz}\Theta_{3W}(z)}{\frac{d}{dz}\Gamma(z)} \tag{18}$$

where

$\frac{d}{dz}\Theta_{3W}(z) = (n+6)z^{n+5}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+(n+5)z^{n+4}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})+(n+4)z^{n+3}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})$
$+(n+3)z^{n+2}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})+(n+2)z^{n+1}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+(n+1)z^nW_{n+1}W_{n+3}+5z^4t^3(W_2-rW_1-sW_0)W_1+4z^3t(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)$
$+3z^2((s^2-rt)W_2^2-t^2(r^2+s)W_0^2+r\left(rt-s^2\right)W_1W_2+(t^2-s^3+2rst)W_0W_2-st(r^2+s)W_0W_1)+2z(-(r^2+s)W_2^2+s(r^2+s)W_1^2+(r^3-t)W_1W_2+s(r^2+s)W_0W_2+t(r^2+s)W_0W_1)+(-sW_1^2-rW_1W_2+(r^2+s)W_0W_2-tW_0W_1)$

and

$\frac{d}{dz}\Gamma(z) = -6z^5t^4+5z^4t^2(s^2-rt)+4z^3t(r^2t-rs^2-st)+3z^2(r^3t-s^3+2t^2+4rst)+2z(r^2s+s^2+rt)+(s+r^2)$

(iii) *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = (z-a_1)^2f(z) = 0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^kW_{k+2}W_k = \frac{\frac{d^2}{dz^2}\Theta_{3W}(z)}{\frac{d^2}{dz^2}\Gamma(z)} \tag{19}$$

where

$\frac{d^2}{dz^2}\Theta_{3W}(z) = (n+5)(n+6)z^{n+4}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+(n+4)(n+5)z^{n+3}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})+(n+3)(n+4)z^{n+2}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})$
$+(n+2)(n+3)z^{n+1}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})+(n+1)(n+2)z^n(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+n(n+1)z^{n-1}W_{n+1}W_{n+3}+20z^3t^3(W_2-rW_1-sW_0)W_1+12z^2t(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)$
$+6z((s^2-rt)W_2^2-t^2(r^2+s)W_0^2+r\left(rt-s^2\right)W_1W_2+(t^2-s^3+2rst)W_0W_2-st(r^2+s)W_0W_1)+2(-(r^2+s)W_2^2+s(r^2+s)W_1^2+(r^3-t)W_1W_2+s(r^2+s)W_0W_2+t(r^2+s)W_0W_1)$

and

$\frac{d^2}{dz^2}\Gamma(z) = -30z^4t^4+20z^3t^2(s^2-rt)+12z^2t(r^2t-rs^2-st)+6z(r^3t-s^3+2t^2+4rst)+2(r^2s+s^2+rt)$

(iv) *If* $\Gamma(z) = (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) = (z-a_1)^3f(z) = 0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^kW_{k+2}W_k = \frac{\frac{d^3}{dz^3}\Theta_{3W}(z)}{\frac{d^3}{dz^3}\Gamma(z)} = \frac{\frac{d^3}{dz^3}\Theta_{3W}(z)}{-120z^3t^4+60z^2t^2(s^2-rt)+24zt(r^2t-rs^2-st)+6(r^3t-s^3+2t^2+4rst)} \tag{20}$$

where

$\frac{d^3}{dz^3}\Theta_{3W}(z) = (n+4)(n+5)(n+6)z^{n+3}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+(n+3)(n+4)(n+5)z^{n+2}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})+(n+2)(n+3)(n+4)z^{n+1}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})$
$+(n+1)(n+2)(n+3)z^n((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})+n(n+1)(n+2)z^{n-1}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+(n-1)n(n+1)z^{n-2}W_{n+1}W_{n+3}$
$+60z^2t^3(W_2-rW_1-sW_0)W_1+24zt(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)+6((s^2-rt)W_2^2-t^2(r^2+s)W_0^2+r\left(rt-s^2\right)W_1W_2+(t^2-s^3+2rst)W_0W_2-st(r^2+s)W_0W_1)$

and

$\frac{d^3}{dz^3}\Gamma(z) = -120z^3t^4+60z^2t^2(s^2-rt)+24zt(r^2t-rs^2-st)+6(r^3t-s^3+2t^2+4rst)$

(v) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^4f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_{k+2}W_k=\frac{\frac{d^4}{dz^4}\Theta_{3W}(z)}{\frac{d^4}{dz^4}\Gamma(z)}=\frac{\frac{d^4}{dz^4}\Theta_{3W}(z)}{-360z^2t^4+120zt^2(s^2-rt)+24t(r^2t-rs^2-st)} \tag{21}$$

where

$\frac{d^4}{dz^4}\Theta_{3W}(z)=(n+3)(n+4)(n+5)(n+6)z^{n+2}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+(n+2)(n+3)(n+4)(n+5)z^{n+1}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})$
$+(n+1)(n+2)(n+3)(n+4)z^n((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})+n(n+1)(n+2)(n+3)z^{n-1}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})$
$+(n-1)n(n+1)(n+2)z^{n-2}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+(n-2)(n-1)n(n+1)z^{n-3}W_{n+1}W_{n+3}+120zt^3(W_2-rW_1-sW_0)W_1+24t(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)$

and

$\frac{d^4}{dz^4}\Gamma(z)=-360z^2t^4+120zt^2(s^2-rt)+24t(r^2t-rs^2-st)$

(vi) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^5f(z)=0$ *for some* $a_1\in\mathbb{C}$ *and a function* f *in* z *with* $f(a_1)\neq 0$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_{k+2}W_k=\frac{\frac{d^5}{dz^5}\Theta_{3W}(z)}{\frac{d^5}{dz^5}\Gamma(z)}=\frac{\frac{d^5}{dz^5}\Theta_{3W}(z)}{-720zt^4+120t^2(s^2-rt)} \tag{22}$$

where

$\frac{d^5}{dz^5}\Theta_{3W}(z)=(n+2)(n+3)(n+4)(n+5)(n+6)z^{n+1}t^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+(n+1)(n+2)(n+3)(n+4)(n+5)z^nt(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})$
$+n(n+1)(n+2)(n+3)(n+4)z^{n-1}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})+(n-1)n(n+1)(n+2)(n+3)z^{n-2}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})$
$+(n-2)(n-1)n(n+1)(n+2)z^{n-3}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+(n-3)(n-2)(n-1)n(n+1)z^{n-4}W_{n+1}W_{n+3}+120t^3(W_2-rW_1-sW_0)W_1$

and

$\frac{d^5}{dz^5}\Gamma(z)=-720zt^4+120t^2(s^2-rt)$

(vii) *If* $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=(z-a_1)^6=0$ *for some* $a_1\in\mathbb{C}$ *then, for* $z=a_1$*, we get*

$$\sum_{k=0}^{n} z^k W_{k+2}W_k=\frac{\frac{d^6}{dz^6}\Theta_{3W}(z)}{\frac{d^6}{dz^6}\Gamma(z)}=\frac{\frac{d^6}{dz^6}\Theta_{3W}(z)}{-720t^4} \tag{23}$$

where

$\frac{d^6}{dz^6}\Theta_{3W}(z)=(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)z^nt^3(-W_{n+3}+rW_{n+2}+sW_{n+1})W_{n+2}+n(n+1)(n+2)(n+3)(n+4)(n+5)z^{n-1}t(r(rt-s^2)W_{n+2}^2+t(rt-s^2)W_{n+1}^2+(s^2-rt)W_{n+2}W_{n+3}-(s^3+t^2)W_{n+1}W_{n+2}+stW_{n+1}W_{n+3})$

$+(n-1)n(n+1)(n+2)(n+3)(n+4)z^{n-2}((rt-s^2)W_{n+3}^2+t^2(r^2+s)W_{n+1}^2+r(s^2-rt)W_{n+2}W_{n+3}+(s^3-2rst-t^2)W_{n+1}W_{n+3}+st(r^2+s)W_{n+2}W_{n+1})+(n-2)(n-1)n(n+1)(n+2)(n+3)z^{n-3}((r^2+s)W_{n+3}^2-s(r^2+s)W_{n+2}^2+(t-r^3)W_{n+2}W_{n+3}-s(r^2+s)W_{n+3}W_{n+1}-t(r^2+s)W_{n+2}W_{n+1})$
$+(n-3)(n-2)(n-1)n(n+1)(n+2)z^{n-4}(sW_{n+2}^2+rW_{n+2}W_{n+3}-(r^2+s)W_{n+1}W_{n+3}+tW_{n+1}W_{n+2})+(n-4)(n-3)(n-2)(n-1)n(n+1)z^{n-5}W_{n+1}W_{n+3}$

and

$$\frac{d^6}{dz^6}\Gamma(z)=-720t^4$$

Remark 2.1.
According to roots of $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=0$, the sum formulas $\sum_{k=0}^n z^kW_k^2$, $\sum_{k=0}^n z^kW_{k+1}W_k$ and $\sum_{k=0}^n z^kW_{k+2}W_k$ can be evaluated by using theorem 2.1. For example,

- If $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=u(z-a_1)(z-a_2)(z-a_3)(z-a_4)(z-a_5)(z-a_6)=0$ for some $u,a_1,a_2,a_3,a_4,a_5,a_6\in\mathbb{C}$ with $u\neq 0$ and $a_1\neq a_2\neq a_3\neq a_4\neq a_5\neq a_6$, i.e., $z=a_1$ or $z=a_2$ or $z=a_3$ or $z=a_4$ or $z=a_5$ or $z=a_6$ then we use eq. (4) in (a)(ii), eq. (11) in (b)(ii) and eq. (18) in (c)(ii) to calculate $\sum_{k=0}^n z^kW_k^2$, $\sum_{k=0}^n z^kW_{k+1}W_k$ and $\sum_{k=0}^n z^kW_{k+2}W_k$, respectively.
- If $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=u(z-a_1)^3(z-a_2)^2(z-a_3)=0$ for some $u,a_1,a_2,a_3\in\mathbb{C}$ with $u\neq 0$ and $a_1\neq a_2\neq a_3$, i.e., $z=a_1$ or $z=a_2$ or $z=a_3$

  then

  - if $z=a_1$ then we use eq. (6) in (a)(iv), eq. (13) in (b)(iv) and eq. (20) in (c)(iv) to calculate $\sum_{k=0}^n z^kW_k^2$, $\sum_{k=0}^n z^kW_{k+1}W_k$ and $\sum_{k=0}^n z^kW_{k+2}W_k$, respectively,
  - if $z=a_2$ then we use eq. (5) in (a)(iii), eq. (12) in (b)(iii) and eq. (19) in (c)(iii) to calculate $\sum_{k=0}^n z^kW_k^2$, $\sum_{k=0}^n z^kW_{k+1}W_k$ and $\sum_{k=0}^n z^kW_{k+2}W_k$, respectively,
  - if $z=a_3$ then we use eq. (4) in (a)(ii), eq. (11) in (b)(ii) and eq. (18) in (c)(ii) to calculate $\sum_{k=0}^n z^kW_k^2$, $\sum_{k=0}^n z^kW_{k+1}W_k$ and $\sum_{k=0}^n z^kW_{k+2}W_k$, respectively.
- If $\Gamma(z)=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)=u(z-a_1)^4(z-a_2)^2=0$ for some $u,a_1,a_2\in\mathbb{C}$ with $u\neq 0$ and $a_1\neq a_2$, i.e., $z=a_1$ or $z=a_2$

  then

  - if $z=a_1$ then we use eq. (7) in (a)(v), eq. (14) in (b)(v) and eq. (21) in (c)(v) to calculate $\sum_{k=0}^n z^kW_k^2$, $\sum_{k=0}^n z^kW_{k+1}W_k$ and $\sum_{k=0}^n z^kW_{k+2}W_k$, respectively,
  - if $z=a_2$ then we use eq. (5) in (a)(iii), eq. (12) in (b)(iii) and eq. (19) in (c)(iii) to calculate $\sum_{k=0}^n z^kW_k^2$, $\sum_{k=0}^n z^kW_{k+1}W_k$ and $\sum_{k=0}^n z^kW_{k+2}W_k$, respectively,

3. Generating Functions

In this section, we present the closed forms of formulas of generating functions $\sum_{n=0}^\infty W_n^2z^n$, $\sum_{n=0}^\infty W_{n+1}W_nz^n$ and $\sum_{n=0}^\infty W_{n+2}W_nz^n$ for the generalized Tribonacci polynomials.

Theorem 3.1.
[25, Theorem 3.1] Assume that $|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2},|\alpha\beta|^{-1},|\alpha\gamma|^{-1},|\beta\gamma|^{-1}\}$. Then

(a) *The ordinary generating function $\sum_{n=0}^\infty W_n^2z^n$ of the sequence $\{W_n^2\}$ is given by*

$$\sum_{n=0}^{\infty}W_n^2z^n=\frac{\Psi_1(z)}{(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)}$$

where

$\Psi_1(z)=z^5\Theta_7+z^4\Theta_8+z^3\Theta_9+z^2\Theta_{10}+z\Theta_{11}+\Theta_{12}$
$=z^5t^2(-W_2+rW_1+sW_0)^2+z^4t(rW_2^2+(t+2rs+r^3)W_1^2+r(rt-s^2)W_0^2-2(s+r^2)W_1W_2-2(rt-s^2)W_0W_1)+z^3(sW_2^2+r(t+rs)W_1^2+(r^3t-s^3+t^2+4rst)W_0^2-2rsW_1W_2-2rtW_0W_2-2stW_0W_1)+z^2(-W_2^2+(r^2+s)W_1^2+s(s+r^2)W_0^2+rtW_0^2)+z(-W_1^2+(r^2+s)W_0^2)-W_0^2$

(b) *The ordinary generating function* $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$ *of the sequence* $\{W_{n+1}W_n\}$ *is given by*

$$\sum_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{\Psi_2(z)}{(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)}$$

where

$$\Psi_2(z) = z^5\Theta_{19}+z^4\Theta_{20}+z^3\Theta_{21}+z^2\Theta_{22}+z\Theta_{23}+\Theta_{24}$$
$$= z^5t^3(W_2-rW_1-sW_0)W_0+z^4t(W_2-rW_1-sW_0)(-sW_2+(rs+t)W_1)+z^3(-s(t+rs)W_1^2-rt^2W_0^2+s^2W_1W_2-r^2tW_0W_2+(r^3t-s^3+t^2+2rst)W_0W_1)+z^2(-rW_2^2+r^2W_1W_2-tW_0W_2+(r^2s+rt+s^2)W_0W_1)+z(-W_2+(r^2+s)W_0)W_1-W_0W_1$$

(c) *The ordinary generating function* $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ *of the sequence* $\{W_{n+2}W_n\}$ *is given by*

$$\sum_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{\Psi_3(z)}{(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)}$$

where

$$\Psi_3(z) = z^5\Theta_{31}+z^4\Theta_{32}+z^3\Theta_{33}+z^2\Theta_{34}+z\Theta_{35}+\Theta_{36}$$
$$= z^5t^3(W_2-rW_1-sW_0)W_1+z^4t(r(s^2-rt)W_1^2+tW_0^2(s^2-rt)+(rt-s^2)W_1W_2-stW_0W_2+(s^3+t^2)W_0W_1)+z^3((s^2-rt)W_2^2-t^2(r^2+s)W_0^2+r\left(rt-s^2\right)W_1W_2+(t^2-s^3+2rst)W_0W_2-st(r^2+s)W_0W_1)+z^2(-(r^2+s)W_2^2+s(r^2+s)W_1^2+(r^3-t)W_1W_2+s(r^2+s)W_0W_2+t(r^2+s)W_0W_1)+z(-sW_1^2-rW_1W_2+(r^2+s)W_0W_2-tW_0W_1)-W_0W_2$$

4. The Sum Formulas $\sum_{k=0}^{n} z^kW_k^2, \sum_{k=0}^{n} z^kW_{k+1}W_k, \sum_{k=0}^{n} z^kW_{k+2}W_k$ and Generating Functions of Special Cases of Generalized Fibonacci Polynomials/Numbers: First Group

In this section, we present sum formulas and generating functions for special cases of generalized Tribonacci polynomials, namely, generalized Tribonacci numbers, generalized third-order Pell numbers, generalized Padovan numbers, generalized Pell-Padovan numbers, generalized Jacobsthal-Padovan numbers, generalized Narayana numbers, generalized third order Jacobsthal numbers. Moreover, we evaluate the infinite sums of special cases of generalized Tribonacci numbers.

4.1. Sum Formulas $\sum_{k=0}^{n} z^kW_k^2$, $\sum_{k=0}^{n} z^kW_{k+1}W_k$, $\sum_{k=0}^{n} z^kW_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_nz^n$, $\sum_{n=0}^{\infty} W_{n+2}W_nz^n$ of Generalized Tribonacci Numbers

In this subsection, we consider the case $r=1$, $s=1$ and $t=1$. A generalized Tribonacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-1}+W_{n-2}+W_{n-3} \tag{24}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)}-W_{-(n-2)}+W_{-(n-3)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence eq. (24) holds for all integer n. For more information on generalized Tribonacci numbers, see Soykan [21].

Binet formula of generalized Tribonacci numbers can be given as

$$W_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{25}$$

where

$$b_1 = W_2-(\beta+\gamma)W_1+\beta\gamma W_0, \tag{26}$$
$$b_2 = W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0, \tag{27}$$
$$b_3 = W_2-(\alpha+\beta)W_1+\alpha\beta W_0. \tag{28}$$

Here, α,β and γ are the roots of the cubic equation

$$x^3-x^2-x-1=0.$$

Moreover

$$\alpha = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},$$
$$\beta = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$
$$\gamma = \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3},$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Two special cases of the sequence $\{W_n\}$ are the well known Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas (Tribonacci-Lucas-Lucas) sequence $\{K_n\}_{n\geq 0}$. Tribonacci sequence$\{T_n\}_{n\geq 0}$, Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad T_0 = 0, T_1 = 1, T_2 = 1, \tag{29}$$
$$K_{n+3} = K_{n+2} + K_{n+1} + K_n, \qquad K_0 = 3, K_1 = 1, K_2 = 3, \tag{30}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$, can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}, \tag{31}$$
$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}, \tag{32}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences eqs. (29) and (30) hold for all integer n.

For all integers n, Tribonacci and Tribonacci-Lucas numbers can be expressed using Binet's formulas as

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$
$$K_n = \alpha^n + \beta^n + \gamma^n,$$

respectively. Here, $G_n = T_n$ and $H_n = K_n$.

Next, we present sum formulas of generalized Tribonacci numbers.

Theorem 4.1.
For $n \geq 0$, we have the following sum formulas for generalized Tribonacci numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{4}(-W_{n+3}^2 - 4W_{n+2}^2 - 5W_{n+1}^2 + 4W_{n+2}W_{n+3} + 2W_{n+1}W_{n+3} + W_2^2 + 4W_1^2 + 5W_0^2 - 4W_1W_2 - 2W_0W_2).$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{4}(W_{n+3}^2 + 2W_{n+2}^2 + W_{n+1}^2 - 2W_{n+2}W_{n+3} - 2W_{n+1}W_{n+2} - W_2^2 - 2W_1^2 - W_0^2 + 2W_1W_2 + 2W_0W_1).$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{4}(W_{n+3}^2 + W_{n+1}^2 - 2W_{n+1}W_{n+3} - W_2^2 - W_0^2 + 2W_0W_2).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - x^2 - x - 1 = 0$ whose roots are α, β, γ with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = s = t = 1$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)\\ &= (-z^3 + z^2 + z + 1)(z^3 + z^2 + 3z - 1)\end{aligned}$$

and $\Gamma(1) \neq 0$.

(a) Use theorem 2.1 (a) (i) with $z = 1$.

(b) Use theorem 2.1 (b) (i) with $z = 1$.

(c) Use theorem 2.1 (c) (i) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 4.1.
For $n \geq 0$, Tribonacci numbers have the following properties.

(a) $\sum_{k=0}^{n} T_k^2 = \frac{1}{4}(-T_{n+3}^2 - 4T_{n+2}^2 - 5T_{n+1}^2 + 4T_{n+2}T_{n+3} + 2T_{n+1}T_{n+3} + 1).$

(b) $\sum_{k=0}^{n} T_{k+1}T_k = \frac{1}{4}(T_{n+3}^2 + 2T_{n+2}^2 + T_{n+1}^2 - 2T_{n+2}T_{n+3} - 2T_{n+1}T_{n+2} - 1).$

(c) $\sum_{k=0}^{n} T_{k+2}T_k = \frac{1}{4}(T_{n+3}^2 + T_{n+1}^2 - 2T_{n+1}T_{n+3} - 1).$

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last Theorem, we have the following Corollary which gives sum formulas of Tribonacci-Lucas numbers.

Corollary 4.2.
For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} K_k^2 = \frac{1}{4}(-K_{n+3}^2 - 4K_{n+2}^2 - 5K_{n+1}^2 + 4K_{n+2}K_{n+3} + 2K_{n+1}K_{n+3} + 28).$

(b) $\sum_{k=0}^{n} K_{k+1}K_k = \frac{1}{4}(K_{n+3}^2 + 2K_{n+2}^2 + K_{n+1}^2 - 2K_{n+2}K_{n+3} - 2K_{n+1}K_{n+2} - 8).$

(c) $\sum_{k=0}^{n} K_{k+2}K_k = \frac{1}{4}(K_{n+3}^2 + K_{n+1}^2 - 2K_{n+1}K_{n+3}).$

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 4.2.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} \simeq 0.295597$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum_{n=0}^{\infty} W_n^2 z^n = \frac{1}{z^6 + z^4 - 6z^3 - 3z^2 - 2z + 1}(-z^2(z^3 + z^2 + z - 1)W_2^2 - z(z^4 + 4z^3 + 2z^2 + 2z - 1)W_1^2 - (z^5 + 5z^3 + 3z^2 + 2z - 1)W_0^2 + 2z^3(z+1)^2 W_2W_1 + 2z^3(z^2+1)W_2W_0 - 2z^3(z^2-1)W_0W_1).$

(b) $\sum_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{z^6 + z^4 - 6z^3 - 3z^2 - 2z + 1}(z^2(z^2+1)W_2^2 + 2z^3(z+1)W_1^2 + z^3(z^2+1)W_0^2 - z(3z^3 + z^2 + z - 1)W_2W_1 - z^2(z-1)(z+1)^2 W_2W_0 + (z^5 + 2z^4 - 3z^3 - 3z^2 - 2z + 1)W_1W_0).$

(c) $\sum_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{z^6 + z^4 - 6z^3 - 3z^2 - 2z + 1}(2z^2W_2^2 + z(z^4 - 2z + 1)W_1^2 + 2z^3W_0^2 - z(z^4 - 1)W_2W_1 + (z^4 - 2z^3 - 2z^2 - 2z + 1)W_2W_0 + (z^5 - 2z^4 + 2z^3 - 2z^2 + z)W_1W_0).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 4.3.
Assume that $|z| < |\alpha|^{-2} \simeq 0.295597$. The ordinary generating functions of the sequences $\{T_n^2\}$, $\{T_{n+1}T_n\}$, $\{T_{n+2}T_n\}$ and $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ are given as follows:

(a)

$$\sum_{n=0}^{\infty} T_n^2 z^n = \frac{-z^4 - z^3 - z^2 + z}{z^6 + z^4 - 6z^3 - 3z^2 - 2z + 1},$$
$$\sum_{n=0}^{\infty} K_n^2 z^n = \frac{-z^5 - z^4 - 26z^3 - 20z^2 - 17z + 9}{z^6 + z^4 - 6z^3 - 3z^2 - 2z + 1}.$$

(b)

$$\sum_{n=0}^{\infty} T_{n+1}T_n z^n = \frac{z^3+z}{z^6+z^4-6z^3-3z^2-2z+1},$$
$$\sum_{n=0}^{\infty} K_{n+1}K_n z^n = \frac{3z^5-z^4+8z^3+6z^2-3z+3}{z^6+z^4-6z^3-3z^2-2z+1}.$$

(c)

$$\sum_{n=0}^{\infty} T_{n+2}T_n z^n = \frac{2z}{z^6+z^4-6z^3-3z^2-2z+1},$$
$$\sum_{n=0}^{\infty} K_{n+2}K_n z^n = \frac{z^5+3z^4+6z^3-8z^2-11z+9}{z^6+z^4-6z^3-3z^2-2z+1}.$$

From the last corollary, we obtain the following results for Tribonacci and Tribonacci-Lucas numbers.

Corollary 4.4.
Some infinite sums of $\{T_n^2\}$, $\{T_{n+1}T_n\}$, $\{T_{n+2}T_n\}$ *and* $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ *are given as follows:*

(a) $z=\frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{T_n^2}{4^n} = \frac{688}{913},$$
$$\sum_{n=0}^{\infty} \frac{K_n^2}{4^n} = \frac{12652}{913}.$$

(b) $z=\frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{T_{n+1}T_n}{4^n} = \frac{1088}{913},$$
$$\sum_{n=0}^{\infty} \frac{K_{n+1}K_n}{4^n} = \frac{11260}{913}.$$

(c) $z=\frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{T_{n+2}T_n}{4^n} = \frac{2048}{913},$$
$$\sum_{n=0}^{\infty} \frac{K_{n+2}K_n}{4^n} = \frac{23988}{913}.$$

4.2. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Third-Order Pell Numbers

In this subsection, we consider the case $r=2$, $s=1$ and $t=1$. A generalized third order Pell sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 2W_{n-1} + W_{n-2} + W_{n-3} \tag{33}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} - 2W_{-(n-2)} + W_{-(n-3)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence eq. (24) holds for all integer n. For more information on generalized third order Pell numbers, see Soykan [23].

Binet formula of generalized third order Pell numbers can be given as

$$W_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0,\ b_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0,\ b_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0. \tag{34}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - 2x^2 - x - 1 = 0$. Moreover

$$\alpha = \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}$$
$$\beta = \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}$$
$$\gamma = \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Now, we define two special cases of the sequence $\{W_n\}$. Third-order Pell sequence$\{P_n\}_{n\geq 0}$ and third-order Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$P_{n+3} = 2P_{n+2} + P_{n+1} + P_n, \quad P_0 = 0, P_1 = 1, P_2 = 2, \tag{35}$$
$$Q_{n+3} = 2Q_{n+2} + Q_{n+1} + Q_n, \quad Q_0 = 3, Q_1 = 2, Q_2 = 6 \tag{36}$$

The sequences $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} - 2P_{-(n-2)} + P_{-(n-3)}$$
$$Q_{-n} = -Q_{-(n-1)} - 2Q_{-(n-2)} + Q_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences eq. (35) and eq. (36) hold for all integer n.

Note that P_n is the sequence A077939 in [8] associated with the expansion of $1/(1-2x-x^2-x^3)$, Q_n is the sequence A276225 in [8].

For all integers n, third-order Pell and Pell-Lucas numbers can be expressed using Binet's formulas as

$$P_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$
$$Q_n = \alpha^n + \beta^n + \gamma^n,$$

respectively.

Next, we present sum formulas of generalized third-order Pell numbers.

Theorem 4.3.
For $n \geq 0$, we have the following sum formulas for generalized third-order Pell numbers:

(a) $\sum\limits_{k=0}^{n} W_k^2 = \frac{1}{9}(-W_{n+3}^2 - 9W_{n+2}^2 - 10W_{n+1}^2 + 6W_{n+2}W_{n+3} + 2W_{n+1}W_{n+3} + W_2^2 + 9W_1^2 + 10W_0^2 - 6W_1W_2 - 2W_0W_2).$

(b) $\sum\limits_{k=0}^{n} W_{k+1}W_k = \frac{1}{9}(W_{n+3}^2 + 3W_{n+2}^2 + W_{n+1}^2 - 6W_{n+1}W_{n+2} + W_{n+1}W_{n+3} - 3W_{n+2}W_{n+3} - W_2^2 - 3W_1^2 - W_0^2 + 3W_1W_2 - W_0W_2 + 6W_0W_1).$

(c) $\sum\limits_{k=0}^{n} W_{k+2}W_k = \frac{1}{9}(2W_{n+3}^2 + 2W_{n+1}^2 - 3W_{n+2}W_{n+3} - 4W_{n+1}W_{n+3} - 2W_2^2 + 3W_1W_2 - 2W_0^2 + 4W_0W_2).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - 2x^2 - x - 1 = 0$ whose roots are α, β, γ with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 2$, $s = 1$ and $t = 1$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) \\ &= (-z^3 + 2z^2 + z + 1)(z^3 + 3z^2 + 6z - 1)\end{aligned}$$

and $\Gamma(1) \neq 0$.

(a) Use theorem 2.1 (a) (i) with $z = 1$.

(b) Use theorem 2.1 (b) (i) with $z=1$.

(c) Use theorem 2.1 (c) (i) with $z=1$. $\square$

From the last Theorem, we have the following Corollary which gives sum formulas of third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2$).

Corollary 4.5.
For $n \geq 0$, third-order Pell numbers have the following properties.

(a) $\sum\limits_{k=0}^{n} P_k^2 = \frac{1}{9}(-P_{n+3}^2 - 9P_{n+2}^2 - 10P_{n+1}^2 + 6P_{n+2}P_{n+3} + 2P_{n+1}P_{n+3} + 1)$.

(b) $\sum\limits_{k=0}^{n} P_{k+1}P_k = \frac{1}{9}(P_{n+3}^2 + 3P_{n+2}^2 + P_{n+1}^2 - 6P_{n+1}P_{n+2} + P_{n+1}P_{n+3} - 3P_{n+2}P_{n+3} - 1)$.

(c) $\sum\limits_{k=0}^{n} P_{k+2}P_k = \frac{1}{9}(2P_{n+3}^2 + 2P_{n+1}^2 - 3P_{n+2}P_{n+3} - 4P_{n+1}P_{n+3} - 2)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last Theorem, we have the following Corollary which gives sum formulas of third-order Pell-Lucas numbers.

Corollary 4.6.
For $n \geq 0$, third-order Pell numbers have the following properties:

(a) $\sum\limits_{k=0}^{n} Q_k^2 = \frac{1}{9}(-Q_{n+3}^2 - 9Q_{n+2}^2 - 10Q_{n+1}^2 + 6Q_{n+2}Q_{n+3} + 2Q_{n+1}Q_{n+3} + 54)$.

(b) $\sum\limits_{k=0}^{n} Q_{k+1}Q_k = \frac{1}{9}(Q_{n+3}^2 + 3Q_{n+2}^2 + Q_{n+1}^2 - 6Q_{n+1}Q_{n+2} + Q_{n+1}Q_{n+3} - 3Q_{n+2}Q_{n+3} - 3)$.

(c) $\sum\limits_{k=0}^{n} Q_{k+2}Q_k = \frac{1}{9}(2Q_{n+3}^2 + 2Q_{n+1}^2 - 3Q_{n+2}Q_{n+3} - 4Q_{n+1}Q_{n+3} + 18)$.

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 4.4.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} \simeq 0.154171$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-z^6 - z^5 + z^4 + 17z^3 + 7z^2 + 5z - 1}((z^5 + 2z^4 + z^3 - z^2)W_2^2 + (4z^5 + 13z^4 + 6z^3 + 5z^2 - z)W_1^2 + (z^5 + 2z^4 + 16z^3 + 7z^2 + 5z - 1)W_0^2 - 2(2z^5 + 5z^4 + 2z^3)W_1W_2 - 2(z^5 + 2z^3)W_0W_2 + 2(2z^5 - z^4 - z^3)W_0W_1)$.

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-z^6 - z^5 + z^4 + 17z^3 + 7z^2 + 5z - 1}(-(z^4 + 2z^2)W_2^2 - 3(2z^4 + z^3)W_1^2 - (z^5 + 2z^3)W_0^2 + (5z^4 + z^3 + 4z^2 - z)W_1W_2 + (z^5 + z^4 - 4z^3 - z^2)W_0W_2 + (-2z^5 - 3z^4 + 12z^3 + 7z^2 + 5z - 1)W_0W_1)$.

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-z^6 - z^5 + z^4 + 17z^3 + 7z^2 + 5z - 1}(-(z^3 + 5z^2)W_2^2 + (-2z^5 - 2z^4 + 5z^2 - z)W_1^2 - (z^4 + 5z^3)W_0^2 + (z^5 + z^4 + 2z^3 + 7z^2 - 2z)W_1W_2 + (-z^4 + 4z^3 + 5z^2 + 5z - 1)W_0W_2 + (-z^5 + 2z^4 - 5z^3 + 5z^2 - z)W_0W_1)$.

Proof. Use theorem 3.1. $\square$

Now, we consider special cases of the last Theorem.

Corollary 4.7.
Assume that $|z| < |\alpha|^{-2} \simeq 0.154171$. The ordinary generating functions of the sequences $\{P_n^2\}$, $\{P_{n+1}P_n\}$, $\{P_{n+2}P_n\}$ and $\{Q_n^2\}$, $\{Q_{n+1}Q_n\}$, $\{Q_{n+2}Q_n\}$ are given as follows:

(a)

$$\sum_{n=0}^{\infty} P_n^2 z^n = \frac{z^4+2z^3+z^2-z}{-z^6-z^5+z^4+17z^3+7z^2+5z-1},$$
$$\sum_{n=0}^{\infty} Q_n^2 z^n = \frac{z^5+10z^4+72z^3+47z^2+41z-9}{-z^6-z^5+z^4+17z^3+7z^2+5z-1}.$$

(b)

$$\sum_{n=0}^{\infty} P_{n+1}P_n z^n = \frac{-z^3-2z}{-z^6-z^5+z^4+17z^3+7z^2+5z-1},$$
$$\sum_{n=0}^{\infty} Q_{n+1}Q_n z^n = \frac{-3z^5-18z^3+18z-6}{-z^6-z^5+z^4+17z^3+7z^2+5z-1}.$$

(c)

$$\sum_{n=0}^{\infty} P_{n+2}P_n z^n = \frac{-z^2-5z}{-z^6-z^5+z^4+17z^3+7z^2+5z-1},$$
$$\sum_{n=0}^{\infty} Q_{n+2}Q_n z^n = \frac{-2z^5-11z^4-15z^3+44z^2+56z-18}{-z^6-z^5+z^4+17z^3+7z^2+5z-1}.$$

From the last corollary, we obtain the following results for third-order Pell and third-order Pell-Lucas numbers.

Corollary 4.8.
Some infinite sums of $\{P_n^2\}$, $\{P_{n+1}P_n\}$, $\{P_{n+2}P_n\}$ *and* $\{Q_n^2\}$, $\{Q_{n+1}Q_n\}$, $\{Q_{n+2}Q_n\}$ *are given as follows:*

(a) $z=\frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{P_n^2}{8^n} = \frac{27584}{60873},$$
$$\sum_{n=0}^{\infty} \frac{Q_n^2}{8^n} = \frac{261928}{20291}.$$

(b) $z=\frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{P_{n+1}P_n}{8^n} = \frac{22016}{20291},$$
$$\sum_{n=0}^{\infty} \frac{Q_{n+1}Q_n}{8^n} = \frac{330760}{20291}.$$

(c) $z=\frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{P_{n+2}P_n}{8^n} = \frac{167936}{60873},$$
$$\sum_{n=0}^{\infty} \frac{Q_{n+2}Q_n}{8^n} = \frac{903920}{20291}.$$

4.3. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Padovan Numbers

In this subsection, we consider the case $r=0$, $s=1$ and $t=1$. A generalized Padovan sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-2} + W_{n-3} \tag{37}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + W_{-(n-3)}$$

for $n = 1,2,3,....$ Therefore, recurrence eq. (24) holds for all integer n. For more details on the generalized Padovan numbers, see for example Soykan [22].

Binet's formula of generalized Padovan numbers can be given as

$$W_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0,\ b_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0,\ b_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0.$$

Here, α, β and γ are the roots of the cubic equation $x^3 - x - 1 = 0$. Moreover

$$\alpha = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} = 1.32471795724$$

$$\beta = \omega\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}$$

$$\gamma = \omega^2\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3)$$

Now we define three special cases of the sequence $\{W_n\}$. Padovan sequence $\{P_n\}_{n\geq 0}$ (OEIS: A000931, [8]), Perrin (Padovan-Lucas) sequence $\{E_n\}_{n\geq 0}$ (OEIS: A001608, [8]) and adjusted Padovan sequence $\{U_n\}_{n\geq 0}$ (a variant of the sequence $\{P_n\}$) are defined, respectively, by the third-order recurrence relations

$$P_{n+3} = P_{n+1} + P_n, \quad P_0 = 1, P_1 = 1, P_2 = 1, \tag{38}$$

$$E_{n+3} = E_{n+1} + E_n, \quad E_0 = 3, E_1 = 0, E_2 = 2, \tag{39}$$

$$U_{n+3} = U_{n+1} + U_n, \quad U_0 = 0, U_1 = 1, U_2 = 0. \tag{40}$$

The sequences $\{P_n\}_{n\geq 0}$, $\{E_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} + P_{-(n-3)},$$

$$E_{-n} = -E_{-(n-1)} + E_{-(n-3)},$$

$$U_{-n} = -U_{-(n-1)} + U_{-(n-3)},$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences eq. (38) - eq. (40) hold for all integer n.

For all integers n, Padovan, Perrin, Padovan-Perrin and modified Padovan numbers can be expressed using Binet's formulas as

$$P_n = \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)},$$

$$E_n = \alpha^n + \beta^n + \gamma^n,$$

$$U_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

respectively.

Next, we present sum formulas of generalized Padovan numbers.

Theorem 4.5.
For $n \geq 0$, we have the following sum formulas for generalized Padovan numbers:

(a) $\sum_{k=0}^{n} W_k^2 = -W_{n+3}^2 - W_{n+2}^2 - 2W_{n+1}^2 + 2W_{n+2}W_{n+3} + 2W_{n+1}W_{n+3} + W_2^2 + W_1^2 + 2W_0^2 - 2W_1W_2 - 2W_0W_2.$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_{n+2}W_{n+3} - W_{n+1}W_{n+3} - W_2^2 - W_1^2 - W_0^2 + W_1W_2 + W_0W_2.$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = W_{n+2}W_{n+3} - W_1W_2.$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3-x-1=0$ whose roots are α,β,γ with $\alpha\neq\beta\neq\gamma$. In theorem 2.1, for $r=0$, $s=1$ and $t=1$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)\\ &= (z^3-z-1)(-z^3+z^2-2z+1)\end{aligned}$$

and $\Gamma(1)\neq 0$.

(a) Use theorem 2.1 (a) (i) with $z=1$.

(b) Use theorem 2.1 (b) (i) with $z=1$.

(c) Use theorem 2.1 (c) (i) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Padovan numbers (take $W_n=P_n$ with $P_0=1,P_1=1,P_2=1$).

Corollary 4.9.
For $n\geq 0$, Padovan numbers have the following properties.

(a) $\sum\limits_{k=0}^{n} P_k^2 = -P_{n+3}^2-P_{n+2}^2-2P_{n+1}^2+2P_{n+2}P_{n+3}+2P_{n+1}P_{n+3}.$

(b) $\sum\limits_{k=0}^{n} P_{k+1}P_k = P_{n+3}^2+P_{n+2}^2+P_{n+1}^2-P_{n+2}P_{n+3}-P_{n+1}P_{n+3}-1.$

(c) $\sum\limits_{k=0}^{n} P_{k+2}P_k = P_{n+2}P_{n+3}-1.$

Taking $W_n=E_n$ with $E_0=3,E_1=0,E_2=2$ in the last Theorem, we have the following Corollary which gives sum formulas of Perrin (Padovan-Lucas) numbers.

Corollary 4.10.
For $n\geq 0$, Perrin (Padovan-Lucas) numbers have the following properties:

(a) $\sum\limits_{k=0}^{n} E_k^2 = -E_{n+3}^2-E_{n+2}^2-2E_{n+1}^2+2E_{n+2}E_{n+3}+2E_{n+1}E_{n+3}+10.$

(b) $\sum\limits_{k=0}^{n} E_{k+1}E_k = E_{n+3}^2+E_{n+2}^2+E_{n+1}^2-E_{n+2}E_{n+3}-E_{n+1}E_{n+3}-7.$

(c) $\sum\limits_{k=0}^{n} E_{k+2}E_k = E_{n+2}E_{n+3}.$

From the last Theorem, we have the following Corollary which gives sum formulas of adjusted Padovan numbers (take $W_n=U_n$ with $U_0=0,U_1=1,U_2=0$).

Corollary 4.11.
For $n\geq 0$, adjusted Padovan numbers have the following properties.

(a) $\sum\limits_{k=0}^{n} U_k^2 = -U_{n+3}^2-U_{n+2}^2-2U_{n+1}^2+2U_{n+2}U_{n+3}+2U_{n+1}U_{n+3}+1.$

(b) $\sum\limits_{k=0}^{n} U_{k+1}U_k = U_{n+3}^2+U_{n+2}^2+U_{n+1}^2-U_{n+2}U_{n+3}-U_{n+1}U_{n+3}-1.$

(c) $\sum\limits_{k=0}^{n} U_{k+2}U_k = U_{n+2}U_{n+3}.$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_nz^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_nz^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 4.6.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = 0.5698402$. *Then the ordinary generating functions of the sequences* $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ *are given as follows:*

(a) $\sum_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-z^6+z^5-z^4+z^3+z^2+z-1}((z^5+z^3-z^2)W_2^2+(z^4+z^2-z)W_1^2+(z^5+z^2+z-1)W_0^2-2z^4W_1W_2-2z^5W_0W_2+2(z^4-z^3)W_0W_1)$.

(b) $\sum_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-z^6+z^5-z^4+z^3+z^2+z-1}(-z^4W_2^2-z^3W_1^2-z^5W_0^2+(z^4+z^3-z)W_1W_2+(z^5+z^4-z^2)W_0W_2+(-z^4+z^2+z-1)W_0W_1)$.

(c) $\sum_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-z^6+z^5-z^4+z^3+z^2+z-1}((z^3-z^2)W_2^2+(z^2-z)W_1^2+(z^4-z^3)W_0^2+(z^5-z^4-z^2)W_1W_2+(-z^4+z^2+z-1)W_0W_2+(-z^5+2z^4-z^3+z^2-z)W_0W_1)$.

Proof. Use theorem 3.1. □

Now, we consider special cases of the last .

Corollary 4.12.
Assume that $|z| < |\alpha|^{-2} = 0.5698402$. *The ordinary generating functions of the sequences* $\{P_n^2\}$, $\{P_{n+1}P_n\}$, $\{P_{n+2}P_n\}$ *and* $\{E_n^2\}$, $\{E_{n+1}E_n\}$, $\{E_{n+2}E_n\}$ *and* $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} P_n^2 z^n = \frac{z^4-z^3+z^2-1}{-z^6+z^5-z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} E_n^2 z^n = \frac{z^5+4z^3+5z^2+9z-9}{-z^6+z^5-z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} U_n^2 z^n = \frac{z^4+z^2-z}{-z^6+z^5-z^4+z^3+z^2+z-1}.$$

(b)

$$\sum_{n=0}^{\infty} P_{n+1}P_n z^n = \frac{-1}{-z^6+z^5-z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} E_{n+1}E_n z^n = \frac{-3z^5+2z^4-6z^2}{-z^6+z^5-z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} U_{n+1}U_n z^n = \frac{-z^3}{-z^6+z^5-z^4+z^3+z^2+z-1}.$$

(c)

$$\sum_{n=0}^{\infty} P_{n+2}P_n z^n = \frac{z^4-z^3+z^2-z-1}{-z^6+z^5-z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} E_{n+2}E_n z^n = \frac{3z^4-5z^3+2z^2+6z-6}{-z^6+z^5-z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} U_{n+2}U_n z^n = \frac{z^2-z}{-z^6+z^5-z^4+z^3+z^2+z-1}.$$

From the last corollary, we obtain the following results for Padovan and Perrin (Padovan-Lucas) and adjusted Padovan numbers.

Corollary 4.13.
Some infinite sums of $\{P_n^2\}$, $\{P_{n+1}P_n\}$, $\{P_{n+2}P_n\}$ *and* $\{E_n^2\}$, $\{E_{n+1}E_n\}$, $\{E_{n+2}E_n\}$ *and* $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ *are given as follows:*

(a) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{P_n^2}{2^n} = \frac{52}{11},$$
$$\sum_{n=0}^{\infty} \frac{E_n^2}{2^n} = \frac{174}{11},$$
$$\sum_{n=0}^{\infty} \frac{U_n^2}{2^n} = \frac{12}{11}.$$

(b) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{P_{n+1}P_n}{2^n} = \frac{64}{11},$$
$$\sum_{n=0}^{\infty} \frac{E_{n+1}E_n}{2^n} = \frac{94}{11},$$
$$\sum_{n=0}^{\infty} \frac{U_{n+1}U_n}{2^n} = \frac{8}{11}.$$

(c) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{P_{n+2}P_n}{2^n} = \frac{84}{11},$$
$$\sum_{n=0}^{\infty} \frac{E_{n+2}E_n}{2^n} = \frac{188}{11},$$
$$\sum_{n=0}^{\infty} \frac{U_{n+2}U_n}{2^n} = \frac{16}{11}.$$

4.4. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1} W_k$, $\sum_{k=0}^{n} z^k W_{k+2} W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1} W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2} W_n z^n$ of Generalized Pell-Padovan Numbers

In this subsection, we consider the case $r = 0$, $s = 2$ and $t = 1$. A generalized Pell-Padovan sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 2W_{n-2} + W_{n-3} \tag{41}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -2W_{-(n-1)} + W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (24) holds for all integer n. For more details on the generalized Pell-Padovan numbers, see Soykan [20].

Binet formula of generalized padovan numbers can be given as

$$W_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)}$$

where

$$b_1 = W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \; b_2 = W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \; b_3 = W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0. \tag{42}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - 2x - 1 = 0$. Moreover

$$\alpha = \frac{1 + \sqrt{5}}{2},$$
$$\beta = \frac{1 - \sqrt{5}}{2},$$
$$\gamma = -1.$$

Now we define four special cases of the sequence $\{W_n\}$. Adjusted Pell-Padovan sequence $\{M_n\}_{n\geq 0}$, third order Lucas-Pell sequence $\{B_n\}_{n\geq 0}$ (OEIS: A099925, [8]), third order Fibonacci-Pell sequence $\{G_n\}_{n\geq 0}$ (OEIS: A008346, [8]), Pell-Perrin sequence $\{C_n\}_{n\geq 0}$, Pell-Padovan sequence$\{R_n\}_{n\geq 0}$ (OEIS: A066983, [8]), are defined, respectively, by the third-order recurrence relations

$$M_{n+3} = 2M_{n+1} + M_n, \qquad M_0 = 0, M_1 = 1, M_2 = 0, \tag{43}$$
$$B_{n+3} = 2B_{n+1} + B_n, \qquad B_0 = 3, B_1 = 0, B_2 = 4 \tag{44}$$
$$G_{n+3} = 2G_{n+1} + G_n, \qquad G_0 = 1, G_1 = 0, G_2 = 2, \tag{45}$$
$$C_{n+3} = 2C_{n+1} + C_n, \qquad C_0 = 3, C_1 = 0, C_2 = 2, \tag{46}$$
$$R_{n+3} = 2R_{n+1} + R_n, \qquad R_0 = 1, R_1 = 1, R_2 = 1. \tag{47}$$

The sequences $\{M_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, $\{G_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -2M_{-(n-1)} + M_{-(n-3)},$$
$$B_{-n} = -2B_{-(n-1)} + B_{-(n-3)},$$
$$G_{-n} = -2G_{-(n-1)} + G_{-(n-3)},$$
$$C_{-n} = -2C_{-(n-1)} + C_{-(n-3)},$$
$$R_{-n} = -2R_{-(n-1)} + R_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences eq. (43) - eq. (47) hold for all integer n.

Note that for all integers n, adjusted Pell-Padovan, third order Lucas-Pell, third order Fibonacci-Pell, Pell-Perrin, Pell-Padovan numbers can be expressed using Binet's formulas as

$$\begin{aligned} M_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1} \\ &= (\frac{1}{2} - \frac{1}{10}\sqrt{5})\alpha^n + (\frac{1}{2} + \frac{1}{10}\sqrt{5})\beta^n - \gamma^n, \\ B_n &= \alpha^n + \beta^n + \gamma^n, \\ G_n &= \frac{1}{\sqrt{5}}\alpha^n - \frac{1}{\sqrt{5}}\beta^n + \gamma^n, \\ C_n &= (2 - \frac{3}{\sqrt{5}})\alpha^n + (2 + \frac{3}{\sqrt{5}})\beta^n - \gamma^n, \\ R_n &= (1 - \frac{1}{\sqrt{5}})\alpha^n + (1 + \frac{1}{\sqrt{5}})\beta^n - \gamma^n, \end{aligned}$$

respectively, see Soykan [20] for more details.

B_n is the sequence A099925 in [8] associated with the relation

$$B_n = L_n + (-1)^n$$

where L_n is Lucas sequence which is given as

$$L_n = L_{n-1} + L_{n-2} \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

G_n is the sequence A008346 in [8] associated with the relation

$$G_n = F_n + (-1)^n$$

where F_n is Fibonacci sequence which is given as

$$F_n = F_{n-1} + F_{n-2} \text{ with } F_0 = 0 \text{ and } F_1 = 1.$$

C_n is not indexed in [8].

R_n is the sequence A066983 in [8] associated with the relation

$$R_{n+2} = R_{n+1} + R_n + (-1)^n, \text{ with } R_1 = R_2 = 1.$$

Since

$$F_{-n} = (-1)^{n+1}F_n \text{ and } L_{-n} = (-1)^n L_n$$

we get

$$G_{-n} = (-1)^{n+1}G_n + 1 + (-1)^n = (-1)^n(1 - F_n)$$

and

$$B_{-n} = (-1)^n B_n - 1 + (-1)^n = (-1)^n(L_n + 1).$$

Next, we present sum formulas of generalized Pell-Padovan numbers.

Theorem 4.7.
For $n \geq 0$, we have the following sum formulas for generalized Pell-Padovan numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{2}((2n+11)W_{n+3}^2 + (2n+9)W_{n+2}^2 + (2n+11)W_{n+1}^2 - 4(n+5)W_{n+3}W_{n+2} - 4(n+6)W_{n+3}W_{n+1} + 4(n+6)W_{n+2}W_{n+1} - 9W_2^2 - 7W_1^2 - 9W_0^2 + 16W_1W_2 + 20W_0W_2 - 20W_0W_1)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{2}(-2(n+5)W_{n+3}^2 - 2(n+4)W_{n+2}^2 - 2(n+6)W_{n+1}^2 + 19W_{n+2}W_{n+3} + 23W_{n+1}W_{n+3} - 23W_{n+1}W_{n+2} + 4nW_{n+2}W_{n+3} + 4nW_{n+1}W_{n+3} - 4nW_{n+1}W_{n+2} + 8W_2^2 + 6W_1^2 + 10W_0^2 - 15W_1W_2 - 19W_0W_2 + 19W_0W_1)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{2}(2(n+5)W_{n+3}^2 + 2(n+4)W_{n+2}^2 + 2(n+6)W_{n+1}^2 - (4n+17)W_{n+3}W_{n+2} - (4n+23)W_{n+3}W_{n+1} + (4n+21)W_{n+2}W_{n+1} - 8W_2^2 - 6W_1^2 - 10W_0^2 + 13W_1W_2 + 19W_0W_2 - 17W_0W_1)$.

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - 2x - 1 = 0 = 0$ whose roots are

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, \gamma = -1$$

with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 0$, $s = 2$ and $t = 1$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) \\ &= -(z-1)(z+1)(z^2 - z - 1)(z^2 - 3z + 1)\end{aligned}$$

and $\Gamma(1) = 0$. Here 1 is the root of $\Gamma(z) = 0$ with multiplicity 1.

(a) Use theorem 2.1 (a) (ii) with $z = 1$.

(b) Use theorem 2.1 (b) (ii) with $z = 1$.

(c) Use theorem 2.1 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of adjusted Pell-Padovan (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 0$).

Corollary 4.14.
For $n \geq 0$, adjusted Pell-Padovan numbers have the following properties.

(a) $\sum_{k=0}^{n} M_k^2 = \frac{1}{2}((2n+11)M_{n+3}^2 + (2n+9)M_{n+2}^2 + (2n+11)M_{n+1}^2 - 4(n+5)M_{n+3}M_{n+2} - 4(n+6)M_{n+3}M_{n+1} + 4(n+6)M_{n+2}M_{n+1} - 7)$.

(b) $\sum_{k=0}^{n} M_{k+1}M_k = \frac{1}{2}(-2(n+5)M_{n+3}^2 - 2(n+4)M_{n+2}^2 - 2(n+6)M_{n+1}^2 + 19M_{n+2}M_{n+3} + 23M_{n+1}M_{n+3} - 23M_{n+1}M_{n+2} + 4nM_{n+2}M_{n+3} + 4nM_{n+1}M_{n+3} - 4nM_{n+1}M_{n+2} + 6)$.

(c) $\sum_{k=0}^{n} M_{k+2}M_k = \frac{1}{2}(2(n+5)M_{n+3}^2 + 2(n+4)M_{n+2}^2 + 2(n+6)M_{n+1}^2 - (4n+17)M_{n+3}M_{n+2} - (4n+23)M_{n+3}M_{n+1} + (4n+21)M_{n+2}M_{n+1} - 6)$.

Taking $W_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in the last Theorem, we have the following Corollary which gives sum formulas of third order Lucas-Pell numbers.

Corollary 4.15.
For $n \geq 0$, third order Lucas-Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} B_k^2 = \frac{1}{2}((2n+11)B_{n+3}^2 + (2n+9)B_{n+2}^2 + (2n+11)B_{n+1}^2 - 4(n+5)B_{n+3}B_{n+2} - 4(n+6)B_{n+3}B_{n+1} + 4(n+6)B_{n+2}B_{n+1} + 15)$.

(b) $\sum_{k=0}^{n} B_{k+1}B_k = \frac{1}{2}(-2(n+5)B_{n+3}^2 - 2(n+4)B_{n+2}^2 - 2(n+6)B_{n+1}^2 + 19B_{n+2}B_{n+3} + 23B_{n+1}B_{n+3} - 23B_{n+1}B_{n+2} + 4nB_{n+2}B_{n+3} + 4nB_{n+1}B_{n+3} - 4nB_{n+1}B_{n+2} - 10)$.

(c) $\sum_{k=0}^{n} B_{k+2}B_k = \frac{1}{2}(2(n+5)B_{n+3}^2 + 2(n+4)B_{n+2}^2 + 2(n+6)B_{n+1}^2 - (4n+17)B_{n+3}B_{n+2} - (4n+23)B_{n+3}B_{n+1} + (4n+21)B_{n+2}B_{n+1} + 10)$.

From the last Theorem, we have the following Corollary which gives sum formulas of third order Fibonacci-Pell numbers (take $W_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

Corollary 4.16.
For $n \geq 0$, third order Fibonacci-Pell numbers have the following properties.

(a) $\sum_{k=0}^{n} G_k^2 = \frac{1}{2}((2n+11)G_{n+3}^2 + (2n+9)G_{n+2}^2 + (2n+11)G_{n+1}^2 - 4(n+5)G_{n+3}G_{n+2} - 4(n+6)G_{n+3}G_{n+1} + 4(n+6)G_{n+2}G_{n+1} - 5)$.

(b) $\sum_{k=0}^{n} G_{k+1}G_k = \frac{1}{2}(-2(n+5)G_{n+3}^2 - 2(n+4)G_{n+2}^2 - 2(n+6)G_{n+1}^2 + 19G_{n+2}G_{n+3} + 23G_{n+1}G_{n+3} - 23G_{n+1}G_{n+2} + 4nG_{n+2}G_{n+3} + 4nG_{n+1}G_{n+3} - 4nG_{n+1}G_{n+2} + 4)$.

(c) $\sum_{k=0}^{n} G_{k+2}G_k = \frac{1}{2}(2(n+5)G_{n+3}^2 + 2(n+4)G_{n+2}^2 + 2(n+6)G_{n+1}^2 - (4n+17)G_{n+3}G_{n+2} - (4n+23)G_{n+3}G_{n+1} + (4n+21)G_{n+2}G_{n+1} - 4)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last Theorem, we have the following Corollary which gives sum formulas of Pell-Perrin numbers.

Corollary 4.17.
For $n \geq 0$, Pell-Perrin numbers have the following properties:

(a) $\sum_{k=0}^{n} C_k^2 = \frac{1}{2}((2n+11)C_{n+3}^2 + (2n+9)C_{n+2}^2 + (2n+11)C_{n+1}^2 - 4(n+5)C_{n+3}C_{n+2} - 4(n+6)C_{n+3}C_{n+1} + 4(n+6)C_{n+2}C_{n+1} + 3)$.

(b) $\sum_{k=0}^{n} C_{k+1}C_k = \frac{1}{2}(-2(n+5)C_{n+3}^2 - 2(n+4)C_{n+2}^2 - 2(n+6)C_{n+1}^2 + 19C_{n+2}C_{n+3} + 23C_{n+1}C_{n+3} - 23C_{n+1}C_{n+2} + 4nC_{n+2}C_{n+3} + 4nC_{n+1}C_{n+3} - 4nC_{n+1}C_{n+2} + 8)$.

(c) $\sum_{k=0}^{n} C_{k+2}C_k = \frac{1}{2}(2(n+5)C_{n+3}^2 + 2(n+4)C_{n+2}^2 + 2(n+6)C_{n+1}^2 - (4n+17)C_{n+3}C_{n+2} - (4n+23)C_{n+3}C_{n+1} + (4n+21)C_{n+2}C_{n+1} - 8)$.

From the last Theorem, we have the following Corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 4.18.
For $n \geq 0$, Pell-Padovan numbers have the following properties.

(a) $\sum_{k=0}^{n} R_k^2 = \frac{1}{2}((2n+11)R_{n+3}^2 + (2n+9)R_{n+2}^2 + (2n+11)R_{n+1}^2 - 4(n+5)R_{n+3}R_{n+2} - 4(n+6)R_{n+3}R_{n+1} + 4(n+6)R_{n+2}R_{n+1} - 9)$.

(b) $\sum_{k=0}^{n} R_{k+1}R_k = \frac{1}{2}(-2(n+5)R_{n+3}^2 - 2(n+4)R_{n+2}^2 - 2(n+6)R_{n+1}^2 + 19R_{n+2}R_{n+3} + 23R_{n+1}R_{n+3} - 23R_{n+1}R_{n+2} + 4nR_{n+2}R_{n+3} + 4nR_{n+1}R_{n+3} - 4nR_{n+1}R_{n+2} + 9)$.

(c) $\sum_{k=0}^{n} R_{k+2}R_k = \frac{1}{2}(2(n+5)R_{n+3}^2 + 2(n+4)R_{n+2}^2 + 2(n+6)R_{n+1}^2 - (4n+17)R_{n+3}R_{n+2} - (4n+23)R_{n+3}R_{n+1} + (4n+21)R_{n+2}R_{n+1} - 9)$.

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 4.8.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = 0.381966$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1}((z^5+2z^3-z^2)W_2^2+(z^4+2z^2-z)W_1^2+(4z^5-7z^3+4z^2+2z-1)W_0^2-4z^4W_1W_2-4z^5W_0W_2+(8z^4-4z^3)W_0W_1).$

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1}(-2z^4W_2^2-2z^3W_1^2-2z^5W_0^2+(z^4+4z^3-z)W_1W_2+(z^5+4z^4-z^2)W_0W_2+(-2z^4-7z^3+4z^2+2z-1)W_0W_1).$

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1}(2(2z^3-z^2)W_2^2+2(2z^2-z)W_1^2+2(2z^4-z^3)W_0^2+(z^5-4z^4-z^2)W_1W_2+(-2z^4-7z^3+4z^2+2z-1)W_0W_2+(-2z^5+9z^4-4z^3+2z^2-z)W_0W_1).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 4.19.
Assume that $|z| < |\alpha|^{-2} = 0.381966$. *The ordinary generating functions of the sequences* $\{M_n^2\}, \{M_{n+1}M_n\}, \{M_{n+2}M_n\}$ *and* $\{B_n^2\}, \{B_{n+1}B_n\}, \{B_{n+2}B_n\}$ *and* $\{G_n^2\}, \{G_{n+1}G_n\}, \{G_{n+2}G_n\}$ *and* $\{C_n^2\}, \{C_{n+1}C_n\}, \{C_{n+2}C_n\}$ *and* $\{R_n^2\}, \{R_{n+1}R_n\}, \{R_{n+2}R_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} M_n^2 z^n = \frac{z^4+2z^2-z}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} B_n^2 z^n = \frac{4z^5-31z^3+20z^2+18z-9}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} G_n^2 z^n = \frac{z^3+2z-1}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} C_n^2 z^n = \frac{16z^5-55z^3+32z^2+18z-9}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} R_n^2 z^n = \frac{z^5+5z^4-9z^3+5z^2+z-1}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1}.$$

(b)

$$\sum_{n=0}^{\infty} M_{n+1}M_n z^n = \frac{-2z^3}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} B_{n+1}B_n z^n = \frac{-6z^5+16z^4-12z^2}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} G_{n+1}G_n z^n = \frac{-2z^2}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} C_{n+1}C_n z^n = \frac{-12z^5+16z^4-6z^2}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} R_{n+1}R_n z^n = \frac{-z^5+z^4-5z^3+3z^2+z-1}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1}.$$

(c)

$$\sum_{n=0}^{\infty} M_{n+2}M_n z^n = \frac{4z^2-2z}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} B_{n+2}B_n z^n = \frac{12z^4-38z^3+16z^2+24z-12}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} G_{n+2}G_n z^n = \frac{4z-2}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} C_{n+2}C_n z^n = \frac{24z^4-44z^3+16z^2+12z-6}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} R_{n+2}R_n z^n = \frac{-z^5+7z^4-9z^3+7z^2-z-1}{-z^6+4z^5-2z^4-6z^3+4z^2+2z-1}.$$

From the last corollary, we obtain the following results for adjusted Pell-Padovan, third order Lucas-Pell sequence, third order Fibonacci-Pell sequence, Pell-Perrin and Pell-Padovan numbers.

Corollary 4.20.
Some infinite sums of $\{M_n^2\}$, $\{M_{n+1}M_n\}$, $\{M_{n+2}M_n\}$ *and* $\{B_n^2\}$, $\{B_{n+1}B_n\}$, $\{B_{n+2}B_n\}$ *and* $\{G_n^2\}$, $\{G_{n+1}G_n\}$, $\{G_{n+2}G_n\}$ *and* $\{C_n^2\}$, $\{C_{n+1}C_n\}$, $\{C_{n+2}C_n\}$ *and* $\{R_n^2\}$, $\{R_{n+1}R_n\}$, $\{R_{n+2}R_n\}$ *are given as follows:*

(a) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{M_n^2}{4^n} = \frac{496}{1425},$$
$$\sum_{n=0}^{\infty} \frac{B_n^2}{4^n} = \frac{3056}{285},$$
$$\sum_{n=0}^{\infty} \frac{G_n^2}{4^n} = \frac{1984}{1425},$$
$$\sum_{n=0}^{\infty} \frac{C_n^2}{4^n} = \frac{13696}{1425},$$
$$\sum_{n=0}^{\infty} \frac{R_n^2}{4^n} = \frac{2284}{1425}.$$

(b) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{M_{n+1}M_n}{4^n} = \frac{128}{1425},$$
$$\sum_{n=0}^{\infty} \frac{B_{n+1}B_n}{4^n} = \frac{568}{285},$$
$$\sum_{n=0}^{\infty} \frac{G_{n+1}G_n}{4^n} = \frac{512}{1425},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+1}C_n}{4^n} = \frac{1328}{1425},$$
$$\sum_{n=0}^{\infty} \frac{R_{n+1}R_n}{4^n} = \frac{2612}{1425}.$$

(c) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{M_{n+2}M_n}{4^n} = \frac{1024}{1425},$$
$$\sum_{n=0}^{\infty} \frac{B_{n+2}B_n}{4^n} = \frac{4544}{285},$$
$$\sum_{n=0}^{\infty} \frac{G_{n+2}G_n}{4^n} = \frac{4096}{1425},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+2}C_n}{4^n} = \frac{10624}{1425},$$
$$\sum_{n=0}^{\infty} \frac{R_{n+2}R_n}{4^n} = \frac{3796}{1425}.$$

4.5. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Jacobsthal-Padovan Numbers

In this subsection, we consider the case $r = 0$, $s = 1$ and $t = 2$. A generalized Jacobsthal-Padovan sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-2} + 2W_{n-3} \quad (48)$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (24) holds for all integer n. For more information on Jacobsthal-Padovan sequence, see Soykan [19].

Binet formula of generalized Jacobsthal-Padovan numbers can be given as

$$W_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, \tag{49}$$
$$b_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, \tag{50}$$
$$b_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0. \tag{51}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - x - 2 = 0.$$

Moreover

$$\alpha = \sqrt[3]{1+\frac{\sqrt{78}}{9}} + \sqrt[3]{1-\frac{\sqrt{78}}{9}} \simeq 1.521379706804568,$$
$$\beta = \omega\sqrt[3]{1+\frac{\sqrt{78}}{9}} + \omega^2\sqrt[3]{1-\frac{\sqrt{78}}{9}},$$
$$\gamma = \omega^2\sqrt[3]{1+\frac{\sqrt{78}}{9}} + \omega\sqrt[3]{1-\frac{\sqrt{78}}{9}},$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Adjusted Jacobsthal-Padovan sequence $\{K_n\}_{n\geq 0}$ (OEIS: A159287, [8]), Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) sequence $\{L_n\}_{n\geq 0}$ (OEIS: A072328, [8]), Jacobsthal-Padovan sequence $\{Q_n\}_{n\geq 0}$ (OEIS: A159284, [8]), and modified Jacobsthal-Padovan sequence $\{M_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$K_{n+3} = K_{n+1} + 2K_n, \quad K_0 = 0, K_1 = 1, K_2 = 0, \tag{52}$$
$$L_{n+3} = L_{n+1} + 2L_n, \quad L_0 = 3, L_1 = 0, L_2 = 2, \tag{53}$$
$$Q_{n+3} = Q_{n+1} + 2Q_n, \quad Q_0 = 1, Q_1 = 1, Q_2 = 1. \tag{54}$$

The sequences $\{Q_n\}_{n\geq 0}$, $\{L_n\}_{n\geq 0}$, and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$K_{-n} = -\frac{1}{2}K_{-(n-1)} + \frac{1}{2}K_{-(n-3)},$$
$$L_{-n} = -\frac{1}{2}L_{-(n-1)} + \frac{1}{2}L_{-(n-3)},$$
$$Q_{-n} = -\frac{1}{2}Q_{-(n-1)} + \frac{1}{2}Q_{-(n-3)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences eq. (52))- eq. (54) hold for all integer n.

Note that for all integers n, adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers can be expressed using Binet's formulas as

$$K_n = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1},$$
$$L_n = \alpha^n + \beta^n + \gamma^n,$$
$$Q_n = \frac{(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{(\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1} + \frac{(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1},$$

respectively, see Soykan [19] for more details.

Next, we present sum formulas of generalized Jacobsthal-Padovan numbers.

Theorem 4.9.
For $n \geq 0$, we have the following sum formulas for generalized Jacobsthal-Padovan numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{2}(W_{n+3}^2 + W_{n+2}^2 + 2W_{n+1}^2 - W_{n+2}W_{n+3} - 2W_{n+1}W_{n+3} - W_2^2 - W_1^2 - 2W_0^2 + W_1W_2 + 2W_0W_2)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{4}(-W_{n+3}^2 - W_{n+2}^2 - 4W_{n+1}^2 + 2W_{n+2}W_{n+3} + 4W_{n+1}W_{n+3} + W_2^2 + W_1^2 + 4W_0^2 - 2W_1W_2 - 4W_0W_2)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{2}(W_{n+2}W_{n+3} - W_1W_2)$.

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - x - 2 = 0$ whose roots are α, β, γ with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 0$, $s = 1$ and $t = 2$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) \\ &= (4z^3 - z - 1)(-4z^3 + z^2 - 2z + 1)\end{aligned}$$

and $\Gamma(1) \neq 0$.

(a) Use theorem 2.1 (a) (i) with $z = 1$.

(b) Use theorem 2.1 (b) (i) with $z = 1$.

(c) Use theorem 2.1 (c) (i) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of adjusted Jacobsthal-Padovan numbers (take $W_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$).

Corollary 4.21.
For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties.

(a) $\sum_{k=0}^{n} K_k^2 = \frac{1}{2}(K_{n+3}^2 + K_{n+2}^2 + 2K_{n+1}^2 - K_{n+2}K_{n+3} - 2K_{n+1}K_{n+3} - 1)$.

(b) $\sum_{k=0}^{n} K_{k+1}K_k = \frac{1}{4}(-K_{n+3}^2 - K_{n+2}^2 - 4K_{n+1}^2 + 2K_{n+2}K_{n+3} + 4K_{n+1}K_{n+3} + 1)$.

(c) $\sum_{k=0}^{n} K_{k+2}K_k = \frac{1}{2}K_{n+2}K_{n+3}$.

Taking $W_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) numbers.

Corollary 4.22.
For $n \geq 0$, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) numbers have the following properties:

(a) $\sum_{k=0}^{n} L_k^2 = \frac{1}{2}(L_{n+3}^2 + L_{n+2}^2 + 2L_{n+1}^2 - L_{n+2}L_{n+3} - 2L_{n+1}L_{n+3} - 10)$.

(b) $\sum_{k=0}^{n} L_{k+1}L_k = \frac{1}{4}(-L_{n+3}^2 - L_{n+2}^2 - 4L_{n+1}^2 + 2L_{n+2}L_{n+3} + 4L_{n+1}L_{n+3} + 16)$.

(c) $\sum_{k=0}^{n} L_{k+2}L_k = \frac{1}{2}L_{n+2}L_{n+3}$.

From the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 4.23.
For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties.

(a) $\sum_{k=0}^{n} Q_k^2 = \frac{1}{2}(Q_{n+3}^2 + Q_{n+2}^2 + 2Q_{n+1}^2 - Q_{n+2}Q_{n+3} - 2Q_{n+1}Q_{n+3} - 1)$.

(b) $\sum\limits_{k=0}^{n} Q_{k+1}Q_k = \frac{1}{4}(-Q_{n+3}^2 - Q_{n+2}^2 - 4Q_{n+1}^2 + 2Q_{n+2}Q_{n+3} + 4Q_{n+1}Q_{n+3}).$

(c) $\sum\limits_{k=0}^{n} Q_{k+2}Q_k = \frac{1}{2}(Q_{n+2}Q_{n+3} - 1).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 4.10.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = 0.4320408$. *Then the ordinary generating functions of the sequences* $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ *are given as follows:*

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1}((4z^5+z^3-z^2)W_2^2 + (4z^4+z^2-z)W_1^2 + (4z^5+3z^3+z^2+z-1)W_0^2 - 4z^4W_1W_2 - 8z^5W_0W_2 + 4(z^4-z^3)W_0W_1).$

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1}(-2z^4W_2^2 - 2z^3W_1^2 - 8z^5W_0^2 + (4z^4+z^3-z)W_1W_2 + (8z^5+2z^4-2z^2)W_0W_2 + (-4z^4+3z^3+z^2+z-1)W_0W_1).$

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1}((z^3-z^2)W_2^2 + (z^2-z)W_1^2 + (4z^4-4z^3)W_0^2 + 2(4z^5-z^4-z^2)W_1W_2 + (-4z^4+3z^3+z^2+z-1)W_0W_2 + (-8z^5+10z^4-2z^3+2z^2-2z)W_0W_1).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 4.24.
Assume that $|z| < |\alpha|^{-2} = 0.4320408$. *The ordinary generating functions of the sequences* $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ *and* $\{L_n^2\}$, $\{L_{n+1}L_n\}$, $\{L_{n+2}L_n\}$ *and* $\{Q_n^2\}$, $\{Q_{n+1}Q_n\}$, $\{Q_{n+2}Q_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} K_n^2 z^n = \frac{4z^4+z^2-z}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} L_n^2 z^n = \frac{4z^5+31z^3+5z^2+9z-9}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} Q_n^2 z^n = \frac{4z^4+z^2-1}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1}.$$

(b)

$$\sum_{n=0}^{\infty} K_{n+1}K_n z^n = \frac{-2z^3}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} L_{n+1}L_n z^n = \frac{-24z^5+4z^4-12z^2}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} Q_{n+1}Q_n z^n = \frac{2z^3-z^2-1}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1}.$$

(c)

$$\sum_{n=0}^{\infty} K_{n+2}K_n z^n = \frac{z^2-z}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} L_{n+2}L_n z^n = \frac{12z^4-14z^3+2z^2+6z-6}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} Q_{n+2}Q_n z^n = \frac{8z^4-2z^3+z^2-2z-1}{-16z^6+4z^5-4z^4+7z^3+z^2+z-1}.$$

From the last corollary, we obtain the following results for adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers.

Corollary 4.25.
Some infinite sums of $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ *and* $\{L_n^2\}$, $\{L_{n+1}L_n\}$, $\{L_{n+2}L_n\}$ *and* $\{Q_n^2\}$, $\{Q_{n+1}Q_n\}$, $\{Q_{n+2}Q_n\}$ *are given as follows:*

(a) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{K_n^2}{4^n} = \frac{11}{38}$$
$$\sum_{n=0}^{\infty} \frac{L_n^2}{4^n} = \frac{1523}{152}$$
$$\sum_{n=0}^{\infty} \frac{Q_n^2}{4^n} = \frac{59}{38}$$

(b) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{K_{n+1}K_n}{4^n} = \frac{1}{19}$$
$$\sum_{n=0}^{\infty} \frac{L_{n+1}L_n}{4^n} = \frac{97}{76}$$
$$\sum_{n=0}^{\infty} \frac{Q_{n+1}Q_n}{4^n} = \frac{33}{19}$$

(c) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{K_{n+2}K_n}{4^n} = \frac{6}{19}$$
$$\sum_{n=0}^{\infty} \frac{L_{n+2}L_n}{4^n} = \frac{291}{38}$$
$$\sum_{n=0}^{\infty} \frac{Q_{n+2}Q_n}{4^n} = \frac{46}{19}$$

4.6. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Narayana Numbers

In this subsection, we consider the case $r = 1$, $s = 0$ and $t = 1$. A generalized Narayana sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-1} + W_{n-3} \tag{55}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (24) holds for all integer n. For more information on Narayana sequence, see Soykan [18].

Binet formula of generalized Narayana numbers can be given as

$$W_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0,\ b_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0,\ b_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0. \tag{56}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - x^2 - 1 = 0$.

Moreover

$$\alpha = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}$$
$$\beta = \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}$$
$$\gamma = \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Narayana sequence$\{N_n\}_{n\geq 0}$ (OEIS: A000930, [8]) and Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ (OEIS: A001609, [8]) are defined, respectively, by the third-order recurrence relations

$$N_{n+3} = N_{n+2} + N_n, \quad N_0 = 0, N_1 = 1, N_2 = 1, \tag{57}$$
$$U_{n+3} = U_{n+2} + U_n, \quad U_0 = 3, U_1 = 1, U_2 = 1. \tag{58}$$

The sequences $\{N_n\}_{n\geq 0}$ and $\{U_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$N_{-n} = -N_{-(n-2)} + N_{-(n-3)},$$
$$U_{-n} = -U_{-(n-2)} + U_{-(n-3)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences eq. (57) - eq. (58) hold for all integer n. For more information on generalized Narayana numbers, see Soykan [18].

Binet's formulas of Narayana and Narayana-Lucas numbers, respectively, are

$$N_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$
$$U_n = \alpha^n + \beta^n + \gamma^n.$$

Next, we present sum formulas of generalized Narayana numbers.

Theorem 4.11.
For $n \geq 0$, we have the following sum formulas for generalized Narayana numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{3}(-W_{n+3}^2 - 4W_{n+2}^2 - 4W_{n+1}^2 + 4W_{n+2}W_{n+3} + 2W_{n+1}W_{n+3} + 2W_{n+1}W_{n+2} + W_2^2 + 4W_1^2 + 4W_0^2 - 2W_0W_2 - 4W_1W_2 - 2W_0W_1)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{3}(W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_{n+2}W_{n+3} + W_{n+1}W_{n+3} - 2W_{n+1}W_{n+2} - W_2^2 - W_1^2 - W_0^2 + W_1W_2 - W_0W_2 + 2W_0W_1)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{3}(2W_{n+3}^2 + 2W_{n+2}^2 + 2W_{n+1}^2 - 2W_{n+2}W_{n+3} - W_{n+1}W_{n+3} - W_{n+1}W_{n+2} - 2W_2^2 - 2W_1^2 - 2W_0^2 + 2W_1W_2 + W_0W_2 + W_0W_1)$.

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - x^2 - 1 = 0$ whose roots are α, β, γ with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 1$, $s = 0$ and $t = 1$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)\\ &= (-z^3 + z^2 + 1)(z^3 + 2z^2 + z - 1)\end{aligned}$$

and $\Gamma(1) \neq 0$.

(a) Use theorem 2.1 (a) (i) with $z = 1$.

(b) Use theorem 2.1 (b) (i) with $z = 1$.

(c) Use theorem 2.1 (c) (i) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 4.26.
For $n \geq 0$, Narayana numbers have the following properties.

(a) $\sum\limits_{k=0}^{n} N_k^2 = \frac{1}{3}(-N_{n+3}^2 - 4N_{n+2}^2 - 4N_{n+1}^2 + 4N_{n+2}N_{n+3} + 2N_{n+1}N_{n+3} + 2N_{n+1}N_{n+2} + 1).$

(b) $\sum\limits_{k=0}^{n} N_{k+1}N_k = \frac{1}{3}(N_{n+3}^2 + N_{n+2}^2 + N_{n+1}^2 - N_{n+2}N_{n+3} + N_{n+1}N_{n+3} - 2N_{n+1}N_{n+2} - 1).$

(c) $\sum\limits_{k=0}^{n} N_{k+2}N_k = \frac{1}{3}(2N_{n+3}^2 + 2N_{n+2}^2 + 2N_{n+1}^2 - 2N_{n+2}N_{n+3} - N_{n+1}N_{n+3} - N_{n+1}N_{n+2} - 2).$

Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last Theorem, we have the following Corollary which gives sum formulas of Narayana-Lucas numbers.

Corollary 4.27.
For $n \geq 0$, Narayana-Lucas numbers have the following properties:

(a) $\sum\limits_{k=0}^{n} U_k^2 = \frac{1}{3}(-U_{n+3}^2 - 4U_{n+2}^2 - 4U_{n+1}^2 + 4U_{n+2}U_{n+3} + 2U_{n+1}U_{n+3} + 2U_{n+1}U_{n+2} + 25).$

(b) $\sum\limits_{k=0}^{n} U_{k+1}U_k = \frac{1}{3}(U_{n+3}^2 + U_{n+2}^2 + U_{n+1}^2 - U_{n+2}U_{n+3} + U_{n+1}U_{n+3} - 2U_{n+1}U_{n+2} - 7).$

(c) $\sum\limits_{k=0}^{n} U_{k+2}U_k = \frac{1}{3}(2U_{n+3}^2 + 2U_{n+2}^2 + 2U_{n+1}^2 - 2U_{n+2}U_{n+3} - U_{n+1}U_{n+3} - U_{n+1}U_{n+2} - 14).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 4.12.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = 0.465571$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1}((z^5 + z^4 - z^2)W_2^2 + (z^5 + 2z^4 + z^3 + z^2 - z)W_1^2 + (z^4 + 2z^3 + z^2 + z - 1)W_0^2 - 2(z^5 + z^4)W_1W_2 - 2z^3W_0W_2 - 2z^4W_0W_1).$

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1}(-z^2W_2^2 - z^4W_1^2 - z^3W_0^2 + (z^4 + z^2 - z)W_1W_2 + (z^5 - z^3 - z^2)W_0W_2 + (-z^5 + 2z^3 + z^2 + z - 1)W_0W_1).$

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1}(-(z^3 + z^2)W_2^2 - (z^5 + z^4)W_1^2 - (z^4 + z^3)W_0^2 + (z^5 + z^4 + z^3 - z)W_1W_2 + (z^3 + z - 1)W_0W_2 + (z^4 + z^2 - z)W_0W_1).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 4.28.
Assume that $|z| < |\alpha|^{-2} = 0.465571$. The ordinary generating functions of the sequences $\{N_n^2\}$, $\{N_{n+1}N_n\}$, $\{N_{n+2}N_n\}$ and $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ are given as follows:

(a)

$$\sum_{n=0}^{\infty} N_n^2 z^n = \frac{z^4 + z^3 - z}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1},$$
$$\sum_{n=0}^{\infty} U_n^2 z^n = \frac{4z^4 + 13z^3 + 9z^2 + 8z - 9}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1}.$$

(b)

$$\sum_{n=0}^{\infty} N_{n+1}N_n z^n = \frac{-z}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1},$$
$$\sum_{n=0}^{\infty} U_{n+1}U_n z^n = \frac{-6z^3 + 2z - 3}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1}.$$

(c)

$$\sum_{n=0}^{\infty} N_{n+2}N_n z^n = \frac{-z^2 - z}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1},$$
$$\sum_{n=0}^{\infty} U_{n+2}U_n z^n = \frac{-6z^4 - 6z^3 + 2z^2 - z - 3}{-z^6 - z^5 + z^4 + 3z^3 + z^2 + z - 1}.$$

From the last corollary, we obtain the following results for Narayana and Narayana-Lucas numbers.

Corollary 4.29.
Some infinite sums of $\{N_n^2\}$, $\{N_{n+1}N_n\}$, $\{N_{n+2}N_n\}$ *and* $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ *are given as follows:*

(a) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{N_n^2}{4^n} = \frac{944}{2613},$$
$$\sum_{n=0}^{\infty} \frac{U_n^2}{4^n} = \frac{25472}{2613}.$$

(b) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{N_{n+1}N_n}{4^n} = \frac{1024}{2613},$$
$$\sum_{n=0}^{\infty} \frac{U_{n+1}U_n}{4^n} = \frac{10624}{2613}.$$

(c) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{N_{n+2}N_n}{4^n} = \frac{1280}{2613},$$
$$\sum_{n=0}^{\infty} \frac{U_{n+2}U_n}{4^n} = \frac{13280}{2613}.$$

4.7. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Third Order Jacobsthal Numbers

In this subsection, we consider the case $r = 1$, $s = 1$ and $t = 2$. A generalized third order Jacobsthal sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-1} + W_{n-2} + 2W_{n-3} \tag{59}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-1)} - \frac{1}{2}W_{-(n-2)} + \frac{1}{2}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (24) holds for all integer n. For more information on generalized third order Jacobsthal sequence, see Soykan [7].

Binet formula of generalized third order Jacobsthal numbers can be given as

$$W_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0,\ b_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0,\ b_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0. \tag{60}$$

Here, α, β and $\gamma$are the roots of the cubic equation $x^3 - x^2 - x - 2 = 0$. Moreover

$$\alpha = 2,$$
$$\beta = \frac{-1+i\sqrt{3}}{2},$$
$$\gamma = \frac{-1-i\sqrt{3}}{2}.$$

Third-order Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (OEIS: A077947, [8]), modified third-order Jacobsthal sequence $\{K_n\}_{n\geq 0}$ (OEIS: A186575, [8]) and third-order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ (OEIS: A226308, [8]) are defined, respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2} + J_{n+1} + 2J_n,\ \ J_0 = 0, J_1 = 1, J_2 = 1, \tag{61}$$
$$K_{n+3} = K_{n+2} + K_{n+1} + 2K_n,\ \ K_0 = 3, K_1 = 1, K_2 = 3. \tag{62}$$
$$j_{n+3} = j_{n+2} + j_{n+1} + 2j_n,\ \ j_0 = 2, j_1 = 1, j_2 = 5, \tag{63}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ are defined in [3] and $\{K_n\}_{n\geq 0}$ is given in [1]. For more details on the generalized third-order Jacobsthal numbers and its special cases, see [7].

The sequences $\{J_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n} = -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} + \frac{1}{2}J_{-(n-3)},$$
$$K_{-n} = -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)},$$
$$j_{-n} = -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} + \frac{1}{2}j_{-(n-3)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences eq. (61))- eq. (63) hold for all integer n.

Note that for all integers n, third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$J_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{64}$$
$$K_n = \alpha^n + \beta^n + \gamma^n, \tag{65}$$
$$j_n = \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{66}$$

respectively.

Next, we present sum formulas of generalized third order Jacobsthal numbers.

Theorem 4.13.
For $n \geq 0$, we have the following sum formulas for generalized third order Jacobsthal numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{63}((6n+35)W_{n+3}^2 + 18(n+5)W_{n+2}^2 + (24n+101)W_{n+1}^2 - 6(3n+16)W_{n+3}W_{n+2} - 4(3n+16)W_{n+3}W_{n+1} + 12W_{n+1}W_{n+2} - 29W_2^2 - 72W_1^2 - 77W_0^2 + 78W_1W_2 + 52W_0W_2 - 12W_0W_1).$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{63}(-(3n+13)W_{n+3}^2 - 3(3n+14)W_{n+2}^2 - 4(3n+16)W_{n+1}^2 + 9(n+5)W_{n+2}W_{n+3} + 2(3n+22)W_{n+3}W_{n+1} - 27W_{n+1}W_{n+2} + 10W_2^2 + 33W_1^2 + 52W_0^2 - 36W_1W_2 - 38W_0W_2 + 27W_0W_1).$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{63}(-(3n+10)W_{n+3}^2 - 9(n+6)W_{n+2}^2 - 4(3n+13)W_{n+1}^2 + 3(3n+19)W_{n+3}W_{n+2} + (6n+17)W_{n+3}W_{n+1} - 6W_{n+1}W_{n+2} + 7W_2^2 + 45W_1^2 + 40W_0^2 - 48W_1W_2 - 11W_0W_2 + 6W_0W_1).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - x^2 - x - 2 = 0$ whose roots are α, β, γ with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 1$, $s = 1$ and $t = 2$,

$$\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)$$
$$= -(z-1)(4z-1)(z^2+z+1)(4z^2+2z+1)$$

and $\Gamma(1) = 0$. Here 1 is the root of $\Gamma(z) = 0$ with multiplicity 1.

(a) Use theorem 2.1 (a) (ii) with $z=1$.

(b) Use theorem 2.1 (b) (ii) with $z=1$.

(c) Use theorem 2.1 (c) (ii) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of third-order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

Corollary 4.30.
For $n \geq 0$, third-order Jacobsthal numbers have the following properties.

(a) $\sum_{k=0}^{n} J_k^2 = \frac{1}{63}((6n+35)J_{n+3}^2+18(n+5)J_{n+2}^2+(24n+101)J_{n+1}^2-6(3n+16)J_{n+3}J_{n+2}-4(3n+16)J_{n+3}J_{n+1}+12J_{n+1}J_{n+2}-23)$.

(b) $\sum_{k=0}^{n} J_{k+1}J_k = \frac{1}{63}(-(3n+13)J_{n+3}^2-3(3n+14)J_{n+2}^2-4(3n+16)J_{n+1}^2+9(n+5)J_{n+2}J_{n+3}+2(3n+22)J_{n+3}J_{n+1}-27J_{n+1}J_{n+2}+7)$.

(c) $\sum_{k=0}^{n} J_{k+2}J_k = \frac{1}{63}(-(3n+10)J_{n+3}^2-9(n+6)J_{n+2}^2-4(3n+13)J_{n+1}^2+3(3n+19)J_{n+3}J_{n+2}+(6n+17)J_{n+3}J_{n+1}-6J_{n+1}J_{n+2}+4)$.

Taking $J_n = W_n$ with $J_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last Theorem, we have the following Corollary which gives sum formulas of modified third-order Jacobsthal numbers.

Corollary 4.31.
For $n \geq 0$, modified third-order Jacobsthal numbers have the following properties:

(a) $\sum_{k=0}^{n} K_k^2 = \frac{1}{63}((6n+35)K_{n+3}^2+18(n+5)K_{n+2}^2+(24n+101)K_{n+1}^2-6(3n+16)K_{n+3}K_{n+2}-4(3n+16)K_{n+3}K_{n+1}+12K_{n+1}K_{n+2}-360)$.

(b) $\sum_{k=0}^{n} K_{k+1}K_k = \frac{1}{63}(-(3n+13)K_{n+3}^2-3(3n+14)K_{n+2}^2-4(3n+16)K_{n+1}^2+9(n+5)K_{n+2}K_{n+3}+2(3n+22)K_{n+3}K_{n+1}-27K_{n+1}K_{n+2}+222)$.

(c) $\sum_{k=0}^{n} K_{k+2}K_k = \frac{1}{63}(-(3n+10)K_{n+3}^2-9(n+6)K_{n+2}^2-4(3n+13)K_{n+1}^2+3(3n+19)K_{n+3}K_{n+2}+(6n+17)K_{n+3}K_{n+1}-6K_{n+1}K_{n+2}+243)$.

From the last Theorem, we have the following Corollary which gives sum formulas of third-order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$).

Corollary 4.32.
For $n \geq 0$, third-order Jacobsthal-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n} j_k^2 = \frac{1}{63}((6n+35)j_{n+3}^2+18(n+5)j_{n+2}^2+(24n+101)j_{n+1}^2-6(3n+16)j_{n+3}j_{n+2}-4(3n+16)j_{n+3}j_{n+1}+12j_{n+1}j_{n+2}-219)$.

(b) $\sum_{k=0}^{n} j_{k+1}j_k = \frac{1}{63}(-(3n+13)j_{n+3}^2-3(3n+14)j_{n+2}^2-4(3n+16)j_{n+1}^2+9(n+5)j_{n+2}j_{n+3}+2(3n+22)j_{n+3}j_{n+1}-27j_{n+1}j_{n+2}-15)$.

(c) $\sum_{k=0}^{n} j_{k+2}j_k = \frac{1}{63}(-(3n+10)j_{n+3}^2-9(n+6)j_{n+2}^2-4(3n+13)j_{n+1}^2+3(3n+19)j_{n+3}j_{n+2}+(6n+17)j_{n+3}j_{n+1}-6j_{n+1}j_{n+2}+42)$.

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 4.14.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = 0.25$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1}((4z^5+2z^4+z^3-z^2)W_2^2+(4z^5+10z^4+3z^3+2z^2-z)W_1^2+(4z^5+2z^4+13z^3+4z^2+2z-1)W_0^2-2(4z^5+4z^4+z^3)W_1W_2-4(2z^5+z^3)W_0W_2+4(2z^5-z^4-z^3)W_0W_1).$

(b) $\sum_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1}(-(2z^4+z^2)W_2^2-3(2z^4+z^3)W_1^2-4(2z^5+z^3)W_0^2+(8z^4+z^3+z^2-z)W_1W_2+2(4z^5+z^4-z^3-z^2)W_0W_2+(-8z^5-6z^4+9z^3+4z^2+2z-1)W_0W_1).$

(c) $\sum_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1}(-(z^3+2z^2)W_2^2+(-8z^5-2z^4+2z^2-z)W_1^2-4(z^4+2z^3)W_0^2+(8z^5+2z^4+z^3-z^2-z)W_1W_2+(-4z^4+7z^3+2z^2+2z-1)W_0W_2+2(-4z^5+5z^4-2z^3+2z^2-z)W_0W_1).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 4.33.
Assume that $|z| < |\alpha|^{-2} = 0.25$. The ordinary generating functions of the sequences $\{J_n^2\}$, $\{J_{n+1}J_n\}$, $\{J_{n+2}J_n\}$ and $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ and $\{j_n^2\}$, $\{j_{n+1}j_n\}$, $\{j_{n+2}j_n\}$ are given as follows:

(a)

$$\sum_{n=0}^{\infty} J_n^2 z^n = \frac{4z^4+2z^3+z^2-z}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} K_n^2 z^n = \frac{4z^5+10z^4+75z^3+29z^2+17z-9}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} j_n^2 z^n = \frac{16z^5+20z^4+22z^3-7z^2+7z-4}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1}.$$

(b)

$$\sum_{n=0}^{\infty} J_{n+1}J_n z^n = \frac{-2z^3-z}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} K_{n+1}K_n z^n = \frac{-24z^5-27z^3-12z^2+3z-3}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} j_{n+1}j_n z^n = \frac{32z^5-8z^4-16z^3-32z^2-z-2}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1}.$$

(c)

$$\sum_{n=0}^{\infty} J_{n+2}J_n z^n = \frac{-z^2-2z}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} K_{n+2}K_n z^n = \frac{-8z^5-38z^4-27z^3+11z^2+8z-9}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1},$$
$$\sum_{n=0}^{\infty} j_{n+2}j_n z^n = \frac{16z^5-28z^4+10z^3-25z^2+10z-10}{-16z^6-4z^5-2z^4+17z^3+4z^2+2z-1}.$$

From the last corollary, we obtain the following results for third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers.

Corollary 4.34.
Some infinite sums of $\{J_n^2\}$, $\{J_{n+1}J_n\}$, $\{J_{n+2}J_n\}$ and $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ and $\{j_n^2\}$, $\{j_{n+1}j_n\}$, $\{j_{n+2}j_n\}$ are given as follows:

(a) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{J_n^2}{8^n} = \frac{1712}{10731},$$
$$\sum_{n=0}^{\infty} \frac{K_n^2}{8^n} = \frac{4894}{511},$$
$$\sum_{n=0}^{\infty} \frac{j_n^2}{8^n} = \frac{17400}{3577}.$$

(b) $z=\frac{1}{8}$.

$$\sum_{n=0}^{\infty}\frac{J_{n+1}J_n}{8^n}=\frac{704}{3577},$$
$$\sum_{n=0}^{\infty}\frac{K_{n+1}K_n}{8^n}=\frac{2236}{511},$$
$$\sum_{n=0}^{\infty}\frac{j_{n+1}j_n}{8^n}=\frac{14512}{3577}.$$

(c) $z=\frac{1}{8}$.

$$\sum_{n=0}^{\infty}\frac{J_{n+2}J_n}{8^n}=\frac{4352}{10731},$$
$$\sum_{n=0}^{\infty}\frac{K_{n+2}K_n}{8^n}=\frac{6156}{511},$$
$$\sum_{n=0}^{\infty}\frac{j_{n+2}j_n}{8^n}=\frac{49848}{3577}.$$

4.8. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Graham Numbers

In this subsection, we consider the case $r=2$, $s=3$ and $t=5$. A generalized Graham sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$W_n=2W_{n-1}+3W_{n-2}+5W_{n-3} \tag{67}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=-\frac{3}{5}W_{-(n-1)}-\frac{2}{5}W_{-(n-2)}+\frac{1}{5}W_{-(n-3)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence eq. (67) holds for all integer n. For more information on generalized Graham numbers, see Soykan [24].

Binet formula of generalized Graham numbers can be given as

$$W_n=\frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1=W_2-(\beta+\gamma)W_1+\beta\gamma W_0,\ b_2=W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0,\ b_3=W_2-(\alpha+\beta)W_1+\alpha\beta W_0. \tag{68}$$

Here, α,β and $\gamma$are the roots of the cubic equation $x^3-2x^2-3x-5=0$. Moreover

$$\alpha=\frac{2}{3}+\left(\frac{205}{54}+\sqrt{\frac{1231}{108}}\right)^{1/3}+\left(\frac{205}{54}-\sqrt{\frac{1231}{108}}\right)^{1/3}$$
$$\beta=\frac{2}{3}+\omega\left(\frac{205}{54}+\sqrt{\frac{1231}{108}}\right)^{1/3}+\omega^2\left(\frac{205}{54}-\sqrt{\frac{1231}{108}}\right)^{1/3}$$
$$\gamma=\frac{2}{3}+\omega^2\left(\frac{205}{54}+\sqrt{\frac{1231}{108}}\right)^{1/3}+\omega\left(\frac{205}{54}-\sqrt{\frac{1231}{108}}\right)^{1/3}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3)$$

Now we define two special cases of the sequence $\{W_n\}$. Graham sequence $\{G_n\}_{n\geq 0}$ and Graham-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3}=2G_{n+2}+3G_{n+1}+5G_n,\qquad G_0=0,G_1=1,G_2=2, \tag{69}$$
$$H_{n+3}=2H_{n+2}+3H_{n+1}+5H_n,\qquad H_0=3,H_1=2,H_2=10, \tag{70}$$

The sequences $\{G_n\}_{n\geq0}$ and $\{H_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{3}{5}G_{-(n-1)} - \frac{2}{5}G_{-(n-2)} + \frac{1}{5}G_{-(n-3)},$$
$$H_{-n} = -\frac{3}{5}H_{-(n-1)} - \frac{2}{5}H_{-(n-2)} + \frac{1}{5}H_{-(n-3)}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences eq. (69) and eq. (70) hold for all integer n.For all integers n, Graham and Graham-Lucas numbers can be expressed using Binet's formulas as

$$G_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$
$$H_n = \alpha^n + \beta^n + \gamma^n,$$

respectively.

Next, we present sum formulas of generalized Graham numbers.

Theorem 4.15.
For $n \geq 0$, we have the following sum formulas for generalized Graham numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{495}(37W_{n+3}^2 + 253W_{n+2}^2 + 430W_{n+1}^2 - 182W_{n+2}W_{n+3} - 170W_{n+1}W_{n+3} + 260W_{n+1}W_{n+2} - 37W_2^2 - 253W_1^2 - 430W_0^2 + 182W_1W_2 + 170W_0W_2 - 260W_0W_1)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{495}(-17W_{n+3}^2 - 143W_{n+2}^2 - 425W_{n+1}^2 + 97W_{n+2}W_{n+3} + 145W_{n+1}W_{n+3} - 280W_{n+1}W_{n+2} + 17W_2^2 + 143W_1^2 + 425W_0^2 - 97W_1W_2 - 145W_0W_2 + 280W_0W_1)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{495}(-8W_{n+3}^2 - 242W_{n+2}^2 - 200W_{n+1}^2 + 133W_{n+2}W_{n+3} + 10W_{n+1}W_{n+3} - 190W_{n+1}W_{n+2} + 8W_2^2 + 242W_1^2 + 200W_0^2 - 133W_1W_2 - 10W_0W_2 + 190W_0W_1)$.

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - 2x^2 - 3x - 5 = 0$ whose roots are α, β, γ with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 2$, $s = 3$ and $t = 5$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)\\ &= (-25z^3 + 10z^2 + 3z + 1)(25z^3 + 11z^2 + 10z - 1)\end{aligned}$$

and $\Gamma(1) \neq 0$.

(a) Use theorem 2.1 (a) (i) with $z = 1$.

(b) Use theorem 2.1 (b) (i) with $z = 1$.

(c) Use theorem 2.1 (c) (i) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Graham numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 4.35.
For $n \geq 0$, Graham numbers have the following properties.

(a) $\sum_{k=0}^{n} G_k^2 = \frac{1}{495}(37G_{n+3}^2 + 253G_{n+2}^2 + 430G_{n+1}^2 - 182G_{n+2}G_{n+3} - 170G_{n+1}G_{n+3} + 260G_{n+1}G_{n+2} - 37)$.

(b) $\sum_{k=0}^{n} G_{k+1}G_k = \frac{1}{495}(-17G_{n+3}^2 - 143G_{n+2}^2 - 425G_{n+1}^2 + 97G_{n+2}G_{n+3} + 145G_{n+1}G_{n+3} - 280G_{n+1}G_{n+2} + 17)$.

(c) $\sum_{k=0}^{n} G_{k+2}G_k = \frac{1}{495}(-8G_{n+3}^2 - 242G_{n+2}^2 - 200G_{n+1}^2 + 133G_{n+2}G_{n+3} + 10G_{n+1}G_{n+3} - 190G_{n+1}G_{n+2} + 8)$.

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last Theorem, we have the following Corollary which gives sum formulas of Graham-Lucas numbers.

Corollary 4.36.
For $n \geq 0$, Graham-Lucas numbers have the following properties:

(a) $\sum\limits_{k=0}^{n} H_k^2 = \frac{1}{495}(37H_{n+3}^2 + 253H_{n+2}^2 + 430H_{n+1}^2 - 182H_{n+2}H_{n+3} - 170H_{n+1}H_{n+3} + 260H_{n+1}H_{n+2} - 1402).$

(b) $\sum\limits_{k=0}^{n} H_{k+1}H_k = \frac{1}{495}(-17H_{n+3}^2 - 143H_{n+2}^2 - 425H_{n+1}^2 + 97H_{n+2}H_{n+3} + 145H_{n+1}H_{n+3} - 280H_{n+1}H_{n+2} + 1487).$

(c) $\sum\limits_{k=0}^{n} H_{k+2}H_k = \frac{1}{495}(-8H_{n+3}^2 - 242H_{n+2}^2 - 200H_{n+1}^2 + 133H_{n+2}H_{n+3} + 10H_{n+1}H_{n+3} - 190H_{n+1}H_{n+2} + 1748).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 4.16.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} \simeq 0.089417$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1}((25z^5 + 10z^4 + 3z^3 - z^2)W_2^2 + (100z^5 + 125z^4 + 22z^3 + 7z^2 - z)W_1^2 + (225z^5 + 10z^4 + 158z^3 + 31z^2 + 7z - 1)W_0^2 - 2(50z^5 + 35z^4 + 6z^3)W_1W_2 - 10(15z^5 + 2z^3)W_0W_2 + 10(30z^5 - z^4 - 3z^3)W_0W_1).$

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1}(-(15z^4 + 2z^2)W_2^2 - 11(10z^4 + 3z^3)W_1^2 - 25(15z^5 + 2z^3)W_0^2 + (85z^4 + 9z^3 + 4z^2 - z)W_1W_2 + 5(25z^5 + 9z^4 - 4z^3 - z^2)W_0W_2 + (-250z^5 - 165z^4 + 98z^3 + 31z^2 + 7z - 1)W_0W_1).$

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1}(-(z^3 + 7z^2)W_2^2 + (-250z^5 - 10z^4 + 21z^2 - 3z)W_1^2 - 25(z^4 + 7z^3)W_0^2 + (125z^5 + 5z^4 + 2z^3 + 3z^2 - 2z)W_1W_2 + (-75z^4 + 58z^3 + 21z^2 + 7z - 1)W_0W_2 + (-375z^5 + 260z^4 - 105z^3 + 35z^2 - 5z)W_0W_1).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 4.37.
Assume that $|z| < |\alpha|^{-2} \simeq 0.089417$. The ordinary generating functions of the sequences $\{G_n^2\}$, $\{G_{n+1}G_n\}$, $\{G_{n+2}G_n\}$ and $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ are given as follows:

(a)

$$\sum_{n=0}^{\infty} G_n^2 z^n = \frac{25z^4 + 10z^3 + 3z^2 - z}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1},$$
$$\sum_{n=0}^{\infty} H_n^2 z^n = \frac{225z^5 + 130z^4 + 790z^3 + 207z^2 + 59z - 9}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1}.$$

(b)

$$\sum_{n=0}^{\infty} G_{n+1}G_n z^n = \frac{-15z^3 - 2z}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1},$$
$$\sum_{n=0}^{\infty} H_{n+1}H_n z^n = \frac{-1125z^5 + 120z^4 - 414z^3 - 84z^2 + 22z - 6}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1}.$$

(c)

$$\sum_{n=0}^{\infty} G_{n+2}G_n z^n = \frac{-z^2 - 7z}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1},$$
$$\sum_{n=0}^{\infty} H_{n+2}H_n z^n = \frac{-750z^5 - 855z^4 - 525z^3 + 284z^2 + 128z - 30}{-625z^6 - 25z^5 - 65z^4 + 183z^3 + 31z^2 + 7z - 1}.$$

From the last corollary, we obtain the following results for Graham and Graham-Lucas numbers.

Corollary 4.38.
Some infinite sums of $\{G_n^2\}$, $\{G_{n+1}G_n\}$, $\{G_{n+2}G_n\}$ *and* $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ *are given as follows:*

(a) $z = \frac{1}{16}$.

$$\sum_{n=0}^{\infty} \frac{G_n^2}{16^n} = \frac{804608}{6673665},$$
$$\sum_{n=0}^{\infty} \frac{H_n^2}{16^n} = \frac{72290288}{6673665}.$$

(b) $z = \frac{1}{16}$.

$$\sum_{n=0}^{\infty} \frac{G_{n+1}G_n}{16a^n} = \frac{2158592}{6673665},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+1}H_n}{16^n} = \frac{84782672}{6673665}.$$

(c) $z = \frac{1}{16}$.

$$\sum_{n=0}^{\infty} \frac{G_{n+2}G_n}{16^n} = \frac{7405568}{6673665},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+2}H_n}{16^n} = \frac{352867808}{6673665}.$$

5. The Sum Formulas $\sum_{k=0}^{n} z^k W_k^2, \sum_{k=0}^{n} z^k W_{k+1}W_k, \sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions of Special Cases of Generalized Fibonacci Polynomials/Numbers: Second Group

In this section, we present sum formulas and generating functions for special cases of generalized Tribonacci polynomials, namely, generalized Leonardo numbers, generalized Ernst numbers, generalized Edouard numbers, generalized John numbers, generalized Pisano numbers, generalized Bigollo numbers, generalized Guglielmo numbers, generalized Woodall numbers. Moreover, we evaluate the infinite sums of special cases of generalized Tribonacci numbers.

5.1. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Leonardo Numbers

In this subsection, we consider the case $r = 2, s = 0, t = -1$. A generalized Leonardo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-3} \tag{71}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-2)} - W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (71) holds for all integer n. For more information on generalized Leonardo numbers, see Soykan [9].

Binet formula of generalized Leonardo numbers can be given as

$$W_n = \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{72}$$
$$= \frac{z_1\alpha^{n+1} - z_2\beta^{n+1}}{(\alpha-\beta)} - z_3$$

where

$$z_1 = W_2 - (2-\alpha)W_1 + (1-\alpha)W_0, \tag{73}$$
$$z_2 = W_2 - (2-\beta)W_1 + (1-\beta)W_0, \tag{74}$$
$$z_3 = W_2 - W_1 - W_0. \tag{75}$$

Here, α,β and γ are the roots of the cubic equation

$$x^3-2x^2+1=\left(x^2-x-1\right)(x-1)=0.$$

Moreover

$$\alpha=\frac{1+\sqrt{5}}{2},$$
$$\beta=\frac{1-\sqrt{5}}{2},$$
$$\gamma=1.$$

Note that

$$\alpha+\beta+\gamma=2,$$
$$\alpha\beta+\alpha\gamma+\beta\gamma=0,$$
$$\alpha\beta\gamma=-1,$$

or

$$\alpha+\beta=1,\ \alpha\beta=-1.$$

Now we define three special cases of the sequence $\{W_n\}$. Modified Leonardo sequence $\{G_n\}_{n\geq 0}$, Leonardo-Lucas sequence $\{H_n\}_{n\geq 0}$ and Leonardo sequence $\{l_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_n=2G_{n-1}-G_{n-3},\quad G_0=0,G_1=1,G_2=2, \tag{76}$$
$$H_n=2H_{n-1}-H_{n-3},\quad H_0=3,H_1=2,H_2=4, \tag{77}$$
$$l_n=2l_{n-1}-l_{n-3},\quad l_0=1,l_1=1,l_2=3, \tag{78}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{l_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n}=2G_{-(n-2)}-G_{-(n-3)}$$
$$H_{-n}=2H_{-(n-2)}-H_{-(n-3)}$$
$$l_{-n}=2l_{-(n-2)}-l_{-(n-3)}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences eq. (76) - eq. (78) hold for all integer n.

G_n,H_n and l_n are the sequences A000071, A001612, A001595 in [8] , respectively.

For all integers n, modified Leonardo, Leonardo-Lucas and Leonardo numbers can be expressed using Binet's formulas as

$$G_n=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}=\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}-1$$
$$H_n=\alpha^n+\beta^n+\gamma^n=\alpha^n+\beta^n+1$$
$$l_n=\frac{2(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta}-1$$

respectively. Here, $G_n:=G_n$ and $H_n:=H_n$.

Next, we present sum formulas of generalized Leonardo numbers.

Theorem 5.1.
For $n\geq 0$, we have the following sum formulas for generalized Leonardo numbers:

(a) $\sum_{k=0}^{n}W_k^2=\frac{1}{2}((2n+7)W_{n+3}^2+(2n+9)W_{n+2}^2+(2n+7)W_{n+1}^2-4(n+4)W_{n+3}W_{n+2}-4(n+4)W_{n+3}W_{n+1}+4(n+5)W_{n+2}W_{n+1}-5W_2^2-7W_1^2-5W_0^2+12W_1W_2+12W_0W_2-16W_0W_1).$

(b) $\sum_{k=0}^{n}W_{k+1}W_k=\frac{1}{2}(2(n+3)W_{n+3}^2+2(n+5)W_{n+2}^2+2(n+4)W_{n+1}^2-(4n+15)W_{n+2}W_{n+3}-(4n+13)W_{n+3}W_{n+1}+(4n+15)W_{n+2}W_{n+1}-4W_2^2-8W_1^2-6W_0^2+11W_1W_2+9W_0W_2-11W_0W_1).$

(c) $\sum_{k=0}^{n}W_{k+2}W_k=\frac{1}{2}(2(n+2)W_{n+3}^2+2(n+4)W_{n+2}^2+2(n+3)W_{n+1}^2-(4n+11)W_{n+3}W_{n+2}-(4n+11)W_{n+3}W_{n+1}+(4n+15)W_{n+2}W_{n+1}-2W_2^2-6W_1^2-4W_0^2+7W_1W_2+7W_0W_2-11W_0W_1).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3-2x^2+1=\left(x^2-x-1\right)(x-1)=0$ whose roots are

$$\alpha=\frac{1+\sqrt{5}}{2},\ \beta=\frac{1-\sqrt{5}}{2},\ \gamma=1,$$

with $\alpha\neq\beta\neq\gamma$. In theorem 2.1, for $r=2,\ s=0,\ t=-1$,

$$\begin{aligned}\Gamma(z)&=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)\\&=-(z-1)(z+1)(z^2+z-1)(z^2-3z+1)\end{aligned}$$

and $\Gamma(1)=0$. Here 1 is the root of $\Gamma(z)=0$ with multiplicity 1.

(a) Use theorem 2.1 (a) (ii) with $z=1$.

(b) Use theorem 2.1 (b) (ii) with $z=1$.

(c) Use theorem 2.1 (c) (ii) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of modified Leonardo numbers (take $W_n=G_n$ with $G_0=0,G_1=1,G_2=2$).

Corollary 5.1.
For $n\geq 0$, modified Leonardo numbers have the following properties.

(a) $\sum\limits_{k=0}^{n}G_k^2=\frac{1}{2}((2n+7)G_{n+3}^2+(2n+9)G_{n+2}^2+(2n+7)G_{n+1}^2-4(n+4)G_{n+3}G_{n+2}-4(n+4)G_{n+3}G_{n+1}+4(n+5)G_{n+2}G_{n+1}-3).$

(b) $\sum\limits_{k=0}^{n}G_{k+1}G_k=\frac{1}{2}(2(n+3)G_{n+3}^2+2(n+5)G_{n+2}^2+2(n+4)G_{n+1}^2-(4n+15)G_{n+2}G_{n+3}-(4n+13)G_{n+3}G_{n+1}+(4n+15)G_{n+2}G_{n+1}-2).$

(c) $\sum\limits_{k=0}^{n}G_{k+2}G_k=\frac{1}{2}(2(n+2)G_{n+3}^2+2(n+4)G_{n+2}^2+2(n+3)G_{n+1}^2-(4n+11)G_{n+3}G_{n+2}-(4n+11)G_{n+3}G_{n+1}+(4n+15)G_{n+2}G_{n+1}).$

Taking $W_n=H_n$ with $H_0=3,H_1=2,H_2=4$ in the last Theorem, we have the following Corollary which gives sum formulas of Leonardo-Lucas numbers.

Corollary 5.2.
For $n\geq 0$, Leonardo-Lucas numbers have the following properties:

(a) $\sum\limits_{k=0}^{n}H_k^2=\frac{1}{2}((2n+7)H_{n+3}^2+(2n+9)H_{n+2}^2+(2n+7)H_{n+1}^2-4(n+4)H_{n+3}H_{n+2}-4(n+4)H_{n+3}H_{n+1}+4(n+5)H_{n+2}H_{n+1}-9).$

(b) $\sum\limits_{k=0}^{n}H_{k+1}H_k=\frac{1}{2}(2(n+3)H_{n+3}^2+2(n+5)H_{n+2}^2+2(n+4)H_{n+1}^2-(4n+15)H_{n+2}H_{n+3}-(4n+13)H_{n+3}H_{n+1}+(4n+15)H_{n+2}H_{n+1}-20).$

(c) $\sum\limits_{k=0}^{n}H_{k+2}H_k=\frac{1}{2}(2(n+2)H_{n+3}^2+2(n+4)H_{n+2}^2+2(n+3)H_{n+1}^2-(4n+11)H_{n+3}H_{n+2}-(4n+11)H_{n+3}H_{n+1}+(4n+15)H_{n+2}H_{n+1}-18).$

From the last Theorem, we have the following Corollary which gives sum formulas of Leonardo numbers (take $W_n=l_n$ with $l_0=1,l_1=1,l_2=3$).

Corollary 5.3.
For $n\geq 0$, Leonardo numbers have the following properties.

(a) $\sum\limits_{k=0}^{n}l_k^2=\frac{1}{2}((2n+7)l_{n+3}^2+(2n+9)l_{n+2}^2+(2n+7)l_{n+1}^2-4(n+4)l_{n+3}l_{n+2}-4(n+4)l_{n+3}l_{n+1}+4(n+5)l_{n+2}l_{n+1}-1).$

(b) $\sum\limits_{k=0}^{n}l_{k+1}l_k=\frac{1}{2}(2(n+3)l_{n+3}^2+2(n+5)l_{n+2}^2+2(n+4)l_{n+1}^2-(4n+15)l_{n+2}l_{n+3}-(4n+13)l_{n+3}l_{n+1}+(4n+15)l_{n+2}l_{n+1}-1).$

(c) $\sum\limits_{k=0}^{n} l_{k+2}l_k = \frac{1}{2}(2(n+2)l_{n+3}^2+2(n+4)l_{n+2}^2+2(n+3)l_{n+1}^2-(4n+11)l_{n+3}l_{n+2}-(4n+11)l_{n+3}l_{n+1}+(4n+15)l_{n+2}l_{n+1}+3).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 5.2.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = 0.381966$. *Then the ordinary generating functions of the sequences* $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ *are given as follows:*

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1}((z^5-2z^4-z^2)W_2^2+(4z^5-7z^4-2z^3+4z^2-z)W_1^2+(4z^4-7z^3-2z^2+4z-1)W_0^2+(8z^4-4z^5)W_2W_1++4z^3W_0W_2-4z^4W_0W_1).$

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1}(-2z^2W_2^2-2z^4W_1^2-2z^3W_0^2+(z^4+4z^2-z)W_1W_2+(-z^5+4z^3+z^2)W_0W_2+(2z^5-7z^3-2z^2+4z-1)W_0W_1).$

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1}(2(z^3-2z^2)W_2^2+2(z^5-2z^4)W_1^2+2(z^4-2z^3)W_0^2+(-z^5+2z^4-4z^3+9z^2-2z)W_1W_2+(z^3+4z-1)W_0W_2+(-z^4-4z^2+z)W_0W_1).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 5.4.
Assume that $|z| < |\alpha|^{-2} = 0.381966$. *The ordinary generating functions of the sequences* $\{G_n^2\}$, $\{G_{n+1}G_n\}$, $\{G_{n+2}G_n\}$ *and* $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ *and* $\{l_n^2\}$, $\{l_{n+1}l_n\}$, $\{l_{n+2}l_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} G_n^2 z^n = \frac{z^4-2z^3-z}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1},$$
$$\sum_{n=0}^{\infty} H_n^2 z^n = \frac{16z^4-23z^3-18z^2+32z-9}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1},$$
$$\sum_{n=0}^{\infty} l_n^2 z^n = \frac{z^5-z^4+3z^3-7z^2+3z-1}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1}.$$

(b)

$$\sum_{n=0}^{\infty} G_{n+1}G_n z^n = \frac{-2z}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1},$$
$$\sum_{n=0}^{\infty} H_{n+1}H_n z^n = \frac{-12z^3+16z-6}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1},$$
$$\sum_{n=0}^{\infty} l_{n+1}l_n z^n = \frac{-z^5+z^4+3z^3-5z^2+z-1}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1}.$$

(c)

$$\sum_{n=0}^{\infty} G_{n+2}G_n z^n = \frac{2z^2-4z}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1},$$
$$\sum_{n=0}^{\infty} H_{n+2}H_n z^n = \frac{12z^4-24z^3-16z^2+38z-12}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1},$$
$$\sum_{n=0}^{\infty} l_{n+2}l_n z^n = \frac{-z^5+3z^4+5z^3-13z^2+7z-3}{-z^6+2z^5+4z^4-6z^3-2z^2+4z-1}.$$

From the last corollary, we obtain the following results for modified Leonardo, Leonardo-Lucas and Leonardo numbers.

Corollary 5.5.
Some infinite sums of $\{G_n^2\}$, $\{G_{n+1}G_n\}$, $\{G_{n+2}G_n\}$ *and* $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ *and* $\{l_n^2\}$, $\{l_{n+1}l_n\}$, $\{l_{n+2}l_n\}$ *are given as follows:*

(a) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{G_n^2}{4^n} = \frac{1136}{825},$$
$$\sum_{n=0}^{\infty} \frac{H_n^2}{4^n} = \frac{1984}{165},$$
$$\sum_{n=0}^{\infty} \frac{l_n^2}{4^n} = \frac{2636}{825}.$$

(b) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{G_{n+1}G_n}{4^n} = \frac{2048}{825},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+1}H_n}{4^n} = \frac{1792}{165},$$
$$\sum_{n=0}^{\infty} \frac{l_{n+1}l_n}{4^n} = \frac{4148}{825}.$$

(c) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{G_{n+2}G_n}{4^n} = \frac{3584}{825},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+2}H_n}{4^n} = \frac{3136}{165},$$
$$\sum_{n=0}^{\infty} \frac{l_{n+2}l_n}{4^n} = \frac{8084}{825}.$$

5.2. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Ernst Numbers

In this subsection, we consider the case $r = 2, s = 1, t = -2$. A generalized Ernst sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} + W_{n-2} - 2W_{n-3} \tag{79}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{1}{2}W_{-(n-1)} + W_{-(n-2)} - \frac{1}{2}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (79) holds for all integer n. For more information on generalized Ernst numbers, see Soykan [10].

The Binet formula of generalized Ernst numbers can be given as

$$W_n = \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{80}$$
$$= \frac{z_1}{3}\alpha^n + \frac{z_2}{6}\beta^n - \frac{z_3}{2}$$

where

$$z_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 = W_2 - W_0$$
$$z_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 = W_2 - 3W_1 + 2W_0$$
$$z_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 = W_2 - W_1 - 2W_0$$

i.e.,

$$W_n = \frac{W_2 - W_0}{3}\alpha^n + \frac{W_2 - 3W_1 + 2W_0}{6}\beta^n - \frac{W_2 - W_1 - 2W_0}{2}.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 2x^2 - x + 2 = (x^2 - x - 2)(x - 1) = (x - 2)(x + 1)(x - 1) = 0.$$

Moreover

$$\alpha = 2,$$
$$\beta = -1,$$
$$\gamma = 1.$$

Note that

$$\alpha + \beta + \gamma = 2,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -1,$$
$$\alpha\beta\gamma = -2,$$
$$\alpha - \beta = 3.$$

Now we define two special cases of the sequence $\{W_n\}$. Ernst sequence $\{E_n\}_{n\geq 0}$ and Ernst-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$E_n = 2E_{n-1} + E_{n-2} - 2E_{n-3}, \; E_0 = 0, E_1 = 1, \; E_3 = 2, \tag{81}$$

$$H_n = 2H_{n-1} + H_{n-2} - 2H_{n-3}, \;\; H_0 = 3, H_1 = 2, \; H_0 = 6. \tag{82}$$

The sequences $\{E_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$E_{-n} = \frac{1}{2}E_{-(n-1)} + E_{-(n-2)} - \frac{1}{2}E_{-(n-3)}$$
$$H_{-n} = \frac{1}{2}H_{-(n-1)} + H_{-(n-2)} - \frac{1}{2}H_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences eq. (81)- eq. (82) hold for all integer n.

For all integers n, Ernst and Ernst-Lucas numbers can be expressed using Binet's formulas as

$$E_n = \frac{2}{3}\alpha^n - \frac{1}{6}\beta^n - \frac{1}{2}$$
$$H_n = \alpha^n + \beta^n + 1$$

respectively. Here, $G_n = E_n$ and $H_n := H_n$.

Next, we present sum formulas of generalized Ernst numbers.

Theorem 5.3.
For $n \geq 0$, we have the following sum formulas for generalized Ernst numbers:

(a) $\sum\limits_{k=0}^{n} W_k^2 = \frac{1}{18}((5n+23)W_{n+3}^2 + 9(n+6)W_{n+2}^2 + 2(10n+47)W_{n+1}^2 - 4(3n+16)W_{n+3}W_{n+2} + 4(3n+16)W_{n+2}W_{n+1} - 8(2n+9)W_{n+3}W_{n+1} - 18W_2^2 - 45W_1^2 - 74W_0^2 + 52W_1W_2 + 56W_0W_2 - 52W_0W_1).$

(b) $\sum\limits_{k=0}^{n} W_{k+1}W_k = \frac{1}{18}(2(2n+7)W_{n+3}^2 + 8(2n+9)W_{n+1}^2 - (6n+23)W_{n+2}W_{n+3} - 2(10n+43)W_{n+3}W_{n+1} + 2(12n+61)W_{n+2}W_{n+1} - 10W_2^2 - 56W_0^2 + 17W_1W_2 + 66W_0W_2 - 98W_0W_1).$

(c) $\sum\limits_{k=0}^{n} W_{k+2}W_k = \frac{1}{18}(5(n+3)W_{n+3}^2 + 9(n+6)W_{n+2}^2 + 20(n+4)W_{n+1}^2 - 4(3n+13)W_{n+2}W_{n+3} - 2(8n+25)W_{n+3}W_{n+1} + 4(3n+13)W_{n+2}W_{n+1} - 10W_2^2 - 45W_1^2 - 60W_0^2 + 40W_1W_2 + 34W_0W_2 - 40W_0W_1).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation

$$x^3 - 2x^2 - x + 2 = (x^2 - x - 2)(x - 1) = (x - 2)(x + 1)(x - 1) = 0$$

whose roots are

$$\alpha = 2,\ \beta = -1,\ \gamma = 1.$$

with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 2, s = 1, t = -2$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)\\ &= -(z-1)^2(4z-1)(2z+1)(2z-1)(z+1)\end{aligned}$$

and $\Gamma(1) = 0$. Here 1 is the root of $\Gamma(z) = 0$ with multiplicity 2.

(a) Use theorem 2.1 (a) (iii) with $z = 1$.

(b) Use theorem 2.1 (b) (iii) with $z = 1$.

(c) Use theorem 2.1 (c) (iii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Ernst numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_3 = 2$).

Corollary 5.6.
For $n \geq 0$, Ernst numbers have the following properties.

(a) $\sum_{k=0}^{n} E_k^2 = \frac{1}{18}((5n+23)E_{n+3}^2 + 9(n+6)E_{n+2}^2 + 2(10n+47)E_{n+1}^2 - 4(3n+16)E_{n+3}E_{n+2} + 4(3n+16)E_{n+2}E_{n+1} - 8(2n+9)E_{n+3}E_{n+1} - 13)$.

(b) $\sum_{k=0}^{n} E_{k+1}E_k = \frac{1}{18}(2(2n+7)E_{n+3}^2 + 8(2n+9)E_{n+1}^2 - (6n+23)E_{n+2}E_{n+3} - 2(10n+43)E_{n+3}E_{n+1} + 2(12n+61)E_{n+2}E_{n+1} - 6)$.

(c) $\sum_{k=0}^{n} E_{k+2}E_k = \frac{1}{18}(5(n+3)E_{n+3}^2 + 9(n+6)E_{n+2}^2 + 20(n+4)E_{n+1}^2 - 4(3n+13)E_{n+2}E_{n+3} - 2(8n+25)E_{n+3}E_{n+1} + 4(3n+13)E_{n+2}E_{n+1} - 5)$.

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_0 = 6$ in the last Theorem, we have the following Corollary which gives sum formulas of Ernst-Lucas numbers.

Corollary 5.7.
For $n \geq 0$, Ernst-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{18}((5n+23)H_{n+3}^2 + 9(n+6)H_{n+2}^2 + 2(10n+47)H_{n+1}^2 - 4(3n+16)H_{n+3}H_{n+2} + 4(3n+16)H_{n+2}H_{n+1} - 8(2n+9)H_{n+3}H_{n+1} - 174)$.

(b) $\sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{18}(2(2n+7)H_{n+3}^2 + 8(2n+9)H_{n+1}^2 - (6n+23)H_{n+2}H_{n+3} - 2(10n+43)H_{n+3}H_{n+1} + 2(12n+61)H_{n+2}H_{n+1} - 60)$.

(c) $\sum_{k=0}^{n} H_{k+2}H_k = \frac{1}{18}(5(n+3)H_{n+3}^2 + 9(n+6)H_{n+2}^2 + 20(n+4)H_{n+1}^2 - 4(3n+13)H_{n+2}H_{n+3} - 2(8n+25)H_{n+3}H_{n+1} + 4(3n+13)H_{n+2}H_{n+1} - 228)$.

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 5.4.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = 0.25$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1}((4z^5-4z^4+z^3-z^2)W_2^2+(16z^5-20z^4+5z^2-z)W_1^2+(4z^5+20z^4-29z^3+z^2+5z-1)W_0^2+(-16z^5+20z^4-4z^3)W_1W_2+(8z^3-8z^5)W_0W_2+4(4z^5-5z^4+z^3)W_0W_1).$

(b) $\sum_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1}(2(z^4-z^2)W_2^2+8(z^5-z^3)W_0^2+(-4z^4+z^3+4z^2-z)W_1W_2+2(-4z^5-z^4+4z^3+z^2)W_0W_2+(16z^5-21z^3+z^2+5z-1)W_0W_1).$

(c) $\sum_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1}(5(z^3-z^2)W_2^2+(16z^5-20z^4+5z^2-z)W_1^2+20(z^4-z^3)W_0^2+2(-4z^5+5z^4-5z^3+5z^2-z)W_1W_2+(-4z^4-5z^3+5z^2+5z-1)W_0W_2+2(4z^5-5z^4+5z^3-5z^2+z)W_0W_1).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 5.8.

Assume that $|z| < |\alpha|^{-2} = |\alpha|^{-2} = 0.25$. The ordinary generating functions of the sequences $\{E_n^2\}$, $\{E_{n+1}E_n\}$, $\{E_{n+2}E_n\}$ and $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ are given as follows:

(a)

$$\sum_{n=0}^{\infty} E_n^2 z^n = \frac{4z^4-4z^3+z^2-z}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1},$$
$$\sum_{n=0}^{\infty} H_n^2 z^n = \frac{4z^5+76z^4-105z^3-7z^2+41z-9}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1}.$$

(b)

$$\sum_{n=0}^{\infty} E_{n+1}E_n z^n = \frac{2z^3-2z}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1},$$
$$\sum_{n=0}^{\infty} H_{n+1}H_n z^n = \frac{24z^5-12z^4-42z^3+18z^2+18z-6}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1}.$$

(c)

$$\sum_{n=0}^{\infty} E_{n+2}E_n z^n = \frac{5z^2-5z}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1},$$
$$\sum_{n=0}^{\infty} H_{n+2}H_n z^n = \frac{16z^5+88z^4-150z^3-10z^2+74z-18}{-16z^6+20z^5+16z^4-25z^3+z^2+5z-1}.$$

From the last corollary, we obtain the following results for Ernst and Ernst-Lucas numbers.

Corollary 5.9.

Some infinite sums of $\{E_n^2\}$, $\{E_{n+1}E_n\}$, $\{E_{n+2}E_n\}$ and $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ are given as follows:

(a) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{E_n^2}{8^n} = \frac{272}{945},$$
$$\sum_{n=0}^{\infty} \frac{H_n^2}{8^n} = \frac{3254}{315}.$$

(b) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{E_{n+1}E_n}{8^n} = \frac{64}{105},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+1}H_n}{8^n} = \frac{44}{5}.$$

(c) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{E_{n+2}E_n}{8^n} = \frac{256}{189},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+2}H_n}{8^n} = \frac{1432}{63}.$$

5.3. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1} W_k$, $\sum_{k=0}^{n} z^k W_{k+2} W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1} W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2} W_n z^n$ of Generalized Edouard Numbers

In this subsection, we consider the case $r = 7, s = -7, t = 1$. A generalized Edouard sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3} \tag{83}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 7W_{-(n-1)} - 7W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (83) holds for all integer n. For more information on generalized Edouard numbers, see Soykan [11].

Binet's formula of generalized Edouard numbers can be given as

$$\begin{aligned} W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} - \frac{z_3}{4} \end{aligned}$$

where

$$\begin{aligned} z_1 &= W_2 - (\beta+1)W_1 + \beta W_0, \\ z_2 &= W_2 - (\alpha+1)W_1 + \alpha W_0, \\ z_3 &= W_2 - 6W_1 + W_0. \end{aligned}$$

i.e.,

$$W_n = \frac{(W_2 - (\beta+1)W_1 + \beta W_0)\alpha^n}{(\alpha-\beta)(\alpha-1)} + \frac{(W_2 - (\alpha+1)W_1 + \alpha W_0)\beta^n}{(\beta-\alpha)(\beta-1)} - \frac{(W_2 - 6W_1 + W_0)}{4}.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 7x^2 + 7x - 1 = \left(x^2 - 6x + 1\right)(x-1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2}, \\ \beta &= 3 - 2\sqrt{2}, \\ \gamma &= 1, \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 7, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 7, \\ \alpha\beta\gamma &= 1, \end{aligned}$$

or

$$\alpha + \beta = 6, \ \alpha\beta = 1.$$

Now we define two special cases of the sequence $\{W_n\}$. Edouard sequence $\{E_n\}_{n\geq 0}$ and Edouard-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \quad E_0 = 0, E_1 = 1, E_2 = 7, \tag{84}$$

$$K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, \quad K_0 = 3, K_1 = 7, K_2 = 35. \tag{85}$$

The sequences $\{E_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} E_{-n} &= 7E_{-(n-1)} - 7E_{-(n-2)} + E_{-(n-3)}, \\ K_{-n} &= 7K_{-(n-1)} - 7K_{-(n-2)} + K_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences eq. (84)- eq. (85) hold for all integer n.

E_n and K_n are the sequences A053142, A081555 in [8], respectively.

For all integers n, Edouard and Edouard-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned} E_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}, \\ K_n &= \alpha^n + \beta^n + 1 \end{aligned}$$

respectively. Here, $G_n = E_n$ and $H_n = K_n$.

Next, we present sum formulas of generalized Edouard numbers.

Theorem 5.5.
For $n \geq 0$, we have the following sum formulas for generalized Edouard numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{64}(5(n+4)W_{n+3}^2 + 4(38n+163)W_{n+2}^2 + (5n-39)W_{n+1}^2 - 7(8n+33)W_{n+3}W_{n+2} + 7(2n+9)W_{n+3}W_{n+1} - 7(8n+39)W_{n+2}W_{n+1} - 15W_2^2 - 500W_1^2 + 44W_0^2 + 175W_1W_2 - 49W_0W_2 + 217W_0W_1)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{128}(7(2n+7)W_{n+3}^2 + 336(n+4)W_{n+2}^2 + 7(2n+9)W_{n+1}^2 - 2(72n+265)W_{n+3}W_{n+2} + 52(n+4)W_{n+3}W_{n+1} - 6(24n+125)W_{n+2}W_{n+1} - 35W_2^2 - 1008W_1^2 - 49W_0^2 + 386W_1W_2 - 156W_0W_2 + 606W_0W_1)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{64}(21(n+3)W_{n+3}^2 + 140(2n+7)W_{n+2}^2 + 21(n+4)W_{n+1}^2 - 23(8n+25)W_{n+3}W_{n+2} + (110n+353)W_{n+3}W_{n+1} - 23(8n+31)W_{n+2}W_{n+1} - 42W_2^2 - 700W_1^2 - 63W_0^2 + 391W_1W_2 - 243W_0W_2 + 529W_0W_1)$.

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - 7x^2 + 7x - 1 = \left(x^2 - 6x + 1\right)(x-1) = 0$ whose roots are

$$\alpha = 3 + 2\sqrt{2},\ \beta = 3 - 2\sqrt{2},\ \gamma = 1,$$

with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 7, s = -7, t = 1$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) \\ &= -(z-1)^2(z^2 - 6z + 1)(z^2 - 34z + 1)\end{aligned}$$

and $\Gamma(1) = 0$. Here 1 is the root of $\Gamma(z) = 0$ with multiplicity 2.

(a) Use theorem 2.1 (a) (iii) with $z = 1$.

(b) Use theorem 2.1 (b) (iii) with $z = 1$.

(c) Use theorem 2.1 (c) (iii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Edouard numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 7$).

Corollary 5.10.
For $n \geq 0$, Edouard numbers have the following properties.

(a) $\sum_{k=0}^{n} E_k^2 = \frac{1}{64}(5(n+4)E_{n+3}^2 + 4(38n+163)E_{n+2}^2 + (5n-39)E_{n+1}^2 - 7(8n+33)E_{n+3}E_{n+2} + 7(2n+9)E_{n+3}E_{n+1} - 7(8n+39)E_{n+2}E_{n+1} - 10)$.

(b) $\sum_{k=0}^{n} E_{k+1}E_k = \frac{1}{128}(7(2n+7)E_{n+3}^2 + 336(n+4)E_{n+2}^2 + 7(2n+9)E_{n+1}^2 - 2(72n+265)E_{n+3}E_{n+2} + 52(n+4)E_{n+3}E_{n+1} - 6(24n+125)E_{n+2}E_{n+1} - 21)$.

(c) $\sum_{k=0}^{n} E_{k+2}E_k = \frac{1}{64}(21(n+3)E_{n+3}^2 + 140(2n+7)E_{n+2}^2 + 21(n+4)E_{n+1}^2 - 23(8n+25)E_{n+3}E_{n+2} + (110n+353)E_{n+3}E_{n+1} - 23(8n+31)E_{n+2}E_{n+1} - 21)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 7, K_2 = 35$ in the last Theorem, we have the following Corollary which gives sum formulas of Edouard-Lucas numbers.

Corollary 5.11.
For $n \geq 0$, Edouard-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} K_k^2 = \frac{1}{64}(5(n+4)K_{n+3}^2 + 4(38n+163)K_{n+2}^2 + (5n-39)K_{n+1}^2 - 7(8n+33)K_{n+3}K_{n+2} + 7(2n+9)K_{n+3}K_{n+1} - 7(8n+39)K_{n+2}K_{n+1} - 192)$.

(b) $\sum_{k=0}^{n} K_{k+1}K_k = \frac{1}{128}(7(2n+7)K_{n+3}^2 + 336(n+4)K_{n+2}^2 + 7(2n+9)K_{n+1}^2 - 2(72n+265)K_{n+3}K_{n+2} + 52(n+4)K_{n+3}K_{n+1} - 6(24n+125)K_{n+2}K_{n+1} - 1792)$.

(c) $\sum\limits_{k=0}^{n} K_{k+2}K_k = \frac{1}{64}(21(n+3)K_{n+3}^2 + 140(2n+7)K_{n+2}^2 + 21(n+4)K_{n+1}^2 - 23(8n+25)K_{n+3}K_{n+2} + (110n+353)K_{n+3}K_{n+1} - 23(8n+31)K_{n+2}K_{n+1} - 4928).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 5.6.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = (3+2\sqrt{2})^{-2} \simeq 0.029437$. *Then the ordinary generating functions of the sequences* $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ *are given as follows:*

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1}((z^5+7z^4-7z^3-z^2)W_2^2 + (49z^5+246z^4-336z^3+42z^2-z)W_1^2 + (49z^5-294z^4+491z^3-287z^2+42z-1)W_0^2 + 14(-z^5-6z^4+7z^3)W_1W_2 + 14(z^5-z^3)W_0W_2 + 14(-7z^5+6z^4+z^3)W_0W_1).$

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1}(7(z^4-z^2)W_2^2 + 336(z^4-z^3)W_1^2 + 7(z^5-z^3)W_0^2 + (-97z^4+49z^3+49z^2-z)W_1W_2 + (z^5+49z^4-49z^3-z^2)W_0W_2 + (-7z^5-336z^4+589z^3-287z^2+42z-1)W_0W_1).$

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1}(42(z^3-z^2)W_2^2 + 7(-z^5+42z^4-42z^2+z)W_1^2 + 42(z^4-z^3)W_0^2 + (z^5-42z^4-294z^3+342z^2-7z)W_1W_2 + (7z^4+246z^3-294z^2+42z-1)W_0W_2 + (7z^5-342z^4+294z^3+42z^2-z)W_0W_1).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 5.12.
Assume that $|z| < |\alpha|^{-2} \simeq 0.029437$. *The ordinary generating functions of the sequences* $\{E_n^2\}$, $\{E_{n+1}E_n\}$, $\{E_{n+2}E_n\}$ *and* $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} E_n^2 z^n = \frac{z^4+7z^3-7z^2-z}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1},$$
$$\sum_{n=0}^{\infty} K_n^2 z^n = \frac{49z^5-833z^4+2214z^3-1750z^2+329z-9}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1}.$$

(b)

$$\sum_{n=0}^{\infty} E_{n+1}E_n z^n = \frac{7z^3-7z}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1},$$
$$\sum_{n=0}^{\infty} K_{n+1}K_n z^n = \frac{21z^5-637z^4+2702z^3-2702z^2+637z-21}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1}.$$

(c)

$$\sum_{n=0}^{\infty} E_{n+2}E_n z^n = \frac{42z^2-42z}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1},$$
$$\sum_{n=0}^{\infty} K_{n+2}K_n z^n = \frac{49z^5-1953z^4+11046z^3-12054z^2+3017z-105}{-z^6+42z^5-287z^4+492z^3-287z^2+42z-1}.$$

From the last corollary, we obtain the following results for Edouard and Edouard-Lucas numbers.

Corollary 5.13.
Some infinite sums of $\{E_n^2\}$, $\{E_{n+1}E_n\}$, $\{E_{n+2}E_n\}$ *and* $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ *are given as follows:*

(a) $z = \frac{1}{50}$.

$$\sum_{n=0}^{\infty} \frac{E_n^2}{50^n} = \frac{2417500}{28795683},$$
$$\sum_{n=0}^{\infty} \frac{K_n^2}{50^n} = \frac{329764150}{28795683}.$$

(b) $z = \frac{1}{50}$.

$$\sum_{n=0}^{\infty} \frac{E_{n+1}E_n}{50^n} = \frac{2125000}{4113669},$$
$$\sum_{n=0}^{\infty} \frac{K_{n+1}K_n}{50^n} = \frac{141510050}{4113669}.$$

(c) $z = \frac{1}{50}$.

$$\sum_{n=0}^{\infty} \frac{E_{n+2}E_n}{50^n} = \frac{12500000}{4113669},$$
$$\sum_{n=0}^{\infty} \frac{K_{n+2}K_n}{50^n} = \frac{750023450}{4113669}.$$

5.4. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized John Numbers

In this subsection, we consider the case $r = 3, s = -1, t = -1$. A generalized John sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-3} \tag{86}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + 3W_{-(n-2)} - W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence eq. (86) holds for all integer n. For more information on generalized John numbers, see Soykan [12].

Binet formula of generalized John numbers can be given as

$$\begin{aligned} W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^n + z_2\beta^n}{4} - \frac{z_3}{2} \\ &= \frac{z_1\alpha^n + z_2\beta^n - 2z_3}{4} \end{aligned} \tag{87}$$

where

$$\begin{aligned} z_1 &= W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 = W_2 - (\beta+1)W_1 + \beta W_0, \\ z_2 &= W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 = W_2 - (\alpha+1)W_1 + \alpha W_0, \\ z_3 &= W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 = W_2 - 2W_1 - W_0, \end{aligned}$$

i.e.,

$$W_n = \frac{(W_2 - (\beta+1)W_1 + \beta W_0)\alpha^n + (W_2 - (\alpha+1)W_1 + \alpha W_0)\beta^n - 2(W_2 - 2W_1 - W_0)}{4}.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 3x^2 + x + 1 = (x^2 - 2x - 1)(x - 1) = 0.$$

Moreover

$$\alpha = 1+\sqrt{2},$$
$$\beta = 1-\sqrt{2},$$
$$\gamma = 1.$$

Note that

$$\alpha+\beta+\gamma = 3,$$
$$\alpha\beta+\alpha\gamma+\beta\gamma = 1,$$
$$\alpha\beta\gamma = -1,$$

i.e

$$\alpha+\beta = 2,\ \alpha\beta = -1,\ \alpha-\beta = 2\sqrt{2}$$

and

$$(\alpha-\beta)(\alpha-\gamma) = 4,$$
$$(\beta-\alpha)(\beta-\gamma) = 4,$$
$$(\gamma-\alpha)(\gamma-\beta) = -2.$$

Now we define two special cases of the sequence $\{W_n\}$. John sequence $\{J_n\}_{n\geq 0}$ and John-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$J_n = 3J_{n-1} - J_{n-2} - J_{n-3}, \quad J_0 = 0, J_1 = 1, J_2 = 3, \tag{88}$$
$$H_n = 3H_{n-1} - H_{n-2} - H_{n-3}, \quad H_0 = 3, H_1 = 3, H_2 = 7. \tag{89}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n} = -J_{-(n-1)} + 3J_{-(n-2)} - J_{-(n-3)}$$
$$H_{-n} = -H_{-(n-1)} + 3H_{-(n-2)} - H_{-(n-3)}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences eq. (88) - eq. (89) hold for all integer n.

For all integers n, John and John-Lucas numbers can be expressed using Binet's formulas as

$$J_n = \frac{\alpha^{n+1}+\beta^{n+1}-2}{4},$$
$$H_n = \alpha^n + \beta^n + 1,$$

respectively. Here, $G_n = J_n$ and $H_n := H_n$.

Next, we present sum formulas of generalized John numbers.

Theorem 5.7.
For $n \geq 0$, we have the following sum formulas for generalized John numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{4}((n+4)W_{n+3}^2 + 2(2n+9)W_{n+2}^2 + (n+1)W_{n+1}^2 - (4n+17)W_{n+3}W_{n+2} - (2n+9)W_{n+3}W_{n+1} + (4n+21)W_{n+2}W_{n+1} - 3W_2^2 - 14W_1^2 + 13W_1W_2 + 7W_0W_2 - 17W_0W_1).$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{8}((2n+7)W_{n+3}^2 + 2(4n+19)W_{n+2}^2 + (2n+9)W_{n+1}^2 - 8(n+4)W_{n+3}W_{n+2} - 2(2n+7)W_{n+3}W_{n+1} + 4(2n+7)W_{n+2}W_{n+1} - 5W_2^2 - 30W_1^2 - 7W_0^2 + 24W_1W_2 + 10W_0W_2 - 20W_0W_1).$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{4}((n+2)W_{n+3}^2 + 4(n+4)W_{n+2}^2 + (n+3)W_{n+1}^2 - (4n+11)W_{n+2}W_{n+3} - (2n+7)W_{n+3}W_{n+1} + (4n+15)W_{n+2}W_{n+1} - W_2^2 - 12W_1^2 - 2W_0^2 + 7W_1W_2 + 5W_0W_2 - 11W_0W_1).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3-3x^2+x+1 = (x^2-2x-1)(x-1) = 0$ whose roots are

$$\alpha = 1+\sqrt{2},\ \beta = 1-\sqrt{2}, \gamma = 1$$

with $\alpha \neq \beta \neq \gamma$. In theorem 2.1, for $r = 3, s = -1, t = -1,$

$$\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)$$
$$= -(z-1)(z+1)(z^2-6z+1)(z^2+2z-1)$$

and $\Gamma(1) = 0$. Here 1 is the root of $\Gamma(z) = 0$ with multiplicity 1.

(a) Use theorem 2.1 (a) (ii) with $z=1$.

(b) Use theorem 2.1 (b) (ii) with $z=1$.

(c) Use theorem 2.1 (c) (ii) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of John numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 3$).

Corollary 5.14.
For $n \geq 0$, John numbers have the following properties.

(a) $\sum\limits_{k=0}^{n} J_k^2 = \frac{1}{4}((n+4)J_{n+3}^2 + 2(2n+9)J_{n+2}^2 + (n+1)J_{n+1}^2 - (4n+17)J_{n+3}J_{n+2} - (2n+9)J_{n+3}J_{n+1} + (4n+21)J_{n+2}J_{n+1} - 2).$

(b) $\sum\limits_{k=0}^{n} J_{k+1}J_k = \frac{1}{8}((2n+7)J_{n+3}^2 + 2(4n+19)J_{n+2}^2 + (2n+9)J_{n+1}^2 - 8(n+4)J_{n+3}J_{n+2} - 2(2n+7)J_{n+3}J_{n+1} + 4(2n+7)J_{n+2}J_{n+1} - 3).$

(c) $\sum\limits_{k=0}^{n} J_{k+2}J_k = \frac{1}{4}((n+2)J_{n+3}^2 + 4(n+4)J_{n+2}^2 + (n+3)J_{n+1}^2 - (4n+11)J_{n+2}J_{n+3} - (2n+7)J_{n+3}J_{n+1} + (4n+15)J_{n+2}J_{n+1}).$

Taking $W_n = H_n$ with $H_0 = H_0 = 3, H_1 = 3, H_2 = 7$ in the last Theorem, we have the following Corollary which gives sum formulas of John-Lucas numbers.

Corollary 5.15.
For $n \geq 0$, John-Lucas numbers have the following properties:

(a) $\sum\limits_{k=0}^{n} H_k^2 = \frac{1}{4}((n+4)H_{n+3}^2 + 2(2n+9)H_{n+2}^2 + (n+1)H_{n+1}^2 - (4n+17)H_{n+3}H_{n+2} - (2n+9)H_{n+3}H_{n+1} + (4n+21)H_{n+2}H_{n+1} - 6).$

(b) $\sum\limits_{k=0}^{n} H_{k+1}H_k = \frac{1}{8}((2n+7)H_{n+3}^2 + 2(4n+19)H_{n+2}^2 + (2n+9)H_{n+1}^2 - 8(n+4)H_{n+3}H_{n+2} - 2(2n+7)H_{n+3}H_{n+1} + 4(2n+7)H_{n+2}H_{n+1} - 44).$

(c) $\sum\limits_{k=0}^{n} H_{k+2}H_k = \frac{1}{4}((n+2)H_{n+3}^2 + 4(n+4)H_{n+2}^2 + (n+3)H_{n+1}^2 - (4n+11)H_{n+2}H_{n+3} - (2n+7)H_{n+3}H_{n+1} + (4n+15) H_{n+2}H_{n+1} - 22).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 5.8.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = 0.171572$. *Then the ordinary generating functions of the sequences* $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ *are given as follows:*

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-z^6+4z^5+13z^4-12z^3-11z^2+8z-1}((z^5-3z^4-z^3-z^2)W_2^2 + (9z^5-20z^4-12z^3+8z^2-z)W_1^2 + (z^5+12z^4-13z^3-11z^2+8z-1)W_0^2 + 2(-3z^5+8z^4+3z^3)W_1W_2 + 2(z^5+3z^3)W_0W_2 - 2(3z^5+4z^4+z^3)W_0W_1).$

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-z^6+4z^5+13z^4-12z^3-11z^2+8z-1}(-(z^4+3z^2)W_2^2 - 4(3z^4+z^3)W_1^2 - (z^5+3z^3)W_0^2 + (7z^4+z^3+9z^2-z)W_1W_2 + (-z^5-z^4+9z^3+z^2)W_0W_2 + (3z^5+4z^4-19z^3-11z^2+8z-1)W_0W_1).$

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-z^6+4z^5+13z^4-12z^3-11z^2+8z-1}(4(z^3-2z^2)W_2^2 + (3z^5-12z^4-8z^2+z)W_1^2 + 4(z^4-2z^3)W_0^2 + (-z^5+4z^4-12z^3+28z^2-3z)W_1W_2 + (z^4+8z^3-8z^2+8z-1)W_0W_2 + (-z^5-8z^3-8z^2+z)W_0W_1)).$

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 5.16.
Assume that $|z| < |\alpha|^{-2} = 0.171572$. *The ordinary generating functions of the sequences* $\{J_n^2\}$, $\{J_{n+1}J_n\}$, $\{J_{n+2}J_n\}$ *and* $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} J_n^2 z^n = \frac{z^4 - 3z^3 - z^2 - z}{-z^6 + 4z^5 + 13z^4 - 12z^3 - 11z^2 + 8z - 1},$$
$$\sum_{n=0}^{\infty} H_n^2 z^n = \frac{z^5 + 45z^4 - 40z^3 - 76z^2 + 63z - 9}{-z^6 + 4z^5 + 13z^4 - 12z^3 - 11z^2 + 8z - 1}.$$

(b)

$$\sum_{n=0}^{\infty} J_{n+1}J_n z^n = \frac{-z^3 - 3z}{-z^6 + 4z^5 + 13z^4 - 12z^3 - 11z^2 + 8z - 1},$$
$$\sum_{n=0}^{\infty} H_{n+1}H_n z^n = \frac{-3z^5 + 5z^4 - 24z^3 - 36z^2 + 51z - 9}{-z^6 + 4z^5 + 13z^4 - 12z^3 - 11z^2 + 8z - 1}.$$

(c)

$$\sum_{n=0}^{\infty} J_{n+2}J_n z^n = \frac{4z^2 - 8z}{-z^6 + 4z^5 + 13z^4 - 12z^3 - 11z^2 + 8z - 1},$$
$$\sum_{n=0}^{\infty} H_{n+2}H_n z^n = \frac{-3z^5 + 33z^4 - 32z^3 - 116z^2 + 123z - 21}{-z^6 + 4z^5 + 13z^4 - 12z^3 - 11z^2 + 8z - 1}.$$

From the last corollary, we obtain the following results for John and John-Lucas numbers.

Corollary 5.17.
Some infinite sums of $\{J_n^2\}$, $\{J_{n+1}J_n\}$, $\{J_{n+2}J_n\}$ *and* $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ *are given as follows:*

(a) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{J_n^2}{8^n} = \frac{38336}{50337},$$
$$\sum_{n=0}^{\infty} \frac{H_n^2}{8^n} = \frac{623800}{50337}.$$

(b) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{J_{n+1}J_n}{8^n} = \frac{98816}{50337},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+1}H_n}{8^n} = \frac{847576}{50337}.$$

(c) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{J_{n+2}J_n}{8^n} = \frac{81920}{16779},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+2}H_n}{8^n} = \frac{654664}{16779}.$$

5.5. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1} W_k$, $\sum_{k=0}^{n} z^k W_{k+2} W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1} W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2} W_n z^n$ of Generalized Pisano Numbers

In this subsection, we consider the case $r = 2, s = -\frac{5}{4}, t = \frac{1}{4}$. A generalized Pisano sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} - \frac{5}{4}W_{n-2} + \frac{1}{4}W_{n-3} \tag{90}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 5W_{-(n-1)} - 8W_{-(n-2)} + 4W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (90) holds for all integers n. For more information on generalized Pisano numbers, see [13].

Characteristic equation of the third-order recurrence sequence W_n is the cubic equation

$$x^3 - 2x^2 + \frac{5}{4}x - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 (x-1) = 0$$

whose roots are

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{2}, \quad \gamma = 1.$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 2, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= \frac{5}{4}, \\ \alpha\beta\gamma &= \frac{1}{4}, \end{aligned}$$

or

$$\alpha + \beta = 1, \; \alpha\beta = \frac{1}{4}.$$

Binet formula of generalized Pisano numbers can be given as

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3 \gamma^n$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2} = -4W_2 + 4W_1, \\ A_2 &= \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)} = -4W_2 + 6W_1 - 2W_0, \\ A_3 &= \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2} = 4W_2 - 4W_1 + W_0, \end{aligned}$$

i.e.,

$$W_n = ((-4W_2 + 4W_1) + (-4W_2 + 6W_1 - 2W_0)n) \times \left(\frac{1}{2}\right)^n + (4W_2 - 4W_1 + W_0).$$

Now, we define two special cases of the sequence $\{W_n\}$. Pisano sequence $\{P_n\}_{n\geq 0}$ and Pisano–Lucas sequence $\{R_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$P_n = 2P_{n-1} - \frac{5}{4}P_{n-2} + \frac{1}{4}P_{n-3}, \quad P_0 = 0, P_1 = 1, P_2 = 2, \tag{91}$$

$$R_n = 2R_{n-1} - \frac{5}{4}R_{n-2} + \frac{1}{4}R_{n-3}, \quad R_0 = 3, R_1 = 2, R_2 = \frac{3}{2}. \tag{92}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = 5P_{-(n-1)} - 8P_{-(n-2)} + 4P_{-(n-3)},$$
$$R_{-n} = 5R_{-(n-1)} - 8R_{-(n-2)} + 4R_{-(n-3)},$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences eq. (91) - eq. (92) hold for all integer n.

For all integers n, Pisano and Pisano–Lucas numbers can be expressed using Binet's formulas as

$$P_n = -(n+2)\times 2^{-n+1} + 4$$
$$R_n = 2^{-n+1} + 1$$

respectively. Here, $G_n = P_n$ and $H_n = R_n$.

Next, we present sum formulas of generalized Pisano numbers.

Theorem 5.9.
For $n \geq 0$, we have the following sum formulas for generalized Pisano numbers:

(a) $\sum\limits_{k=0}^{n} W_k^2 = \frac{1}{27}(16(27n+82)W_{n+3}^2 + 144(3n+11)W_{n+2}^2 + (27n+82)W_{n+1}^2 - 96(9n+31)W_{n+3}W_{n+2} + 8(27n+98)W_{n+3}W_{n+1} - 24(9n+35)W_{n+2}W_{n+1} - 880W_2^2 - 1152W_1^2 - 55W_0^2 + 2112W_1W_2 - 568W_0W_2 + 624W_0W_1).$

(b) $\sum\limits_{k=0}^{n} W_{k+1}W_k = \frac{1}{27}(16(27n+71)W_{n+3}^2 + 144(3n+10)W_{n+2}^2 + (27n+98)W_{n+1}^2 - 48(18n+55)W_{n+3}W_{n+2} + 4(54n+173)W_{n+3}W_{n+1} - 12(18n+65)W_{n+2}W_{n+1} - 704W_2^2 - 1008W_1^2 - 71W_0^2 + 1776W_1W_2 - 476W_0W_2 + 564W_0W_1).$

(c) $\sum\limits_{k=0}^{n} W_{k+2}W_k = \frac{1}{27}(16(27n+64)W_{n+3}^2 + 72(6n+19)W_{n+2}^2 + (27n+91)W_{n+1}^2 - 12(72n+203)W_{n+3}W_{n+2} + 4(54n+151)W_{n+3}W_{n+1} - 3(72n+235)W_{n+2}W_{n+1} - 592W_2^2 - 936W_1^2 - 64W_0^2 + 1572W_1W_2 - 388W_0W_2 + 489W_0W_1).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - 2x^2 + \frac{5}{4}x - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 (x-1) = 0$ whose roots are

$$\alpha = \frac{1}{2},\ \beta = \frac{1}{2},\ \gamma = 1,$$

with $\alpha = \beta \neq \gamma$. In theorem 2.1, for $r = 2, s = -\frac{5}{4}, t = \frac{1}{4}$,

$$\Gamma(z) = (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)$$
$$= -\frac{1}{256}(z-1)(z-2)^2(z-4)^3$$

and $\Gamma(1) = 0$. Here 1 is the root of $\Gamma(z) = 0$ with multiplicity 1.

(a) Use theorem 2.1 (a) (ii) with $z = 1$.

(b) Use theorem 2.1 (b) (ii) with $z = 1$.

(c) Use theorem 2.1 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Pisano numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2$).

Corollary 5.18.
For $n \geq 0$, Pisano numbers have the following properties.

(a) $\sum\limits_{k=0}^{n} P_k^2 = \frac{1}{27}(16(27n+82)P_{n+3}^2 + 144(3n+11)P_{n+2}^2 + (27n+82)P_{n+1}^2 - 96(9n+31)P_{n+3}P_{n+2} + 8(27n+98)P_{n+3}P_{n+1} - 24(9n+35)P_{n+2}P_{n+1} - 448).$

(b) $\sum\limits_{k=0}^{n} P_{k+1}P_k = \frac{1}{27}(16(27n+71)P_{n+3}^2 + 144(3n+10)P_{n+2}^2 + (27n+98)P_{n+1}^2 - 48(18n+55)P_{n+3}P_{n+2} + 4(54n+173)P_{n+3}P_{n+1} - 12(18n+65)P_{n+2}P_{n+1} - 272).$

(c) $\sum\limits_{k=0}^{n} P_{k+2}P_k = \frac{1}{27}(16(27n+64)P_{n+3}^2 + 72(6n+19)P_{n+2}^2 + (27n+91)P_{n+1}^2 - 12(72n+203)P_{n+3}P_{n+2} + 4(54n+151)P_{n+3}P_{n+1} - 3(72n+235)P_{n+2}P_{n+1} - 160).$

Taking $W_n = R_n$ with $R_0 = 3, R_1 = 2, R_2 = \frac{3}{2}$ in the last Theorem, we have the following Corollary which gives sum formulas of Pisano-Lucas numbers.

Corollary 5.19.
For $n \geq 0$, Pisano-Lucas numbers have the following properties:

(a) $\sum\limits_{k=0}^{n} R_k^2 = \frac{1}{27}(16(27n+82)R_{n+3}^2 + 144(3n+11)R_{n+2}^2 + (27n+82)R_{n+1}^2 - 96(9n+31)R_{n+3}R_{n+2} + 8(27n+98)R_{n+3}R_{n+1} - 24(9n+35)R_{n+2}R_{n+1} + 441).$

(b) $\sum\limits_{k=0}^{n} R_{k+1}R_k = \frac{1}{27}(16(27n+71)R_{n+3}^2 + 144(3n+10)R_{n+2}^2 + (27n+98)R_{n+1}^2 - 48(18n+55)R_{n+3}R_{n+2} + 4(54n+173)R_{n+3}R_{n+1} - 12(18n+65)R_{n+2}R_{n+1} + 315).$

(c) $\sum\limits_{k=0}^{n} R_{k+2}R_k = \frac{1}{27}(16(27n+64)R_{n+3}^2 + 72(6n+19)R_{n+2}^2 + (27n+91)R_{n+1}^2 - 12(72n+203)R_{n+3}R_{n+2} + 4(54n+151)R_{n+3}R_{n+1} - 3(72n+235)R_{n+2}R_{n+1} + 252).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 5.10.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\gamma|^{-2} = 1$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-\frac{1}{256}z^6 + \frac{17}{256}z^5 - \frac{29}{64}z^4 + \frac{101}{64}z^3 - \frac{47}{16}z^2 + \frac{11}{4}z - 1}(\frac{1}{16}(z^5 + 8z^4 - 20z^3 - 16z^2)W_2^2 + \frac{1}{16}(4z^5 + 13z^4 - 72z^3 + 44z^2 - 16z)W_1^2 + \frac{1}{256}(25z^5 - 136z^4 + 388z^3 - 752z^2 + 704z - 256)W_0^2 + \frac{1}{8}(-2z^5 - 11z^4 + 40z^3)W_1W_2 + \frac{1}{32}(5z^5 - 32z^3)W_0W_2 + \frac{1}{32}(-10z^5 + 17z^4 + 20z^3)W_0W_1).$

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-\frac{1}{256}z^6 + \frac{17}{256}z^5 - \frac{29}{64}z^4 + \frac{101}{64}z^3 - \frac{47}{16}z^2 + \frac{11}{4}z - 1}(\frac{1}{16}(5z^4 - 32z^2)W_2^2 + \frac{9}{16}(2z^4 - 5z^3)W_1^2 + \frac{1}{256}(5z^5 - 32z^3)W_0^2 + \frac{1}{16}(-19z^4 + 25z^3 + 64z^2 - 16z)W_1W_2 + \frac{1}{64}(z^5 + 25z^4 - 64z^3 - 16z^2)W_0W_2 + \frac{1}{64}(-2z^5 - 45z^4 + 177z^3 - 188z^2 + 176z - 64)W_0W_1).$

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-\frac{1}{256}z^6 + \frac{17}{256}z^5 - \frac{29}{64}z^4 + \frac{101}{64}z^3 - \frac{47}{16}z^2 + \frac{11}{4}z - 1}(\frac{1}{16}(17z^3 - 44z^2)W_2^2 + \frac{1}{32}(-z^5 + 17z^4 - 110z^2 + 40z)W_1^2 + \frac{1}{256}(17z^4 - 44z^3)W_0^2 + \frac{1}{64}(z^5 - 17z^4 - 136z^3 + 496z^2 - 128z)W_1W_2 + \frac{1}{64}(5z^4 + 49z^3 - 220z^2 + 176z - 64)W_0W_2 + \frac{1}{256}(5z^5 - 121z^4 + 220z^3 + 176z^2 - 64z)W_0W_1).$

Proof. Use theorem 3.1. □
Now, we consider special cases of the last Theorem.

Corollary 5.20.
Assume that $|z| < |\gamma|^{-2} = 1$. The ordinary generating functions of the sequences $\{P_n^2\}$, $\{P_{n+1}P_n\}$, $\{P_{n+2}P_n\}$ and $\{R_n^2\}$, $\{R_{n+1}R_n\}$, $\{R_{n+2}R_n\}$ are given as follows:

(a)

$$\sum_{n=0}^{\infty} P_n^2 z^n = \frac{16z^4 + 128z^3 - 320z^2 - 256z}{-z^6 + 17z^5 - 116z^4 + 404z^3 - 752z^2 + 704z - 256},$$
$$\sum_{n=0}^{\infty} R_n^2 z^n = \frac{25z^2 - 94z + 72}{-z^3 + 7z^2 - 14z + 8}.$$

(b)

$$\sum_{n=0}^{\infty} P_{n+1}P_n z^n = \frac{80z^3 - 512z}{-z^6 + 17z^5 - 116z^4 + 404z^3 - 752z^2 + 704z - 256},$$
$$\sum_{n=0}^{\infty} R_{n+1}R_n z^n = \frac{15z^2 - 60z + 48}{-z^3 + 7z^2 - 14z + 8}.$$

(c)

$$\sum_{n=0}^{\infty} P_{n+2}P_n z^n = \frac{16(17z^2-44z)}{-z^6+17z^5-116z^4+404z^3-752z^2+704z-256},$$
$$\sum_{n=0}^{\infty} R_{n+2}R_n z^n = \frac{10z^2-43z+36}{-z^3+7z^2-14z+8}.$$

From the last corollary, we obtain the following results for Pisano and Pisano-Lucas numbers.

Corollary 5.21.
Some infinite sums of $\{P_n^2\}$, $\{P_{n+1}P_n\}$, $\{P_{n+2}P_n\}$ *and* $\{R_n^2\}$, $\{R_{n+1}R_n\}$, $\{R_{n+2}R_n\}$ *are given as follows:*

(a) $z=\frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{P_n^2}{2^n} = \frac{12224}{3087},$$
$$\sum_{n=0}^{\infty} \frac{R_n^2}{2^n} = \frac{250}{21}.$$

(b) $z=\frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{P_{n+1}P_n}{2^n} = \frac{5248}{1029},$$
$$\sum_{n=0}^{\infty} \frac{R_{n+1}R_n}{2^n} = \frac{58}{7}.$$

(c) $z=\frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{P_{n+2}P_n}{2^n} = \frac{18176}{3087},$$
$$\sum_{n=0}^{\infty} \frac{R_{n+2}R_n}{2^n} = \frac{136}{21}.$$

5.6. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Bigollo Numbers

In this subsection, we consider the case $r=4, s=-5, t=2$. A generalized Bigollo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 4W_{n-1} - 5W_{n-2} + 2W_{n-3} \tag{93}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{5}{2}W_{-(n-1)} - 2W_{-(n-2)} + \frac{1}{2}W_{-(n-3)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence eq. (93) holds for all integer n. For more information on generalized Bigollo numbers, see [14].

Binet formula of generalized Bigollo numbers (two distinct roots case: $\alpha \neq \beta = \gamma$) can be given as

$$W_n = (A_1 + A_2 n)\times \beta^n + A_3 \times \alpha^n = (A_1 + A_2 n) + A_3 \times 2^n$$

where

$$A_1 = \frac{-W_2 + 2\beta W_1 - \alpha(2\beta-\alpha)W_0}{(\beta-\alpha)^2} = -W_2 + 2W_1,$$
$$A_2 = \frac{W_2 - (\beta+\alpha)W_1 + \beta\alpha W_0}{\beta(\beta-\alpha)} = -W_2 + 3W_1 - 2W_0,$$
$$A_3 = \frac{W_2 - 2\beta W_1 + \beta^2 W_0}{(\beta-\alpha)^2} = W_2 - 2W_1 + W_0,$$

i.e.

$$W_n = ((-W_2 + 2W_1) + (-W_2 + 3W_1 - 2W_0)n) + (W_2 - 2W_1 + W_0) \times 2^n.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 4x^2 + 5x - 2 = (x^2 - 3x + 2)(x - 1) = (x - 2)(x - 1)(x - 1) = 0.$$

Moreover

$$\alpha = 2,$$
$$\beta = 1,$$
$$\gamma = 1.$$

Note that

$$\alpha + \beta + \gamma = 4,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = 5,$$
$$\alpha\beta\gamma = 2.$$

Now we define two special cases of the sequence $\{W_n\}$. Bigollo sequence $\{B_n\}_{n\geq 0}$ and Bigollo-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, \quad B_0 = 0, B_1 = 1, B_2 = 4, \tag{94}$$
$$C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, \quad C_0 = 3, C_1 = 4, C_2 = 6. \tag{95}$$

The sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$B_{-n} = \frac{5}{2}B_{-(n-1)} - 2B_{-(n-2)} + \frac{1}{2}B_{-(n-3)},$$
$$C_{-n} = \frac{5}{2}C_{-(n-1)} - 2C_{-(n-2)} + \frac{1}{2}C_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences eq. (94) - eq. (95) hold for all integer n.

B_n and C_n are the sequences A000295 (Eulerian numbers), A052548 in [**?**], respectively.

For all integers n, Bigollo and Bigollo-Lucas numbers can be expressed using Binet's formulas as

$$B_n = 2^{n+1} - n - 2,$$
$$C_n = 2^n + 2,$$

respectively. Here, $G_n = B_n$ and $H_n = C_n$.

Next, we present sum formulas of generalized Bigollo numbers.

Theorem 5.11.
For $n \geq 0$, we have the following sum formulas for generalized Bigollo numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{6}((2n^3+35n^2+177n+278)W_{n+3}^2 + (18n^3+333n^2+1809n+3046)W_{n+2}^2 + 2(2n+9)(2n^2+32n+109)W_{n+1}^2 - 4(n+4)(3n^2+42n+115)W_{n+3}W_{n+2} + 8(n+4)(n^2+15n+46)W_{n+3}W_{n+1} - 4(n+4)(6n^2+93n+305)W_{n+1}W_{n+2} - 134W_2^2 - 1552W_1^2 - 1106W_0^2 + 912W_1W_2 - 768W_0W_2 + 2616W_0W_1).$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{3}((n+3)(n^2+13n+32)W_{n+3}^2 + 9(n+10)(n+4)(n+3)W_{n+2}^2 + 4(n+4)(n^2+15n+46)W_{n+1}^2 - (6n^3+99n^2+458n+644)W_{n+3}W_{n+2} + 2(2n^3+35n^2+174n+265)W_{n+3}W_{n+1} - (12n^3+216n^2+1120n+1779)W_{n+2}W_{n+1} - 40W_2^2 - 486W_1^2 - 384W_0^2 + 279W_1W_2 - 248W_0W_2 + 863W_0W_1).$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{6}((2n+19)(n+3)(n+2)W_{n+3}^2 + (18n^3+279n^2+1143n+1330)W_{n+2}^2 + 4(2n+21)(n+4)(n+3)W_{n+1}^2 - 4(3n^3+45n^2+175n+195)W_{n+3}W_{n+2} + 2(4n^3+64n^2+272n+341)W_{n+3}W_{n+1} - 4(6n^3+99n^2+443n+579)W_{n+1}W_{n+2} - 34W_2^2 - 448W_1^2 - 456W_0^2 + 248W_1W_2 - 258W_0W_2 + 916W_0W_1).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation

$$x^3-4x^2+5x-2=(x^2-3x+2)(x-1)=(x-2)(x-1)(x-1)=0,$$

whose roots are

$$\alpha=2,\ \beta=1,\ \gamma=1,$$

with $\alpha\neq\beta=\gamma$. In theorem 2.1, for $r=4, s=-5, t=2$,

$$\begin{aligned}\Gamma(z)&=(-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1)\\&=-(z-1)^3(2z-1)^2(4z-1)\end{aligned}$$

and $\Gamma(1)=0$. Here 1 is the root of $\Gamma(z)=0$ with multiplicity 3.

(a) Use theorem 2.1 (a) (iv) with $z=1$.

(b) Use theorem 2.1 (b) (iv) with $z=1$.

(c) Use theorem 2.1 (c) (iv) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Bigollo numbers (take $W_n=B_n$ with $B_0=0, B_1=1, B_2=4$).

Corollary 5.22.
For $n\geq 0$, Bigollo numbers have the following properties.

(a) $\sum\limits_{k=0}^{n}B_k^2=\frac{1}{6}((2n^3+35n^2+177n+278)B_{n+3}^2+(18n^3+333n^2+1809n+3046)B_{n+2}^2+2(2n+9)(2n^2+32n+109)B_{n+1}^2-4(n+4)(3n^2+42n+115)B_{n+3}B_{n+2}+8(n+4)(n^2+15n+46)B_{n+3}B_{n+1}-4(n+4)(6n^2+93n+305)B_{n+1}B_{n+2}-48).$

(b) $\sum\limits_{k=0}^{n}B_{k+1}B_k=\frac{1}{3}((n+3)(n^2+13n+32)B_{n+3}^2+9(n+10)(n+4)(n+3)B_{n+2}^2+4(n+4)(n^2+15n+46)B_{n+1}^2-(6n^3+99n^2+458n+644)B_{n+3}B_{n+2}+2(2n^3+35n^2+174n+265)B_{n+3}B_{n+1}-(12n^3+216n^2+1120n+1779)B_{n+2}B_{n+1}-10).$

(c) $\sum\limits_{k=0}^{n}B_{k+2}B_k=\frac{1}{6}((2n+19)(n+3)(n+2)B_{n+3}^2+(18n^3+279n^2+1143n+1330)B_{n+2}^2+4(2n+21)(n+4)(n+3)B_{n+1}^2-4(3n^3+45n^2+175n+195)B_{n+3}B_{n+2}+2(4n^3+64n^2+272n+341)B_{n+3}B_{n+1}-4(6n^3+99n^2+443n+579)B_{n+1}B_{n+2}).$

Taking $W_n=C_n$ with $C_0=3, C_1=4, C_2=6$ in the last Theorem, we have the following Corollary which gives sum formulas of Bigollo-Lucas numbers.

Corollary 5.23.
For $n\geq 0$, Bigollo-Lucas numbers have the following properties:

(a) $\sum\limits_{k=0}^{n}C_k^2=\frac{1}{6}((2n^3+35n^2+177n+278)C_{n+3}^2+(18n^3+333n^2+1809n+3046)C_{n+2}^2+2(2n+9)(2n^2+32n+109)C_{n+1}^2-4(n+4)(3n^2+42n+115)C_{n+3}C_{n+2}+8(n+4)(n^2+15n+46)C_{n+3}C_{n+1}-4(n+4)(6n^2+93n+305)C_{n+1}C_{n+2}-154).$

(b) $\sum\limits_{k=0}^{n}C_{k+1}C_k=\frac{1}{3}((n+3)(n^2+13n+32)C_{n+3}^2+9(n+10)(n+4)(n+3)C_{n+2}^2+4(n+4)(n^2+15n+46)C_{n+1}^2-(6n^3+99n^2+458n+644)C_{n+3}C_{n+2}+2(2n^3+35n^2+174n+265)C_{n+3}C_{n+1}-(12n^3+216n^2+1120n+1779)C_{n+2}C_{n+1}-84).$

(c) $\sum\limits_{k=0}^{n}C_{k+2}C_k=\frac{1}{6}((2n+19)(n+3)(n+2)C_{n+3}^2+(18n^3+279n^2+1143n+1330)C_{n+2}^2+4(2n+21)(n+4)(n+3)C_{n+1}^2-4(3n^3+45n^2+175n+195)C_{n+3}C_{n+2}+2(4n^3+64n^2+272n+341)C_{n+3}C_{n+1}-4(6n^3+99n^2+443n+579)C_{n+1}C_{n+2}-196).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty}W_n^2z^n$, $\sum\limits_{n=0}^{\infty}W_{n+1}W_nz^n$, $\sum\limits_{n=0}^{\infty}W_{n+2}W_nz^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 5.12.
Assume that $|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2},|\alpha\beta|^{-1},|\alpha\gamma|^{-1},|\beta\gamma|^{-1}\}=|\alpha|^{-2}=0.25$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1}((4z^5+8z^4-5z^3-z^2)W_2^2+(64z^5+52z^4-72z^3+11z^2-z)W_1^2+(100z^5-136z^4+97z^3-47z^2+11z-1)W_0^2+4(-8z^5-11z^4+10z^3)W_1W_2+8(5z^5-2z^3)W_0W_2+4(-40z^5+17z^4+5z^3)W_0W_1)$.

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1}(2(5z^4-2z^2)W_2^2+18(8z^4-5z^3)W_1^2+8(5z^5-2z^3)W_0^2+(-76z^4+25z^3+16z^2-z)W_1W_2+2(4z^5+25z^4-16z^3-z^2)W_0W_2+(-32z^5-180z^4+177z^3-47z^2+11z-1)W_0W_1)$.

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1}((17z^3-11z^2)W_2^2+(-32z^5+136z^4-55z^2+5z)W_1^2+4(17z^4-11z^3)W_0^2+2(4z^5-17z^4-34z^3+31z^2-2z)W_1W_2+(20z^4+49z^3-55z^2+11z-1)W_0W_2+2(20z^5-121z^4+55z^3+11z^2-z)W_0W_1)$.

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 5.24.
Assume that $|z|<|\alpha|^{-2}=0.25$. *The ordinary generating functions of the sequences* $\{B_n^2\}$, $\{B_{n+1}B_n\}$, $\{B_{n+2}B_n\}$ *and* $\{C_n^2\}$, $\{C_{n+1}C_n\}$, $\{C_{n+2}C_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} B_n^2 z^n = \frac{4z^4+8z^3-5z^2-z}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1},$$
$$\sum_{n=0}^{\infty} C_n^2 z^n = \frac{100z^5-344z^4+453z^3-283z^2+83z-9}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1} = \frac{50z^2-47z+9}{-8z^3+14z^2-7z+1}.$$

(b)

$$\sum_{n=0}^{\infty} B_{n+1}B_n z^n = \frac{10z^3-4z}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1},$$
$$\sum_{n=0}^{\infty} C_{n+1}C_n z^n = \frac{120z^5-420z^4+564z^3-360z^2+108z-12}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1} = \frac{12(5z^2-5z+1)}{-8z^3+14z^2-7z+1}.$$

(c)

$$\sum_{n=0}^{\infty} B_{n+2}B_n z^n = \frac{17z^2-11z}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1},$$
$$\sum_{n=0}^{\infty} C_{n+2}C_n z^n = \frac{160z^5-572z^4+786z^3-514z^2+158z-18}{-16z^6+68z^5-116z^4+101z^3-47z^2+11z-1} = \frac{2(40z^2-43z+9)}{-8z^3+14z^2-7z+1}.$$

From the last corollary, we obtain the following results for Bigollo and Bigollo-Lucas numbers.

Corollary 5.25.
Some infinite sums of $\{B_n^2\}$, $\{B_{n+1}B_n\}$, $\{B_{n+2}B_n\}$ *and* $\{C_n^2\}$, $\{C_{n+1}C_n\}$, $\{C_{n+2}C_n\}$ *are given as follows:*

(a) $z=\frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{B_n^2}{8^n} = \frac{3056}{3087},$$
$$\sum_{n=0}^{\infty} \frac{C_n^2}{8^n} = \frac{250}{21}.$$

(b) $z=\frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{B_{n+1}B_n}{8^n} = \frac{2624}{1029},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+1}C_n}{8^n} = \frac{116}{7}.$$

(c) $z=\frac{1}{8}$.

$$\sum_{n=0}^{\infty}\frac{B_{n+2}B_n}{8^n}=\frac{18176}{3087},$$
$$\sum_{n=0}^{\infty}\frac{C_{n+2}C_n}{8^n}=\frac{544}{21}.$$

5.7. Sum Formulas $\sum_{k=0}^{n}z^kW_k^2$, $\sum_{k=0}^{n}z^kW_{k+1}W_k$, $\sum_{k=0}^{n}z^kW_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty}W_n^2z^n$, $\sum_{n=0}^{\infty}W_{n+1}W_nz^n$, $\sum_{n=0}^{\infty}W_{n+2}W_nz^n$ of Generalized Guglielmo Numbers

In this subsection, we consider the case $r=3,s=-3,t=1$. A generalized Guglielmo sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n=3W_{n-1}-3W_{n-2}+W_{n-3} \tag{96}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=3W_{-(n-1)}-3W_{-(n-2)}+W_{-(n-3)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence eq. (96) holds for all integer n. For more information on generalized Guglielmo numbers, see Soykan [15].

Binet formula of generalized Guglielmo numbers can be given as

$$W_n=A_1+A_2n+A_3n^2$$

where

$$A_1=W_0,$$
$$A_2=\frac{1}{2}(-W_2+4W_1-3W_0),$$
$$A_3=\frac{1}{2}(W_2-2W_1+W_0),$$

i.e.,

$$W_n=W_0+\frac{1}{2}(-W_2+4W_1-3W_0)n+\frac{1}{2}(W_2-2W_1+W_0)n^2. \tag{97}$$

Here, we use the roots α,β,γ of the cubic equation

$$x^3-3x^2+3x-1=(x-1)^3=0$$

where $\alpha=\beta=\gamma=1$.

Now we define four special cases of the sequence $\{W_n\}$. Triangular sequence $\{T_n\}_{n\geq 0}$,triangular-Lucas sequence $\{H_n\}_{n\geq 0}$, oblong sequence $\{O_n\}_{n\geq 0}$ and pentagonal sequence $\{p_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$T_n=3T_{n-1}-3T_{n-2}+T_{n-3}, \quad T_0=0,T_1=1,T_2=3, \tag{98}$$
$$H_n=3H_{n-1}-3H_{n-2}+H_{n-3}, \quad H_0=3,H_1=3,H_2=3, \tag{99}$$
$$O_n=3O_{n-1}-3O_{n-2}+O_{n-3}, \quad O_0=0,O_1=2,O_2=6, \tag{100}$$
$$p_n=3p_{n-1}-3p_{n-2}+p_{n-3}, \quad p_0=0,p_1=1,p_2=5. \tag{101}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{O_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n}=3T_{-(n-1)}-3T_{-(n-2)}+T_{-(n-3)},$$
$$H_{-n}=3H_{-(n-1)}-3H_{-(n-2)}+H_{-(n-3)},$$
$$O_{-n}=3O_{-(n-1)}-3O_{-(n-2)}+O_{-(n-3)},$$
$$p_{-n}=3p_{-(n-1)}-3p_{-(n-2)}+p_{-(n-3)},$$

for $n=1,2,3,\ldots$ respectively. Therefore, recurrences eq. (98)- eq. (101) hold for all integer n.

H_n is the constant sequence (the all 3's sequence) A010701 in [8].

For all integers n, triangular, triangular-Lucas, oblong and pentagonal numbers can be expressed using Binet's formulas as

$$
\begin{aligned}
T_n &= \frac{n(n+1)}{2}, \\
H_n &= 3, \\
O_n &= n(n+1), \\
p_n &= \frac{1}{2}n(3n-1),
\end{aligned}
$$

Here, $G_n = T_n$ and $H_n := H_n$.

Next, we present sum formulas of generalized Guglielmo numbers.

Theorem 5.13.
For $n \geq 0$, we have the following sum formulas for generalized Guglielmo numbers:

(a) $\sum\limits_{k=0}^{n} W_k^2 = \frac{1}{60}(n+1)((n+2)(n+3)(3n^2+12n+10)W_{n+3}^2+2(n+2)(6n^3+57n^2+167n+135)W_{n+2}^2+(3n^4+42n^3+223n^2+542n+540)W_{n+1}^2-3(n+2)(n+3)(n+4)(4n+5)W_{n+2}W_{n+3}+3(n+2)(n+3)(n+4)(2n+5)W_{n+1}W_{n+3}-3(n+2)(n+3)(n+4)(4n+15)W_{n+2}W_{n+1})$.

(b) $\sum\limits_{k=0}^{n} W_{k+1}W_k = \frac{1}{120}(n+1)(3n(n+2)(n+3)(2n+3)W_{n+3}^2+24n(n+2)(n+3)(n+4)W_{n+2}^2+3(n+2)(n+3)(n+4)(2n+5)W_{n+1}^2-2n(n+2)(12n^2+69n+89)W_{n+3}W_{n+2}+4(n+2)(n+3)(3n^2+12n+5)W_{n+3}W_{n+1}-2(12n^4+123n^3+437n^2+598n+180)W_{n+2}W_{n+1})$.

(c) $\sum\limits_{k=0}^{n} W_{k+2}W_k = \frac{1}{60}(n+1)(3(n-1)n(n+2)(n+3)W_{n+3}^2+6(n+2)(2n+3)(n^2+3n-5)W_{n+2}^2+3n(n+2)(n+3)(n+4)W_{n+1}^2-(n+2)(12n^3+39n^2-31n-30)W_{n+3}W_{n+2}+n(6n^3+39n^2+61n-16)W_{n+3}W_{n+1}-(n+2)(12n^3+69n^2+59n-90)W_{n+2}W_{n+1})$.

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3-3x^2+3x-1=(x-1)^3=0$ whose roots are $\alpha=1$, $\beta=1$, $\gamma=1$ with $\alpha=\beta=\gamma$. In theorem 2.1, for $r=3, s=-3, t=1$,

$$
\begin{aligned}
\Gamma(z) &= (-t^2z^3+sz+rtz^2+1)(r^2z-s^2z^2+t^2z^3+2sz+2rtz^2-1) \\
&= -(z-1)^6
\end{aligned}
$$

and $\Gamma(1)=0$. Here 1 is the root of $\Gamma(z)=0$ with multiplicity 6.

(a) Use theorem 2.1 (a) (vii) with $z=1$.

(b) Use theorem 2.1 (b) (vii) with $z=1$.

(c) Use theorem 2.1 (c) (vii) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of triangular numbers (take $W_n = T_n$ with $T_0=0, T_1=1, T_2=3$).

Corollary 5.26.
For $n \geq 0$, triangular numbers have the following properties.

(a) $\sum\limits_{k=0}^{n} T_k^2 = \frac{1}{60}(n+1)((n+2)(n+3)(3n^2+12n+10)T_{n+3}^2+2(n+2)(6n^3+57n^2+167n+135)T_{n+2}^2+(3n^4+42n^3+223n^2+542n+540)T_{n+1}^2-3(n+2)(n+3)(n+4)(4n+5)T_{n+2}T_{n+3}+3(n+2)(n+3)(n+4)(2n+5)T_{n+1}T_{n+3}-3(n+2)(n+3)(n+4)(4n+15)T_{n+2}T_{n+1})$.

(b) $\sum\limits_{k=0}^{n} T_{k+1}T_k = \frac{1}{120}(n+1)(3n(n+2)(n+3)(2n+3)T_{n+3}^2+24n(n+2)(n+3)(n+4)T_{n+2}^2+3(n+2)(n+3)(n+4)(2n+5)T_{n+1}^2-2n(n+2)(12n^2+69n+89)T_{n+3}T_{n+2}+4(n+2)(n+3)(3n^2+12n+5)T_{n+3}T_{n+1}-2(12n^4+123n^3+437n^2+598n+180)T_{n+2}T_{n+1})$.

(c) $\sum\limits_{k=0}^{n} T_{k+2}T_k = \frac{1}{60}(n+1)(3(n-1)n(n+2)(n+3)T_{n+3}^2+6(n+2)(2n+3)(n^2+3n-5)T_{n+2}^2+3n(n+2)(n+3)(n+4)T_{n+1}^2-(n+2)(12n^3+39n^2-31n-30)T_{n+3}T_{n+2}+n(6n^3+39n^2+61n-16)T_{n+3}T_{n+1}-(n+2)(12n^3+69n^2+59n-90)T_{n+2}T_{n+1})$.

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 3, H_2 = 3$ in the last Theorem, we have the following Corollary which gives sum formulas of triangular-Lucas numbers.

Corollary 5.27.
For $n \geq 0$, triangular-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{60}(n+1)((n+2)(n+3)(3n^2+12n+10)H_{n+3}^2+2(n+2)(6n^3+57n^2+167n+135)H_{n+2}^2+(3n^4+42n^3+223n^2+542n+540)H_{n+1}^2-3(n+2)(n+3)(n+4)(4n+5)H_{n+2}H_{n+3}+3(n+2)(n+3)(n+4)(2n+5)H_{n+1}H_{n+3}-3(n+2)(n+3)(n+4)(4n+15)H_{n+2}H_{n+1})$.

(b) $\sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{120}(n+1)(3n(n+2)(n+3)(2n+3)H_{n+3}^2+24n(n+2)(n+3)(n+4)H_{n+2}^2+3(n+2)(n+3)(n+4)(2n+5)H_{n+1}^2-2n(n+2)(12n^2+69n+89)H_{n+3}H_{n+2}+4(n+2)(n+3)(3n^2+12n+5)H_{n+3}H_{n+1}-2(12n^4+123n^3+437n^2+598n+180)H_{n+2}H_{n+1})$.

(c) $\sum_{k=0}^{n} H_{k+2}H_k = \frac{1}{60}(n+1)(3(n-1)n(n+2)(n+3)H_{n+3}^2+6(n+2)(2n+3)(n^2+3n-5)H_{n+2}^2+3n(n+2)(n+3)(n+4)H_{n+1}^2-(n+2)(12n^3+39n^2-31n-30)H_{n+3}H_{n+2}+n(6n^3+39n^2+61n-16)H_{n+3}H_{n+1}-(n+2)(12n^3+69n^2+59n-90)H_{n+2}H_{n+1})$.

From the last Theorem, we have the following Corollary which gives sum formulas of oblong numbers (take $W_n = O_n$ with $O_0 = 0, O_1 = 2, O_2 = 6$).

Corollary 5.28.
For $n \geq 0$, oblong numbers have the following properties.

(a) $\sum_{k=0}^{n} O_k^2 = \frac{1}{60}(n+1)((n+2)(n+3)(3n^2+12n+10)O_{n+3}^2+2(n+2)(6n^3+57n^2+167n+135)O_{n+2}^2+(3n^4+42n^3+223n^2+542n+540)O_{n+1}^2-3(n+2)(n+3)(n+4)(4n+5)O_{n+2}O_{n+3}+3(n+2)(n+3)(n+4)(2n+5)O_{n+1}O_{n+3}-3(n+2)(n+3)(n+4)(4n+15)O_{n+2}O_{n+1})$.

(b) $\sum_{k=0}^{n} O_{k+1}O_k = \frac{1}{120}(n+1)(3n(n+2)(n+3)(2n+3)O_{n+3}^2+24n(n+2)(n+3)(n+4)O_{n+2}^2+3(n+2)(n+3)(n+4)(2n+5)O_{n+1}^2-2n(n+2)(12n^2+69n+89)O_{n+3}O_{n+2}+4(n+2)(n+3)(3n^2+12n+5)O_{n+3}O_{n+1}-2(12n^4+123n^3+437n^2+598n+180)O_{n+2}O_{n+1})$.

(c) $\sum_{k=0}^{n} O_{k+2}O_k = \frac{1}{60}(n+1)(3(n-1)n(n+2)(n+3)O_{n+3}^2+6(n+2)(2n+3)(n^2+3n-5)O_{n+2}^2+3n(n+2)(n+3)(n+4)O_{n+1}^2-(n+2)(12n^3+39n^2-31n-30)O_{n+3}O_{n+2}+n(6n^3+39n^2+61n-16)O_{n+3}O_{n+1}-(n+2)(12n^3+69n^2+59n-90)O_{n+2}O_{n+1})$.

Taking $W_n = p_n$ with $p_0 = 0, p_1 = 1, p_2 = 5$ in the last Theorem, we have the following Corollary which gives sum formulas of pentagonal numbers.

Corollary 5.29.
For $n \geq 0$, pentagonal numbers have the following properties:

(a) $\sum_{k=0}^{n} p_k^2 = \frac{1}{60}(n+1)((n+2)(n+3)(3n^2+12n+10)p_{n+3}^2+2(n+2)(6n^3+57n^2+167n+135)p_{n+2}^2+(3n^4+42n^3+223n^2+542n+540)p_{n+1}^2-3(n+2)(n+3)(n+4)(4n+5)p_{n+2}p_{n+3}+3(n+2)(n+3)(n+4)(2n+5)p_{n+1}p_{n+3}-3(n+2)(n+3)(n+4)(4n+15)p_{n+2}p_{n+1})$.

(b) $\sum_{k=0}^{n} p_{k+1}p_k = \frac{1}{120}(n+1)(3n(n+2)(n+3)(2n+3)p_{n+3}^2+24n(n+2)(n+3)(n+4)p_{n+2}^2+3(n+2)(n+3)(n+4)(2n+5)p_{n+1}^2-2n(n+2)(12n^2+69n+89)p_{n+3}p_{n+2}+4(n+2)(n+3)(3n^2+12n+5)p_{n+3}p_{n+1}-2(12n^4+123n^3+437n^2+598n+180)p_{n+2}p_{n+1})$.

(c) $\sum_{k=0}^{n} p_{k+2}p_k = \frac{1}{60}(n+1)(3(n-1)n(n+2)(n+3)p_{n+3}^2+6(n+2)(2n+3)(n^2+3n-5)p_{n+2}^2+3n(n+2)(n+3)(n+4)p_{n+1}^2-(n+2)(12n^3+39n^2-31n-30)p_{n+3}p_{n+2}+n(6n^3+39n^2+61n-16)p_{n+3}p_{n+1}-(n+2)(12n^3+69n^2+59n-90)p_{n+2}p_{n+1})$.

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 5.14.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = 1$. *Then the ordinary generating functions of the sequences* $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ *are given as follows:*

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-(z-1)^6}((z^5+3z^4-3z^3-z^2)W_2^2+(9z^5+10z^4-24z^3+6z^2-z)W_1^2+(9z^5-18z^4+19z^3-15z^2+6z-1)W_0^2+6(-z^5-2z^4+3z^3)W_1W_2+6(z^5-z^3)W_0W_2+6(-3z^5+2z^4+z^3)W_0W_1)$.

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-(z-1)^6}(3(z^4-z^2)W_2^2+24(z^4-z^3)W_1^2+3(z^5-z^3)W_0^2+(-17z^4+9z^3+9z^2-z)W_1W_2+(z^5+9z^4-9z^3-z^2)W_0W_2+(-3z^5-24z^4+37z^3-15z^2+6z-1)W_0W_1)$.

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-(z-1)^6}(6(z^3-z^2)W_2^2+(-3z^5+18z^4-18z^2+3z)W_1^2+6(z^4-z^3)W_0^2+(z^5-6z^4-18z^3+26z^2-3z)W_1W_2+(3z^4+10z^3-18z^2+6z-1)W_0W_2+(3z^5-26z^4+18z^3+6z^2-z)W_0W_1)$.

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 5.30.
Assume that $|z| < 1$. *The ordinary generating functions of the sequences* $\{T_n^2\}$, $\{T_{n+1}T_n\}$, $\{T_{n+2}T_n\}$ *and* $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ *and* $\{O_n^2\}$, $\{O_{n+1}O_n\}$, $\{O_{n+2}O_n\}$ *and* $\{p_n^2\}$, $\{p_{n+1}p_n\}$, $\{p_{n+2}p_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} T_n^2 z^n = \frac{z^4+3z^3-3z^2-z}{-(z-1)^6},$$
$$\sum_{n=0}^{\infty} H_n^2 z^n = \frac{9z^5-45z^4+90z^3-90z^2+45z-9}{-(z-1)^6} = -\frac{9}{z-1},$$
$$\sum_{n=0}^{\infty} O_n^2 z^n = \frac{4z^4+12z^3-12z^2-4z}{-(z-1)^6} = -\frac{4(z^3+4z^2+z)}{(z-1)^5},$$
$$\sum_{n=0}^{\infty} p_n^2 z^n = \frac{4z^5+25z^4-9z^3-19z^2-z}{-(z-1)^6} = -\frac{(4z^4+29z^3+20z^2+z)}{(z-1)^5}.$$

(b)

$$\sum_{n=0}^{\infty} T_{n+1}T_n z^n = \frac{3z^3-3z}{-(z-1)^6} = -\frac{3(z^2+z)}{(z-1)^5},$$
$$\sum_{n=0}^{\infty} H_{n+1}H_n z^n = \frac{9z^5-45z^4+90z^3-90z^2+45z-9}{-(z-1)^6} = -\frac{9}{z-1},$$
$$\sum_{n=0}^{\infty} O_{n+1}O_n z^n = \frac{12z^3-12z}{-(z-1)^6} = -\frac{12(z^2+z)}{(z-1)^5},$$
$$\sum_{n=0}^{\infty} p_{n+1}p_n z^n = \frac{14z^4+21z^3-30z^2-5z}{-(z-1)^6} = -\frac{(14z^3+35z^2+5z)}{(z-1)^5}.$$

(c)

$$\sum_{n=0}^{\infty} T_{n+2}T_n z^n = \frac{6z^2-6z}{-(z-1)^6} = -\frac{6z}{(z-1)^5},$$
$$\sum_{n=0}^{\infty} H_{n+2}H_n z^n = \frac{9z^5-45z^4+90z^3-90z^2+45z-9}{-(z-1)^6} = -\frac{9}{z-1},$$
$$\sum_{n=0}^{\infty} O_{n+2}O_n z^n = \frac{24z^2-24z}{-(z-1)^6} = -\frac{24z}{(z-1)^5},$$
$$\sum_{n=0}^{\infty} p_{n+2}p_n z^n = \frac{2z^5-12z^4+60z^3-38z^2-12z}{-(z-1)^6} = -\frac{2(z^4-5z^3+25z^2+6z)}{(z-1)^5}.$$

From the last corollary, we obtain the following results for triangular,triangular-Lucas, oblong and pentagonal numbers.

Corollary 5.31.
Some infinite sums of $\{T_n^2\}$, $\{T_{n+1}T_n\}$, $\{T_{n+2}T_n\}$ *and* $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ *and* $\{O_n^2\}$, $\{O_{n+1}O_n\}$, $\{O_{n+2}O_n\}$ *and* $\{p_n^2\}$, $\{p_{n+1}p_n\}$, $\{p_{n+2}p_n\}$ *are given as follows:*

(a) $z=\frac{1}{2}$.

$$\sum_{n=0}^{\infty}\frac{T_n^2}{2^n}=52,$$
$$\sum_{n=0}^{\infty}\frac{H_n^2}{2^n}=18,$$
$$\sum_{n=0}^{\infty}\frac{O_n^2}{2^n}=208,$$
$$\sum_{n=0}^{\infty}\frac{p_n^2}{2^n}=300.$$

(b) $z=\frac{1}{2}$.

$$\sum_{n=0}^{\infty}\frac{T_{n+1}T_n}{2^n}=72,$$
$$\sum_{n=0}^{\infty}\frac{H_{n+1}H_n}{2^n}=18,$$
$$\sum_{n=0}^{\infty}\frac{O_{n+1}O_n}{2^n}=288,$$
$$\sum_{n=0}^{\infty}\frac{p_{n+1}p_n}{2^n}=416.$$

(c) $z=\frac{1}{2}$.

$$\sum_{n=0}^{\infty}\frac{T_{n+2}T_n}{2^n}=96,$$
$$\sum_{n=0}^{\infty}\frac{H_{n+2}H_n}{2^n}=18,$$
$$\sum_{n=0}^{\infty}\frac{O_{n+2}O_n}{2^n}=384,$$
$$\sum_{n=0}^{\infty}\frac{p_{n+2}p_n}{2^n}=556.$$

5.8. Sum Formulas $\sum_{k=0}^{n}z^kW_k^2$, $\sum_{k=0}^{n}z^kW_{k+1}W_k$, $\sum_{k=0}^{n}z^kW_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty}W_n^2z^n$, $\sum_{n=0}^{\infty}W_{n+1}W_nz^n$, $\sum_{n=0}^{\infty}W_{n+2}W_nz^n$ of Generalized Woodall Numbers

In this subsection, we consider the case $r=5, s=-8, t=4$. A generalized Woodall sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n=5W_{n-1}-8W_{n-2}+4W_{n-3} \tag{102}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=2W_{-(n-1)}-\frac{5}{4}W_{-(n-2)}+\frac{1}{4}W_{-(n-3)}$$

for $n=1,2,3,....$ Therefore, recurrence eq. (102) holds for all integer n. For more information on generalized Woodall numbers, see [16].

Binet formula of generalized Woodall numbers can be given as

(two distinct roots case: $\alpha = \beta \neq \gamma$)

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3 \gamma^n$$

where

$$A_1 = \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma) W_0}{(\alpha - \gamma)^2},$$
$$A_2 = \frac{W_2 - (\alpha + \gamma) W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)},$$
$$A_3 = \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2}.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2 (x-1) = 0.$$

Moreover

$$\alpha = \beta = 2,$$
$$\gamma = 1.$$

So,

$$W_n = (A_1 + A_2 n) \times 2^n + A_3$$

where

$$A_1 = -W_2 + 4W_1 - 3W_0,$$
$$A_2 = \frac{W_2 - 3W_1 + 2W_0}{2},$$
$$A_3 = W_2 - 4W_1 + 4W_0,$$

i.e.,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{103}$$

Now, we define four special cases of the sequence $\{W_n\}$. Modified Woodall sequence $\{G_n\}_{n\geq 0}$, modified Cullen sequence $\{H_n\}_{n\geq 0}$, Woodall sequence $\{R_n\}$ and Cullen sequence $\{C_n\}$ are defined, respectively, by the third-order recurrence relations

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \quad G_0 = 0, G_1 = 1, G_2 = 5, \tag{104}$$
$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \quad H_0 = 3, H_1 = 5, H_2 = 9, \tag{105}$$
$$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \quad R_0 = -1, R_1 = 1, R_2 = 7, \tag{106}$$
$$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \quad C_0 = 1, C_1 = 3, C_2 = 9. \tag{107}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = 2G_{-(n-1)} - \frac{5}{4} G_{-(n-2)} + \frac{1}{4} G_{-(n-3)},$$
$$H_{-n} = 2H_{-(n-1)} - \frac{5}{4} H_{-(n-2)} + \frac{1}{4} H_{-(n-3)},$$
$$R_{-n} = 2R_{-(n-1)} - \frac{5}{4} R_{-(n-2)} + \frac{1}{4} R_{-(n-3)},$$
$$C_{-n} = 2C_{-(n-1)} - \frac{5}{4} C_{-(n-2)} + \frac{1}{4} C_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences eq. (104) - eq. (107) hold for all integer n.

G_n, H_n, R_n and C_n are the sequences A000337, A000051 (and A048578), A003261 and A002064 in [8], respectively.

Note that $\{H_n\}$ satisfies the following second order linear recurrence:

$$H_n = 3H_{n-1} - 2H_{n-2}, \quad H_0 = 3, H_1 = 5$$

and satisfies the following first order non-linear recurrence:

$$H_n = 2H_{n-1} - 1, \quad H_0 = 3.$$

For all integers n, modified Woodall, modified Cullen, Woodall and Cullen numbers can be expressed using Binet's formulas as

$$G_n = (n-1)2^n + 1$$
$$H_n = 2^{n+1} + 1$$
$$R_n = n \times 2^n - 1$$
$$C_n = n \times 2^n + 1$$

respectively. Here, $G_n := G_n$ and $H_n := H_n$.

Next, we present sum formulas of generalized Woodall numbers.

Theorem 5.15.
For $n \geq 0$, we have the following sum formulas for generalized Woodall numbers:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{27}((27n+161)W_{n+3}^2 + 9(48n+301)W_{n+2}^2 + (432n+2981)W_{n+1}^2 - 24(9n+55)W_{n+3}W_{n+2} + 8(27n+172)W_{n+3}W_{n+1} - 96(9n+59)W_{n+2}W_{n+1} - 134W_2^2 - 2277W_1^2 - 2549W_0^2 + 1104W_1W_2 - 1160W_0W_2 + 4800W_0W_1).$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{27}((27n+145)W_{n+3}^2 + 144(3n+17)W_{n+2}^2 + 16(27n+172)W_{n+1}^2 - 3(72n+397)W_{n+3}W_{n+2} + 4(54n+313)W_{n+3}W_{n+1} - 3(288n+1721)W_{n+2}W_{n+1} - 118W_2^2 - 2016W_1^2 - 2320W_0^2 + 975W_1W_2 - 1036W_0W_2 + 4299W_0W_1).$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{27}((27n+125)W_{n+3}^2 + 72(6n+29)W_{n+2}^2 + 16(27n+152)W_{n+1}^2 - 3(72n+341)W_{n+3}W_{n+2} + (216n+1097)W_{n+3}W_{n+1} - 12(72n+373)W_{n+2}W_{n+1} - 98W_2^2 - 1656W_1^2 - 2000W_0^2 + 807W_1W_2 - 881W_0W_2 + 3612W_0W_1).$

Proof. Note that characteristic equation of the third-order recurrence sequence W_n is the cubic equation $x^3 - 5x^2 + 8x - 4 = (x-2)^2(x-1) = 0$ whose roots are $\alpha = 2$, $\beta = 2, \gamma = 1$, with $\alpha = \beta \neq \gamma$. In theorem 2.1, for $r = 5, s = -8, t = 4$,

$$\begin{aligned}\Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1)\\ &= -(z-1)(4z-1)^3(2z-1)^2\end{aligned}$$

and $\Gamma(1) = 0$. Here 1 is the root of $\Gamma(z) = 0$ with multiplicity 1.

(a) Use theorem 2.1 (a) (ii) with $z = 1$.

(b) Use theorem 2.1 (b) (ii) with $z = 1$.

(c) Use theorem 2.1 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of modified Woodall numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 5$).

Corollary 5.32.
For $n \geq 0$, modified Woodall numbers have the following properties.

(a) $\sum_{k=0}^{n} G_k^2 = \frac{1}{27}((27n+161)G_{n+3}^2 + 9(48n+301)G_{n+2}^2 + (432n+2981)G_{n+1}^2 - 24(9n+55)G_{n+3}G_{n+2} + 8(27n+172)G_{n+3}G_{n+1} - 96(9n+59)G_{n+2}G_{n+1} - 107).$

(b) $\sum_{k=0}^{n} G_{k+1}G_k = \frac{1}{27}((27n+145)G_{n+3}^2 + 144(3n+17)G_{n+2}^2 + 16(27n+172)G_{n+1}^2 - 3(72n+397)G_{n+3}G_{n+2} + 4(54n+313)G_{n+3}G_{n+1} - 3(288n+1721)G_{n+2}G_{n+1} - 91).$

(c) $\sum_{k=0}^{n} G_{k+2}G_k = \frac{1}{27}((27n+125)G_{n+3}^2 + 72(6n+29)G_{n+2}^2 + 16(27n+152)G_{n+1}^2 - 3(72n+341)G_{n+3}G_{n+2} + (216n+1097)G_{n+3}G_{n+1} - 12(72n+373)G_{n+2}G_{n+1} - 71).$

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 5, H_2 = 9$ in the last Theorem, we have the following Corollary which gives sum formulas of modified Cullen numbers.

Corollary 5.33.
For $n \geq 0$, modified Cullen numbers have the following properties:

(a) $\sum\limits_{k=0}^{n} H_k^2 = \frac{1}{27}((27n+161)H_{n+3}^2 + 9(48n+301)H_{n+2}^2 + (432n+2981)H_{n+1}^2 - 24(9n+55)H_{n+3}H_{n+2} + 8(27n+172)H_{n+3}H_{n+1} - 96(9n+59)H_{n+2}H_{n+1} - 360)$.

(b) $\sum\limits_{k=0}^{n} H_{k+1}H_k = \frac{1}{27}((27n+145)H_{n+3}^2 + 144(3n+17)H_{n+2}^2 + 16(27n+172)H_{n+1}^2 - 3(72n+397)H_{n+3}H_{n+2} + 4(54n+313)H_{n+3}H_{n+1} - 3(288n+1721)H_{n+2}H_{n+1} - 450)$.

(c) $\sum\limits_{k=0}^{n} H_{k+2}H_k = \frac{1}{27}((27n+125)H_{n+3}^2 + 72(6n+29)H_{n+2}^2 + 16(27n+152)H_{n+1}^2 - 3(72n+341)H_{n+3}H_{n+2} + (216n+1097)H_{n+3}H_{n+1} - 12(72n+373)H_{n+2}H_{n+1} - 630)$.

From the last Theorem, we have the following Corollary which gives sum formulas of Woodall numbers (take $W_n = R_n$ with $R_0 = -1, R_1 = 1, R_2 = 7$).

Corollary 5.34.
For $n \geq 0$, Woodall numbers have the following properties.

(a) $\sum\limits_{k=0}^{n} R_k^2 = \frac{1}{27}((27n+161)R_{n+3}^2 + 9(48n+301)R_{n+2}^2 + (432n+2981)R_{n+1}^2 - 24(9n+55)R_{n+3}R_{n+2} + 8(27n+172)R_{n+3}R_{n+1} - 96(9n+59)R_{n+2}R_{n+1} - 344)$.

(b) $\sum\limits_{k=0}^{n} R_{k+1}R_k = \frac{1}{27}((27n+145)R_{n+3}^2 + 144(3n+17)R_{n+2}^2 + 16(27n+172)R_{n+1}^2 - 3(72n+397)R_{n+3}R_{n+2} + 4(54n+313)R_{n+3}R_{n+1} - 3(288n+1721)R_{n+2}R_{n+1} - 340)$.

(c) $\sum\limits_{k=0}^{n} R_{k+2}R_k = \frac{1}{27}((27n+125)R_{n+3}^2 + 72(6n+29)R_{n+2}^2 + 16(27n+152)R_{n+1}^2 - 3(72n+341)R_{n+3}R_{n+2} + (216n+1097)R_{n+3}R_{n+1} - 12(72n+373)R_{n+2}R_{n+1} - 254)$.

Taking $W_n = C_n$ with $C_0 = 1, C_1 = 3, C_2 = 9$ in the last Theorem, we have the following Corollary which gives sum formulas of Cullen numbers.

Corollary 5.35.
For $n \geq 0$, Cullen numbers have the following properties:

(a) $\sum\limits_{k=0}^{n} C_k^2 = \frac{1}{27}((27n+161)C_{n+3}^2 + 9(48n+301)C_{n+2}^2 + (432n+2981)C_{n+1}^2 - 24(9n+55)C_{n+3}C_{n+2} + 8(27n+172)C_{n+3}C_{n+1} - 96(9n+59)C_{n+2}C_{n+1} - 128)$.

(b) $\sum\limits_{k=0}^{n} C_{k+1}C_k = \frac{1}{27}((27n+145)C_{n+3}^2 + 144(3n+17)C_{n+2}^2 + 16(27n+172)C_{n+1}^2 - 3(72n+397)C_{n+3}C_{n+2} + 4(54n+313)C_{n+3}C_{n+1} - 3(288n+1721)C_{n+2}C_{n+1} - 124)$.

(c) $\sum\limits_{k=0}^{n} C_{k+2}C_k = \frac{1}{27}((27n+125)C_{n+3}^2 + 72(6n+29)C_{n+2}^2 + 16(27n+152)C_{n+1}^2 - 3(72n+341)C_{n+3}C_{n+2} + (216n+1097)C_{n+3}C_{n+1} - 12(72n+373)C_{n+2}C_{n+1} - 146)$.

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

Theorem 5.16.
Assume that $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} = |\beta|^{-2} = 0.25$. Then the ordinary generating functions of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ are given as follows:

(a) $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1}((16z^5+20z^4-8z^3-z^2)W_2^2 + (400z^5+196z^4-180z^3+17z^2-z)W_1^2 + (1024z^5-880z^4+388z^3-116z^2+17z-1)W_0^2 + 8(-20z^5-17z^4+10z^3)W_1W_2 + 8(32z^5-5z^3)W_0W_2 + 32(-40z^5+11z^4+2z^3)W_0W_1)$.

(b) $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1}((32z^4-5z^2)W_2^2+144(5z^4-2z^3)W_1^2+16(32z^5-5z^3)W_0^2+(-304z^4+64z^3+25z^2-z)W_1W_2+4(16z^5+64z^4-25z^3-z^2)W_0W_2+(-320z^5-1152z^4+708z^3-116z^2+17z-1)W_0W_1)$.

(c) $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1}((44z^3-17z^2)W_2^2+(-320z^5+880z^4-136z^2+8z)W_1^2+16(44z^4-17z^3)W_0^2+(64z^5-176z^4-220z^3+121z^2-5z)W_1W_2+(128z^4+208z^3-136z^2+17z-1)W_0W_2+4(128z^5-496z^4+136z^3+17z^2-z)W_0W_1)$.

Proof. Use theorem 3.1. □

Now, we consider special cases of the last Theorem.

Corollary 5.36.

Assume that $|z|<|\alpha|^{-2}=|\beta|^{-2}=0.25$. The ordinary generating functions of the sequences $\{G_n^2\}$, $\{G_{n+1}G_n\}$, $\{G_{n+2}G_n\}$ and $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ are given as follows:

(a)

$$\sum_{n=0}^{\infty} G_n^2 z^n = \frac{16z^4+20z^3-8z^2-z}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1},$$
$$\sum_{n=0}^{\infty} H_n^2 z^n = \frac{1024z^5-2240z^4+1824z^3-700z^2+128z-9}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1} = \frac{32z^2-38z+9}{-8z^3+14z^2-7z+1},$$
$$\sum_{n=0}^{\infty} R_n^2 z^n = \frac{576z^5-1008z^4+592z^3-148z^2+16z-1}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1},$$
$$\sum_{n=0}^{\infty} C_n^2 z^n = \frac{64z^5-112z^4+112z^3-44z^2+8z-1}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1}.$$

(b)

$$\sum_{n=0}^{\infty} G_{n+1}G_n z^n = \frac{32z^3-5z}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1},$$
$$\sum_{n=0}^{\infty} H_{n+1}H_n z^n = \frac{1536z^5-3456z^4+2880z^3-1128z^2+210z-15}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1} = \frac{48z^2-60z+15}{-8z^3+14z^2-7z+1},$$
$$\sum_{n=0}^{\infty} R_{n+1}R_n z^n = \frac{384z^5-480z^4+72z^3+74z^2-24z+1}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1},$$
$$\sum_{n=0}^{\infty} C_{n+1}C_n z^n = \frac{128z^5-288z^4+280z^3-114z^2+24z-3}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1}.$$

(c)

$$\sum_{n=0}^{\infty} G_{n+2}G_n z^n = \frac{44z^2-17z}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1},$$
$$\sum_{n=0}^{\infty} H_{n+2}H_n z^n = \frac{2560z^5-5888z^4+4992z^3-1984z^2+374z-27}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1} = \frac{80z^2-104z+27}{-8z^3+14z^2-7z+1},$$
$$\sum_{n=0}^{\infty} R_{n+2}R_n z^n = \frac{-384z^5+1440z^4-1656z^3+762z^2-142z+7}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1},$$
$$\sum_{n=0}^{\infty} C_{n+2}C_n z^n = \frac{384z^5-928z^4+856z^3-354z^2+78z-9}{-256z^6+704z^5-752z^4+404z^3-116z^2+17z-1}.$$

From the last corollary, we obtain the following results for Modified Woodall, modified Cullen, Woodall and Cullen numbers.

Corollary 5.37.

Some infinite sums of $\{G_n^2\}$, $\{G_{n+1}G_n\}$, $\{G_{n+2}G_n\}$ and $\{H_n^2\}$, $\{H_{n+1}H_n\}$, $\{H_{n+2}H_n\}$ are given as follows:

(a) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{G_n^2}{8^n} = \frac{212}{63},$$
$$\sum_{n=0}^{\infty} \frac{H_n^2}{8^n} = \frac{304}{21},$$
$$\sum_{n=0}^{\infty} \frac{R_n^2}{8^n} = \frac{394}{63},$$
$$\sum_{n=0}^{\infty} \frac{C_n^2}{8^n} = \frac{506}{63}.$$

(b) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{G_{n+1}G_n}{8^n} = \frac{64}{7},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+1}H_n}{8^n} = \frac{176}{7},$$
$$\sum_{n=0}^{\infty} \frac{R_{n+1}R_n}{8^n} = \frac{92}{7},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+1}C_n}{8^n} = \frac{148}{7}.$$

(c) $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{G_{n+2}G_n}{8^n} = \frac{1472}{63},$$
$$\sum_{n=0}^{\infty} \frac{H_{n+2}H_n}{8^n} = \frac{976}{21},$$
$$\sum_{n=0}^{\infty} \frac{R_{n+2}R_n}{8^n} = \frac{1780}{63},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+2}C_n}{8^n} = \frac{3404}{63}.$$

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