Int. J. Adv. Appl. Math. and Mech. 12(3) (2025) pages 73–84 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

Managing Pension Portfolios - Optimal Investment Strategies under Wage and Inflation Uncertainty

Research Article

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Volume 12, Issue 03
Received: 13 November 2024
Accepted (in revised version): 04 December 2024

Abstract: This paper addresses the optimal asset allocation and consumption strategies for a defined contribution pension plan, considering stochastic wage income and inflation risk. The analysis focuses on a pension fund manager managing investments for members with constant relative risk aversion (CRRA) preferences, factoring in external shocks such as the COVID-19 pandemic, fiscal policy shifts, and environmental changes. The portfolio comprises both risk-free assets (e.g., bonds) and risky assets (e.g., equities), with the goal of maximizing the expected utility of members' terminal wealth. The study highlights that key economic variables significantly influence investment and consumption strategies. Increased risk aversion leads to a higher allocation in risky assets, while higher interest rates and growth rates in risky assets promote more aggressive equity investments. On the consumption side, rising wealth and reduced risk aversion encourage greater consumption, as investors anticipate higher returns from riskier assets. By solving the Hamilton-Jacobi-Bellman (HJB) equation using the Dynamic Programming Principle (DPP), the paper offers a robust framework for optimizing pension fund investment and consumption decisions under uncertainty.

MSC: 62B15 • 62C25

Keywords: Pension Fund Manager • Defined Contribution Pension Plan • Inflation risk • Dynamic Programming Principle • Stochastic wage • CRRA investor

1. Introduction

Pension funds can be managed in two primary ways: defined benefit (DB) plans and defined contribution (DC) plans. In DB schemes, the benefits are predetermined by the plan sponsor, while contributions are initially set and later adjusted as needed to keep the fund balanced. In contrast, DC schemes involve fixed contributions, and the benefits depend on the investment returns of the fund's portfolio. The key distinction between DB and DC plans lies in who bears the financial risk: in DB plans, the sponsor assumes the risk, while in DC plans, the contributors bear it. Recently, DC schemes have gained global popularity due to demographic changes and the growth of equity markets.

Defined Contribution (DC) pension plans have become increasingly prevalent, shifting both investment and longevity risks from employers to employees, which makes the optimal asset allocation a crucial issue for pension

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fund managers. These managers must develop strategies that consider the long-term impacts of stochastic wage income and inflation risks. Investors typically exhibit Constant Relative Risk Aversion (CRRA), meaning that their relative risk tolerance remains constant over varying levels of wealth. This behavior necessitates utility-maximizing portfolio choices, particularly in volatile economic environments such as those observed in recent years.

Stochastic wage income refers to the unpredictable fluctuations in wages due to various macroeconomic factors. Historically, wage growth has been influenced by labor market dynamics, globalization, technological innovation, and more recently, external shocks like the COVID-19 pandemic. During the pandemic, economies worldwide experienced wage volatility driven by job losses, changes in employment conditions, and government interventions. These shocks also led to a significant fiscal and monetary response, including stimulus packages and interest rate adjustments, which introduced additional uncertainty into inflation and wage dynamics Kim and Park [1]. Factors like supply chain disruptions, expansive fiscal policies, and geopolitical tensions have contributed to inflationary pressures globally. Climate change, a long-term risk, is another important factor influencing both inflation and economic policy. Environmental risks have triggered policy shifts that could further influence inflation, wages, and overall economic stability Dutta and Morgan [2]. Managing risks effectively is paramount to ensuring the sustainability and adequacy of pension outcomes.

Pension fund managers, responsible for significant capital, must navigate these risks by allocating assets between a risk-free asset, such as a government bond or money market account, and a risky asset, typically equities. The risk-free asset provides stable, predictable returns, while equities offer the potential for higher returns but also carry greater risk. In an uncertain environment, finding the right balance between these two asset classes is critical Zhang and Chen [3] . For instance, more volatile, especially during periods of economic stress, but they may also offer inflation protection over the long term.

The challenge for pension fund managers is further compounded by the long-term nature of pension liabilities. To address this, managers often apply dynamic optimization techniques, such as the Dynamic Programming Principle (DPP), which enables continuous adjustment of the portfolio based on changing market conditions and evolving individual circumstances. In particular, the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) is a common mathematical tool used to solve for the value function, representing the maximum expected utility of terminal wealth. By solving the HJB PDE, pension fund managers can derive the optimal investment strategy that maximizes the expected utility of their clients' retirement wealth Basak and Shapiro [4].

The use of stochastic models with inflation risk has advanced significantly in recent years, with many pension models now incorporating geometric Brownian motion and mean-reverting processes to simulate real-world uncertainties. These models have allowed fund managers to account for risks more accurately and to implement more robust investment strategies that are better suited to the stochastic nature of wage and inflation risks Wang and Zhou [5].

Moreover, ongoing research continues to explore combining stochastic wage and inflation risks with the risk preferences of investors under CRRA utility functions. The goal is to ensure that pension fund managers can maximize the expected utility of retirement wealth by continuously adjusting their portfolio in response to changes in economic conditions. Studies highlight the importance of a dynamic approach that integrates market shocks, policy changes, and long-term economic trends in developing an optimal portfolio strategy for DC pension plans Cairns and Blake [6].

2. Links to the literature

Optimal portfolio allocation in Defined Contribution (DC) pension plans has long been a topic of extensive research within both actuarial science and financial economics. A well-performing defined contribution (DC) pension scheme seeks to provide substantial retirement income, typically in the form of an annuity, making the investment strategy of the pension fund critical. The literature on pension fund investment strategies is extensive, and two main methodological approaches are commonly discussed.

The first approach is stochastic control, initially developed by Merton [7] [8]. Stochastic control methods involve solving the Hamilton–Jacobi–Bellman (HJB) equation through dynamic programming under a real-world probability measure. Recent applications of this method in DC pension fund portfolio management include the work of Vigna and Haberman [9], who applied stochastic dynamic programming to assess financial risk in DC schemes under Gaussian interest rate models. Their study focused on finding an optimal investment strategy to achieve a final target based on the net replacement ratio, while also considering interim targets. Haberman and Vigna [10] expanded upon their earlier work by exploring the investment allocation of a DC scheme that includes multiple assets, analyzing the final net replacement ratio under three distinct risk measures. Devolder et al [11] explored the management of annuity contracts using an interest rate model described by a geometric Brownian motion. Menoncin and Scaillet [12] conducted a similar analysis for life annuity contracts. Xiao et al [13] applied the constant elasticity of variance (CEV) model to annuity contracts and derived a dual solution for logarithmic utility using the Legendre transform. However, these studies generally assume that the short-term interest rate is constant, which may be problematic for pension funds, as the contribution period typically spans 20 to 40 years, making the assumption of constant interest rates unrealistic in DC plan contexts.

The second approach was developed by Cox and Huang [14] in the context of complete markets, utilizing the theory of Lagrange multipliers. Known as the martingale approach, this method typically solves the partial differential equation under a risk-neutral measure. In recent years, this approach has been successfully applied by other researchers. For example, Boulier et al [15] and Deelstra et al [16][17] utilized the martingale method to study the optimal design and asset allocation of pension funds. More recently, Hainaut and Devolder [11] examined the dividend policy and asset allocation of pension funds under the Vasicek interest rate model. Despite the effectiveness of the martingale approach, few other methodologies have been applied to pension fund management in a stochastic interest rate framework.

The shift from Defined Benefit (DB) plans to DC plans has transferred significant investment risk from employers to employees, requiring pension fund managers to make strategic decisions that account for uncertain future wage income, inflation risk, and market volatility Cairns and Blake [6]. The incorporation of stochastic wage income and inflation risk in pension modeling represents an ongoing challenge, as these factors introduce an additional layer of complexity to asset allocation strategies.

The inclusion of wage income and inflation as stochastic processes is crucial for realistic pension fund modeling. Wage income, which influences contributions to pension plans, is inherently uncertain due to macroeconomic shocks, labor market trends, and individual career paths. Empirical studies have shown that wage growth tends to follow stochastic paths, often modeled using geometric Brownian motion or mean-reverting processes Anderson and Liu [19]. These models capture both short-term volatility and long-term trends in wage growth, which are important for projecting pension contributions over time.

Inflation, another critical factor, affects both the purchasing power of retirement benefits and the real return on pension fund investments. The period following the COVID-19 pandemic has underscored the significance of inflation risk, as global economies experienced sudden spikes in inflation due to expansive fiscal policies and supply chain disruptions Kim and Park [1]. Recent research has explored various inflation modeling techniques, with some studies favoring stochastic differential equations to reflect inflation's unpredictable nature. Understanding how inflation evolves is key to constructing portfolios that protect against eroding real returns on pension assets Zhang and Chen [3].

Moreover, the emerging threat of climate change introduces another dimension of inflationary pressure. Environmental risks can disrupt economic stability, influencing both inflation and wage dynamics. Dutta and Morgan [2] highlight the increasing relevance of climate risk in long-term financial planning, particularly for pension funds with a multi-decade investment horizon. Pension fund managers must now account for potential inflationary impacts of climate policy interventions, such as carbon taxes or shifts in global energy markets.

Dynamic portfolio optimization, particularly through the use of the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE), has become the standard approach for solving asset allocation problems under uncertainty. The Dynamic Programming Principle (DPP) provides a framework for continuously adjusting investment strategies in response to changes in market conditions and individual preferences. The derivation of the HJB PDE allows for the optimization of utility functions based on CRRA preferences, a common utility function in financial and actuarial modeling Basak and Shapiro [4]. By solving the HJB PDE, researchers can determine both the value function and optimal investment controls, which specify the proportion of wealth allocated to risky and risk-free assets at any given time.

The literature has focused extensively on the role of dynamic optimization in balancing short-term risks with long-term goals. Cairns and Blake [6] explored the theoretical underpinnings of dynamic portfolio choice, demonstrating that continuous adjustment of asset allocation leads to superior long-term outcomes compared to static strategies. Their work emphasizes the importance of integrating stochastic models for wage growth and inflation into the optimization process, ensuring that pension fund managers account for the unpredictable nature of these risks. Additionally, Merton's seminal contributions to continuous-time finance laid the foundation for much of the current research on dynamic optimization in pension plans. His models illustrate how optimal portfolio choices evolve over time as new information becomes available, a concept that remains central to modern portfolio theory [20].

The assumption of Constant Relative Risk Aversion (CRRA) utility functions is a cornerstone of many models used in pension fund management. CRRA utility assumes that an individual's risk tolerance remains constant relative to their wealth, making it a suitable framework for analyzing investment decisions in DC pension plans. Research has demonstrated that under CRRA preferences, the optimal investment strategy involves a balance between risky and risk-free assets that maximizes the expected utility of terminal wealth Wang and Zhou [5]. This approach is particularly well-suited to long-term investment planning, where fund managers must account for both short-term volatility and long-term growth.

Basak and Shapiro [4] extended this framework by incorporating stochastic wage and inflation risk into the traditional CRRA model, demonstrating that such risks have significant implications for optimal asset allocation. Their findings suggest that the inclusion of wage and inflation risks leads to a more conservative allocation to risky assets, particularly in the early stages of a pension plan's life cycle. This shift reflects the increased uncertainty surrounding future income streams and the potential for inflation to erode real returns on pension assets.

The traditional approach to pension fund asset allocation involves dividing investments between risk-free assets, such as bonds or money market accounts, and risky assets, such as equities. Risk-free assets provide stability and predictable returns, while risky assets offer the potential for higher returns but come with greater volatility Zhang and Chen [3]. Recent studies have examined the relative performance of these asset classes in the context of stochastic wage and inflation risks, finding that equities tend to perform better over the long term, particularly in inflationary environments, but also pose higher short-term risks.

Incorporating stochastic modeling of both wage and inflation risks has furthered the understanding of optimal asset allocation in pension funds. Wang and Zhou [5] used a stochastic control framework to demonstrate that the optimal allocation to risky assets decreases as uncertainty about future wage income increases. Their research highlights the importance of tailoring investment strategies to individual circumstances, including age, risk tolerance, and projected wage growth. The trade-off between short-term stability and long-term growth is a key consideration for pension fund managers, who must ensure that their portfolios can meet both immediate liquidity needs and long-term retirement goals.

The outline of this paper is as follows. Section 2, introduction. Section 3, literature review. Section 4, Problem formulation. In section 5, the wealth model is determined. Section 6 Optimization criterion description. In section 7, the Hamilton-Jacobi-Bellman equation (HJB) for the value function is derived. In section 8, we investigate the value function and optimal policy. In Section 9, numerical examples and simulations are provided. Here, the effect of market parameters on the optimal investment policy is illustrated. In Section 10, the conclusion and suggested possible future research work are stated.

3. Problem formulation

3.1. Financial market model

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered, complete probability space, where the filtration $\mathbb{F} = (\mathscr{F}_t)_{0\le t\le T}$ satisfies the usual conditions, including right-continuity and $\mathbb{P}$-completeness.

We consider a stochastic control problem involving a single investor who manages a portfolio consisting of one risk-free security, such as a money market account or bond, denoted by $B(t)$, and one risky security, such as a stock or stock index, denoted by $S(t)$.

Let the price dynamics of the risk-free security $B(t)$ evolve according to the following stochastic differential equation (SDE):

$$\begin{cases} dB(t) = r(t)B(t)\,dt, \\ B(0) = 1, \end{cases} \tag{1}$$

where $r(t)$ is the constant risk-free interest rate.

The dynamics of the stochastic wage income $\eta(t)$ follow:

$$\begin{cases} d\eta(t) = \mu_\eta(t)\eta(t)\,dt + \sigma_\eta(t)\eta(t)\,dW^\eta(t), \\ \eta(0) = \eta, \end{cases} \tag{2}$$

where $\mu_\eta(t)$ and $\sigma_\eta(t)$ represent the drift and volatility of the wage income, and $W^\eta(t)$ is a standard Wiener process.

Let inflation risk be modeled by a price index $I(t)$ (e.g., the Consumer Price Index, CPI), representing a fixed basket of goods, evolving according to:

$$\begin{cases} dI(t) = \mu_I(t)I(t)\,dt + \sigma_I(t)I(t)\,dW^I(t), \\ I(0) = i, \end{cases} \tag{3}$$

where $\mu_I(t)$ and $\sigma_I(t)$ are the drift and volatility of the price index, and $W^I(t)$ is a standard Wiener process modeling inflation risk.

The price dynamics of the risky security, $S(t)$, are modeled by a Geometric Lévy process, which can be expressed as a Geometric Brownian Motion (GBM) with an additional term accounting for jumps. The SDE for $S(t)$ is given by:

$$\begin{cases} dS(t) = \mu_s S(t)\,dt + \sigma_s S(t)\,dW^S(t), \\ S(0) > 0, \end{cases} \tag{4}$$

where $W^S(t)$ is a Wiener process modeling market risk, μ_s is the mean rate of return, and σ_s is the volatility of the risky security.

3.2. Contribution

Each employee contributes a constant proportion of their salary at the rate $q(t)$. The process of contribution, denoted by $\Gamma(t)$, is modeled as:

$$\Gamma(t) = q(t)\eta(t)\,dt. \tag{5}$$

3.3. The Pension Wealth Model

Consider an investor with initial amount of money $\mathscr{X}(0) > 0$ and a time horizon of interest T. Over the time interval, $[0, T]$, the investor changes his portfolio dynamically. Let $\mathscr{C}(t)$ denote the rate of continuous consumption, $1-\pi(t)$ be the amount invested in the risk-free security and $\pi(t)$ denote the pension amount to be invested in the risky asset, S. Note that the pair $(\mathscr{C}(t), \pi(t))$ is a trading strategy.

Lemma 3.1.
The net pension wealth $X(t)$ for an investor evolve as follows:

$$\begin{cases} \frac{d\mathscr{X}(t)}{\mathscr{X}(t)} & = (1-\pi(t))\frac{dB(t)}{B(t)} + \pi(t)\frac{dS(t)}{S(t)} - C(t)dt + \Gamma(t) \\ & = \left[\pi\mu_s + (1-\pi)r\right]dt + \pi\sigma_s dW^S(t) - \mathscr{C}(t)dt + q(t)\eta(t), \\ \mathscr{X}(0) > 0. \end{cases}$$

Note that the inflation-adjusted real wealth process at time t denoted by $\bar{X}(t)$ is calculated as follows:

$$\bar{\mathscr{X}}(t) = \frac{\mathscr{X}(t)}{I(t)} \tag{6}$$

Applying Itô lemme on eq. (6), we obtain the following real net wealth model

$$\begin{cases} d\bar{\mathscr{X}}(t) & = \frac{1}{I(t)}d\mathscr{X}(t) + \frac{-\mathscr{X}(t)}{[I(t)]^2}dI(t) - \frac{1}{[I(t)]^2}dI(t)d\mathscr{X}(t) + \frac{-\mathscr{X}(t)}{[I(t)]^3}dI(t)dI(t) \\ & -C(t)dt + q(t)\eta(t)dt \\ & = \frac{\mathscr{X}}{I}\left[\frac{d\mathscr{X}(t)}{\mathscr{X}(t)} - \frac{dI(t)}{I(t)} + \frac{dI^2(t)}{I^2(t)} - \frac{d\mathscr{X}(t)dI(t)}{\mathscr{X}(t)I(t)}\right] - \mathscr{C}(t)dt + q(t)\eta(t)dt. \end{cases}$$

Simplifying further gives:

$$\begin{cases} \frac{d\bar{\mathscr{X}}(t)}{\bar{\mathscr{X}}(t)} = \left[\pi\mu_s + (1-\pi)r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 - C(t) + q(t)\right]dt - \sigma_I dW^I(t) + \pi(t)\sigma_s dW^s(t) \\ \bar{\mathscr{X}}(0) > 0, \end{cases} \tag{7}$$

where $\bar{\mathscr{X}}(t)$ is inflation adjusted real wealth, $C(t)$ is consumption and $q(t)$ is the employee contribution at constant rate.

4. The Optimization Criterion

Suppose the set of all admissible strategies is denoted by $\mathscr{A}$.

Definition 4.1.
An investment and consumption strategy $\mathscr{A} = (\pi(t), \mathscr{C}(t))$ is said to be admissible if the following conditions are satisfied.

(i) The pair $(\pi(t), \mathscr{C}(t))$ is progressively $\mathscr{F}_t - measurable$.

(ii) $\int_0^T \pi(t)^2 dt < \infty$, $\int_0^T \mathscr{C}(t)dt < \infty$, for all $T > 0$.

(iii) If $(\pi(t), \mathscr{C})$ is the strategy, the wealth process eq. (7) with $\bar{\mathscr{X}}(0) > 0$ has a path wise unique solution.

Remark 4.1.
The investor's objective is to maximize the net expected discounted utility of terminal real wealth plus consumption.

The objective function for this stochastic control problem is then formulated mathematically as follows:

$$J(t, x; \pi(t), \mathscr{C}(t)) = \mathbb{E}\Big[\int_0^T \phi e^{-\lambda t} U_1(\mathscr{C}(t))dt + (1-\phi)e^{-\lambda T} U_2(\bar{\mathscr{X}}(T))\Big]. \tag{8}$$

Definition 4.2.
The value function is defined as

$$V(t, x) = \sup_{(\pi(t), \mathscr{C}(t)) \in \mathscr{A}} \mathbb{E}\Big[\int_0^T \phi e^{-\lambda t} U_1(\mathscr{C}(t))dt + (1-\phi)e^{-\lambda T} U_2(\bar{\mathscr{X}}(T))\Big], \tag{9}$$

with boundary conditions $V(T, x) = (1-\phi)e^{-\lambda T} U_2(\bar{\mathscr{X}}(T))$, Here, $\bar{\mathscr{X}}(t) \geq 0$ for all t, with T being the date of death, $\bar{\mathscr{X}}(T)$ is the value at time T of a trading strategy. The parameter λ is the subjective discount rate and ϕ determines the relative importance of the intermediate consumption and the bequest. $\mathbb{E}$ denotes the conditional expectation operator. $U_1(\mathscr{C}(t))$ and $U_2(\bar{\mathscr{X}}(T))$ are consumption and bequest functions respectively.

5. The Hamilton-Jacobi-Bellman equation

By applying Dynamic Programming Principle, we obtain the Hamilton-Jacobi-Bellman equation (HJB equation) for the value function. The fully HJB PDE associated with the stochastic control problem eq. (9) is the second-order nonlinear PDE given as follows:

$$V_t + \sup_{(\pi,\mathscr{C}\in\mathscr{A}} \Big[[\pi\mu_s + (1-\pi)r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 - C(t) + q(t)]\bar{\mathscr{X}}V_x + \Big[\frac{1}{2}\pi^2\sigma_s^2 + \sigma_I^2 - 2\pi\sigma_s\sigma_I\rho\Big]\bar{\mathscr{X}}^2 V_{xx}$$
$$+ \mu_\eta\eta(t)V_\eta + \frac{1}{2}\sigma_\eta^2\eta^2(t)V_{\eta\eta} + \pi\sigma_s\sigma_I\eta(t)\bar{\mathscr{X}}V_{x\eta} + \phi e^{-\lambda t}U_1(\mathscr{C})\Big] = 0. \tag{10}$$

where V_t , V_x, V_{xx}, V_η, $V_{\eta\eta}$ and $V_{x\eta}$ denote partial derivatives.

During the phase of retirement, the investor consumes from the accumulated surplus. The HJB PDE in the retirement phase is as follows:

$$V_t + \sup_{(\pi,\mathscr{C}\in\mathscr{A}} \Big[[\pi\mu_s + (1-\pi)r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 - C(t) + q(t)]\bar{\mathscr{X}}V_x + \Big[\frac{1}{2}\pi^2\sigma_s^2 + \sigma_I^2 - 2\pi\sigma_s\sigma_I\rho\Big]\bar{\mathscr{X}}^2 V_{xx}$$
$$+ \phi e^{-\lambda t}U_1(\mathscr{C})\Big] = 0. \tag{11}$$

Thus, the candidate optimal controls for eq. (11) are as follows:

$$\pi^*(t) = -\frac{-(\mu_s - r)\bar{\mathscr{X}}V_x - 2\sigma_s\sigma_I\rho\bar{\mathscr{X}}^2V_{xx}}{\sigma_s^2\bar{\mathscr{X}}V_x} \tag{12}$$

and

$$\mathscr{C}^* = \Big[\frac{\bar{\mathscr{X}}V_x}{\phi e^{-\lambda t}}\Big]^{\frac{-1}{\delta}}. \tag{13}$$

where the utility function is defined as

$$U_1(x) = U_2(x) = \frac{x^{1-\delta}}{1-\delta}, \quad \delta > 0, \quad \delta \neq 1, \tag{14}$$

with δ being the risk aversion factor.

6. The value function and optimal policies

Assume the solution V for eq. (11) take the form

$$V(t,x) = \frac{x^{1-\delta}}{1-\delta}G(t), \quad g(T) = 1. \tag{15}$$

Partial derivatives for V are as follows:

$$V_t = \frac{x^{1-\delta}}{1-\delta}G_t, \quad V_x = x^{-\delta}G, \quad V_{xx} = -\delta x^{-\delta-1}G, \tag{16}$$

Note that eqs. (12) and (13) can be simplified further having known eqs. (15) and (16) as follows:

$$\pi^*(t) = \frac{(\mu_s - r) + 2\sigma_s\sigma_I\rho\delta}{\delta} \tag{17}$$

and

$$\mathscr{C}^* = \Big[\frac{\bar{\mathscr{X}}^{-\delta+1}G}{\phi e^{-\lambda t}}\Big]^{\frac{-1}{\delta}}. \tag{18}$$

Substituting eqs. (16)–(18) into eq. (11) gives:

$$\frac{x^{1-\delta}}{1-\delta}G_t + \sup_{\pi\in\mathscr{A}}\Bigg(\Big[\Big(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Big)\mu_s + \Big(1 - \frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Big)r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2\Big]xx^{-\delta}G$$
$$+\frac{1}{2}\Big[\Big(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Big)^2\sigma_s^2 + \sigma_I^2 - 2\Big(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Big)\sigma_s\sigma_I\rho\Big]x^2(-\delta x^{-\delta-1}G)\Bigg) + \Theta(t)$$
$$= 0. \tag{19}$$

where,

$$\Theta(t) = -C(t)\bar{\mathscr{X}}V_x + \phi e^{-\lambda t}U_1(\mathscr{C}). \tag{20}$$

Equation (19) is still a Differential Equation (DE) which is difficult to solve. Inspired by Liu [19] , we assume G is given by:

$$G = \int_t^T \hat{G}du + \hat{G}. \tag{21}$$

This implies eq. (19) reduces to the following DE with well-defined solutions:

$$\frac{x^{1-\delta}}{1-\delta}\hat{G}_t + \sup_{\pi\in\mathscr{A}}\Bigg(\Big[\Big(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Big)\mu_s + \Big(1 - \frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Big)r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2\Big]xx^{-\delta}\hat{G}$$
$$+\frac{1}{2}\Big[\Big(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Big)^2\sigma_s^2 + \sigma_I^2 - 2\Big(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Big)\sigma_s\sigma_I\rho\Big]x^2(-\delta x^{-\delta-1}\hat{G})\Bigg)$$
$$= 0. \tag{22}$$

Simplifying further gives the following DE which has a well-defined solution:

$$\frac{\hat{G}_t}{\hat{G}} = (1-\delta)\Big[-\frac{[\mu_s - r + \sigma_s\sigma_I\rho]^2}{2\sigma_s^2\delta} - \Big(r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 + \frac{\sigma_I^2\delta}{2}\Big)\Big]. \tag{23}$$

Solving eq. (23), we obtain the following solution:

$$\hat{G}(t) = \exp\Big[(1-\delta)\Big(-\frac{[\mu_s - r + \sigma_s\sigma_I\rho]^2}{2\sigma_s^2\delta} - \Big[r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 + \frac{\sigma_I^2\delta}{2}\Big]\Big)(T-t)\Big]. \tag{24}$$

Therefore, the value function V having solved eq. (23) is given by:

$$V(t,x) = \frac{x^{1-\delta}}{1-\delta}\exp\Big[(1-\delta)\Big(-\frac{[\mu_s - r + \sigma_s\sigma_I\rho]^2}{2\sigma_s^2\delta} - \Big[r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 + \frac{\sigma_I^2\delta}{2}\Big]\Big)(T-t)\Big]. \tag{25}$$

In addition, the optimal controls are given as follows:

$$\pi^*(t) = \frac{(\mu_s - r) + 2\sigma_s\sigma_I\rho\delta}{\delta} \tag{26}$$

and

$$\mathscr{C}^* = \Big[\frac{\bar{\mathscr{X}}^{-\delta+1}\hat{G}}{\phi e^{-\lambda t}}\Big]^{\frac{-1}{\delta}}, \tag{27}$$

where,

$$\hat{G}(t) = \exp\Big[(1-\delta)\Big(-\frac{[\mu_s - r + \sigma_s\sigma_I\rho]^2}{2\sigma_s^2\delta} - \Big[r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 + \frac{\sigma_I^2\delta}{2}\Big]\Big)(T-t)\Big]. \tag{28}$$

7. Numerical simulations

In this section, we determine how parameters affect optimal investment $\pi^*(t)$ and optimal consumption $\mathscr{C}^\star(t)$ controls through curve analysis:

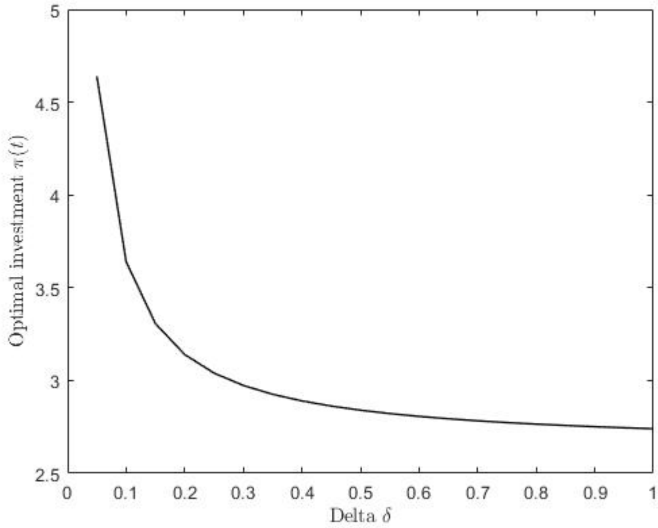

Fig. 1. The effects of risk aversion factor δ on optimal investment $\pi^*(t)$

In fig. 1, the optimal investment strategy, $\pi^*(t)$ shows an increasing trend as the risk aversion parameter, δ, grows. As δ (the risk aversion factor) increases, the pension Manager's relative risk aversion $1-\delta$ decreases. A lower relative risk aversion means the pension Manager is more willing to take on risk. This leads to more aggressive investment, resulting in a larger allocation to risky assets.

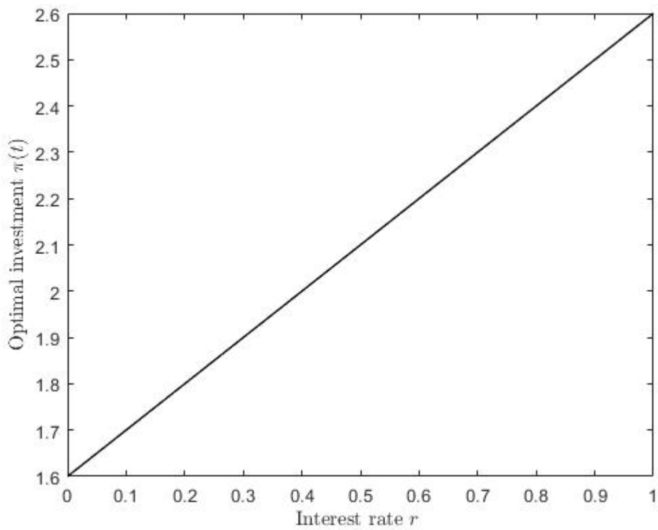

Fig. 2. The effects of interest rate r on optimal investment $\pi^*(t)$

Figure 2 shows the effect of interest rate r on optimal investment $\pi^*(t)$. The curve results indicate that the interest rate r positively influences the optimal investment strategy $\pi^*(t)$. As r rises, the fund manager increases the total allocation to investments, reflecting the enhanced appeal of higher-yielding opportunities in the market.

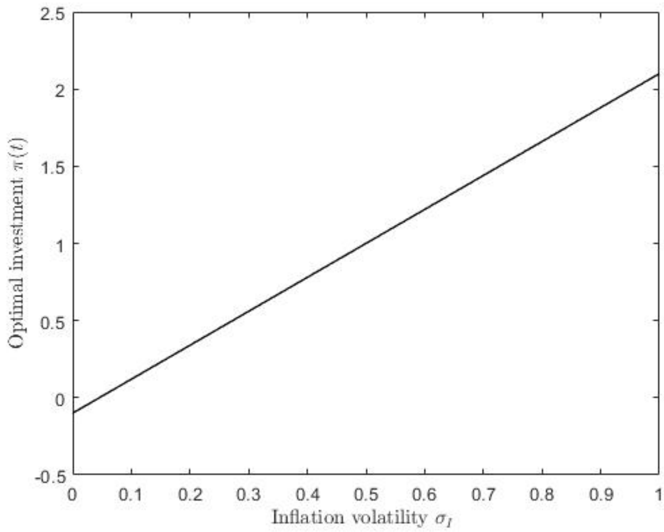

Fig. 3. The effects of growth rate μ_s of the risky asset on optimal investment $\pi^*(t)$

In fig. 3, the optimal investment strategy $\pi^*(t)$ increases with increases in growth rate μ_s of the risky asset. As the growth rate μ_s increases, the pension Manager sees more potential for higher returns. This encourages the pension Manager to allocate more funds to the risky asset (e.g., stocks) to maximize wealth and consumption. This aligns with practical investment behavior, where higher growth opportunities typically lead to more aggressive investment in those assets.

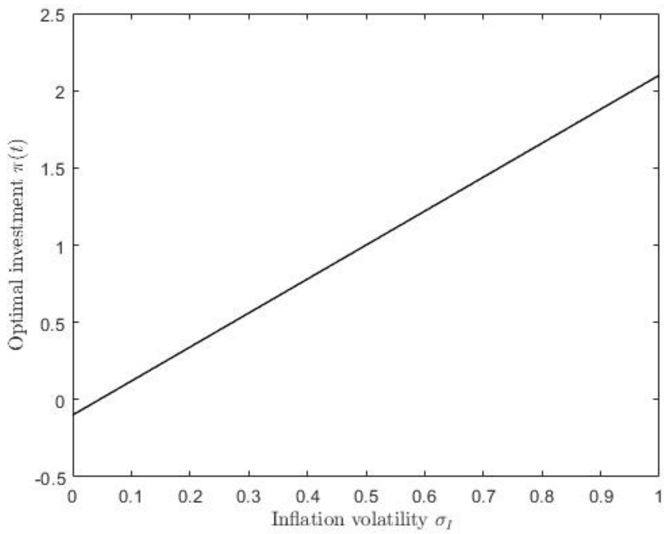

Fig. 4. The effects of inflation volatility σ_I on optimal investment $\pi^*(t)$

In fig. 4, the optimal investment strategy $\pi^*(t)$ increase with an increase in inflation volatility σ_I. As inflation volatility grows, the pension Manager adopts a more aggressive stance, allocating a larger portion of their portfolio to risky assets in pursuit of higher returns to counter potential inflationary risks.

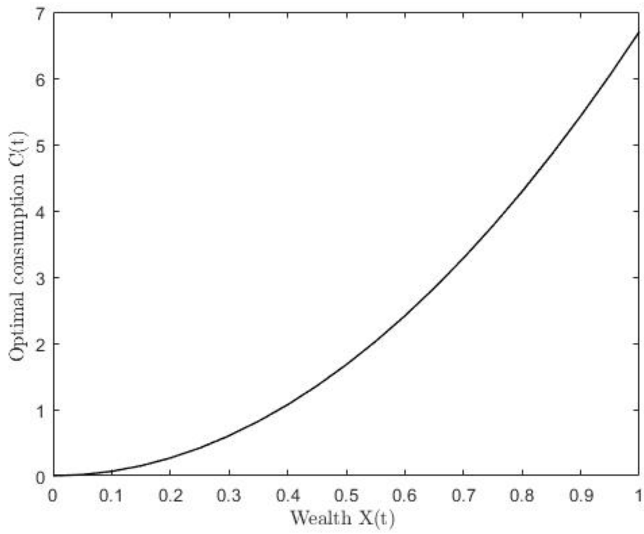

Fig. 5. The effects of wealth $X(t)$ on optimal consumption $X(t)$

In fig. 5, the optimal consumption strategy $C(t)$ increases with increases in wealth $X(t)$. As wealth increases, it logically follows that consumption would rise, especially in the context of a pension manager who adjusts consumption based on available resources. This statement aligns with practical investment behavior, where more wealth generally leads to greater consumption.

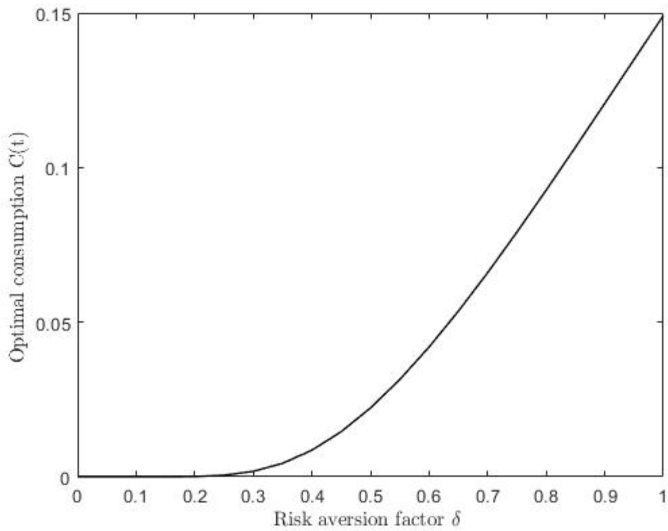

Fig. 6. The effects of risk aversion factor δ on optimal consumption $C(t)$

In fig. 6, the optimal consumption strategy $C(t)$ increases as the risk aversion parameter δ, rises. Increasing the risk aversion factor δ results in a decrease in relative risk aversion $(1-\delta)$, then the pension Manager becomes less risk-averse as δ increases. In this case, the investor would be more willing to take risks, which could justify an increase in consumption since they may expect higher returns from riskier investments.

8. Conclusion

This paper presents a comprehensive solution to the optimal portfolio allocation problem in Defined Contribution (DC) pension plans, accounting for the critical uncertainties of stochastic wage income and inflation risk. By leveraging the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) and Constant Relative Risk Aversion (CRRA) preferences, we provide a robust framework for pension fund managers to maximize the expected utility of terminal wealth and consumption. The analysis of optimal investment and consumption strategies demonstrates a clear relationship between key economic factors and investor behavior. As shown in fig. 1, an increase in the risk aversion parameter δ leads to a decrease in relative risk aversion, resulting in more aggressive investment in risky assets. Similarly, fig. 2 illustrates that higher interest rates positively influence investment decisions, encouraging greater allocation to investments with higher potential returns. The findings in fig. 3 further support this, as investors allocate more to risky assets when the growth rate of those assets rises, reflecting a practical response to higher expected returns. On the consumption side, as demonstrated in figs. 5 and 6 both wealth and risk aversion factor δ significantly impact consumption strategies. Increasing wealth leads to higher consumption, while higher δ reduces relative risk aversion, prompting more consumption as investors anticipate greater returns from riskier investments. Overall, these findings align with established investment behavior, providing insights into how economic conditions drive optimal strategies for wealth and consumption management. The latest publications [21–25] have motivated us to publish this paper in this journal.

9. Acknowledgements

The authors are thankful to the Department of Mathematics and Statistics, Mulungushi University and the Department of Computing, Mathematical and Statistical, University of Namibia for using their resources. Many thanks also go to colleagues who helped in making comments on this paper.

10. Conflicts of Interest

The authors declare no conflicts of interest.

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