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New Approach to Solving Nonlinear Fractional Order Partial Differential Equation Models with Non-Homogeneous Robin Type Conditions

Research Article

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Abstract: In this paper we introduce a new approach to solving fractional nonlinear convection-diffusion-reaction models with non-homogenous mixed (Robin) conditions. This new approach uses fractional derivation in the Caputo sense in time and the classical dervation in space. We can say that this technique enables us to integrate all the conditions imposed on the models.

MSC: 65Nxx • 65Lxx • 65Mxx • 65Qxx

Keywords: SOME BLAISE ABBO Method • Caputo fractional derivation • Fractional Riemmann Liouville integral • Convection diffusion reaction model • Robin's condition.

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1. Introduction

Here we are interested in solving one-dimensional fractionary order partial differential equations with mixed(Robin where Fourier) condition. These equations model a large number of phenomena namely in biomathematics, chemistry, economy and engineering. Unfortunately, in the literature we have only encoutered analytical resolution of models with Robin-type conditions. This problem was raised in [5, 13] prompted us to make our contribution from this angle. This project is structured in four points: the second point is entited elements on fractionals, the third point is the description of the SBA method in Caputo's sense, the fourth is the result of an illustrative and finally a conclusion.

2. Fractional elements

This section presents some definitions of useful function and operators, and a result on the existence and uniqueness of fractional differential equations(EDF) of the Caputo type:

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2.1. Mittag Leffler functions

Basic fractional applications are described in the sub-sections below.

Definition 2.1.
we define the one-parametter Mittag Leffler function as a generalized of the exponential function by:

$$E_{\alpha}(t) = \sum_{n=0}^{+\infty} \frac{t^n}{\Gamma(n\alpha+1)}, \alpha > 0 \tag{1}$$

Definition 2.2.
The two-parameter Mittag Leffler function is defined by the following formula:

$$E_{\alpha,\gamma}(t) = \sum_{n=0}^{+\infty} \frac{t^n}{\Gamma(n\alpha+\gamma)}, \alpha > 0, \gamma > 0$$

2.2. Function of Mellin-Ross

This function is denoted $E_t(v, b)$, and its purpose is to intervene in the in the determination of the generalised integral of an exponential function defined by:

$$E_t(v, b) = t^v e^{bt} \Gamma^*(v, t)$$

With

$$\Gamma^*(v, t) = \frac{1}{t^v \Gamma(v)} \int_0^t y^{v-1} e^{-y} dy,\ Re(v) > 0$$

In addition:

$$E_t(v, b) = t^v \sum_{n=0}^{+\infty} \frac{(bt)^n}{\Gamma(n+v+1)} = t^v E_{1,v+1}(bt)$$

2.3. Caputo derivatives

Definition 2.3.
the following relations define the fractional right and left derivatives respectively :

$${}^{c}C_{d^+}^{\alpha} H(x) = \frac{1}{\Gamma(j-\alpha)} \int_d^x (x-u)^{j-\alpha-1} H^{(j)}(u) du,\ \alpha > 0 \tag{2}$$

$${}^{c}C_{s^-}^{\alpha} H(x) = \frac{1}{\Gamma(j-\alpha)} \int_x^s (u-x)^{n-\alpha-1} H^{(j)}(u) du,\ \alpha > 0 \tag{3}$$

Where $j = E(\alpha) + 1$, $E(\alpha)$ defines the integer of α

2.4. Riemann-liouville integral

Definition 2.4.
be H defined in $C([0, +\infty])$ then that of order $\alpha > 0$ of the function p is defined by:

$$I^{\alpha}(H)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} H(u) du,\ t > 0 \tag{4}$$

$$I^0(H)(x) = H(x) \tag{5}$$

The above expression defines the left-hand integral in the Riemmann-Liouville sense.

2.5. Existence and uniqueness

In this subsection, we state a result that shows the existence and uniqueness of the Caputo-type fractional eq. (6) de type Caputo.

$$^{c}C^{\alpha}u(y) = F(y, u(y)),\ \alpha \in]0,1]. \tag{6}$$

This equation is given with the following initial conditions:

$$u^{i}(0) = w_i\ \ (i = 1,2,3,...n-1).$$

Theorem 2.1.
Let α be a strictly positive rational number, $n = E(\alpha)+1$, X and T^* are strictly positive real number. Moreover, $w_1, w_2, w_3, ..., w_{n-1} \in \mathbb{R}$.

We défine F: $[0, Y^*] \times [w_0 - X, w_0 + X] \to \mathbb{R}$ a continuous function. Then there exits a $T = \min\{T^*, (\frac{X\Gamma(\alpha+1)}{M})^{\frac{1}{\alpha}}\}$(with $M = \sup_{(x,t)\in E} |F(x,t)|$, $E = [0, Y^*] \times [w_0 - X, w_0 + X]$) and a solution $u \in [0,T]$ of eq. (6) with the initial conditions given above.
If, moreover, F is Lipschitzian with respect to the second variable, which means:

$$|F(x,z_1) - F(x,z_2| < L|z_1 - z_2|$$

with L a strictly positive constant independent of x, z_1, z_1, then the solution $u \in C[0,T]$ of model eq. (6) is unique. For further details and the proof of this result, we refer the reader to we [9]

3. Description of the SBA method in the sense of Caputo

The description is based on the procedure description in [1, 3, 5] and solution is defined in a Banach space denoted B.

$$\begin{cases} ^{c}C_t^{\alpha}u = R(u) + N(u),\ t \geq 0,\ \alpha > 0 \\ u^{(i)}(0) = w_i,\ i = 0,1,...,j-1 \end{cases} \tag{7}$$

Where:
R and N are respectively linear and non-linear operators of B in B;
$^{c}C_t^{\alpha}u$ represents the fractional derivation of order α in the sense of Caputo;
$u \in B$ unknown function of the problem to be determined.
Assuming:

$$\begin{cases} L_t(.) =^{c} C_t^{\alpha}u \\ L_t^{-1}(.) = I^{\alpha}(.) \end{cases} \tag{8}$$

Where L_t^{-1} est inverstible in the Adomian sense.The canonical form associated with eq. (7) is given by the following relation:

$$L_t^{-1}L_t u = L_t^{-1}R(u) + L_t^{-1}N(u) \tag{9}$$

$L_t^{-1}L_t u(t) = u(t) - \sum_{i=0}^{j-1} \frac{u^{(i)}(0)}{i!}t^i$, $h = \sum_{i=0}^{j-1} \frac{u^{(i)}(0)}{i!}t^i$
$j = E(\alpha)+1$, $E(\alpha)$ is the integer part of α
$L_t^{-1}L_t u(t)$ is contracting.
Applying the method of successive approximations to eq. (9) we obtain:

$$u^k = h^k + L_t^{-1}R(u^k) + L_t^{-1}N(u^{k-1}),\ k \geq 1 \tag{10}$$

Then the SBA algorithm gives us:

$$\begin{cases} u_0^k = h^k + L_t^{-1}N(u^{k-1}),\ k \geq 1 \\ u_n^k = L_t^{-1}R(u_{n-1}^k),\ n \geq 1 \end{cases} \tag{11}$$

Picard's principle: let u^0 be any solution such that$N(u^0) = 0$ For k=1, calculate the approximate solution at the first iteration, u^1 defined by:

$$u^1 = \sum_{n=0}^{+\infty} u_n^1$$

Where: u_n^1 is an extension of $u_0^1, u_1^1, u_2^1 ... u_{n-1}^1$
For $k \geq 2$ we will first examine $N(u^1)$ as follows:

- Si $N(u^1)$=0, then we can say that the analytical solution in step 1 is the solution to the given problem.
- If not, we are obliged to improve the given problem or resort to other optimal theories for convergence.

At the end of the exact solution u, if it exist, we'll have:

$$u = \lim_{k \longmapsto +\infty} u^k = \lim_{k \longmapsto +\infty} \left(\sum_{n=0}^{+\infty} u_n^1 \right)$$

4. Results

Example 4.1.
Consider the nonlinear fractional convection-diffusion-réaction equation:

$$\begin{cases} {}^c D_t^\alpha u(x,t) - \lambda u(x,t) - \dfrac{\partial u(x,t)}{\partial x} - \dfrac{\partial^2 u(x,t)}{\partial x^2} - P(x,t) u^2(x,t) + \Psi(x,t) \dfrac{\partial u^2(x,t)}{\partial x} = F(x,t), \alpha \in]0,1], \\ t \geq 0 \\ u(x,0) = \cos x, x \in [0, 2\pi] \\ a u(0,t) + b u_x(0,t) = e^{\lambda t}, a, b \in \mathbb{R}, \lambda \in \mathbb{R}^* \\ c u(2\pi, t) + d u_x(2\pi, t) = e^{\lambda t}, c, d \in \mathbb{R} \end{cases} \tag{12}$$

Where :

$$\begin{cases} \Psi(x,t) = \dfrac{1}{p_0(x,t)} \\ P(x,t) = \dfrac{E_{1,\gamma+1}(\lambda t)}{p_0(x,t)} \\ F(x,t) = E_\alpha(\lambda t^\alpha) \left[-\tanh(\ln(\tan(\dfrac{\pi}{4} + \dfrac{x}{2}))) + \cos x (1 + \dfrac{E_t(\gamma, \lambda)}{t^\gamma}) \right] \\ p_0(x,t) = (\cos x) E_\alpha(\lambda t^\alpha) \end{cases}$$

Equation (12) can then be transformed into the following form:

$$L_t u(x,t) = \lambda u(x,t) + N_1(u) \tag{13}$$

$$L_t u(x,t) = L_x u(x,t) + L_{xx} u(x,t) + N_2(u) \tag{14}$$

With:

$$\begin{cases} L_t(.) = {}^c D_t^\alpha(.) \\ L_t^{-1} = I^\alpha(.) = \displaystyle\int_0^t ds \\ L_x = \dfrac{\partial}{\partial x}(.) \\ L_x^{-1} = \displaystyle\int_0^{2\pi} dz \end{cases} \tag{15}$$

By composing eq. (13) with L_t^{-1} and eq. (14) by L_x^{-1}, we find respectively the following equations:

$$u(x,t) = u(x,0) + L_t^{-1}(\lambda u(x,t)) + N_1(u) \tag{16}$$

$$L_x^{-1} L_t u(x,t) + u(0,t) + u_x(0,t) - (u(2\pi,t) + u_x(2\pi,t)) + N_2(u) = 0 \tag{17}$$

Equation (17) can then be transformed into the following form:

$$L_x^{-1} L_t u(x,t) + au(0,t) + bu_x(0,t) - (cu(2\pi,t) + du_x(2\pi,t)) + N_2(u) = 0 \tag{18}$$

With: a=b=c=d=1.
Taking eq. (18) into account, eq. (16) becomes:

$$u(x,t) = u(x,0) + au(0,t) + bu_x(0,t) - (cu(2\pi,t) + du_x(2\pi,t)) + L_t^{-1}(\lambda u(x,t)) + N_1(u) + N_x(u) \tag{19}$$

$$u(x,t) = u(x,0) + R(u) + \bar{N}(u) \tag{20}$$

The SBA method allows us to establish the following successive approximation:

$$u^k(x,t) = u^k(x,0) + au^k(0,t) + bu_x^k(0,t) + cu^k(2\pi,t) + du_x^k(2\pi,t) + R(u^k) + \bar{N}(u^k),\ k \geq 1 \tag{21}$$

The SBA algorithm is therefore as follows:

$$\begin{cases} u_0^k(x,t) = u^k(x,0) + au^k(0,t) + bu_x^k(0,t) + cu^k(1,t) + du_x^k(1,t) + \bar{N}(u^{k-1}),\ k \geq 1 \\ \\ u_n^k(x,t) = R(u_{n-1}^k),\ n \geq 1 \end{cases} \tag{22}$$

For k=1, suppose there exist u^0 such that $N(u^0)$=0, then the SBA algorithm becomes:

$$\begin{cases} u_0^1(x,t) = u^1(x,0) + au^k(0,t) + bu_x^k(0,t) + cu^1(2\pi,t) + du_x^1(2\pi,t) = \cos x \\ \\ u_1^1(x,t) = L^{-1}(R(u_0^1)) \end{cases} \tag{23}$$

Finding the solution in the first iteration

$$\begin{cases} u_1^1(x,t) = L^{-1}(R(u_0^1)) = I^\alpha(\lambda u_0^1) = \dfrac{\lambda(\cos x)}{\Gamma(\alpha+1)} t^\alpha \\ \\ u_2^1(x,t) = L^{-1}(R(u_1^1)) = I^\alpha(\lambda u_1^1) = \dfrac{\lambda^2(\cos x)}{\Gamma(2\alpha+2)} t^{2\alpha} \\ \\ . \\ . \\ . \\ \\ u_n^1(x,t) = L^{-1}(R(u_{n-1}^1)) = I^\alpha(\lambda u_{n-1}^1) = \dfrac{\lambda^n(\cos x)}{\Gamma(n\alpha+1)} t^{n\alpha} \end{cases}$$

Then the solution at rank k=1 is:

$$u^1 = (\cos x) E_\alpha(\lambda t^\alpha)$$

For K=2 it first comes down to calculating $\bar{N}(u^1)$

$$\begin{cases} \bar{N}(u^1) = I^\alpha\left(\dfrac{\partial u^1(x,t)}{\partial x} + \dfrac{\partial^2 u^1(x,t)}{\partial x^2} + P(x,t)u^2(x,t) - \Psi(x,t)\dfrac{\partial (u^1)^2(x,t)}{\partial x} + F(x,t)\right) \\ \\ +L_x^{-1}\left(\lambda u^1(x,t) - L_t u^1(x,t) - \Psi(x,t)\dfrac{\partial (u^1)^2(x,t)}{\partial x} - P(x,t)(u^1)^2(x,t) - F(x,t)\right) \\ \\ = I^\alpha\left((\sin x)E_\alpha(\lambda t^\alpha) - E_\alpha(\lambda t^\alpha)(1 + \dfrac{E_t(\gamma,\lambda)}{t^\gamma})\cos x + F(x,t)\right) \\ \\ +L_x^{-1}\left(\lambda E_\alpha(\lambda t^\alpha)\cos x + (1 + 2E_\alpha(\lambda t^\alpha))\sin x + \dfrac{E_\alpha(\lambda t^\alpha)(E_t(\gamma,\lambda)+1)}{t^\nu}\cos x\right) \\ \\ = I^\alpha(0) + 0 \end{cases} \tag{24}$$

$$\bar{N}(u^1) = 0$$

Hence the analytical solution of problem eq. (12) is:

$$u(x,t) = (\cos x) E_\alpha(\lambda t^\alpha)$$

Example 4.2.
Consider the nonlinear fractional convection-diffusion-reaction model in one-dimensional space:

$$\begin{cases} {}^cD_t^\alpha u(x,t) + \lambda u(x,t) - \dfrac{\partial u(x,t)}{\partial x} - \dfrac{\partial^2 u(x,t)}{\partial x^2} - P(x,t)u^2(x,t) + \Psi(x,t)\dfrac{\partial u^2(x,t)}{\partial x} = F(x,t), \alpha \in]0,1], \\ t \geq 0 \\ u(x,0) = \sin x, x \in [0, 2\pi] \\ au(0,t) + bu_x(0,t) = e^{-\lambda t}, a, b \in \mathbb{R}, \lambda \in \mathbb{R}^* \\ cu(2\pi,t) + du_x(2\pi,t) = e^{-\lambda t}, c, d \in \mathbb{R} \end{cases} \quad (25)$$

Proceeding in the same way as the previous example, we obtain:

$$u(x,t) = \sin x E_\alpha(-\lambda t^\alpha)$$

5. Conclusion

The analytical resolution of fractional convection-diffusion-reaction models with mixed-type conditions has been a success thanks to the new technique using fractional derivation in time and classical differentiation in space in the SBA algorithm. This approach enabled us to obtain a solution integrating all the conditions imposed on the models.

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References

[1] ABAKAR MAHAMAT SEID, MINOUNGOU YOUSSOUF, ABDARAHMAN ABAKAR HIMEDA, BAKARI ABBO, PARE YOUSSOUF, Application of the SBA method to fractional Kolmogorov-Fisher- Piskunov-Petrovsky models, International Journal of Advances in Applied Mathematics and Mechanics 12(2) (2024) pages 129–136.

[2] Hassan Kamil Jassim, Mayada Gassab Mohammed b, Saad Abdul Hussain Khafif, The approximate solutions of time-fractional Burger's and coupled time-fractional Burger's equations, International Journal of Advances in Applied Mathematics and Mechanics 6(4) (2019) pages 64–70.

[3] A. A. HIMEDA, Bakari Abbo and Germain KABORE. Twin Reversal Available at in the Somé Blaise Abbo Procedure Featured to Schrödinger-Type Fractional-Order Evolution Equations. ARJOM ; Volume 20, Issue 9, Page 15-25, 2024.

[4] Jannelli, A.; Ruggieri, M.; Speciale, M.P. Numerical solutions of space fractional advection–diffusion equation, with nonlinear source term. Appl. Numer. Math. 2020, 155, 93–102. [CrossRef] .

[5] B.Abbo New numerical algorithm for solving ordinary differential equations (ODE) and non linear partial differential equations(PDE). Doctorat thesis unique . University of Ouagadougou, January 2007, UFR/ SEA, Mathematics and Cumputer Science Departement(Burkina Faso).

[6] Dwivedi, K.D.; Das, S.; Baleanu, D. Numerical solution of nonlinear space–time fractional-order advection–reaction–diffusion equation. J. Comput. Nonlinear Dynam. 2020, 15, 061005. [CrossRef]

[7] Maged Z. Youssef, M. M. Khader , Ibrahim Al-Dayel , and W. E. Ahmed Solving Fractional Generalized Fisher–Kolmogorov–Petrovsky –Piskunov's Equation Using Compact-Finite Different Methods Together with Spectral Collocation Algorithms, Hindawi Journal of Mathematics, Volume 2022, Article ID 1901131, 9 pages https://doi.org/10.1155/2022/1901131.

[8] Maud El-Hachem, Scott W. McCue , Wang Jin , Yihong Du and Matthew J. Simpson Revisiting the Fisher–Kolmogorov– Petrovsky–Piskunov equation to interpret the spreading–extinction dichotomy, royal society publishing, Volume 475, Article ID 20190378, 19 pages https://doi.org/10.6084/m9.figshare. c.4638110.

[9] Houmor Tarek. Chaos analysis in a system of fractionary differential equations. Thesis defended on 30/09/2014. Constantine1 University.

[10] K. M. Saad, M. M. Khader, J. F. Gómez-Aguilar, and D. Baleanu. Numerical solutions of the fractional Fisher's type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods, Chaos (Woodbury, N.Y.), vol. 29, pp. 23116–23125, 2019.

[11] KEHAILI ABDELKADER (2021), *Numerical resolution of a certain type of partial equations of any order* . Doctoral thesis . University of DJILLALI LIABES, ALGERIA.

[12] Nauman Raza, Muhammad Asad Ullah (2020), *A comparative study of the analyse of heat transfer of fractional Maxwell fluid using Caputo and Caputo-Fabrizio derivatives* . Vol. 98,No. 1, Janvier 2020. doi:http://dx.doi.org/10.1139/cjp-2018-0602.

[13] Virginie Seng. Finding D'Adomian canonical forms for solving nonlinear functional equations by the decompositional method, Doctoral thesis defended on 26/11/1997; PIERRE and MARIE CURIE University (FRANCE).

[14] Y. Pare, J. M. Loufouilou and G. Bissanga. Application of the SBA and Peturbation Methods to solving a Nonlinear Partial Equation; Nonl. Analysis and Differential Equations, Vol.1, 2013, no.3, 101-113; HIKARI Ltd, www.m-hikari.com.

[15] J. B. Yindoula, Y. Pare, G. Bissanga, F. Bassono and B. Somé. Application of the Adomian decomposition method and Laplace transform method to solving the convection diffusion-dissipation equation; International Journal of Applied Mathematical Research, 3(1)(2014)30-35.

[16] J. B. Yindoula, G. Bissanga, Y. Pare, F. Bassono and B. Somé. Application of the Adomian decomposition method (ADM) and Some Blaise Abbo method(SBA) to solving the diffusion-reaction equation; Advances in Theoretical and Applied Mathematics, 9(2)(2014)97-104.

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