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Numerical solution of some wave equations with non-local edge conditions of fractional order in the sense of Caputo by the SOME BLAISE ABBO method (SBA)

Research Article

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Abstract: In this paper, we are interested in solving some one-dimensional wave equations of fractional order β with $1 < \beta \leq 2$ in the Caputo sense by the **SBA** method.
The **SBA** method is based on a combination of the Adomian decomposition method, Picard's principle and the method of successive approximations. This method uses a process that converges rapidly to the exact solution when it exists in a functional space of the problem posed.

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1. Introduction

In this paper, we consider the one-dimensional wave equation of fractional order :

$$\frac{\partial^\alpha v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = q(x,t) \qquad 0 \leq x \leq l, 0 < t \leq T \qquad 1 < \alpha \leq 2. \tag{1}$$

Subject to the following conditions:
Initial conditions

$$v(x,0) = f_1(x) \qquad 0 \leq x \leq l \tag{2}$$

$$v_t(x,0) = f_2(x) \qquad 0 \leq x \leq l \tag{3}$$

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Dirichlet condition

$$v(0,t) = g_1(t) \qquad 0 < t \le T \tag{4}$$

Non-local condition

$$\int_0^l v(x,t)\,dx = g_2(t) \qquad 0 < t \le T \tag{5}$$

If we meet the above conditions, we obtain :

$$\begin{cases} \frac{\partial^\alpha v}{\partial t} - \frac{\partial^2 v}{\partial x^2} &= q(x,t) \\ v(x,0) &= f_1(x) \\ v_t(x,0) &= f_2(x) \quad 0 \le x \le l, 0 < t \le T, \qquad 1 < \alpha \le 2 \\ v(0,t) &= g_1(t) \\ \int_0^l v(x,t)\,dx &= g_2(t) \end{cases} \tag{6}$$

where q, f_1, f_2, g_1, g_2 are known functions such that they satisfy the conditions for the solution of this equation to exist and be unique. The existence and uniqueness of the solution of this problem are discussed in [1]. The notion of a non-local condition is more precise than the classical condition for describing natural phenomena. The importance of the non-local condition in many applications is discussed in [4, 9].
We are interested in the application of the SBA method in [10], to solve the above equations. This document is organised as follows: the second section deals with the preliminaries on fractional calculations. Section 3 is the description of the SBA method, section 4 is devoted to the application of the method to a few one-dimensional wave equations of fractional order in the sense of Caputo. Finally, section 5 is the conclusion.

2. Preliminaries on fractional calculations

For definitions of special functions for fractional calculations, fractional integrals, fractional derivatives, fractional differential equations, please see [15–17, 19, 20].

3. Description et convergence de la méthode SBA

This section is devoted to the description and convergence of the SOME Blaise Abbo method **(SBA)**.

3.1. Description of the SBA method

For the description, we adopt the approach proposed in ([14]). Thus it is done on a linear fractional EDO and the technique adapts to linear fractional EDP_S.
In a complete space H, consider the following non-linear fractional EDO problem:

$$\begin{cases} {}^C D_t^\alpha v &= R(v) \\ \\ v^{(m)}(0) &= \beta_m, \; m = \{0;1;2;...;j-1\} \end{cases} \tag{7}$$

where
R is a linear operator in H
$v \in H$
${}^C\mathcal{D}_t^\alpha$ is the fractional derivative of order α in the sense of Caputo. $t \ge 0$; $\alpha > 0$; $v = v(x,t)$; $v \in C^m(\mathbb{R} \times [0;T])$.
$m \in \mathbb{N}$
$j = [\alpha]$, $[\alpha]$ is the integer part of α.
$v^{(m)}$ is the derivative of order m of the function u.

Posing

$$\begin{cases} L(.) &= {}^c\mathcal{D}^\alpha(.) \\ \\ L^{-1}(.) &= I^\alpha(.) \end{cases}, \tag{8}$$

where L^{-1} is the inverse of L in the Adomian sense and I^{alpha} is the integral in the Riemann-Liouville sense, we have:

$$Lv = Rv. \tag{9}$$

Composing eq. (9) by L^{-1} gives:

$$L^{-1}Lv = L^{-1}Rv \tag{10}$$

where $L^{-1}R$ is a contracting operator.

Or

$$L^{-1}Lv = v - \sum_{m=0}^{j-1} \frac{t^m}{m!} v^{(m)}(0) \tag{11}$$

Equations (10) and (11) and the method of successive approximations give:

$$v^k = \gamma^k(t) + L^{-1}Rv^k \tag{12}$$

with $\gamma^k(t) = \sum_{m=0}^{j-1} \beta_m \frac{t^m}{m!}$
Solving the scheme eq. (12) by the method of approximations consists in determining at each iteration($k = 1;2;...$) approximate solutions v^1, v^2,...v^n, which are series.

Posing

$$v^k = \sum_{n=0}^{+\infty} v_n^k \tag{13}$$

we derive the following algorithm **SBA**:

$$\begin{cases} v_0^k &= \gamma^k(t),\ k \geqslant 1 \\ v_{n+1}^k &= L^{-1}(R(v_n^k)),\ n \geqslant 0 \end{cases} \tag{14}$$

Explicitly, the development of the algorithm eq. (14) consists of first calculating the terms of the sequence $(v_n^k)_n$ for $k \geqslant 1$ fixed, and deducing v^k if the series $displaystylev^k = \sum_{n=0}^{+\infty} v_n^k$ converges.
So, for the first iteration, $k = 1$, we calculate the v_0^1, v_1^1, v_2^1, v_3^1, v_n^1 of the sequence $(v_n^1)_n$ and deduce

$$v^1 = \sum_{n=0}^{+\infty} v_n^1$$

Since the problem eq. (7) is linear then its general solution is given by v^1.

3.2. Convergence of the SBA algorithm

For convergence, we refer the reader to the following documents [10] and [14].

4. Applications

Posing

$$\begin{cases} L = & \frac{\partial^\alpha}{\partial t}(.) \\ L^{-1} = & \mathcal{I}^\alpha(.) \end{cases}$$

By integrating eq. (1) with respect to x, we obtain:

$$\int_0^x \int_0^x \frac{\partial^\alpha v}{\partial t} dt dt - \int_0^x \int_0^x \frac{\partial^2 v}{\partial \mu^2} d\mu d\mu - \int_0^x \int_0^x q(x,t) dt dt = 0 \tag{15}$$

Consider

$$\int_0^x \int_0^x \frac{\partial^2 v}{\partial \mu^2} d\mu d\mu = \int_0^x \left(\frac{\partial v(x,t)}{\partial \mu} - \frac{\partial v(0,t)}{\partial \mu}\right) d\mu$$

Using eqs. (4) and (5), we obtain: $\int_0^x \int_0^x \frac{\partial^2 v}{\partial \mu^2} d\mu d\mu = g_2(t)$.
So,eq. (15) becomes:

$$\int_0^x \int_0^x \frac{\partial^\alpha v}{\partial t} dtdt - g_2(t) - \int_0^x \int_0^x q(x,t)dtdt = 0 \tag{16}$$

By integrating eq. (1) with respect to t, we obtain:

$$\mathcal{I}^\alpha\left(\frac{\partial^\alpha}{\partial t} v\right) = \mathcal{I}^\alpha\left(\frac{\partial^2}{\partial x^2} v\right) + \mathcal{I}^\alpha\left(q(x,t)\right) \tag{17}$$

Or

$$\mathcal{I}^\alpha\left(\frac{\partial^\alpha}{\partial t} v(x,t)\right) = v(x,t) - v(x,0) - t v_t(x,0)$$

Using eqs. (2) and (3), we obtain:

$$\mathcal{I}^\alpha\left(\frac{\partial^\alpha}{\partial t} v(x,t)\right) = v(x,t) - f_1(x) - t f_2(x) \tag{18}$$

Equation (17) becomes:

$$v(x,t) - f_1(x) - t f_2(x) - \mathcal{I}^\alpha\left(\frac{\partial^2}{\partial x^2} v\right) - \mathcal{I}^\alpha\left(q(x,t)\right) = 0 \tag{19}$$

If we add eqs. (16) and (19), we get:

$$\begin{aligned} v(x,t) = f_1(x) + t f_2(x) + g_2(t) + \mathcal{I}^\alpha\left(\frac{\partial^2}{\partial x^2} v\right) - \int_0^x \int_0^x \frac{\partial^\alpha v}{\partial t} dtdt + \\ \mathcal{I}^\alpha\left(q(x,t)\right) + \int_0^x \int_0^x q(x,t)dtdt \end{aligned} \tag{20}$$

Posing

$$\begin{cases} L = & \frac{\partial^\alpha}{\partial t}(.) \\ L^{-1} = & \mathcal{I}^\alpha(.) \\ L^{-1}(Rv) = & \mathcal{I}^\alpha\left(\frac{\partial^2}{\partial x^2} v\right) - \int_0^x \int_0^x \frac{\partial^\alpha v}{\partial t} dtdt \\ h(x,t) = & \mathcal{I}^\alpha\left(q(x,t)\right) + \int_0^x \int_0^x q(x,t)dtdt \\ Nv = & 0 \end{cases},$$

we obtain the following SBA algorithm:

$$\begin{cases} v_0^k = f_1^k(x) + t f_2^k(x) + g_2^k(t) + h^k(x,t); & k \geq 1 \\ v_{n+1}^k = L^{-1}R(v_n^k); & n \geq 0 \end{cases} \tag{21}$$

Application 4.1.

$$Pour \quad q(x,t) = 0,\ f_1(x) = 0,\ f_2(x) = \pi\cos(\pi x),\ g_1(t) = \sin(\pi t),\ g_2(t) = 0.$$

By replacing q, f_1,f_2 and g_2 in eq. (21), we obtain the following SBA algorithm :

$$\begin{cases} v_0^k = \pi t\cos(\pi x) & k \geq 1 \\ v_{n+1}^k = L^{-1}R(v_n^1) & n \geq 0 \end{cases} \tag{22}$$

By developing the algorithm eq. (22) for the different values of the integers k and n, we obtain:

$$\begin{cases} v_0^1 = \pi t \cos(\pi x) \\ v_1^1 = (-1)\pi^{2\times 1+1} \cos\pi x \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} \\ v_2^1 = (-1)^2 \pi^{2\times 2+1} \cos\pi x \frac{\Gamma(2)}{\Gamma(2\alpha+2)} t^{2\alpha+1} \\ v_3^1 = (-1)^3 \pi^{2\times 3+1} \cos\pi x \frac{\Gamma(2)}{\Gamma(3\alpha+2)} t^{2\alpha+1} \\ \vdots \\ v_n^1 = (-1)^n \pi^{2n+1} \cos\pi x \frac{\Gamma(2)}{\Gamma(n\alpha+2)} t^{n\alpha+1} \end{cases} \tag{23}$$

The approximate solution to the first iteration is :

$$v^1 = \sum_{n=0}^{+\infty} v_n^1$$

$$v^1 = \pi t \cos\pi x \mathbb{E}_{\alpha,2}(-\pi^2 t^\alpha)$$

The exact solution to the problem eq. (6) is:

$$v(x,t) = \lim_{k\to+\infty} v^1$$

$$v(x,t) = \pi t \cos\pi x \mathbb{E}_{\alpha,2}(-\pi^2 t^\alpha)$$

Application 4.2.

$$Pour \qquad q(x,t) = 0,\ f_1(x) = \cos(\pi x),\ f_2(x) = 0,\ g_1(t) = \cos(\pi t),\ g_2(t) = 0.$$

By replacing q, f_1,f_2 and g_2 in eq. (21) we obtain the following SBA algorithm:

$$\begin{cases} v_0^k = \cos(\pi x) \quad k \geq 1 \\ v_{n+1}^k = L^{-1} R(v_n^k) \quad n \geq 0 \end{cases} \tag{24}$$

By developing the algorithm eq. (22) for the different values of the integers k and n, we obtain:

$$\begin{cases} v_0^1 = \cos(\pi x) \\ v_1^1 = (-1)\pi^2 \cos\pi x \frac{t^\alpha}{\Gamma(\alpha+1)} \\ v_2^1 = (-1)^2 \frac{\pi^{2\times 2}\cos\pi x}{\Gamma(2\alpha+1)} t^{2\alpha} \\ v_3^1 = (-1)^3 \frac{\pi^{2\times 3}\cos\pi x}{\Gamma(3\alpha+1)} t^{3\alpha} \\ \vdots \\ v_n^1 = (-1)^n \frac{\pi^{2n}\cos\pi x}{\Gamma(n\alpha+1)} t^{n\alpha} \end{cases} \tag{25}$$

The approximate solution to the first iteration is :

$$v^1 = \sum_{n=0}^{+\infty} v_n^1$$

$$v^1 = \cos\pi x \mathbb{E}_\alpha(-\pi^2 t^\alpha)$$

The exact solution to the problem eq. (6) is:

$$v(x,t) = \lim_{k\to+\infty} v^1$$

$$v(x,t) = \cos\pi x \mathbb{E}_\alpha(-\pi^2 t^\alpha)$$

5. Conclusion

In this paper, the **SBA** method has been successfully used to solve some fractional order partial differential equation problems with non-local edge conditions using Caputo's approach. This method is therefore a very effective numerical analysis tool for solving these types of problems.

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