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Finsler manifolds with applications in fluid flow mechanics

Research Article

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Abstract: Flows under load of an incompressible viscous fluid in the gravity field in anisotropic media is studied based on non-Euclidean geometry called the theory of Finsler geometry. According to fermat's variational principle, for a two-dimensional, the paths of fluids flow are described by geodesics in a Finsler metric through the influence of direction dependence on fluids flow called Kropina metric.
Then, in this case, the deviation curvature tensor implies that the trajectory of fluids flow is Jacobi unstable for the deviation of geodesics.

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1. Introduction

The aim of this work is to apply the principles and methods of Finsler geometry to solve the flow equations; which are sometimes differential equations of the second order. According to Fermat's variational principle, the trajectories of fluid flow are described by geodesics on a Finsler manifold. After several decades of relatively slow development, Finsler geometry is now a modern subject with a large body of theorems and techniques, and has mathematical content which are no comparable to any area of differential geometry.
Contrary to these fossilized and outdated opinions (we believe that the world is "Finslerian") in a true sense; the Finslerian world can be the key to the practical resolution of the equations modeling several natural phenomena of physical life, the insights of which are given in the theory of "sprays" and that some authors have tried to show it in several applications in thermodynamic, optic, ecology, evolution and development of biology [3, 6, 14, 18–20].
On the other hand, if the complexity of the subject has not disappeared, the modern theoretical approach to sheaves has considerably increased its comprehensibility
The flow of fluids followed by the phenomenological condition through continuous media is studied by the theory of Finsler geometry. According to Fermat's variational principle, the paths of fluid flow are described by geodesics in a Finsler manifold. For continuous media, the dependence of the flow direction gives a Finsler metric [1, 19, 20].
A geometric approach is much more recommended; we describe the concrete case of physics that we will be consider.

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We distinguish how the properties of wave propagation can be a geometric form. We are interested in the behavior of the solution of the equations in a movement. The geodesics of the metric are the fluid flows in a domain.
The motion of fluid flow is given by the phenomenological law, which states that the flow velocity is equivalent to the hydraulic head gradient [2, 10, 16]. In this case; since the hydraulic conductivity depends on the position through homogeneity, the current line of the fluid is not a straight line. This means that the flow of fluids through nonhomogeneous porous media is formulated geometrically in non-Euclidean space [7].
It has been mentioned that the Finslerian framework of fluid flow which depends on the direction caused by a change in surface is perpendicular to the flow. Therefore, the flow will be discussed on the basis of the fundamental function and geometric objects in the Finslerian framework.
In this work, the flow of fluids through homogeneous or non-homogeneous media is formulated by Finsler geometry. In section 2, we review the basic ideas of Finslerian geometry and the mathematical formalism of the KCC theory and the Jacobi stability analysis of the homogeneous isotropic by using the second order formulation of the dynamics is performed.Always in this section, we introduce the formulation of problem of fluids flow. In section 3, the Finsler metric of fluids flow is obtained, the geodesics as the path of fluids flow are investigated. Thus, The KCC geometric quantities giving the geometric description of fluids flow models; then the stability of fluids flow is discussed on the basis of the curvature tensor and the geodesics equations for fluids flow is resolved. Finally, section 4 is the conclusion.

2. Preliminaries

In this section we present the important objects induced by a Finsler manifold of dimension n and we are briefly present in the section following the propagation model through the Navier-Stockes equations.

2.1. Preliminaries on Finsler geometry

Let M be an n-dimensional smooth manifold and denote dy $TM = \bigcup_{x\in M} T_xM$ a tangent bundle of M. Each element of TM has the form (x, y) for all $x \in M$ and $y \in TM$ $(y \neq 0)$.

Definition 2.1.
A function $F : TM \longrightarrow [0;\infty[$ is called a Finsler structure, if, in local coordinate system (x^i, y^i), F satisfies

1. F is $C^\infty(TM_0)$; where $TM_0 = TM - \{0_x; x \in M\}$;
2. F is positively homogeneous of degree 1 in y^i;
3. The fundamental tensor g_{ij} is given by:

$$g_{ij}(x, y) = \frac{1}{2}\partial_{(i)}\partial_{(j)}F^2(x, y), \tag{1}$$

where $\partial_{(i)} = \dfrac{\partial}{\partial y^i}$, are definite positive.

Proposition 2.1.
[3, 8] Let (M,F) be a Finsler manifold; then the geodesics equations are given by:

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \dot{x}) = 0 \tag{2}$$

$$\textit{where} \quad G^i(x, \dot{x}) = \frac{1}{2}\gamma^i_{jk}y^j y^k. \tag{3}$$

Definition 2.2.
[8] Let (M, g) be a Riemann manifold, then the Christoffel symbols are defined by:

$$\gamma^i_{jk} = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}). \tag{4}$$

2.2. KCC theory and Jacobi stability

The geometrization of the dynamic system consist to associate a geometric structure with a dynamic system defined by eq. (2). We first introduce a nonlinear connection N on the manifold M, with the coefficients N^i_j defined as:

$$N^i_j = \frac{\partial G^i}{\partial y^j} = \partial_{(j)}G^i. \tag{5}$$

The nonlinear connection can be seen as a covariant derivative ∇^N.
For two vector fields v, w set to M. We introduce the covariant derivative ∇^N as follows:

$$\nabla^N_v w = \left[v^j \partial_j w^i + N^i_j(x,y) w^j \right] \partial_i. \tag{6}$$

If $N^i_j(x,y) = \Gamma^i_{jl}(x) y^l$, the eq. (6) reduces to the definition of the covariant derivative for the special case of a linear connection.
In the following, we will try to determine a simplified form of the functions G^i, in order to facilitate the resolution of the geodesic equations or to use the KCC theory for their stability.
For the second order differential equations system, the Jacobi stability which is a trajectory was introduced by the Kosambi-Cartan-Chern (KCC) [11, 12, 15]. In this section, we will consider the stability of the flow path based on KCC theory.
Based on KCC theory [4, 9, 19]; the trajectory $x^i(t)$ of the system eq. (2) roughly changes its trajectory according to $x^i_* = x^i(t) + \epsilon \xi_i$ where ξ^i is a vector field and ϵ is a parameter. In this case, the eq. (2) becomes the variational equation for the limit $\epsilon \longrightarrow 0$

$$\frac{d^2\xi_i}{dt^2} + 2\frac{\partial G^i}{\partial y^i}\frac{d\xi_i}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi_j = 0. \tag{7}$$

Definition 2.3.
The covariant derivative $\frac{D}{Dt}$ is defined as follows:

$$\frac{D\xi_i}{Dt} = \frac{d\xi_i}{dt} + G^i_j \xi_j, \tag{8}$$

where G^i_j are the coefficients of the nonlinear connection $G^i_j = \frac{\partial G^i}{\partial y^j}$.

Using the relation eqs. (2) and (8) becomes:

$$\frac{D\dot{x}^i}{Dt} = G^i_j \dot{x}^j - G^i. \tag{9}$$

Finally eq. (9) defines the first KCC-invariant of eq. (2).
Assume the relation eq. (8), the KCC-covariant derivative; differentiating, we get

$$\frac{D^2\xi_i}{dt^2} = \mathscr{P}^i_r \xi_r, \tag{10}$$

where

$$\mathscr{P}^i_j = -2\partial_j G^i - 2G^k G^i_{jk} + \dot{x}^k \partial_k G^i_j + G^i_k G^k_j, \tag{11}$$

$$\text{with} \quad G^i_{jk} = \frac{\partial G^i_j}{\partial \dot{x}^k}.$$

The tensor $\mathscr{P}^i_j$ is the second KCC-invariant of eq. (2), or deviation curvature tensor and the eq. (10) is called Jacobi's equation; implies the stability of the trajectories: the trajectories of the system eq. (2) are Jacobi stable if and only if the real parts of the eigenvalues of $\mathscr{P}^i_j$ are strictly negative everywhere and on the other hand the Jacobi is unstable.
In Riemann or Finsler geometry, when the system eq. (2) describes the geodesic equations in the given geometry, eq. (10) represents the Jacobi field equation.
The 3^{rd}, 4^{th} and 5^{th} invariants of eq. (2) are defined as:

$$\mathscr{R}^i_{jk} = \frac{1}{3}\left(\partial_{(j)}\mathscr{P}^i_k - \partial_{(k)}\mathscr{P}^i_j\right), \tag{12}$$

$$\mathscr{P}^i_{jkl} = \partial_{(l)}\mathscr{R}^i_{jk}, \tag{13}$$

$$\mathscr{D}^i_{jkl} = \partial_{(l)} G^i_{jk}. \tag{14}$$

The third invariant $\mathscr{R}^i_{jk}$ is interpreted as a torsion.
The fourth and fifth invariants, $\mathscr{P}^i_{jkl}$ and $\mathscr{D}^i_{jkl}$ are called Riemann-Christoffel curvature tensor and Douglas tensor.
The Jacobi field along the geodesic is related by the curvature tensor of the Finsler manifold [8]; if the curvature is positive at ξ, then the geodesics are merged and if the curvature is negative at ξ, they are dispersed. Particularly, for the Finsler manifold of dimension 2, it is a scalar curvature. In this case, the tensor $\mathscr{P}^i_j$ is:

$$\mathscr{P}^i_j = -\tau(x^i, \dot{x}^i) L^2 h^i_j, \tag{15}$$

where τ is the scalar curvature and h^i_j the angular metric

$$h^i_j = g^{ik} h_{kj}, \tag{16}$$

with $h_{kj} = g_{kj} - \frac{\partial L}{\partial y^j}\frac{\partial L}{\partial y^k}$.

2.3. Formulation of problem

The physical modeling of flows requires a certain number of steps to lead to a model representative of reality.[10]
Let consider the above equations come from the Navier-Stockes equations

$$\frac{du}{dt} + \nabla((hu)u) = 0 \tag{17}$$

$$\frac{\partial(hu)}{\partial t} + u\nabla(hu) = -\frac{1}{\rho}\nabla p + \nabla(\lambda\nabla(hu)) + \nabla(\mu D), \tag{18}$$

where h is height of immersion, u is speed, hu is flow per unit width, p is pressure, μ is viscosity and λ denote a conductivity.
In the case of fluid flows, two categories of movement are possible[10]:

1. Flows under load;
2. Free surface flows.

2.3.1. Flows under load of an incompressible viscous fluid in the gravity field

We denote h the coast of a point and an equation of motion (Navier-Stokes) is written by:

$$\begin{cases} \dfrac{du}{dt} = -\dfrac{1}{\rho}\dfrac{\partial}{\partial x}(p+\rho g h) + \dfrac{\mu}{\rho}\nabla^2 u, \\ \dfrac{dv}{dt} = -\dfrac{1}{\rho}\dfrac{\partial}{\partial y}(p+\rho g h) + \dfrac{\mu}{\rho}\nabla^2 v, \\ \dfrac{dw}{dt} = -\dfrac{1}{\rho}\dfrac{\partial}{\partial z}(p+\rho g h) + \dfrac{\mu}{\rho}\nabla^2 w, \end{cases} \tag{19}$$

Then the continuity equation is given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{20}$$

and an equation of state is

$$: \rho = constant. \tag{21}$$

Let's set

$$p^* = p + \rho g h. \tag{22}$$

Proposition 2.2.
The system of equations with the reduced variables becomes

$$\begin{cases} \dfrac{du}{dt} = -\dfrac{\partial}{\partial x}p + \dfrac{1}{Re}\nabla^2 u \\ \dfrac{dv}{dt} = -\dfrac{\partial}{\partial y}p + \dfrac{1}{Re}\nabla^2 v \\ \dfrac{dw}{dt} = -\dfrac{\partial}{\partial z}p + \dfrac{1}{Re}\nabla^2 w \end{cases} \tag{23}$$

where $Re = \dfrac{DV_0\rho}{\mu}$ *and D be a linear dimension characteristic of the flow studied: width diameter.*

Proof. For the proof, we have 4 variables u, v, w, p considered as a function of x, y, z, t. Then Let's set:

$$x = \frac{x'}{D},\ y = \frac{y'}{D},\ z = \frac{z'}{D}$$

V_0 a reference speed and $t = \dfrac{D}{V_0}$, $p = \rho V_0^2$ is the pressure
Then $x = \dfrac{x'}{D}$, $V = \dfrac{V'}{V_0}$, $t = \dfrac{t'V_0}{D}$.
In vectorial form

$$\frac{dV}{dt} = -\nabla p + \frac{1}{Re}\nabla^2 V \tag{24}$$

$$\frac{\partial V}{\partial t} + V\nabla V = -\nabla p + \frac{1}{Re}\nabla^2 V \tag{25}$$

with $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y} + w\dfrac{\partial}{\partial z}$
$p = p_0$ on a horizontal free surface, $V = V_0$ Initial condition on a surface defined by $f(x, y, z) = 0$ □

2.3.2. Free surface flow

We especially want to examine this case. Free surface flow implies the pressure is constant and the velocity cancels. Then the system eq. (23) of equations becomes

$$\frac{d(hu)}{dt} = -\nabla p + \frac{1}{Re}\nabla^2(hu), \tag{26}$$

$$\text{where} \quad \frac{d(hu)}{dt} = \frac{\partial(hu)}{\partial t} + \nabla((hu).u) \quad \text{and} \quad Re = \frac{DV_0\rho}{\mu} \tag{27}$$

3. Finsler Metric

The aim of this section is to determine the Finsler metric using Fermat's variational principle and then to establish the fundamental tensor.[3]

3.1. Variational principle

Proposition 3.1.
Consider the relation eq. (26), we obtain the fondamental Function F which will verify the different properties of a Finsler metric.

$$J = \int_{x_a}^{x_b} F dl = \int_{x_a}^{x_b} pDV_0\frac{\rho}{\mu} dl. \tag{28}$$

Proof. If the height and speed vary over time; then the equation of motion becomes

$$\frac{d(hu)}{dt} = -\nabla p + \frac{\mu}{DV_0\rho}\nabla^2(hu). \tag{29}$$

The phenomenological condition can be written in the local form as:

$$0 = -\nabla p + \frac{\mu}{DV_0\rho}\nabla^2(hu), \tag{30}$$

$$\nabla(hu) = pDV_0\frac{\rho}{\mu}. \tag{31}$$

The path of flow through a porous medium between two points x_a and x_b is given by the variational principle which makes minimal[3, 19]:

$$J = \int_{x_a}^{x_b} \nabla(hu) dl = \int_{x_a}^{x_b} pDV_0\frac{\rho}{\mu} dl. \tag{32}$$

Let M be a manifold of dimensions corresponding to the section or surface..
Then the variational problem will be written by defining the linear dimension of the flow:

$$J = \int_{x_a}^{x_b} pDV_0\frac{\rho}{\mu} dl. \tag{33}$$

Then

$$F = pDV_0\frac{\rho}{\mu}\sqrt{\delta_{ij}y^i y^j}. \tag{34}$$

The dependence on y implies that the flow is formulated by Finsler geometry. Now, everything depends on the linear dimension D of the flow studied to determine the fundamental function F □

3.2. Finsler structure

Theorem 3.1.
We consider a flow in dimension 2; Let $(x^i) = (x^1, x^2)$ be the local coordinates of the area. Here x^2 represents the horizontal direction and x^1 represents the vertical direction.
Based on the above conditions, we derive the Finsler metric of the flow. Let M be the variety of dimension 2 and $(x^i) = (x^1, x^2) \in M$; then the tangent bundle TM, and $(x^i, y^i) \in TM$ and $y^i = \frac{dx^i}{dt}$. Then The dependence on y implies that the flow is formulated by Finsler geometry, the fundamental function is given:

$$F(x,y) = pV_0 D(x)\frac{\rho}{\mu}\sqrt{(y^1)^2+(y^2)^2}. \tag{35}$$

By definition, the fundamental tensor is given through the geometric objects of the Finsler metric by:

$$(g_{ij}) = \left(\frac{pQV_0}{S_0}\frac{\rho}{\mu}\right)^2\begin{pmatrix} 2+6Z^2 & -4Z^3 \\ -4Z^3 & 1+3Z^4\end{pmatrix}, \tag{36}$$

which is positive definite ; where $Z = \frac{y^1}{y^2}$.

Proof. Consider D the linear dimension as the length of the flow

$$dx^1 = u\,dt = \frac{Q}{S}dt, \tag{37}$$

with Q the flow rate and S the flow area.
Fermat's variational principle determines the shape of the flow trajectories. The angle of incidence θ between the flow path and the horizontal axis x^2 are given by:

$$\tan\theta = \frac{dx^1}{dx^2}; \tag{38}$$

$$\cos\theta = \frac{dx^2}{\sqrt{(dx^1)^2+(dx^2)^2}}. \tag{39}$$

When the (constant) area of the flow projected on the axis x^2 is denoted S_0, the variable area S perpendicular to the flow is expressed by $S = S_0\cos\theta$. In this case, according to eq. (38), the area of the section S depends on the direction of the trajectory: So we get:[3, 14]

$$J = \int_{t_a}^{t_b}\frac{pQV_0}{S_0}\frac{\rho}{\mu}\left\{\frac{(y^1)^2+(y^2)^2}{y^2}\right\}dt, \tag{40}$$

$$F = \frac{pQV_0}{S_0}\frac{\rho}{\mu}\left\{\frac{(y^1)^2+(y^2)^2}{y^2}\right\} = \frac{\alpha^2}{\beta}, \tag{41}$$

where the components are given by

$$\alpha = (\alpha_{ij}(x)y^iy^j)^{1/2}, \tag{42}$$

$$\beta = \beta_i(x)y^i, \tag{43}$$

$$\alpha_{ij}(x) = \frac{pQV_0}{S_0}\frac{\rho}{\mu}\delta_{ij}; \quad \beta_1 = 0 \text{ and } \beta_2 = 1.$$

The fundamental function $F: TM_0 \longrightarrow \mathbb{R}$ is homogeneous of degree 1 in y^i.
Furthermore, it verifies the different properties of a Finsler manifold. □

Theorem 3.2.
Assume the conditions above of theorem 3.1, then the equations geodesics eq. (2) through the fundamental tensor of Finsler manifold are given by:

$$\begin{aligned} \frac{d^2x^1}{dt^2} + \frac{(y^2)^2}{2(1+Z^2)}\left[(-1+4Z^2+Z^4)d_1 + 4Zd_2\right] = 0,\\ \frac{d^2x^2}{dt^2} + \frac{(y^2)^2}{1+Z^2}\left[2Zd_1 + (1-Z^2)d_2\right] = 0. \end{aligned} \tag{44}$$

Proof. Use the relations eqs. (2) and (3) of the proposition 2.1 and eq. (4), So the inverse of the matrix is given

$$(g^{ij}) = [2(\frac{pQV_0}{S_0}\frac{\rho}{\mu})^2(1+Z^2)^3]^{-1}\begin{pmatrix} 1+3Z^4 & 4Z^3 \\ 4Z^3 & 2+6Z^2 \end{pmatrix}.$$

After all calculation we obtain, then the Christoffel symbols:

$$\gamma^1_{11} = \{(1+Z^2)^3\}^{-1}[(1+3Z^2+3Z^4-7Z^6)d_1 - 4Z^3(1+3Z^2)d_2]$$
$$\gamma^1_{12} = \{(1+Z^2)^3\}^{-1}[(1+3Z^2+3Z^4+9Z^6)d_2 + 2Z^3(1+3Z^4)d_1]$$
$$\gamma^1_{22} = \frac{1}{2}\{(1+Z^2)^3\}^{-1}[-4Z^3(1+3Z^4)d_2 - (1+3Z^4)^2 d_1]$$
$$\gamma^2_{11} = \frac{1}{2}\{(1+Z^2)^3\}^{-1}[-4Z^3(2+6Z^2)d_1 - (2+6Z^2)^2 d_2]$$
$$\gamma^2_{12} = \frac{1}{2}\{(1+Z^2)^3\}^{-1}[4Z^3(2+6Z^2)d_2 + (2+6Z^2)(1+3Z^4)d_1]$$
$$\gamma^2_{22} = \{(1+Z^2)^3\}^{-1}[\{(1+3Z^2)(1+3Z^4) - 16Z^6\}d_2 - 2Z^3(1+3Z^4)d_1]$$

where $d_i = \partial_i \log(\frac{pQV_0}{S_0}\frac{\rho}{\mu})$ and thus the flow geodesics through of eq. (2) are defined by the following functions:

$$G^1(x,\dot{x}) = \frac{(y^2)^2}{2(1+Z^2)}[(-1+4Z^2+Z^4)d_1 + 4Zd_2],$$
$$G^2(x,\dot{x}) = \frac{(y^2)^2}{1+Z^2}[2Zd_1 + (1-Z^2)d_2].$$

Hence the results □

Theorem 3.3.
The geodesics flows de eq. (44) is stable in accordance with the Lyapunov stability.

Proof. Use the proposition proposition 2.1 and the theorem theorem 3.2. Note that, the geodesics flow of eq. (44) can be rewritten as:

$$\begin{aligned} &\frac{dy^1}{dt} + \frac{1}{2(1+Z^2)}[(-1+4Z^2+Z^4)d_1 + 4Zd_2](y^2)^2 = 0 \\ &\frac{dy^2}{dt} + \frac{1}{1+Z^2}[2Zd_1 + (1-Z^2)d_2](y^2)^2 = 0 \end{aligned} \tag{45}$$

Moreover, for study the stability of system eq. (45) who is differential system; then it's applied the Lyapunov stability [6] of this system is completely determined by the sign of the curvature eq. (12).
The object $\mathscr{R}^i_{jk}$ are:

$$\mathscr{R}^1_{11} = \mathscr{R}^1_{22} = \mathscr{R}^2_{11} = \mathscr{R}^2_{22} = 0 \tag{46}$$

$$\begin{aligned} \mathscr{R}^1_{12} = \mathscr{R}^1_{21} =& (1+Z^2)\{[y^2(3+5Z^2-19Z^4-21Z^5)]d_{11} \\ &+ [y^1(9-2Z-10Z^2+2Z^3-16Z^4-4Z^5+2Z^6)]d_{12} + [4y^2(1+2Z^2-9Z^6)]d_{22}\} \\ &+ (1-Z^2)\{[y^2(9-20Z^2-21Z^4+33Z^6)](d_1)^2 + [y^1(-36+24Z^2+60Z^4)]d_1d_2\} \\ &+ [y^1(36Z-108Z^2+60Z^3-84Z^5)](d_2)^2 \end{aligned} \tag{47}$$

$$\begin{aligned} \mathscr{R}^2_{12} = -\mathscr{R}^2_{21} =& (1+Z^2)\{[-3-12Z^2-9Z^4]d_{11} + Z[-7-4Z^2+Z^4]d_{12} + [-4-2Z^2]d_{22}\} \\ &+ [-9+11Z^2+9Z^4-9Z^6](d_1)^2 + [36Z-36Z^5]d_1d_2 + [-36Z^2-36Z^4](d_2)^2 \end{aligned} \tag{48}$$

This implies that the tensor $\mathscr{R}^i_{jk}$ are negatives or vanishes, hence the sytem is stable.' □

Proposition 3.2.
The solution of the system eq. (42) of the theorem 3.2 is:

$$x^1 = \phi(x^2), \tag{49}$$

$$\begin{aligned} \text{where } \phi(x^2) =& \frac{1}{d_1}\Big\{2\Big[\frac{\sqrt{1+(\frac{d_2}{d_1})^2}}{1+(\frac{d_2}{d_1})^2}\{(1+(\frac{d_2}{d_1})^2) + (x^2+c)(\frac{d_2}{(d_1)^2}d_{22})\}\Big]^{-1} \times \\ &\times \ln\cosh\Big[\frac{1}{2}d_1(x^2+c)\sqrt{1+(\frac{d_2}{d_1})^2}\Big] - \log(\frac{pV_0Q}{S_0})\Big\} \end{aligned} \tag{50}$$

Proof. Assume the relation eq. (45) define by:

$$\begin{cases} \dfrac{dy^1}{dt} + \dfrac{1}{2(1+Z^2)}\big[(-1+4Z^2+Z^4)d_1 + 4Zd_2\big](y^2)^2 = 0 \\ \dfrac{dy^2}{dt} + \dfrac{1}{1+Z^2}\big[2Zd_1 + (1-Z^2)d_2\big](y^2)^2 = 0 \end{cases} \tag{51}$$

As $Z = \dfrac{y^1}{y^2}$; then $\dfrac{dZ}{dt} = \dfrac{1}{y^2}\Big(\dfrac{dy^1}{dt} - Z\dfrac{dy^2}{dt}\Big)$. This implies that

$$\frac{dZ}{dt} = y^2\Big[\frac{1}{2}(1-Z^2)d_1 - Zd_2\Big] \tag{52}$$

$$dx^2 = \frac{2dZ}{(1-Z^2)d_1 - 2Zd_2} = \frac{-2dZ}{d_1\Big[(Z+\frac{d_2}{d_1})^2 - (\frac{d_2}{d_1})^2\Big] - d_1} \tag{53}$$

$$= \frac{\frac{-2dZ}{d_1}}{(Z+\frac{d_2}{d_1})^2 - (1+(\frac{d_2}{d_1})^2)} \tag{54}$$

$$= \frac{\dfrac{2dZ}{d_1(1+(\frac{d_2}{d_1})^2)}}{1-\Big[\sqrt{\dfrac{1}{1+(\frac{d_2}{d_1})^2}}(Z^2+\frac{d_2}{d_1})\Big]^2} \tag{55}$$

$$\text{Let} \quad \ell = \sqrt{\frac{1}{1+(\frac{d_2}{d_1})^2}}(Z+\frac{d_2}{d_1}) \Longrightarrow d\ell = \sqrt{\frac{1}{1+(\frac{d_2}{d_1})^2}}dZ \tag{56}$$

$$\text{Then} \quad dx^2 = \frac{\dfrac{2d\ell}{d_1\sqrt{1+(\frac{d_2}{d_1})^2}}}{1-\ell^2} = \frac{2}{d_1\sqrt{1+(\frac{d_2}{d_1})^2}}\frac{d\ell}{1-\ell^2} \tag{57}$$

$$\text{In integrating} \quad x^2 + c = \frac{2}{d_1\sqrt{1+(\frac{d_2}{d_1})^2}} argtanh\,\ell(Z) \tag{58}$$

$$argtanh\,\ell(Z) = \frac{1}{2}d_1(x^2+c)\sqrt{1+(\frac{d_2}{d_1})^2} \tag{59}$$

$$\ell(Z) = \tanh\Big[\frac{1}{2}d_1(x^2+c)\sqrt{1+(\frac{d_2}{d_1})^2}\Big] \tag{60}$$

$$\sqrt{\frac{1}{1+(\frac{d_2}{d_1})^2}}(Z+\frac{d_2}{d_1}) = \tanh\Big[\frac{1}{2}d_1(x^2+c)\sqrt{1+(\frac{d_2}{d_1})^2}\Big] \tag{61}$$

$$Z+\frac{d_2}{d_1} = \sqrt{1+(\frac{d_2}{d_1})^2}\tanh\Big[\frac{1}{2}d_1(x^2+c)\sqrt{1+(\frac{d_2}{d_1})^2}\Big] \tag{62}$$

$$Z = \sqrt{1+(\frac{d_2}{d_1})^2}\tanh\Big[\frac{1}{2}d_1(x^2+c)\sqrt{1+(\frac{d_2}{d_1})^2}\Big] - \frac{d_2}{d_1} \tag{63}$$

$$dx^1 = \varphi(x^2)dx^2 \quad \text{où}\,\varphi(x^2) = \sqrt{1+(\frac{d_2}{d_1})^2}\tanh\Big[\frac{1}{2}d_1(x^2+c)\sqrt{1+(\frac{d_2}{d_1})^2}\Big] - \frac{d_2}{d_1} \tag{64}$$

□

To predict the trajectory of a mobile, once we know its speed at an instant, it is enough to transport the trajectory along itself to form little by little the movement of the object. This type of trajectory that we form by transporting the speed along itself is what we call a ***geodesic****. In the universe, all objects naturally move geodesically. On a variety, a geodesic between two points of the manifold is a path between the two points minimizing its length*

4. Conclusion

The Finsler geometric properties of flow,through to studing of the fundamental tensor, we show that the Finsler metric is obtained from the study of the Navier-Stokes expressed of the Flood problem.
From the Finslerian geometric objects, the influence of direction dependence on flux is geometrically shown by the geodesic in Finsler space. Then, based on the KCC-theory, it is shown that the geodesic of flow is the Jacobi unstable for the deviation of whole trajectory.

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References

[1] A. M. Al-Qashbari, A. A. Abdallah, S. M. Baleedi, A Study on The Extensions and Developments of Generalized BK-Recurrent Finsler Space, Int. J. Adv. Appl. Math. and Mech. 12(1) (2024) 38 – 45 (ISSN: 2347-2529).
[2] C. Ancey, Hydraulique à surface libre, EPFL, ENAC/IIC/LHE,Ecublens, CH-1015 Lausanne, Suisse
[3] P.L. Antonelli, R.S. Ingarden and M. Matsumoto, The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Kluwers Academic Publishers, Dordretch, (1993).
[4] P.L. Antonelli, I. Bucataru, KCC theory of a system of second order differential equations, in: P.L. Antonelli (Ed.), Handbook of Finsler Geometry, Vol. 1, Kluwer Academic Publishers, Dordrecht, 2003, pp. 83–174.
[5] P.L. Antonelli, I. Bucataru, S.F. Rutz, Computer algebra and two and three dimensional Finsler geometry, Publ. Math. Debrecen, 2872 (2003), 1–24.
[6] P.L. Antonelli, L. Bevilacqua, S.F. Rutz, Theories and models in symbiogenesis, Nonlinear Analysis: Real
[7] V. Arnold, Sur la géométrie différentielle des goupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. Annales de l'institut Fourier, tome 16, no 1 (1966), p. 319-361
[8] D. Bao, S.-S. Chern, Z. Shen, An Introduction to Riemann-Finsler Geometry, Springer-Verlag, New York, 2000.
[9] I. Bucataru, R. Miron, Finsler-Lagrange Geometry. Applications to Dynamical Systems, Editura Academiei Române, Bucureşti, 2007.
[10] J.P. Caltagirone, Physique des écoulements continus, Springer-Verlag, Berlin Heidelberg, 2013.
[11] E. Cartan, Observations sur le mémoire précédent, Math. Z. 37 (1933) 619–622.
[12] S.-S. Chern, Sur la géométrie d'un système d'équations différentielles du second ordre, Bull. Sci. Math. 63 (1939) 206–212.
[13] B. Dănilă, T. Harko, M. Kwong Mak, P. Pantaragphong, and S. V. Sabau Jacobi stability analysis of scalar field models with minimal coupling to gravity in a cosmological background, arXuv: 1609.05636v1, [gr-qc], (2016).
[14] H. R. Dehkordi, Mathematical modeling the wildfire propagation in a Randers space, (2010)
[15] D.D. Kosambi, Parallelism and path-spaces, Math. Z. 37 (1933) 608–618.
[16] O. M. Ongonga, M. Kinyanjui, K. Giterere Hydromagnetic non-newtonian fluid flow between parallel semi infinite vertical plates, Int. J. Adv. Appl. Math. and Mech. 9(3) (2022) 21 – 32 (ISSN: 2347-2529).
[17] Q.I. Rahman and G; Schmeisser, Analytic theory of polynomials, London Mathematical Society Monograps. New services 26. Oxford, Oxford University Press (2002).
[18] J. Surawala, M. G. Timol, Analysis of thermal boundary layer flow of viscous fluids by new similarity method, Int. J. Adv. Appl. Math. and Mech. 6(1) (2018) 69 – 77 (ISSN: 2347-2529)
[19] T. Yajima, H. Nagahama, Finsler geometry for nonlinear path of fluids flow through inhomogeneous media, Nonlinear Analysis: Real World Applications 25 (2015) 1–8.

[20] T. Yajima, H. Nagahama, Finsler geometry of seismic ray path in anisotropic media, Proc. R. Soc. A, 2008.