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On Analytical and Numerical Solution of the Telegraph Equation in Power Transmission Lines

Research Article

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Abstract: In this paper, the one-dimensional linear hyperbolic telegraph equation is derived from the equations of transmission line theory. Numerical and analytical solutions of the telegraph equation are then obtained by employing the separation of variables method, which is used for analytic treatment of the telegraph equation. The finite difference scheme explicit method is applied to the telegraph equation to obtain numerical solutions. The techniques are tested on three numerical examples for various parameter values appearing in the telegraph equation and discretization steps. Efficiency and reliability of the methods are determined by an error analysis. The results showed that the finite difference scheme explicit method is highly accurate, efficient and very easy to apply in the solution of telegraph equations.

MSC: 35A20 • 35L10 • 35L70 • 39A14

Keywords: Differential equations • Telegraph equations • Separation of Variables Method • Finite Difference Scheme Explicit Method

1. Introduction

Differential equations are a powerful tool for modelling and analyzing many physical and engineering problems and are an important branch of applied mathematics. In particular, they occur in network design, fluid dynamics, wave motion, telecommunications, electromagnetic, wave distribution, and electronic dynamics [1]. They are used not only in engineering and physical systems, but also in economics, risk theory and many other social sciences. On the other hand, the telegraph equation, a special kind of hyperbolic equations, is a partial differential equation that frequently appears in electrical engineering. In particular, power transmission lines are defined and designed using telegraph equations [2]. Many different problems in electrical, electronics and communication engineering can be modelled by telegraph equations [2–15]. Mathematical modelling of problems in communication systems and transmission lines and their solvability have great importance in today's world in which technology and communication tools regarding them have developed and spread with an increasing velocity. Depending on whether the terminations are short or open circuits and whether they are fed by current or voltage sources, there are many forms of this equation, including local or non-local boundary conditions.

Heaviside [16] was the first to work on telegraph equations and studied in detail the second-order hyperbolic PDEs that arise from the mathematical modelling of the voltage and current on an electrical transmission line with distance.

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This model explains that the wave sequence that can form with the line and the electromagnetic waves can be reflected on the wire. Recently, it has been discovered that the telegraph equation is the best candidate as an alternative to the diffusion equation while investigating turbulence transport in its early phase. This is because, in early times, the movement of the particles was ballistic, and they scattered with the passage of time. The coefficient of diffusion during the running time, was determined numerically for instance, by tracing the mean square displacement of ensemble particles. The reflection of this behaviour is increasing linearly with respect to ballistic motion. The other problem is that the speed of the particles is finitely propagated, which is also considered as there is limited or no magnetostatic turbulence at all. Due to the fixed energy of the particles, it cannot have a finite probability of filling the space when the source is at a large distance.

In contrast, the telegraph equation has great potential to differentiate between the early ballistic motion of particles and the diffusive transport at a later stage. This is due to the fact that the telegraph equation consists of an additional scale of time, which produces behavior like a wave. For this reason, the telegraph equation is used for the mathematical description of pulse propagation along a wire. At the beginning, at least, this behavior looks to have excellent agreement with the propagation of particle speed charged by the available energy [17]. The telegraph equation is most commonly used for the transmission of electrical signals and their propagation in signal analysis. In biological sciences, the telegraph equation can be used for the linearization of neurons of nerves and in muscle cells, the telegraph equations lead to how the pressure waves of pulsating blood flow in the arteries are reproduced. The movement of an insect through a fence in one dimension can also be modelled by the telegraph equation [18]. The telegraph equation has been solved using several methods such as the meshless local radial point interpolation (MLRPI) method [19] and the local fractional function decomposition method which is a blend of the Adomian decomposition method and the Yang-Laplace transform [20]. More recently, the telegraph equation has been solved by Kasumo and Hapunda [21] using the Sumudu and Elzaki transform methods and by Kasumo [22] using the semi analytic iterative method, initially proposed by Temimi and Ansari [23].

In this paper, we propose applications of two methods for solving linear telegraph equations. These are the separation of variables method (SVM) and the finite difference scheme explicit method (FDSEM). The rest of the paper is structured as follows: Section section 2 deals with problem formulation. The proposed methods of solution are reviewed in Section section 3, while Section section 4 presents some numerical examples and discussions and Section section 5 offers some conclusions and future research directions.

2. Problem Formulation

We consider an infinitesimal piece of telegraph cable as an electrical circuit (refer to fig. 1 below). Furthermore,

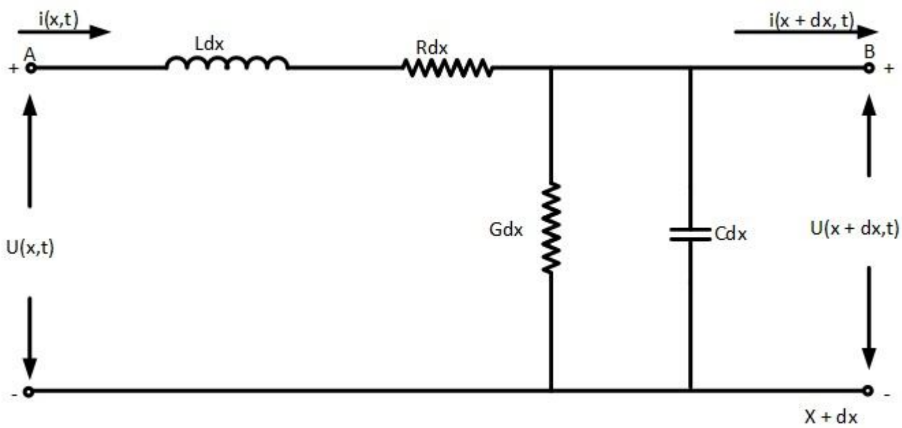

Fig. 1. Schematic diagram of telegraph transmission line with leakage

assume that the cable is imperfectly insulated so that there are both capacitance and current leakage into the ground [24]. Suppose x is the distance from sending end of the cable, $U(x,t)$ the voltage at any point x on the cable and at any time t and $i(x,t)$ the current at any point x on the cable and time t. R denotes the resistance of a cable, C the capacitance to the ground, L the inductance of the cable and G the conductance to the ground. Then by Ohm's Law, the voltage across the resistor is given by

$$U = iR. \tag{1}$$

Further, the voltage drop across the inductor is given as

$$U = L\frac{di}{dt}. \tag{2}$$

The voltage drop across the capacitor is given by

$$U = \frac{1}{C}\int i dt. \tag{3}$$

The voltage at terminal B is equal to the voltage at terminal A, minus the drop in voltage along the element AB, so if eqs. (1)–(3) are combined, we have

$$U(x+dx,t) - U(x,t) = -[Rdx]i - [Ldx]\frac{\partial i}{\partial t}. \tag{4}$$

Let $dx \to 0$. Then differentiating eq. (4) partially with respect to x, we have

$$\frac{\partial U}{\partial x} = -Ri - L\frac{\partial i}{\partial t}. \tag{5}$$

Similarly, the current at terminal B is equal to the current at terminal A minus the current through leakage to the ground, so that we obtain

$$i(x+dx,t) = i(x,t) - [Gdx]U - i_c dx. \tag{6}$$

The current through the capacitor is given by

$$i_C = C\frac{\partial U}{\partial t}. \tag{7}$$

Differentiating eq. (4) with respect to t and eq. (7) with respect to x and eliminating the derivatives of U gives

$$C^2\frac{\partial^2 i}{\partial x^2} = \frac{\partial^2 i}{\partial t^2} + (\alpha+\beta)\frac{\partial i}{\partial t} + (\alpha\beta)i. \tag{8}$$

Similarly, we have

$$C^2\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2} + (\alpha+\beta)\frac{\partial U}{\partial t} + (\alpha\beta)U, \tag{9}$$

where

$$\alpha = \frac{G}{C}, \beta = \frac{R}{L}, C^2 = \frac{1}{LC}.$$

Equations (8) and (9) are known as one-dimensional hyperbolic second-order linear telegraph equations. In compact form the telegraph equation is

$$C^2U_{xx} = U_{tt} + (\alpha+\beta)U_t + \alpha\beta U. \tag{10}$$

This equation describes a telegraph line along a wire that serves as a transmission medium for a signal [25].

3. Review of the Proposed Methods

In this section, the proposed methods of solution will be briefly reviewed or outlined. These are the finite difference scheme explicit method (FDSEM) and the separation of variables method (SVM).

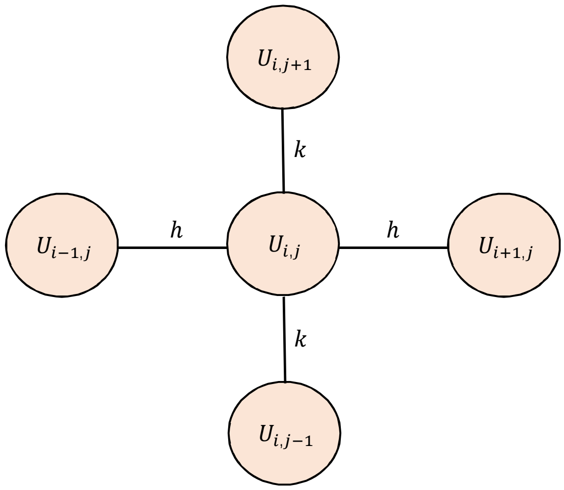

Fig. 2. Representation of coordinates in finite difference form

3.1. Preliminary Concepts of FDSEM

The FDSEM was proposed to find solutions to linear telegraph equation. To explain the basic idea of the FDSEM, let us consider the finite difference scheme used to obtain a numerical solution to a partial differential equation in a bounded domain. The solution to the PDE is replaced with an approximation using a finite number of points in the domain. Increasing the number of points generally increases the accuracy of the numerical solution. The finite difference method involves three methods of approximation, namely, forward, backward and central differences. The forward difference is an explicit method used to approximate f_i using the current point x_i and the next grid point (refer to fig. 2 below).

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i,j}}{h} + O(h),$$

$$\frac{\partial U}{\partial y} \approx \frac{U_{i,j+1} - U_{i,j}}{k} + O(k).$$

The backward difference uses the current and previous grid points.

$$\frac{\partial U}{\partial x} \approx \frac{U_{i,j} - U_{i-1,j}}{h} + O(h),$$

$$\frac{\partial U}{\partial y} \approx \frac{U_{i,j} - U_{i,j-1}}{k} + O(k).$$

The central difference uses the grid points on either side of the point x_i.

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2h} + O(2h),$$

$$\frac{\partial U}{\partial y} \approx \frac{U_{i,j+1} - U_{i,j-1}}{2k} + O(2k).$$

For second order derivative approximation we use central differences

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + O(h^2),$$

$$\frac{\partial^2 U}{\partial y^2} \approx \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} + O(k^2).$$

3.2. Application of FDSEM to the Telegraph Equation

We apply the finite difference scheme explicit method to the solution of the one-dimensional linear telegraph equation.

$$U_{tt} + (\alpha+\beta)U_t + (\alpha\beta)U = C^2 U_{xx}. \tag{11}$$

Discretizing the PDE eq. (11) at the node (i, j), approximating both derivatives by central differences and neglecting the truncation error, we have

$$\frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{k^2} + (\alpha+\beta)\frac{U_{i,j+1} - U_{i,j-1}}{2k} + (\alpha\beta)U_{i,j} = C^2\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2},$$

which implies that

$$U_{i,j-1} - 2U_{i,j} + U_{i,j+1} + \frac{(\alpha+\beta)k}{2}(U_{i,j+1} - U_{i,j-1}) + (\alpha\beta)k^2 U_{i,j} = \frac{C^2k^2}{h^2}U_{i-1,j} - \frac{2C^2k^2}{h^2}U_{i,j} + \frac{C^2k^2}{h^2}U_{i+1,j}.$$

If we let $r = \frac{k}{h}$, then

$$U_{i,j-1} - 2U_{i,j} + U_{i,j+1} + \frac{(\alpha+\beta)k}{2}(U_{i,j+1} - U_{i,j-1}) + (\alpha\beta)k^2 U_{i,j} = C^2r^2U_{i-1,j} - 2C^2r^2U_{i,j} + c^2r^2U_{i+1,j}. \tag{12}$$

Rearranging eq. (12), we have

$$\left[1 + \frac{(\alpha+\beta)k}{2}\right]U_{i,j+1} = C^2r^2(U_{i-1,j} + U_{i+1,j}) + (2 + \alpha\beta k^2 - 2C^2r^2)U_{i,j} + \left(\frac{(\alpha+\beta)k}{2} - 1\right)U_{i,j-1}.$$

Dividing both sides by $[1 + \frac{(\alpha+\beta)k}{2}]$, we have

$$U_{i,j+1} = \frac{C^2r^2}{1 + \frac{(\alpha+\beta)k}{2}}(U_{i-1,j} + U_{i+1,j}) + \frac{2 + \alpha\beta k^2 - 2C^2r^2}{1 + \frac{(\alpha+\beta)k}{2}}U_{i,j} + \frac{(\frac{(\alpha+\beta)k}{2} - 1)}{1 + \frac{(\alpha+\beta)k}{2}}U_{i,j-1},$$

which implies that

$$U_{i,j+1} = \frac{2C^2r^2}{2 + (\alpha+\beta)k}(U_{i-1,j} + U_{i+1,j}) + \frac{2(2 + \alpha\beta k^2 - 2C^2r^2)}{2 + (\alpha+\beta)k}U_{i,j} + \frac{(\alpha+\beta)k - 2}{2 + (\alpha+\beta)k}U_{i,j-1}. \tag{13}$$

The eq. (13) is called the explicit method because each nodal value at the new time level may be explicitly computed from three values at the previous time level, i.e., the nodal value at time $j+1$ depends explicitly on the value at time j.

3.3. Preliminary Concepts of Separation of Variables Method

We consider the second-order homogeneous partial differential equation

$$aU_{xx} + bU_{xy} + cU_{yy} + dU_x + eU_y + fU = 0, \tag{14}$$

where a,b,c,d,e and f are functions of x and y. We can always transform eq. (14) into canonical form

$$a(x,y)u_{xx} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = 0 \tag{15}$$

which when

1. $a = -c$ is hyperbolic,
2. $a = 0$ or $c = 0$ is parabolic,
3. $a = c$ is elliptic.

We assume a separable solution of eq. (15) in the form

$$u(x,y) = X(x)Y(y) \neq 0, \tag{16}$$

where X and Y are, respectively, functions of x and y only, and are twice continuously differentiable. Substituting eq. (16) into eq. (15), we obtain

$$aX''Y + cXY'' + dX'Y + eXY' + fXY = 0. \tag{17}$$

Let there exist a function $p(x,y)$, such that, if we divide eq. (17) by $p(x,y)$, we obtain

$$a_1(x)X''Y + b_1(y)XY'' + a_2(x)X'Y + b_2(y)XY' + [a_3(x) + b_3(y)]XY = 0. \tag{18}$$

Dividing eq. (18) again by XY, we obtain

$$\left[a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3\right] = -\left[b_1\frac{Y''}{Y} + b_2\frac{Y'}{Y} + b_3\right]. \tag{19}$$

The left hand side of eq. (19) is a function of x only and the right hand side depends only on y. Thus, we differentiate eq. (19) with respect to x to obtain

$$\frac{d}{dx}\left[a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3\right] = 0. \tag{20}$$

Integration of eq. (20) yields

$$a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3 = \lambda. \tag{21}$$

From eqs. (19) and (21), we have

$$b_1\frac{Y''}{Y} + b_2\frac{Y'}{Y} + b_3 = -\lambda. \tag{22}$$

We may rewrite eqs. (21) and (22) in the form

$$a_1X'' + a_2X' + (a_3 - \lambda)X = 0 \tag{23}$$

and

$$b_1Y'' + b_2Y' + (b_3 + \lambda)Y = 0. \tag{24}$$

Thus, $u(x,y)$ is the solution of eq. (15) if $X(x)$ and $Y(y)$ are the solutions of the ordinary differential eqs. (23) and (24), respectively.

3.4. Application of Separation of Variables Method to the Telegraph Equation

In this section, we apply the separation of variables method to find the exact solutions. Let us consider the boundary value problem (20) with boundary conditions $U(0,t) = U(l,t) = 0$, and initial conditions $U(x,0) = \delta(x-a)$, $U_t(x,0) = 0$, for all $t > 0$, $0 < x < l$. This models a telegraph wire of length l having the voltage at both ends $x = 0$ and $x = l$ clamped at zero. The initial conditions represent an idealized signal consisting of a spike at $x = a$ that is stationary at time zero. In the separation of variables, we first use a trial solution of the form $U(x,t) = X(x)T(t)$. Equation (11) becomes

$$XT'' + (\alpha+\beta)XT' + \alpha\beta XT = C^2X''T \Rightarrow \frac{1}{C^2}\left[\frac{T''}{T} + (\alpha+\beta)\frac{T'}{T} + \alpha\beta\right] = \frac{X''}{X} = \sigma.$$

Imposing the boundary conditions $X(0) = X(l) = 0$, the ODEs to be solved for $X(x)$ and $T(t)$ are, respectively,

$$X'' - \sigma X = 0$$

and

$$T'' + (\alpha+\beta)T' + (\alpha\beta - \sigma C^2)T = 0,$$

with respective solutions

$$X(x) = C_2\sin\left(\frac{n\pi x}{l}\right),\ \sigma = -\left(\frac{n\pi}{l}\right)^2$$

and

$$T(t) = C_3e^{r_1t} + C_4e^{r_2t} \text{ with } r_i = \frac{1}{2}(-\alpha - \beta \pm \sqrt{(\alpha+\beta)^2 - 4\alpha\beta + 4\sigma C^2}).$$

Let $4\omega_n^2 = 4(\frac{n\pi}{l})^2C^2 - (\alpha+\beta)^2$ and $d = \frac{\alpha+\beta}{2}$. Then $r_i = -d \pm i\omega_n$ and general solution $T(t)$ can be written as

$$T(t) = A_n e^{-dt}\cos(\omega_n t - \phi_n) \tag{25}$$

with arbitrary amplitude A_n and phase ϕ_n. So, for all A_n and ϕ_n,

$$U(x,t) = \sum_{n=1}^{\infty} A_n e^{-dt}\cos(\omega_n t - \phi_n)\sin\left(\frac{n\pi x}{l}\right) \tag{26}$$

satisfies the PDE eq. (11) and the associated boundary conditions. It remains to choose the amplitudes and phases to satisfy the initial conditions:

$U_t(x,0) = 0 \Rightarrow 0 = \sum_{n=1}^{\infty} A_n\left[-de^{-dt}\cos(\omega_n t-\phi_n) - \omega_n e^{-dt}\sin(\omega_n t - \phi_n)\right]\sin\left(\frac{n\pi x}{l}\right)|_{t=0}$

$= \sum_{n=1}^{\infty} A_n\left[-d\cos(\phi_n) + \omega_n\sin(\phi_n)\right]\sin\left(\frac{n\pi x}{l}\right) = 0,$

$U(x,0) = \delta(x-a) \Rightarrow \delta(x-a) = \sum_{n=1}^{\infty} A_n\cos(\phi_n)\sin\left(\frac{n\pi x}{l}\right) = \delta(x-a) \Rightarrow A_n\cos\phi_n = \frac{2}{l}\int_0^l \delta(x-a)\sin\left(\frac{n\pi x}{l}\right) = \frac{2}{l}\sin\left(\frac{n\pi a}{l}\right).$

The final solution is

$$U(x,t) = \sum_{n=1}^{\infty} A_n e^{-dt}\cos(\omega_n t - \phi_n)\sin\left(\frac{n\pi x}{l}\right), \tag{27}$$

where $d = (\alpha+\beta)/2$, $\omega_n = \sqrt{\left(\frac{n\pi C}{l}\right)^2 - \frac{1}{4}(\alpha-\beta)^2}$, $\phi_n = \arctan(\frac{d}{\omega_n})$ and $A_n = \dfrac{2}{l\cos\phi_n}\sin\left(\frac{n\pi a}{l}\right)$. To interpret this result, we rewrite it as

$$U(x,t) = \sum_{n=1}^{\infty}\frac{1}{2}A_n e^{-dt}\left[\sin\left(\frac{n\pi x}{l} - \omega_n t + \phi_n\right) + \sin\left(\frac{n\pi x}{l} + \omega_n t - \phi_n\right)\right]. \tag{28}$$

Suppose that we carefully tune the wire so that $\alpha = \beta$. Then $d = \alpha$, $\omega_n = \dfrac{n\pi C}{l}$ and

$$U(x,t) = \sum_{n=1}^{\infty}\frac{1}{2}A_n e^{-\alpha t}\left[\sin\left(\frac{n\pi x}{l} - \frac{n\pi C}{l}t + \phi_n\right) + \sin\left(\frac{n\pi x}{l} + \frac{n\pi C}{l}t - \phi_n\right)\right] = e^{-\alpha t}f(x-Ct) + e^{-\alpha t}g(x+Ct), \tag{29}$$

where

$$f(z) = \sum_{n=1}^{\infty}\frac{1}{2}A_n\sin\left(\frac{n\pi}{l}z + \phi_n\right),$$

$$g(z) = \sum_{n=1}^{\infty}\frac{1}{2}A_n\sin\left(\frac{n\pi}{l}z - \phi_n\right).$$

Thus, assuming $\alpha = \beta > 0$, $U(x,t)$ is the sum of two signals, one moving to the right and the other moving to the left. They move without changing shape, but their amplitudes decrease with time, due to factors $e^{-\alpha t}$. If $\alpha \neq \beta$, different frequency components of $U(x,t)$ move with different speeds, because ω_n depends on n, and the signals distort as they propagate.

4. Numerical Examples

In this section we present three examples illustrating the applicability of both SVM and FDSEM for solving linear telegraph equations. The results are presented in tables and figures accompanying the discussion. All the computations associated with these examples were performed using a Lenovo Thinkpad DESKTOP-N5OF54M PC with an Intel Core i5-8250U CPU at 1.60GHz with 8.0GB internal memory and 64-bit operating system x64-based processor (Windows 11 Pro, Version 23H2). All the figures in this section were constructed using MATLAB R2023a.

Example 4.1.
Consider the linear telegraph equation [24]:

$$U_{xx} = U_{tt} + 2U_t + U \tag{30}$$

subject to the boundary and initial conditions:

$$U(0,t) = U(2,t) = 0,$$
$$U(x,0) = e^x,\ U_t(x,0) = -2e^x.$$

Solution by SVM:

To solve this telegraph eq. (30), separation of variables will be applied as follows:
Let

$$U(x,t) = X(x)T(t).$$

Then

$$U_x = X'T,\ U_{xx} = X''T,\ U_t = XT' \text{ and } U_{tt} = XT''.$$

Substituting these partial derivatives in the PDE gives

$$X''T = XT'' + 2XT' + XT \Rightarrow \frac{X''}{X} = \frac{T''}{T} + 2\frac{T'}{T} + 1 = \lambda,$$

where λ is a constant.

$$\frac{X''}{X} = \lambda \Rightarrow X'' - \lambda X = 0,$$
$$\frac{T''}{T} + 2\frac{T'}{T} + 1 = \lambda \Rightarrow T'' + 2T' - (\lambda - 1)T = 0.$$

Solving for $X(x)$ and applying boundary conditions we have

$$a\pi = n\pi \Rightarrow a = n \text{ and}$$
$$X(x) = \textstyle\sum_{n=1}^{\infty} C_2 \sin(nx) = 0.$$

Solving for $T(t)$ with $\lambda = -a^2$, we have

$$T(t) = e^{-t}\left[C_3\cos(nt) + C_4\sin(nt)\right].$$

Applying initial conditions, the general solution is

$$U(x,t) = \sum_{n=1}^{\infty} \sin(nx)e^{-t}\left[\frac{1}{1+n^2}\left[e^2(\sin(2n) - n\cos(2n)) + n\right]\cos(nt) + \left[\frac{1}{n(1+n^2)}\left[e^2(\sin(2n) - n\cos(2n)) + n\right] - \frac{2e^x}{n}\right]\sin(nt)\right]. \quad (31)$$

This series converges to the exact solution

$$U(x,t) = e^x e^{-2t} = e^{x-2t}.$$

Solution by FDSEM:

To solve eq. (30), the FDSEM is going to be applied as follows. Choose $n = 20$, $h = \frac{2}{20} = 0.1$, $C^2 = 1, \Rightarrow C = \pm 1$, $\alpha = \beta = 1$ and $0 \le t \le 0.5$, $k = \frac{0.5}{20} = 0.025$. The discretizations are

$x_i = ih$	$t_j = jk$	$x_i = ih$	$t_j = jk$
$x_1 = 0.1$	$t_1 = 0.025$	$x_{11} = 1.1$	$t_{11} = 0.275$
$x_2 = 0.2$	$t_2 = 0.05$	$x_{12} = 1.2$	$t_{12} = 0.3$
$x_3 = 0.3$	$t_3 = 0.075$	$x_{13} = 1.3$	$t_{13} = 0.325$
$x_4 = 0.4$	$t_4 = 0.1$	$x_{14} = 1.4$	$t_{14} = 0.35$
$x_5 = 0.5$	$t_5 = 0.125$	$x_{15} = 1.5$	$t_{15} = 0.375$
$x_6 = 0.6$	$t_6 = 0.15$	$x_{16} = 1.6$	$t_{16} = 0.4$
$x_7 = 0.7$	$t_7 = 0.175$	$x_{17} = 1.7$	$t_{17} = 0.425$
$x_8 = 0.8$	$t_8 = 0.2$	$x_{18} = 1.8$	$t_{18} = 0.45$
$x_9 = 0.9$	$t_9 = 0.225$	$x_{19} = 1.9$	$t_{19} = 0.475$
$x_{10} = 1.0$	$t_{10} = 0.25$	$x_{20} = 2.0$	$t_{20} = 0.5$

Using initial conditions:

$U(x_i,0) = e^{x_i}$	$U(x_i,0) = e^{x_i}$
$U_{1,0} = 1.105170918$	$U_{11,0} = 3.004166024$
$U_{2,0} = 1.221402758$	$U_{12,0} = 3.320116923$
$U_{3,0} = 1.349858808$	$U_{13,0} = 3.669296668$
$U_{4,0} = 1.491824698$	$U_{14,0} = 4.055199967$
$U_{5,0} = 1.648721271$	$U_{15,0} = 4.48168907$
$U_{6,0} = 1.8221188$	$U_{16,0} = 4.9532032424$
$U_{7,0} = 2.01375707$	$U_{17,0} = 5.473947392$
$U_{8,0} = 2.225540928$	$U_{18,0} = 6.049647464$
$U_{9,0} = 2.459603111$	$U_{19,0} = 6.685894442$
$U_{10,0} = 2.718281828$	

Using another initial condition $U_t(x,0) = -2e^x$, we have $U_t = \frac{1}{2k}(U_{i,j+1} - U_{i,j-1})$. At $t = 0$, $jk = 0$, $j = 0$, we have $\left.\frac{\partial U}{\partial t}\right|_{t=0} = -e^x \Rightarrow -2e^{x_i} = \frac{1}{2k}(U_{i,j+1} - U_{i,j-1})$. This means that

$$U_{i,j-1} = U_{i,j+1} + 0.1e^{x_i}. \tag{32}$$

Discretizing the PDE eq. (30) at the mesh point (i, j) and approximating both the derivatives by central differences gives

$$U_{i,j+1} = \frac{2C^2r^2}{2+(\alpha+\beta)k}(U_{i-1,j} + U_{i+1,j}) + \frac{4+2\alpha\beta k^2 - 4C^2r^2}{2+(\alpha+\beta)k}U_{i,j} + \frac{(\alpha+\beta)k-2}{2+(\alpha+\beta)k}U_{i,j-1}. \tag{33}$$

For $r = \frac{k}{h} = \frac{0.025}{0.1} = 0.25$, we have the finite difference equation

$$U_{i,j+1} = 0.061U_{i-1,j} + 0.061U_{i+1,j} + 1.8299U_{i,j} - 0.9512U_{i,j-1}. \tag{34}$$

Substituting eq. (32) into eq. (34) gives

$$U_{i,j+1} = 0.0313U_{i-1,j} + 0.0313U_{i+1,j} + 0.9378U_{i,j} - 0.0487e^{x_i}. \tag{35}$$

At $j = 0$

$$U_{i,1} = 0.0313U_{i-1,0} + 0.0313U_{i+1,0} + 0.9378U_{i,0} - 0.0487e^{x_i} \tag{36}$$

for $i = 1, 2, \ldots, 19$.

Comparison of SVM and FDSEM Solutions for example 4.1:

We now validate the authenticity and efficacy of both FDSEM and SVM in finding solutions to the telegraph equation by comparing results of the FDSEM and SVM for $\alpha = \beta = 1$, $C = 1$ at $t = 0$ and $t = 0.025$. We present the analytical and numerical results of the given linear telegraph equation in tables 1 and 2, and figs. 3 and 4.

Table 1. Comparison of FDSEM and SVM solutions for example 4.1 ($\alpha = \beta = 1$, $C = 1$ and $t = 0$)

x	$U_{\text{SVM}}(x,t)$	$U_{\text{FDSEM}}(x,t)$	Absolute Error
0.2	1.221402758	1.221402758	0
0.4	1.491824698	1.491824698	0
0.6	1.822118800	1.822118800	0
0.8	2.225540928	2.225540928	0
1.0	2.718281828	2.718281828	0
1.2	3.320116923	3.320116923	0
1.4	4.055199967	4.055199967	0
1.6	4.953203242	4.953203242	0
1.8	6.049647464	6.049647464	0

Example 4.2.
Consider the linear telegraph equation [26]:

$$U_{xx} = U_{tt} + 4U_t + 4U \tag{37}$$

subject to the boundary and initial conditions

$$U(0,t) = U(4,t) = 0,$$
$$U(x,0) = e^x,\ U_t(x,0) = -e^x.$$

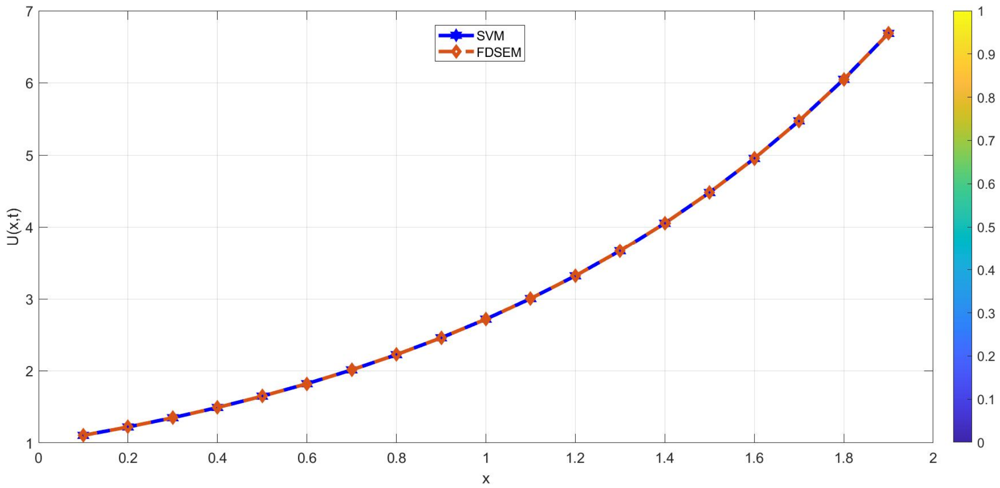

Fig. 3. Comparison of SVM and FDSEM solutions for example 4.1 ($\alpha = \beta = 1$, $C = 1$ and $t = 0$)

Table 2. Comparison of approximate and exact solutions for example 4.1 ($\alpha = \beta = 1$, $C = 1$ and $t = 0.025$)

x	$U_{\text{SVM}}(x,t)$	$U_{\text{FDSEM}}(x,t)$	Absolute Error
0.2	1.161834243	1.162791623	0.000957380
0.4	1.419067549	1.420236895	0.001169346
0.6	1.733253018	1.734681397	0.001428379
0.8	2.117000017	2.118744613	0.001744596
1.0	2.585709659	2.587840350	0.002130691
1.2	3.158192910	3.160795157	0.002602247
1.4	3.857425531	3.860604144	0.003178613
1.6	4.711470183	4.715352549	0.003882366
1.8	5.754602676	5.759344610	0.004741934

Solution by SVM:

To solve the telegraph eq. (37), the separation of variables method will now be applied as follows:
Let

$$U(x,t) = X(x)T(t) \tag{38}$$

Then

$$U_x = X'T,\ U_{xx} = X''T,\ U_t = XT' \text{ and } U_{tt} = XT''.$$

Substituting these derivatives in the PDE gives

$$X''T = XT'' + 4XT' + 4XT,$$
$$\frac{X''}{X} = \frac{T''}{T} + 4\frac{T'}{T} + 4 = \lambda.$$

where λ is a constant. We then have the ODEs

$$\frac{X''}{X} = \lambda \Rightarrow X'' - \lambda X = 0,$$
$$\frac{T''}{T} + 4\frac{T'}{T} + 4 = \lambda \Rightarrow T'' + 4T' - (\lambda - 4)T = 0.$$

Solving for $X(x)$ and applying boundary conditions gives

$$a\pi = n\pi \Rightarrow a = n \text{ and}$$
$$X(x) = \sum_{n=1}^{\infty} C_2 \sin(nx) = 0.$$

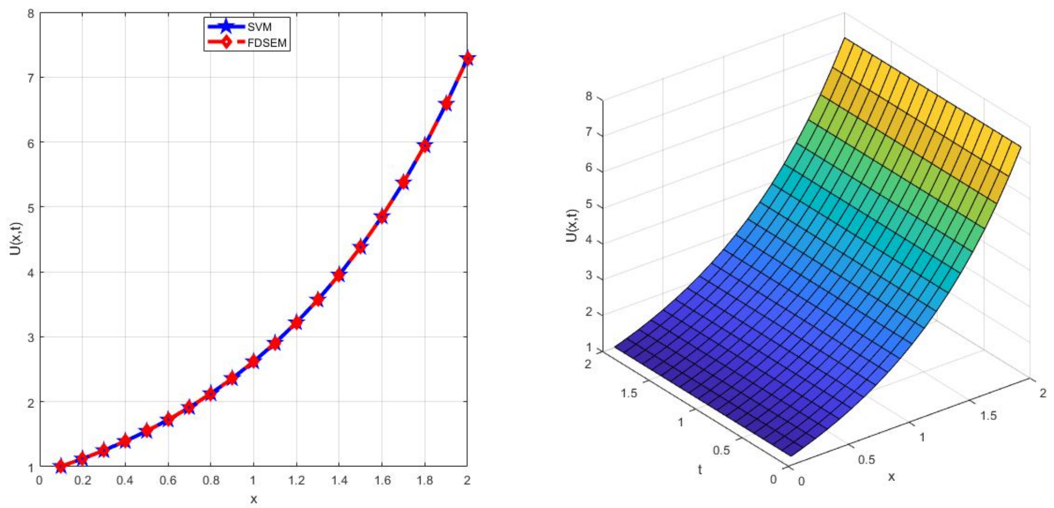

Fig. 4. Comparison of SVM and FDSEM solutions for example 4.1 ($\alpha = \beta = 1$, $C = 1$, $0 \le x \le 2$ for $t = 0.025$ and $0 \le t \le 2$, respectively)

Solving for $T(t)$ with $\lambda = -a^2$, we have;

$$T(t) = e^{-2t}[C_3 \cos(nt) + C_4 \sin(nt)]. \tag{39}$$

Applying initial conditions, the general solution of the given telegraph equation is

$$U(x,t) = \sum_{n=1}^{\infty} \sin(nx) e^{-2t} \left[\frac{1}{2+2n^2} \left[e^4 (\sin(4n) - n\cos(4n)) + n \right] \cos(nt) + \left[\frac{2}{n} \left[\frac{1}{2+2n^2} \left[e^4 (\sin(4n) - n\cos(4n)) + n \right] \right] - \frac{e^x}{n} \right] \sin(nt) \right]. \tag{40}$$

This series converges to the exact solution

$$U(x,t) = e^x e^{-t} = e^{x-t}. \tag{41}$$

Solution by FDSEM:

To solve eq. (37), the FDSEM will be applied as follows. Choose $n = 1, 2, \ldots, 20$. From eq. (37), $C^2 = 1, \Rightarrow C = \pm 1$, $\alpha = \beta = 2$, $h = \frac{x}{n} = \frac{4}{20} = 0.2$, $0 \le t \le 0.5$ and $k = \frac{0.5}{20} = 0.025$. For $r = \frac{k}{h} = \frac{0.025}{0.2} = 0.125$, the discretizations are

$x_i = ih$	$t_j = jk$	$x_i = ih$	$t_j = jk$
$x_1 = 0.2$	$t_1 = 0.025$	$x_{11} = 2.2$	$t_{11} = 0.275$
$x_2 = 0.4$	$t_2 = 0.050$	$x_{12} = 2.4$	$t_{12} = 0.300$
$x_3 = 0.6$	$t_3 = 0.075$	$x_{13} = 2.6$	$t_{13} = 0.325$
$x_4 = 0.8$	$t_4 = 0.100$	$x_{14} = 2.8$	$t_{14} = 0.350$
$x_5 = 1.0$	$t_5 = 0.125$	$x_{15} = 3.0$	$t_{15} = 0.375$
$x_6 = 1.2$	$t_6 = 0.150$	$x_{16} = 3.2$	$t_{16} = 0.400$
$x_7 = 1.4$	$t_7 = 0.175$	$x_{17} = 3.4$	$t_{17} = 0.425$
$x_8 = 1.6$	$t_8 = 0.200$	$x_{18} = 3.6$	$t_{18} = 0.450$
$x_9 = 1.8$	$t_9 = 0.225$	$x_{19} = 3.8$	$t_{19} = 0.475$
$x_{10} = 2.0$	$t_{10} = 0.250$	$x_{20} = 4.0$	$t_{20} = 0.500$

Using initial conditions:

$U(x_i,0)=e^{x_i}$	$U(x_i,0)=e^{x_i}$
$U_{1,0}=1.2214027580$	$U_{11,0}=9.025013499$
$U_{2,0}=1.4918246980$	$U_{12,0}=11.023177638$
$U_{3,0}=1.8221188000$	$U_{13,0}=13.46373804$
$U_{4,0}=2.2255409280$	$U_{14,0}=16.44464677$
$U_{5,0}=2.7182818280$	$U_{15,0}=20.08553692$
$U_{6,0}=3.3201169230$	$U_{16,0}=24.532530200$
$U_{7,0}=4.0551999670$	$U_{17,0}=29.96410005$
$U_{8,0}=4.9532032424$	$U_{18,0}=36.59823444$
$U_{9,0}=6.0496474640$	$U_{19,0}=44.70118449$
$U_{10,0}=7.389056099$	

Using another initial condition $U_t(x,0)=-e^x$, we have $U_t=\frac{1}{2k}(U_{i,j+1}-U_{i,j-1})$. At $t=0$, $jk=0$, $j=0$, $\left.\frac{\partial U}{\partial t}\right|_{t=0}=-e^x \Rightarrow -e^{x_i}=\frac{1}{2k}(U_{i,j+1}-U_{i,j-1})$. Thus,

$$U_{i,j-1}=U_{i,j+1}+0.05e^{x_i}. \tag{42}$$

Discretizing the PDE eq. (37) gives

$$U_{i,j+1}=\frac{2C^2r^2}{2+(\alpha+\beta)k}(U_{i-1,j}+U_{i+1,j})+\frac{4+2\alpha\beta k^2-4C^2r^2}{2+(\alpha+\beta)k}U_{i,j}+\frac{(\alpha+\beta)k-2}{2+(\alpha+\beta)k}U_{i,j-1}.$$

If $r=\frac{k}{h}=\frac{0.025}{0.2}=0.125$, then

$$U_{i,j+1}=0.0149U_{i-1,j}+0.0149U_{i+1,j}+1.8774U_{i,j}-0.9048U_{i,j-1}. \tag{43}$$

Substituting eq. (42) into eq. (43) gives the difference equation

$$U_{i,j+1}=0.0078U_{i-1,j}+0.0078U_{i+1,j}+0.9856U_{i,j}+0.0237e^{x_i}. \tag{44}$$

At $j=0$

$$U_{i,1}=0.0078U_{i-1,0}+0.0078U_{i+1,0}+0.9856U_{i,0}+0.0237e^{x_i} \tag{45}$$

for $i=1,2,\ldots,19$.

Comparison of SVM and FDSEM Solutions for example 4.2:

Here, we validate the authenticity and efficacy of both FDSEM and SVM in solving equation eq. (37) by comparing SVM and FDSEM results for $\alpha=\beta=2$, $C=1$, $0\le x\le 4$ at $t=0$ and $t=0.025$. The numerical and analytical solutions of eq. (37) are shown in tables 3 and 4 and figs. 5 and 6.

Table 3. Comparison of FDSEM and SVM solutions for example 4.2 ($\alpha=\beta=1$, $C=1$, $0\le x\le 2$ and $t=0$)

x	$U_{\text{SVM}}(x,t)$	$U_{\text{FDSEM}}(x,t)$	Absolute Error
0.2	1.221402758	1.221402758	0
0.4	1.491824698	1.491824698	0
0.6	1.822118800	1.822118800	0
0.8	2.225540928	2.225540928	0
1.0	2.718281828	2.718281828	0
1.2	3.320116923	3.320116923	0
1.4	4.055199967	4.055199967	0
1.6	4.953203242	4.953203242	0
1.8	6.049647464	6.049647464	0
2.0	7.389056099	7.389056099	0

Example 4.3.
Consider the linear telegraph equation [24]:

$$U_{xx}=U_{tt}+4U_t+4U \tag{46}$$

subject to the boundary and initial conditions

$$U(0,t)=U(2,t)=0,$$
$$U(x,0)=1+e^{2x},\ U_t(x,0)=-2.$$

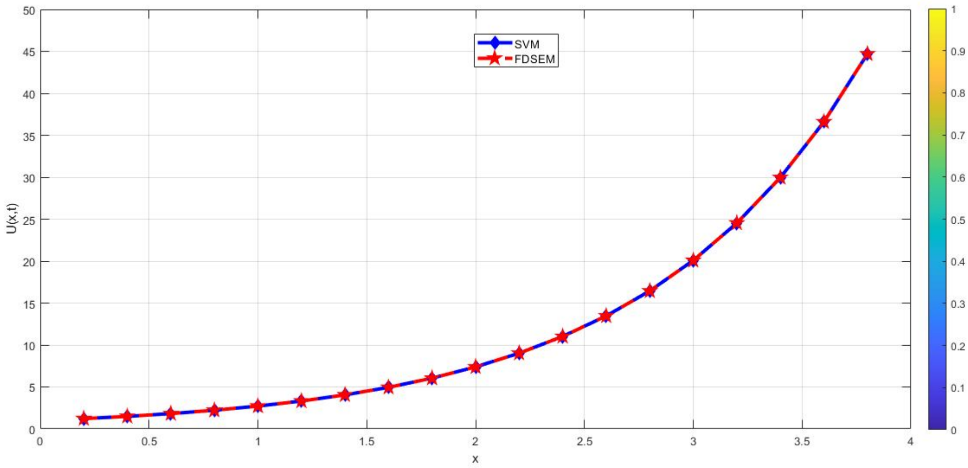

Fig. 5. Comparison of SVM and FDSEM solutions for example 4.2 ($\alpha = \beta = 2$, $C = 1$, $0 \le x \le 4$ and $t = 0$)

Table 4. Comparison of FDSEM and SVM solutions for example 4.2 ($\alpha = \beta = 2$, $C = 1$, $0 \le x \le 2$ and $t = 0.025$)

x	$U_{\text{SVM}}(x,t)$	$U_{\text{FDSEM}}(x,t)$	Absolute Error
0.2	1.191246217	1.186503545	0.004742672
0.4	1.454991415	1.458725645	0.003734230
0.6	1.777130527	1.781691526	0.004560999
0.8	2.170592127	2.176162944	0.005570817
1.0	2.651167211	2.657971422	0.006804211
1.2	3.238142944	3.246453626	0.008310682
1.4	3.955076723	3.965227413	0.010150690
1.6	4.830741618	4.843139699	0.012398081
1.8	5.900281136	5.915424186	0.015143050
2.0	7.206619654	7.225115417	0.018495763

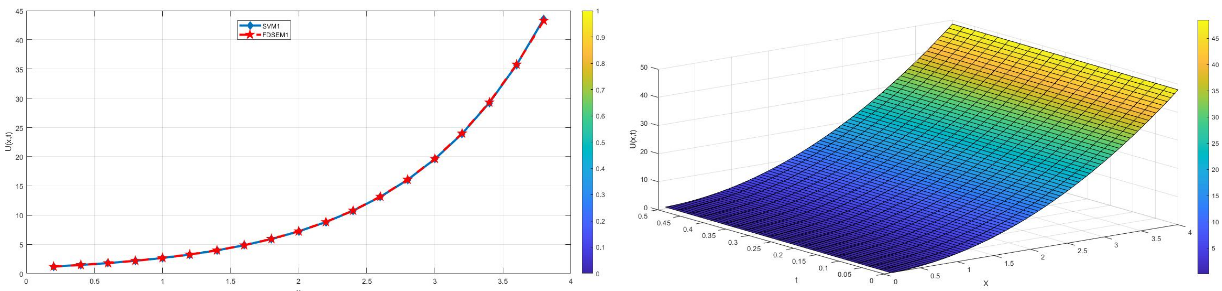

Fig. 6. Comparison of FDSEM and SVM solutions for example 4.2 ($\alpha = \beta = 2$, $C = 1$, $0 \le x \le 4$ for $t = 0.025$ and $0 \le t \le 0.5$, respectively)

Solution by SVM:

To solve this telegraph eq. (46), the separation of variables method will be applied as follows. Let the trial solution be

$$U(x,t) = X(x)T(t). \tag{47}$$

Then

$$U_x = X'T,\ U_{xx} = X''T,\ U_t = XT' \text{ and } U_{tt} = XT''.$$

Substituting into eq. (46):

$$X''T = XT'' + 4XT' + 4XT,$$
$$\frac{X''}{X} = \frac{T''}{T} + 4\frac{T'}{T} + 4 = \lambda,$$

where λ is a constant. The corresponding ODEs to be solved are

$$\frac{X''}{X} = \lambda \Rightarrow X'' - \lambda X = 0,$$
$$T'' + 4T' = (\lambda - 4)T \Rightarrow T'' + 4T' - (\lambda - 4)T = 0.$$

Solving for $X(x)$ and applying boundary conditions gives

$$a\pi = n\pi \Rightarrow a = n,$$
$$X(x) = \textstyle\sum_{n=1}^{\infty} C_2 \sin(nx) = 0.$$

Solving for $T(t)$ with $\lambda = -a^2$, we have

$$T(t) = e^{-2t}[C_3 \cos(nt) + C_4 \sin(nt)]. \tag{48}$$

Applying initial conditions leads the general solution

$$U(x,t) = \sum_{n=1}^{\infty} \sin(nx)e^{-2t}\left[\left[\frac{e^2}{4+n^2}(\sin(2n) - n\cos(2n)) - \frac{1}{n}\cos(2n) + \frac{4+2n^2}{4n+n^3}\right]\cos(nt)\right.$$
$$\left. + \left[\frac{2}{n}\left[\frac{e^2}{4+n^2}(\sin(2n) - n\cos(2n)) - \frac{1}{n}\cos(2n) + \frac{4+2n^2}{4n+n^3}\right] - \frac{2}{n}\right]\sin(nt)\right]. \tag{49}$$

This series converges to exact solution

$$U(x,t) = e^{2x} + e^{-2t}.$$

Solution by FDSEM:

To solve the given telegraph equation by the FDSEM, we proceed as follows. Choose $n = 1, 2, \ldots, 20$, from equation eq. (46) $C = \pm 1$, $\alpha = \beta = 2$, $h = \frac{x}{n} = \frac{2}{20} = 0.1$, $0 \le t \le 0.25$ and $k = \frac{0.25}{20} = 0.0125$. The discretizations are

$x_i = ih$	$t_j = jk$	$x_i = ih$	$t_j = jk$
$x_1 = 0.1$	$t_1 = 0.0125$	$x_{11} = 1.1$	$t_{11} = 0.1375$
$x_2 = 0.2$	$t_2 = 0.0250$	$x_{12} = 1.2$	$t_{12} = 0.1500$
$x_3 = 0.3$	$t_3 = 0.0375$	$x_{13} = 1.3$	$t_{13} = 0.1625$
$x_4 = 0.4$	$t_4 = 0.0500$	$x_{14} = 1.4$	$t_{14} = 0.1750$
$x_5 = 0.5$	$t_5 = 0.0625$	$x_{15} = 1.5$	$t_{15} = 0.1875$
$x_6 = 0.6$	$t_6 = 0.0750$	$x_{16} = 1.6$	$t_{16} = 0.2000$
$x_7 = 0.7$	$t_7 = 0.0875$	$x_{17} = 1.7$	$t_{17} = 0.2125$
$x_8 = 0.8$	$t_8 = 0.1000$	$x_{18} = 1.8$	$t_{18} = 0.2250$
$x_9 = 0.9$	$t_9 = 0.1125$	$x_{19} = 1.9$	$t_{19} = 0.2375$
$x_{10} = 1.0$	$t_{10} = 0.1250$	$x_{20} = 2.0$	$t_{20} = 0.2500$

Using initial conditions:

$U(x_i,0) = 1 + e^{2x_i}$	$U(x_i,0) = 1 + e^{2x_i}$
$U_{1,0} = 2.2214027580$	$U_{11,0} = 10.025013499$
$U_{2,0} = 2.4918246980$	$U_{12,0} = 12.023177638$
$U_{3,0} = 2.8221188000$	$U_{13,0} = 14.46373804$
$U_{4,0} = 3.2255409280$	$U_{14,0} = 17.44464677$
$U_{5,0} = 3.7182818280$	$U_{15,0} = 21.08553692$
$U_{6,0} = 4.3201169230$	$U_{16,0} = 25.532530200$
$U_{7,0} = 5.0551999670$	$U_{17,0} = 30.96410005$
$U_{8,0} = 5.9532032424$	$U_{18,0} = 37.59823444$
$U_{9,0} = 7.0496474640$	$U_{19,0} = 45.70118449$
$U_{10,0} = 8.389056099$	

Table 5. Comparison of FDSEM and SVM solutions for example 4.3 ($\alpha = \beta = 2$, $C = 1$, $0 \le x \le 2$ and $t = 0$)

x	$U_{\text{SVM}}(x,t)$	$U_{\text{FDSEM}}(x,t)$	Absolute Error
0.2	2.4918246980	2.4918246980	0
0.4	3.2255409280	3.2255409280	0
0.6	4.3201169230	4.3201169230	0
0.8	5.9532032424	5.9532032424	0
1.0	8.3890560990	8.3890560990	0
1.2	12.023176380	12.023176380	0
1.4	17.444646770	17.444646770	0
1.6	25.532530200	25.532530200	0
1.8	37.598234440	37.598234440	0

Using another initial condition $U_t(x,0) = -2$, we have $U_t = \frac{1}{2k}(U_{i,j+1} - U_{i,j-1})$. At $t = 0$, $jk = 0$, $j = 0$, we have $\left.\frac{\partial U}{\partial t}\right|_{t=0} = -e^x \Rightarrow -2 = \frac{1}{2k}(U_{i,j+1} - U_{i,j-1})$. Thus,

$$U_{i,j-1} = U_{i,j+1} + 0.05. \tag{50}$$

Discretize the telegraph eq. (46) to obtain

$$U_{i,j+1} = \frac{2C^2r^2}{2+(\alpha+\beta)k}(U_{i-1,j} + U_{i+1,j}) + \frac{4+2\alpha\beta k^2 - 4C^2r^2}{2+(\alpha+\beta)k}U_{i,j} + \frac{(\alpha+\beta)k-2}{2+(\alpha+\beta)k}U_{i,j-1}. \tag{51}$$

With $r = \frac{k}{h} = \frac{0.0125}{0.1} = 0.125$, the difference equation is

$$U_{i,j+1} = 0.0152U_{i-1,j} + 0.0152U_{i+1,j} + 1.9213U_{i,j} - 0.9512U_{i,j-1}. \tag{52}$$

Substituting eq. (50) into eq. (52) gives

$$U_{i,j+1} = 0.0078U_{i-1,j} + 0.0078U_{i+1,j} + 0.9847U_{i,j} - 0.0244. \tag{53}$$

At $j = 0$ the difference equation is

$$U_{i,1} = 0.0078U_{i-1,0} + 0.0078U_{i+1,0} + 0.9847U_{i,0} - 0.0244 \tag{54}$$

for $i = 1,2,\ldots,19$.

Comparison of SVM and FDSEM Solutions for example 4.3:

We now validate the authenticity and efficacy of both FDSEM and SVM for solving the telegraph equation and compare the two methods for $\alpha = \beta = 2$, $C = 1$, $0 \le x \le 2$ at $t = 0$ and $t = 0.0125$. The results are shown in tables 5 and 6 and figs. 7 and 8. From the above findings, FDSEM provides highly accurate numerical solutions and is very reliable and effective. These results show that the solution obtained by the FDSEM technique is nearly identical with the exact solution. For FDSEM, the iterative procedure has an advantage in that each solution is an improvement of the previous iterate and as more iterations are taken, the solution converges to the exact solution of the equation.

Table 6. Comparison of FDSEM and SVM solutions for example 4.3 ($\alpha = \beta = 2$, $C = 1$, $0 \le x \le 2$ and $t = 0.0125$)

x	$U_{\text{SVM}}(x,t)$	$U_{\text{FDSEM}}(x,t)$	Absolute Error
0.2	2.467134610	2.468639248	0.001504638
0.4	3.200850840	3.202805277	0.001954437
0.6	4.295426835	4.298052292	0.002625457
0.8	5.928342336	5.932137042	0.003794706
1.0	8.364366011	8.369485896	0.005119885
1.2	11.99848629	12.00583527	0.007348980
1.4	17.41995668	17.43062802	0.010671340
1.6	25.50784011	25.52346966	0.015629550
1.8	37.57354436	37.59657067	0.023026310

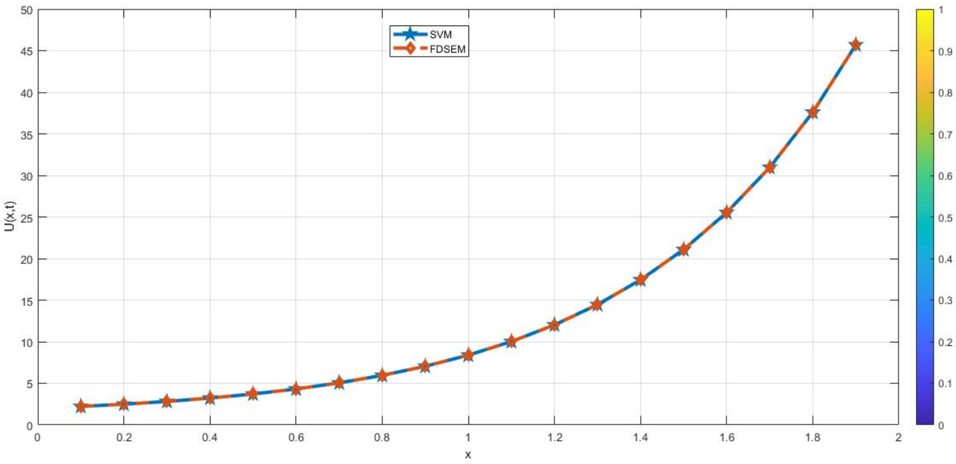

Fig. 7. Comparison of SVM and FDSEM solutions for example 4.3 ($\alpha = \beta = 2$, $C = 1$, $0 \leq x \leq 2$ and $t = 0$)

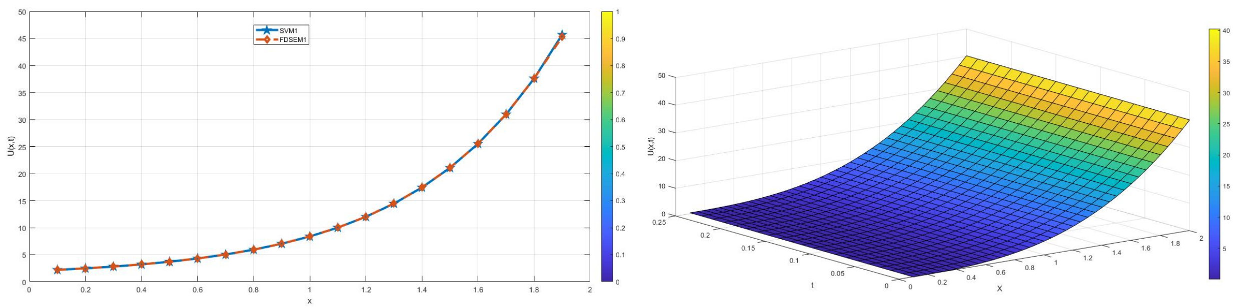

Fig. 8. Comparison of FDSEM and SVM solutions for example 4.3 ($\alpha = \beta = 2$, $C = 1$, $0 \leq x \leq 2$ for $t = 0.0125$ and $0 \leq t \leq 0.25$, respectively)

5. Conclusion

In this work, we applied the separation of variables method for finding the analytic solution and the finite difference scheme explicit method for finding the numerical solution to one dimensional linear hyperbolic telegraph equations. Three examples have been solved to illustrate the efficiency and accuracy of the finite difference scheme explicit method. An error analysis has been achieved by comparing numerical solutions (from the FDSEM) with exact solutions (from the SVM). The numerical results have shown that the FDSEM is highly accurate, efficient and very easy to apply to various engineering problems. Possible future studies may include investigations on convergence, stability and consistency of the present methods using complex values of α and β, application of the present methods to nonlinear telegraph equations as well as to 2D and 3D telegraph equations and checking applicability of the methods to some fractional-stochastic PDEs.

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