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Helicoidal surfaces in 3-dimensional Lorentz-Minkowski space satisfying the condition $\Delta^{II} G = f(G+C)$

Research Article

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Abstract: In this work, we introduce the generalized dual Woodall numbers. As special cases, we study with dual Woodall, dual modified Woodall, dual Cullen numbers and dual modified Cullen numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Also, we give Catalan's and Cassini's identities and present matrices related with these sequences.

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1. Introduction

The study of finitely generated submanifolds began in the late 1970s when B-Y Chen attempted to find a better estimate of the mean total curvature of a compact submanifold of a Euclidean space and to find a notion of "degree" for submanifolds of a Euclidean space.

In the late 1970s, B-y. Chen introduced the notion of finitely generated Euclidean immersion. Basically, it is a submanifold whose immersion $r : M \to \mathbb{R}^m$ is constructed using a finite number of proper functions with valus in $\mathbb{R}^m$ of their Laplacian [2, 4]. Much work have been done to characterize the classification of finitely generated submanifolds and many interesting results have been obtained (see for example [6]).

The theory of submanifolds forms an important and useful class of theories in differential geometry. The mathematical models are often used to describe many different branch of science such as engineering, chemistry, biology, physical physical problems, etc [1, 9, 10, 14].

Then in [3, 5] submanifolds with a type-1 Gaussian map are studied. On the other hand, it can be observed that the Laplacian of the Gaussian map of a helicoid, a catenoid and a right cone in $\mathbb{R}^3$ is of the form

$$\Delta \mathbf{G} = f(\mathbf{G}+C), \tag{1}$$

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where Δ is the Laplace operator associated with the first fundamental form, f is the smooth function in the surface and C a constant vector. This has led several geometers to classify all submanifolds of a Euclidean m-space $\mathbb{R}^m$ (or in the Lorenetz-Minkowski space $\mathbb{R}^3_1$) which satisfies the condition eq. (1).
A submanifold M of a Euclidean space $\mathbb{R}^m$ is said to have a Gaussian map of type 1 if its Gaussian map $\mathbf{G}$ satisfies the condition eq. (1) for a non-zero smooth function f on M and a constant vector.
In [7, 8] M. Choi and Y.H. Kim gave a characterization of the helicoid as a ruled surface with a type 1 Gaussian map in $\mathbb{R}^3$. Furthermore, in the article [12], A. Niang studied the rotation surfaces in $\mathbb{R}^3$ under the condition

$$\Delta\mathbf{G} = f\mathbf{G},$$

and obtained a characterization theorem for rotation surfaces of constant mean curvature. In [13] he studied under certain conditions the version of Lorentz rotation surfaces $\mathbb{R}^3_1$. B-Y.Chen also proved that the surfaces of revolution in $\mathbb{R}^3$ with a first-order pointwise type-1 Gaussian map coincide with the surfaces of revolution with constant mean curvature. Furthermore, they characterized rational surfaces of revolution with a pointwise type-1 Gaussian map in $\mathbb{R}^3$.
If a surface in $\mathbb{R}^3$ has non-parabolic points, then the second fundamental form can be considered a new Riemannian metric.
In this work, we consider the helicoidal surfaces without parabolic points in the Minkowski 3-space $\mathbb{R}^3$ satisfying the following condition

$$\Delta^{II}\mathbf{G} = f(\mathbf{G}+C) \tag{2}$$

where Δ^{II} is the Laplace operator associated with the second fundamental form, f is a smooth function on the surface and C is a constant vector.
Our main results show that the helicoidal surfaces without parabolic points in $\mathbb{R}^3_1$ which satisfy the condition

$$\Delta^{II}\mathbf{G} = f\mathbf{G} \tag{3}$$

for a non-zero function f, coincide with surfaces of revolution with constant non-zero Gaussian curvature. Furthermore, we claim that there are no surfaces of revolution that satisfy the condition $\Delta^{II}\mathbf{G} = 0$.
The assumption throughout this document is that all surfaces are bonded and all objects are smooth, unless otherwise stated.

In Section 2, we will review some basic facts on Euclidean space $\mathbb{R}^3$ (respectively of the Lorentz-Minkowski space $\mathbb{R}^3_1$). In Section 3, we show that the helicoidal surfaces without parabolic points which satisfy the condition $\Delta^{II}G = f(G+C)$ coincide with surfaces of revolution with constant non-zero Gaussian curvature.

2. Preliminaries

In this part we will give some definitions and basic concepts necessary for this work.

Definition 2.1.
We call a pseudo-Euclidean space of dimension m, of signature $(s, m-s)$, the vector space $\mathbb{R}^m$ equipped with the metric

$$g = -dx_1^2 - \dots - dx_s^2 + dx_{s+1}^2 + \dots + dx_m^2, \tag{4}$$

where $(x_1, x_2, \dots, x_m)$ are the coordinates of $\mathbb{R}^m$. We denote it as $\mathbb{R}^m_s$. In particular, for $m > 2, \mathbb{R}^m_1$ is called Lorentz-Minkowski m-space.

We will be particularly interested in the Lorentz-Minkowski 3-space, which is the vector space $\mathbb{R}^3$ equipped with the Lorentzian metric

$$\mathbf{g} = ds^2 = -dx^2 + dy^2 + dz^2, \tag{5}$$

where (x, y, z) is the coordinate system of $\mathbb{R}^3_1$.
We associate with this metric the Lorentz scalar product of two vectors $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ of $\mathbb{R}^3_1$, defined by

$$\langle X, Y\rangle = -X_1Y_1 + X_2Y_2 + X_3Y_3. \tag{6}$$

Definition 2.2.
The Lorentz vector product of two vectors X and Y is denoted as $X \times Y$. It is the unique vector in $\mathbb{R}^3_1$ which is defined by:

$$X \times Y = (-X_2Y_3 + X_3Y_2, X_3Y_1 - X_1Y_3, X_1Y_2 - X_2Y_1) \tag{7}$$

2.1. Surfaces of $\mathbb{R}^3$

Definition 2.3.
Let $\mathbf{M}^2$ be a regular surface of the Euclidean space $\mathbb{R}^3$ (respectively of the Lorentz-Minkowski space $\mathbb{R}^3_1$). We define the functions E, F and $G : \omega \to \mathbb{R}$ by

$$E = \langle r_u, r_u \rangle, F = \langle r_u, r_v \rangle, G = \langle r_v, r_v \rangle \tag{8}$$

where r_u (respectively r_v) denotes the partial derivative of r with respect to u (respectively v). $\{r_u, r_v\}_{|(u,v)}$ is a basis of the tangent space $T_p\mathbf{M}^2$ on the surface M^2 at $p = r(u, v)$..

The functions E, F and G are called coefficients of the first fundamental form associated with the surface $\mathbf{M}^2$.

Let $\mathbf{M}^2$ be a regular surface. Let $\mathcal{N}$ be the unit normal vector to this surface, which is defined by

$$\mathcal{N} = \frac{r_u \times r_v}{||r_u \times r_v||}. \tag{9}$$

The application $\mathcal{N} : M^2 \to Q^2(\varepsilon) \subset \mathbb{R}^3_1$, which at any point of M^2 associates the unit normal vector with $\mathbf{M}^2$ at this point, is called the Gaussian application of the surface $\mathbf{M}^2$, where $\varepsilon(= \pm 1)$ denotes the sign of the vector field $\mathcal{N}$ of $Q^2(\varepsilon)$, the two-dimensional form space.
We then obtain new functions L, M and $N : \omega \to \mathbb{R}$ defined by

$$L = \langle \mathcal{N}, r_{uu} \rangle, M = \langle \mathcal{N}, r_{vu} \rangle, N = \langle \mathcal{N}, r_{vv} \rangle, \tag{10}$$

they are called coefficients of the second fundamental form associated with the surface $\mathbf{M}^2$.

2.2. Ruled surfaces

Definition 2.4.
A ruled surface $\mathbf{M}^2$ of $\mathbb{R}^3$ (or $\mathbb{R}^3_1$) is a surface parametrized by

$$r(u, v) = \alpha(u) + v\beta(u) \tag{11}$$

where α is a curve of $\mathbb{R}^3$ (respectively $\mathbb{R}^3_1$) and β is a non-zero vector field along α.

For a surface $\mathbf{M}^2$ of $\mathbb{R}^3_1$ and a local coordinate system (u, v) in $\mathbf{M}^2$.
The Laplacian associated with the first fundamental form of the surface $\mathbf{M}^2$ is written:

$$\Delta f = -\frac{1}{\sqrt{|EG - F^2|}} \left[\left(\frac{G f_u - F f_v}{\sqrt{|EG - F^2|}} \right)_u - \left(\frac{F f_u - E f_v}{\sqrt{|EG - F^2|}} \right)_v \right] \tag{12}$$

The Laplacian associated with the second fundamental form of the surface $\mathbf{M}^2$ [11] is written:

$$\Delta^{II} f = -\frac{1}{\sqrt{|LN - M^2|}} \left[\left(\frac{N f_u - M f_v}{\sqrt{|LN - M^2|}} \right)_u - \left(\frac{M f_u - L f_v}{\sqrt{|LN - M^2|}} \right)_v \right] \tag{13}$$

where $LN - M^2 \neq 0$.

3. Helicoidal surfaces verifying $\Delta^{II} G = f(G + C)$

In this section we will study only three types of spaces de $\mathbb{R}^3_1$, which are:

1. Type I helicoidal surfaces are described by
$$r(u, v) = (u \sinh v, u \cosh v, g(u) + kv), \quad u \in I, v \in \mathbb{R}, k > 0$$
2. Type II helicoidal surfaces are described by
$$r(u, v) = (u \cosh v, u \sinh v, g(u) + kv), \quad u \in I, v \in \mathbb{R}, k > 0)$$
3. Type III helicoidal surfaces are described by
$$r(u, v) = (g(u) + kv, u \cos v, u \sin v), \quad u \in I, v \in \mathbb{R}, k > 0.$$

We will try to answer the following question:
What are the helicoidal surfaces in 3-dimensional Lorentz-Minkowski space satisfying

$$\Delta^{II} G = f(G + C)? \tag{14}$$

3.1. Type I helicoidal surface

Suppose that the surface $\mathbf{M}^2$ is expressed by:

$$r(u,v) = (u\sinh v, u\cosh v, g(u)+kv). \tag{15}$$

Let us calculate the derivatives of order 1 and 2 with respect to u and v of $r(u;v)$.

$$\begin{aligned} r_u(u,v) &= (\sinh v, \cosh v, g'(u));\\ r_v(u,v) &= (u\cosh v, u\sinh v, k);\\ r_{uu} &= (0,0,g'');\\ r_{vu} &= (\cosh v, \sinh v, 0);\\ r_{vv} &= (u\sinh v, u\cosh v, 0). \end{aligned}$$

The elements of the first and second fundamental form are:

$$E = 1+g'^2; F = cg'; G = k^2 - u^2 \tag{16}$$

$$L = -\frac{ug''}{w}; M = \frac{k}{w}; N = \frac{u^2g'}{w} \tag{17}$$

where $w = |EG-F^2|^{\frac{1}{2}} = |u^2(1+g'^2)-k^2|^{\frac{1}{2}}$

$$LN - M^2 \neq 0 \Rightarrow k^2 + u^3g'g'' \neq 0. \tag{18}$$

We set $R = LN - M^2 = |k^2 + u^3g'g''|$. The unit normal vector or Gaussian map is given by:

$$\mathbf{G} = \frac{1}{w}(-k\cosh v + g'u\sinh v, -k\sinh v + g'u\cosh v, -u). \tag{19}$$

Let us now calculate the Laplacian $\Delta^{II}\mathbf{G}$ for each component of $\mathbf{G}$.
The first component $\mathbf{G}_1$ of $\mathbf{G}$, is

$$\mathbf{G}_1 = \frac{1}{w}(-k\cosh v + g'u\sinh v) \tag{20}$$

The component $\mathbf{G}_1$

$$\begin{aligned} \Delta^{II}G_1 = &\Big(\frac{R'w+6w'R}{2R^2w^3}\left[ku^4g'^2g'' + k^3ug'\right] - \frac{1}{Rw^2}\Big[4ku^3g'^2g'' + 2ku^4g''^2\\ &+ku^4g'^2g'' + k^3ug' + k^3u^3g'^2g'' + 2k^3g'\Big]\Big)\cosh v\\ &+\Big(\frac{R'w+6w'R}{2R^2w^3}\left[u^5g'g'' - k^2u^3g'g'' + k^2u^2 - k^4\right] - \frac{1}{Rw^2}\Big[5u^4g'g'' + u^5g''^2 + u^5g'g'''\\ &-3k^2u^2g'g'' - k^2u^3g''^2 - k^2u^3g'g''' + k^2u - k^2ug'^2 - u^4g'g'' - u^4g'^3g''\Big]\Big)\sinh v. \end{aligned} \tag{21}$$

We can set

$$\Delta^{II}G_1 = A_1\cosh v + B_1\sinh v \tag{22}$$

The component G_2

$$\begin{aligned} \Delta^{II}G_2 = &\Big(\frac{R'w+6w'R}{2R^2w^3}\left[ku^4g'^2g'' + k^3ug'\right] - \frac{1}{Rw^2}\Big[5ku^3g'^2g'' + 2ku^4g'g''^2 + ku^4g'^2g'' + k^3ug'' + 2k^3g'\Big]\Big)\sinh v\\ &+\Big(\frac{R'w+6w'R}{2R^2w^3}\left[u^5g'g'' - k^2u^3g'g'' + k^2u^2 - k^4\right] - \frac{1}{Rw^2}[5u^4g'g'' + u^5g''^2 + u^5g'g'''\\ &-3k^2u^2g'g'' - k^2u^3g''^2 - k^2u^3g'g''' + k^2u - k^2ug'^2 - u^4g'g'' - u^4g'^3g'']\Big)\cosh v. \end{aligned} \tag{23}$$

We can also put

$$\Delta^{II}\mathbf{G}_2 = A_1\sinh v + B_1\cosh v. \tag{24}$$

The Laplacian of the second fundamental form is:

$$\Delta^{II}\mathbf{G} = (A_1 \cosh v + B_1 \sinh v; A_1 \sinh v + B_1 \cosh v; D_1) \tag{25}$$

where

$$A_1 = \frac{R'w + 6w'R}{2R^2w^3}\left[ku^4g'^2g'' + k^3ug'\right] - \frac{1}{Rw^2}\left[5ku^3g'^2g'' + 2ku^4g'g''^2 + ku^4g'^2g'' + k^3ug'' + 2k^3g'\right], \tag{26}$$

$$\begin{aligned} B_1 = {} & \frac{R'w + 6w'R}{2R^2w^3}\left[u^5g'g'' - k^2u^3g'g'' + k^2u^2 - k^4\right] - \frac{1}{Rw^2}[5u^4g'g'' + u^5g''^2 + u^5g'g''' \\ & -3k^2u^2g'g'' - k^2u^3g''^2 - k^2u^3g'g''' + k^2u - k^2ug'^2 - u^4g'g'' - u^4g'^3g''], \end{aligned} \tag{27}$$

$$D_1 = \frac{R'w + 6w'R}{2R^2w^3}(k^2u^2g' + u^5g'^2g'') - \frac{1}{Rw^2}(2k^2ug' + k^2u^2g'' + 5u^4g'^2g'' + 2u^5g'g''^2 + u^5g'^2g'''). \tag{28}$$

So

$$D_1 = \frac{uA_1}{k} \tag{29}$$

Let $(\Delta^{II}\mathbf{G})_i$ be the i^{th} component of $\Delta^{II}\mathbf{G}$ for $i = 1;2;3$. So we have

$$\left\{\begin{array}{rcl} \Delta^{II}\mathbf{G}_1 & = & f(u,v)\left(\dfrac{-k\cosh v + ug'\sinh v}{\sqrt{|u^2(1+g'^2) - k^2|}} + c_1\right) \\ & = & A_1 \cosh v + B_1 \sinh v \\ \Delta^{II}\mathbf{G}_2 & = & f(u,v)\left(\dfrac{-k\sinh v + ug'\cosh v}{\sqrt{|u^2(1+g'^2) - k^2|}} + c_2\right) \\ & = & A_1 \sinh v + B_1 \cosh v \\ \Delta^{II}\mathbf{G}_3 & = & f(u,v)\left(\dfrac{-u}{\sqrt{|u^2(1+g'^2) - k^2|}} + c_3\right) \\ & = & \dfrac{uA_1}{k} \end{array}\right. \tag{30}$$

where $C = (c_1, c_2, c_3)$.
By identification we can easily determine $f(u,v); c_1; c_2$ and c_3.

$$\left\{\begin{array}{rcl} -\dfrac{kf(u,v)}{w} & = & A_1 \\ \dfrac{ug'f(u,v)}{w} & = & B_1 \\ c_1 f(u,v) = 0 & & \end{array}\right. \tag{31}$$

$$\left\{\begin{array}{rcl} -\dfrac{kf(u,v)}{w} & = & A_1 \\ \dfrac{ug'f(u,v)}{w} & = & B_1 \\ c_2 f(u,v) = 0 & & \end{array}\right. \tag{32}$$

So from eqs. (31) and (32), we can deduce that $c_1 = c_2 = 0$ and
$f(u,v) = -\dfrac{wA_1}{k}$ or $f(u,v) = \dfrac{wB_1}{ug'}$

Case 1: If $f(u,v) = -\dfrac{wA_1}{k}$, we have the expression for $f(u,v)$:

$$f(u,v) = -\frac{R'w + 6w'R}{2R^2w^3}\left[u^4g'^2g'' + k^2ug'\right] - \frac{1}{Rw^2}\left[5u^3g'^2g'' + 2u^4g''^2 + u^4g'^2g'' + k^2ug'' + 2k^2g'\right] \tag{33}$$

To determine c_3 we can use

$$\Delta^{II}\mathbf{G}_3 = f(u,v)\left(\frac{-u}{\sqrt{|u^2(1+g'^2) - k^2|}} + c_3\right) = \frac{uA_1}{k} \tag{34}$$

$$-uf + c_3 fw = \frac{uwA_1}{k} \Longrightarrow c_3 = \frac{uA_1}{kf} + \frac{u}{w} \Longrightarrow c_3 = 0 \tag{35}$$

Case 2: If $f(u,v) = \dfrac{wB_1}{ug'}$, we can easily express $f(u,v)$ by replacing B_1 with its expression

$$f(u,v) = \frac{R'w+6w'R}{2ug'R^2w^2}\left[u^5g'g''-k^2u^3g'g''+k^2u^2-k^4\right] - \frac{1}{ug'Rw}[5u^4g'g''+u^5g''^2 \\ +u^5g'g'''-3k^2u^2g'g''-k^2u^3g''^2-k^2u^3g'g'''+k^2u-k^2ug'^2-u^4g'g''-u^4g'^3g''] \tag{36}$$

From eqs. (34) and (36) we have:

$$C_3 = \frac{uA_1}{kf} + \frac{u}{w}. \tag{37}$$

Thus we have just demonstrated the following theorem

Theorem 3.1.
Let $r: \boldsymbol{M}^2 \to \mathbb{R}^3_1$ be a parametrization of the helical surface given by

$$r(u,v) = \left(u\sin v, u\cos v, g(u)+kv\right)$$

where $v \in \mathbb{R}$, $k>0$, $u \in I \subset \mathbb{R}$ and G is a Gaussian map.

1. *If $f(u,v) = -\dfrac{wA_1}{k}$, where*

$$A_1 = \frac{R'w+6w'R}{2R^2w^3}\left[ku^4g'^2g''+k^3ug'\right] - \frac{1}{Rw^2}\left[5ku^3g'^2g''+2ku^4g'g''^2+ku^4g'^2g''+k^3ug''+2k^3g'\right]. \tag{38}$$

   Then the condition $\Delta^{II}G = fG$ is verified.

2. *If $f(u,v) = \dfrac{wB_1}{ug'}$, where*

$$B_1 = \frac{R'w+6w'R}{2R^2w^3}\left[u^5g'g''-k^2u^3g'g''+k^2u^2-k^4\right] - \frac{1}{Rw^2}[5u^4g'g''+u^5g''^2+u^5g'g''' \\ -3k^2u^2g'g''-k^2u^3g''^2-k^2u^3g'g'''+k^2u-k^2ug'^2-u^4g'g''-u^4g'^3g'']. \tag{39}$$

   Then the condition $\Delta^{II}G = f(G+C)$ is verified, where $C = (0,0,c_3)$ according to eq. (37).

Corollary 3.1.
Let $\boldsymbol{M}^2$ be a helicoidal surface parametrized by

$$r(u,v) = (u\sinh v, u\cosh v, g(u)+kv)$$

in $\mathbb{R}^3_1$. If $g''=0$

1. *If $f(u,v) = -\dfrac{wA_1}{k}$, where*

$$A_1 = \frac{1}{w^3}[kau^2(1+a^2)+2k^3a] \tag{40}$$

   Then the condition $\Delta^{II}G = fG$ is verified.

2. *If $f(u,v) = \dfrac{wB_1}{ug'}$, where*

$$B_1 = \frac{1}{w^4}[2u^3-2uk^2+3u^3a^2+u^3a^4-4ua^2k^2] \tag{41}$$

   Then the condition $\Delta^{II}G = f(G+C)$ is verified, where $C = (0,0,c_3)$

with $c_3 = \dfrac{u}{w}\left[1+\dfrac{uA_1g'}{kB_1}\right] \Longrightarrow c_3 = \dfrac{u}{w}\left[1+\dfrac{au^2g'(1+a^2)+2k^2ag'}{2u^2-2k^2+3u^2a^1+u^2a^4-4a^2k^2}\right]$

3.2. Type II helicoidal surfaces

Suppose that the surface **M** is expressed equivalently by:

$$r(u,v) = (u\cosh v, u\sinh v, g(u) + kv), \quad u \in I, v \in \mathbb{R}, k > 0 \tag{42}$$

the result is identical to the previous case. There is no point in studying it.

3.3. Type III helicoidal surfaces

Suppose that the helicodal surfaces **M** are described by:

$$r(u,v) = (g(u) + kv, u\cos v, u\sin v), \quad u \in I, v \in \mathbb{R}, k > 0. \tag{43}$$

We can verify that

$$E = 1 - g'^2; F = -kg'; G = u^2 - k^2 \tag{44}$$

$$L = \frac{ug''}{w}; M = -\frac{k}{w}; N = \frac{u^2 g'}{w}. \tag{45}$$

The unit vector is:

$$\mathbf{G} = \frac{1}{w}\left(u; -ug'\cos v + k\sin v; ug'\sin v + k\cos v\right). \tag{46}$$

Now let us compute the Laplacian $\Delta^{II}G$ for each of the components of **G**.
The first component $\mathbf{G}_1$ of **G**, is $\mathbf{G}_1 = \frac{u}{w}$. The Laplacian of the second fundamental form for $\mathbf{G}_1$ is:

$$\Delta^{II}\mathbf{G}_1 = \frac{R'w + 6w'R}{2R^2w^3}(k^2u^2g' - u^5(g')^2g'') - \frac{1}{Rw^2}(2k^2g' + k^2u^2g'' - 5u^4g'^2g'' - 2u^5g'(g'')^2 - u^5(g')^2g'''). \tag{47}$$

Put $\Delta^{II}\mathbf{G}_1 = D_2$.
The component $\mathbf{G}_2$ is

$$G_2 = \frac{1}{w}(-ug'\cos v + k\sin v). \tag{48}$$

The Laplacian of the second fundamental form for the component $\mathbf{G}_2$ is:

$$\begin{aligned}\Delta^{II}\mathbf{G}_2 = \Big[&\frac{R'w + 6w'R}{2R^2w^3}(k^4 + k^2u^2 - k^2u^3g'g'' - u^5g'g'') - \frac{1}{Rw^2}(k^2u - k^2ug'^2 - 3k^2u^2g'g'' - k^2u^3g'g'''\\ &-4u^4g'g'' + u^4g'^3g'' - u^5g''^2 - u^5g'g''')\Big]\cos v + \Big[\frac{R'w + 6w'R}{2R^2w^3}(k^3ug' - ku^4g'^2g'')\\ &-\frac{1}{Rw^2}(2k^3g' + k^3ug'' - 5ku^3g'^2g'' - 2ku^4g'g''^2 - ku^4g'^2g''')\Big]\sin v.\end{aligned} \tag{49}$$

We pose

$$\Delta^{II}\mathbf{G}_2 = A_2\cos v + B_2\sin v. \tag{50}$$

The component $\mathbf{G}_3$

$$\mathbf{G}_3 = \frac{1}{w}(ug'\sin v + k\cos v). \tag{51}$$

The Laplacian of the second fundamental form for the component $\mathbf{G}_3$ is:

$$\begin{aligned}\Delta^{II}\mathbf{G}_3 = \Big[&\frac{R'w + 6w'R}{2R^2w^3}\Big(-k^4 - k^2u^2 + k^2u^3g'g'' + u^5g'g''\Big) - \frac{1}{Rw^2}\Big(-k^2u + k^2ug'^2 + 3k^2u^2g'g''\\ &+k^2u^3g'g''' + 4u^4g'g'' - u^4g'^3g'' - u^5g''^2 + u^5g'g'''\Big)\Big]\sin v + \Big[\frac{R'w + 6w'R}{2R^2w^3}\Big(k^3ug' - ku^4g'^2g''\Big)\\ &-\frac{1}{Rw^2}\Big(2k^3g' + k^3ug'' - 5ku^3g'^2g'' - 2ku^4g'g''^2 - ku^4g'^2g'''\Big)\Big]\cos v.\end{aligned} \tag{52}$$

Then

$$\Delta^{II}\mathbf{G}_3 = -A_2\sin v + B_2\cos v. \tag{53}$$

The Laplacian of the second fundamental form is

$$\Delta^{II}\mathbf{G} = (D_2; A_2\cos v + B_2\sin v; -A_2\sin v + B_2\cos v), \tag{54}$$

where

$$\begin{aligned} A_2 = &-\frac{1}{Rw^2}\Big(k^2u - k^2ug'^2 - 3k^2u^2g'g'' - k^2u^3g'g''' - 4u^4g'g'' + u^4(g')^3g'' \\ &-u^5(g'')^2 - u^5g'g'''\Big) + \frac{R'w+6w'R}{2R^2w^3}\Big(k^4 + k^2u^2 - k^2u^3g'g'' - u^5g'g''\Big), \end{aligned} \tag{55}$$

$$B_2 = \frac{R'w+6w'R}{2R^2w^3}\Big(k^3ug' - ku^4(g')^2g''\Big) - \frac{1}{Rw^2}\Big(2k^3g' + k^3ug'' - 5ku^3(g')^2g'' - 2ku^4g'(g'')^2 - ku^4g'^2g'''\Big), \tag{56}$$

$$D_2 = \frac{R'w+6w'R}{2R^2w^3}\Big(k^2u^2g' - u^5(g')^2g''\Big) - \frac{1}{Rw^2}\Big(2k^2g' + k^2u^2g'' - 5u^4(g')^2g'' - 2u^5g'(g'')^2 - u^5(g')^2g'''\Big). \tag{57}$$

We note that $D_2 = \dfrac{u}{k}B_2$

Let $\Delta^{II}G_i$ be the i^{th} component of $\Delta^{II}\mathbf{G}$ for $i = 1;2;3$. Then we have

$$\begin{cases} \Delta^{II}\mathbf{G}_1 &= f(u,v)\left(\dfrac{u}{\sqrt{|u^2(1+g'^2)+k^2|}} + C_1\right) \\ &= D_2 \\ \Delta^{II}\mathbf{G}_2 &= f(u,v)\left(\dfrac{-ug'\cos v + k\sin v}{\sqrt{|u^2(1+g'^2)+k^2|}} + C_2\right) \\ &= A_2\cos v + B_2\sin v \\ \Delta^{II}\mathbf{G}_3 &= f(u,v)\left(\dfrac{ug'\sin v + c\cos v}{\sqrt{|u^2(1+g'^2)+k^2|}} + C_3\right) \\ &= -A_2\sin v + B_2\cos v \end{cases}$$

where $C = (C_1, C_2, C_3)$.

By identification we can easily determine $f(u,v); C_1; C_2$ and C_3.

$$\begin{cases} -\dfrac{ug'f(u,v)}{w} &= A_2 \\ \dfrac{kf(u,v)}{w} &= B_2 \\ C_2f(u,v) = 0 \end{cases} \tag{58}$$

$$\begin{cases} -\dfrac{ug'f(u,v)}{w} &= -A_2 \\ \dfrac{kf(u,v)}{w} &= B_2 \\ C_3f(u,v) &= 0. \end{cases} \tag{59}$$

So from eqs. (58) and (59), we can deduce that $C_2 = C_3 = 0$ and

$$f(u,v) = -\frac{w}{ug'}A_2$$

or

$$f(u,v) = \frac{w}{k}B_2.$$

Case 1: If $f(u,v) = \dfrac{w}{k}B_2$, then the expression of $f(u,v)$ is

$$f(u,v) = \frac{1}{Rw}\Big(-2k^2g' - k^2ug'' + 5u^3(g')^2g'' + 2u^4g'g''^2 + u^4(g')^2g'''\Big) + \frac{R'w+6w'R}{2R^2w^2}\Big(k^2ug' - u^4g'^2g''\Big).$$

We can determine C_1 from

$$\Delta^{II}\mathbf{G}_1 = f(u,v)\left(\frac{u}{\sqrt{|u^2(1+g'^2)+k^2|}} + C_1\right) = D_2 \tag{60}$$

Then $f(u,v)\left(\frac{u}{w}+C_1\right) = D_2 \Longrightarrow C_1 = \frac{D_2}{f} - \frac{u}{w} \Longrightarrow C_1 = \frac{u}{w} - \frac{u}{w} \Longrightarrow C_1 = 0$

Case 2: If $f(u,v) = -\frac{w}{ug'}A_2$, We can express $f(u,v)$ by replacing A_2 with its expression.

$$\begin{aligned} f(u,v) = &\frac{1}{Rwug'}\Big(k^2u - k^2u(g')^2 - 3k^2u^2g'g'' - k^2u^3g'g''' - 4u^4g'g'' + u^4(g')^3g'' \\ &-u^5(g'')^2 - u^5g'g'''\Big) - \frac{R'w+6w'R}{2R^2w^2ug'}\Big(k^4+k^2u^2-k^2u^3g'g''-u^5g'g''\Big) \end{aligned} \tag{61}$$

$C_1 = \frac{D_2}{f} - \frac{u}{w} \Longrightarrow C_1 = -\frac{u}{w}\left[\frac{ug'}{k}\frac{B_2}{A_2}+1\right].$
Thus, we have just established the following theorem

Theorem 3.2.
Let $r: \boldsymbol{M}^2 \to \mathbb{R}^3_1$ be a parametrization of the helical surface given by

$$r(u,v) = (g(u)+kv, u\cos v, u\sin v), \quad u \in I, v \in \mathbb{R}, k > 0$$

and G a Gaussian map.

1. *If $f(u,v) = \frac{w}{k}B_2$, where*

$$B_2 = \frac{R'w+6w'R}{2R^2w^3}\Big(k^3ug' - ku^4(g')^2g''\Big) - \frac{1}{Rw^2}\Big(2k^3g' + k^3ug'' - 5ku^3(g')^2g'' - 2ku^4g'(g'')^2 - ku^4(g')^2g'''\Big). \tag{62}$$

   Then the condition $\Delta^{II}G = fG$ is verified.

2. *If $f(u,v) = -\frac{w}{ug'}A_2$, where*

$$\begin{aligned} A_2 = &-\frac{1}{Rw^2}\Big(k^2u - k^2u(g')^2 - 3k^2u^2g'g'' - k^2u^3g'g''' - 4u^4g'g'' + u^4(g')^3g'' - u^5(g'')^2 \\ &-u^5g'g'''\Big) + \frac{R'w+6w'R}{2R^2w^3}\Big(k^4+k^2u^2-k^2u^3g'g''-u^5g'g''\Big) \end{aligned} \tag{63}$$

   then the condition $\Delta^{II}G = f(G+C)$ is verified with $C=(c,0,0)$, where $c = C_1 = -\frac{u}{w}\left[\frac{ug'}{k}\frac{B_2}{A_2}+1\right]$.

Corollary 3.2.
Let $\boldsymbol{M}^2$ be a helical surface parametrized by $r(u,v) = (g(u)+kv, u\cos v, u\sin v)$ in $\mathbb{R}^3_1$. If $g''=0$

1. *If $f(u,v) = -\frac{wB_2}{k}$, where*

$$B_2 = \frac{1}{w^3}[kau^2(1+a^2)+2k^3a] \tag{64}$$

   Then the condition $\Delta^{II}G = fG$ is verified.

2. *If $f(u,v) = \frac{wA_2}{ug'}$, where*

$$A_2 = \frac{1}{w^4}[2u^3 - 2uk^2 + 3u^3a^2 + u^3a^4 - 4ua^2k^2] \tag{65}$$

   Then the condition $\Delta^{II}G = f(G+C)$ is verified, where $C=(c,0,0)$

with $c = c_1 = -\frac{u}{w}\left[1+\frac{uB_2g'}{kA_2}\right] \Longrightarrow c_1 = -\frac{u}{w}\left[1+\frac{au^2g'(1+a^2)+2k^2ag'}{2u^2-2k^2+3u^2a^1+u^2a^4-4a^2k^2}\right]$

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References

[1] A. A. Abdallah, A. A. Navlekar and K. P. Ghadle, B. Hardan, Fundamentals and Recent Studies of Finsler Geometry, Int. J. Adv. Appl. Math. and Mech. 10(2), (2022) 27-38.
[2] Difi Sid Ahmed, Géométrie des surfaces dans l'espace Euclidien et l'espace semi-Euclidien, thèse, Université Djiliali Liabes de Sidi Bel Abbes, département de Mathématiques, 15 Janvier 2019.
[3] C. Baba-Hamed and Mohammed Bekkar, Surfaces of revolution satisfying $\Delta^{II}\mathbf{G} = f(\mathbf{G}+C)$, Bull. Korean Math. Soc. 50, (2013), No.4, pp. 1061-1067.
[4] C. Baba-hamed and M. Bekkar, Helicoidal surfaces of revolution in the 3-Demensional Lorentz-Minkowski space satisfaying $\Delta^{II}r_i = \lambda_i r_i$, Int.J. Geom. 100, (2011),1-10.
[5] C. Baikoussis and D. E. Blair, On the Gauss map of ruled surfaces, Glasggow Math. J. 34, (1992), no.3, 355-359.
[6] B-y. Chen, M. Choi and Y.H. Kim, Surfaces of revolution with pointwise 1-type Gauss map, J. Korean Math. Soc. 42, (2005), No. 3,447-455.
[7] M. Choiand Y.H. Kim , Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map, Bull, Korean Math. Soc. 38, (2001), No.4, 753-761.
[8] M. Choi, D-. Kim and Y. H. Kim, Helicoidal surfaces with pointwise 1-type Gauss map, J. korean Math. Soc. 46, (2009),no.1,215-223.
[9] YAYÉ ARBI I., OUSMANE TOUDOU I., HASSIROU M., Finsler manifolds with applications in fluid flow mechanics, Int. J. Adv. Appl. Math. and Mech. 12(3), (2025) pages 110-11.
[10] Adetola J., Houédanou K. W., Bello W. A., Edou R. J., Generation of B-spline Surfaces for the simulation of mechanical structures, Int. J. Adv. Appl. Math. and Mech. 12(2), (2024) pages 22–34
[11] B. O'Neill, emi-remannian Geometry, academic Press, New York, 1983.
[12] A. Niang, Rotation surfaces with 1-Gauss map, Bull. korean Math. Soc 42, (2005), no. 1, 23-27.
[13] A. Niang, On rotation surfaces in the Minkowski 3-dimensional spacewith pointwise 1-type Gauss map, J.Korean Math. Soc. 41, (2004), no. 6, 1007-1021.
[14] Tevfik Sahina and Merve Okur, *Special Smarandache curves with respect to Darboux frame in Galilean* 3-*Space*, Int. J. Adv. Appl. Math. and Mech. 5(3), (2018) 15-26.