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On Generalized co-Tribonacci Numbers

Research Article

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Abstract: In this paper, we introduce and investigate a new third order recurrence sequence so called generalized co-Tribonacci sequence and its two special subsequences which are related to generalized Tribonacci numbers and its two subsequences. There are close interrelations between recurrence equations of and roots of characteristic equations of generalized Tribonacci and generalized co-Tribonacci numbers. We present Binet's formulas, generating functions, some identities, Simson's formulas, recurrence properties, sum formulas and matrices related with these sequences.

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1. Introduction: Generalized Tribonacci Numbers

The generalized Tribonacci numbers

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or $\{W_n\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s and t are real numbers with $t \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence eq. (1) holds for all integers n.

For r, s, t satisfying eq. (1), the generalized co-Tribonacci numbers

$$\{Y_n(Y_0, Y_1, Y_2; -s, -rt, t^2)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -sY_{n-1} - rtY_{n-2} + t^2Y_{n-3}, \quad Y_0 = d, Y_1 = e, Y_2 = f, \ \ n \geq 3 \tag{2}$$

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i.e.,

$$Y_n = r_1 Y_{n-1} + s_1 Y_{n-2} + t_1 Y_{n-3}, \quad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3$$

where $Y_0, Y_1, Y_2$are arbitrary complex (or real) numbers and $r_1 = -s,\ s_1 = -rt,\ t_1 = t^2$.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} Y_{-n} &= -\frac{-rt}{t^2} Y_{-(n-1)} - \frac{-s}{t^2} Y_{-(n-2)} + \frac{1}{t^2} Y_{-(n-3)} \\ &= -\frac{s_1}{t_1} Y_{-(n-1)} - \frac{r_1}{t_1} Y_{-(n-2)} + \frac{1}{t_1} Y_{-(n-3)} \end{aligned}$$

for $n = 1,2,3,...$ when $t \neq 0$. Therefore, recurrence eq. (2)holds for all integer n. For more information on generalized Tribonacci and co-Tribonacci numbers, see [14].

Note that we can easily use and modify the results given for r, s, t in [14] by substituting r_1, s_1, t_1 for r, s, t and we will do this in this paper.

There are close interrelations between roots of characteristic equations of generalized Tribonacci and generalized co-Tribonacci numbers, see [14], Lemma 17.]: If α, β, γ are the roots of characteristic equation of $\{W_n\}$ which is given as

$$z^3 - rz^2 - sz - t = 0,$$

and if $\theta_1, \theta_2, \theta_3$ are the roots of characteristic equation of $\{Y_n\}$ which is given as

$$y^3 - r_1 y^2 - s_1 y - t_1 = y^3 + sy^2 + rty - t^2 = 0,$$

then we get

$$\begin{aligned} \theta_1 &= \beta\gamma, \\ \theta_2 &= \alpha\beta, \\ \theta_3 &= \alpha\gamma. \end{aligned}$$

There are also close connections and relations between recurrence equations of generalized Tribonacci and generalized co-Tribonacci numbers, see, for example, Lemma 32 in [14].

In this paper, we consider the case $r = 1,\ s = 1,\ t = 1$ so that $r_1 = -s = -1,\ s_1 = -rt = -1,\ t_1 = t^2 = 1$.

In the next section, we also use the notation $r = -1,\ s = -1,\ t = 1$ for $r_1 = -1,\ s_1 = -1,\ t_1 = 1$ to use results in the paper [14]. Now, in this section, for the case $r = 1,\ s = 1,\ t = 1$ we present some well known results. We also call the case $r = 1,\ s = 1,\ t = 1$ as generalized Tribonacci numbers as usual.

A generalized Tribonacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-1} + W_{n-2} + W_{n-3} \tag{3}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} - W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1,2,3,...$. Therefore, recurrence eq. (3) holds for all integer n. For more information on generalized Tribonacci numbers, see Soykan [12]. Tribonacci concept was introduced by 14 year old student M. Feinberg [6] in 1963. Basic properties of these sequences are given in [1–5, 7–12, 16–19].

As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation (cubic equation) is

$$z^3 - z^2 - z - 1 = (z-\alpha)(z-\beta)(z-\gamma) = 0.$$

The roots α, β, γ of characteristic equation of $\{W_n\}$ are given as

$$\begin{aligned} \alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}, \end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\begin{cases} \alpha+\beta+\gamma = 1, \\ \alpha\beta+\alpha\gamma+\beta\gamma = -1, \\ \alpha\beta\gamma = 1. \end{cases}$$

The sequence $\{W_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of W_n, Binet's formula of W_n can be given as follows:

Theorem 1.1.
For all integers n, Binet's formula of generalized Tribonacci numbers is given as follows.

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= A_1\alpha^n + A_2\beta^n + A_3\gamma^n, \end{aligned}$$

where

$$p_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, \tag{4}$$
$$p_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, \tag{5}$$
$$p_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0. \tag{6}$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)} \\ &= \frac{(\alpha W_2 + \alpha(-1+\alpha)W_1 + W_0)}{\alpha^2+2\alpha+3}, \\ A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)} \\ &= \frac{(\beta W_2 + \beta(-1+\beta)W_1 + W_0)}{\beta^2+2\beta+3}, \\ A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{(\gamma W_2 + \gamma(-1+\gamma)W_1 + W_0)}{\gamma^2+2\gamma+3}. \end{aligned}$$

Proof. Take $r = 1, s = 1, t = 1$ in [[14], Theorem 3 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 1.1.
Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Tribonacci numbers $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1-W_0)z + (W_2-W_1-W_0)z^2}{1-z-z^2-z^3}. \tag{7}$$

Proof. Set $r = 1, s = 1, t = 1$ in [[14], Lemma 9.]. □

Two special cases of the sequence $\{W_n\}$ are the well known Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$. Tribonacci sequence, Tribonacci-Lucas sequence are defined, respectively, by the third-order recurrence relations

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad T_0 = 0, T_1 = 1, T_2 = 1, \tag{8}$$
$$K_{n+3} = K_{n+2} + K_{n+1} + K_n, \qquad K_0 = 3, K_1 = 1, K_2 = 3, \tag{9}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$, can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}, \tag{10}$$
$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}, \tag{11}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences eqs. (8) and (9) hold for all integer n.

For all integers n, Binet's formula of Tribonacci and Tribonacci-Lucas numbers (using initial conditions eq. (4)–eq. (6) in theorem 1.1 can be expressed as follows:

Theorem 1.2.
For all integers n, Binet's formulas of Tribonacci and Tribonacci-Lucas numbers are

$$\begin{aligned}
T_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\
&= \frac{\alpha^{n+2}}{\alpha^2+2\alpha+3} + \frac{\beta^{n+2}}{\beta^2+2\beta+3} + \frac{\gamma^{n+2}}{\gamma^2+2\gamma+3}, \\
K_n &= \alpha^n + \beta^n + \gamma^n,
\end{aligned}$$

respectively.

Lemma 1.1 gives the following results as particular examples (generating functions of Tribonacci and Tribonacci-Lucas numbers).

Corollary 1.1.
Generating functions of Tribonacci and Tribonacci-Lucas numbers are

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n z^n &= \frac{z}{1-z-z^2-z^3}, \\
\sum_{n=0}^{\infty} K_n z^n &= \frac{3-2z-z^2}{1-z-z^2-z^3},
\end{aligned}$$

respectively.

Now, we present some identities of Tribonacci and Tribonacci-Lucas numbers.

Lemma 1.2.
The following equalities are true:

(a) $K_n = 5T_{n+4} - 6T_{n+3} - 5T_{n+2}$.

(b) $K_n = -T_{n+3} + 5T_{n+1}$.

(c) $K_n = -T_{n+2} + 4T_{n+1} - T_n$.

(d) $K_n = 3T_{n+1} - 2T_n - T_{n-1}$.

(e) $K_n = T_n + 2T_{n-1} + 3T_{n-2}$.

(f) $44T_n = 2K_{n+4} - 10K_{n+3} + 16K_{n+2}$.

(g) $44T_n = -8K_{n+3} + 18K_{n+2} + 2K_{n+1}$.

(h) $44T_n = 10K_{n+2} - 6K_{n+1} - 8K_n$.

(i) $44T_n = 2K_n + 10K_{n-1} + 4K_{n+1}$.

(j) $44T_n = 6K_n + 14K_{n-1} + 4K_{n-2}$.

2. Generalized co-Tribonacci Numbers

If $r = 1,\ s = 1,\ t = 1$, then we get $r_1 = -1,\ s_1 = -1,\ t_1 = 1$. From now on, throughout the paper, we also use the notation $r = -1,\ s = -1,\ t = 1$ for $r_1 = -1,\ s_1 = -1,\ t_1 = 1$ and we consider the case $r_1 = -1,\ s_1 = -1,\ t_1 = 1$ to use results in the paper [14].

In this section, we define and investigate a new sequence and its two special cases, namely the generalized co-Tribonacci, co-Tribonacci and co-Tribonacci-Lucas numbers. The generalized co-Tribonacci numbers

$$\{Y_n(Y_0, Y_1, Y_2; -1, -1, 1)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -Y_{n-1} - Y_{n-2} + Y_{n-3}, \quad Y_0 = d, Y_1 = e, Y_2 = f, \ \ n \geq 3 \tag{12}$$

i.e.,

$$Y_{n+3} = -Y_{n+2} - Y_{n+1} + Y_n, \quad Y_0 = d, Y_1 = e, Y_2 = f, \ \ n \geq 0$$

where Y_0, Y_1, Y_2 are arbitrary complex (or real) numbers with real coefficients.

The sequence $\{Y_n(z)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$Y_{-n} = Y_{-(n-1)} + Y_{-(n-2)} + Y_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence eq. (12) holds for all integer n.

The first few generalized co-Tribonacci numbers with positive subscript and negative subscript are given in the following table 1.

Table 1. A few generalized co-Tribonacci numbers

n	Y_n	Y_{-n}
0	Y_0	Y_0
1	Y_1	$Y_0 + Y_1 + Y_2$
2	Y_2	$2Y_0 + 2Y_1 + Y_2$
3	$Y_0 - Y_1 - Y_2$	$4Y_0 + 3Y_1 + 2Y_2$
4	$2Y_1 - Y_0$	$7Y_0 + 6Y_1 + 4Y_2$
5	$2Y_2 - Y_1$	$13Y_0 + 11Y_1 + 7Y_2$
6	$2Y_0 - 2Y_1 - 3Y_2$	$24Y_0 + 20Y_1 + 13Y_2$
7	$5Y_1 - 3Y_0 + Y_2$	$44Y_0 + 37Y_1 + 24Y_2$
8	$Y_0 - 4Y_1 + 4Y_2$	$81Y_0 + 68Y_1 + 44Y_2$
9	$4Y_0 - 3Y_1 - 8Y_2$	$149Y_0 + 125Y_1 + 81Y_2$
10	$12Y_1 - 8Y_0 + 5Y_2$	$274Y_0 + 230Y_1 + 149Y_2$
11	$5Y_0 - 13Y_1 + 7Y_2$	$504Y_0 + 423Y_1 + 274Y_2$
12	$7Y_0 - 2Y_1 - 20Y_2$	$927Y_0 + 778Y_1 + 504Y_2$
13	$27Y_1 - 20Y_0 + 18Y_2$	$1705Y_0 + 1431Y_1 + 927Y_2$

Remark 2.1.
In this paper we will extensively use the paper [14]. Note that in the notation of [14], here we have $r = 1,\ s = 1,\ t = 1$ and $r_1 = -1,\ s_1 = -1,\ t_1 = 1$. For simplicity, we can use the result of [14] by taking and replacing $r = -1,\ s = -1,\ t = 1$.

As $\{Y_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation (cubic equation) is

$$y^3 + y^2 + y - 1 = 0. \tag{13}$$

The roots of characteristic equation of $\{Y_n\}$ are

$$\theta_1 = \frac{-1}{3} + \left(\frac{17}{27} + \sqrt{\frac{11}{27}}\right)^{1/3} - \left(\sqrt{\frac{11}{27}} - \frac{17}{27}\right)^{1/3}$$

$$\theta_2 = \frac{-1}{3} + \omega\left(\frac{17}{27} + \sqrt{\frac{11}{27}}\right)^{1/3} - \omega^2\left(\sqrt{\frac{11}{27}} - \frac{17}{27}\right)^{1/3}$$

$$\theta_3 = \frac{-1}{3} + \omega^2\left(\frac{17}{27} + \sqrt{\frac{11}{27}}\right)^{1/3} - \omega\left(\sqrt{\frac{11}{27}} - \frac{17}{27}\right)^{1/3}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\begin{cases} \theta_1+\theta_2+\theta_3 = -1, \\ \theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3 = 1, \\ \theta_1\theta_2\theta_3 = 1. \end{cases}$$

Note that there are an important relation between $\theta_1, \theta_2, \theta_3$ and α, β, γ:

$$\theta_1 = \beta\gamma,$$
$$\theta_2 = \alpha\beta,$$
$$\theta_3 = \alpha\gamma.$$

The sequence $\{Y_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of Y_n, Binet's formula of Y_n can be given as follows:

Theorem 2.1.
For all integers n, Binet's formula of generalized co-Tribonacci numbers is given as follows.

$$\begin{aligned} Y_n &= \frac{p_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{p_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{p_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= A_1\theta_1^n + A_2\theta_2^n + A_3\theta_3^n, \end{aligned}$$

where

$$p_1 = Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0,\ p_2 = Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0,\ p_3 = Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0,$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\theta_1-\theta_2)(\theta_1-\theta_3)} = \frac{Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0}{(\theta_1-\theta_2)(\theta_1-\theta_3)} \\ &= \frac{(\theta_1Y_2+\theta_1(-r+\theta_1)Y_1+tY_0)}{r\theta_1^2+2s\theta_1+3t}, \\ A_2 &= \frac{p_2}{(\theta_2-\theta_1)(\theta_2-\theta_3)} = \frac{Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0}{(\theta_2-\theta_1)(\theta_2-\theta_3)} \\ &= \frac{(\theta_2Y_2+\theta_2(-r+\theta_2)Y_1+tY_0)}{r\theta_2^2+2s\theta_2+3t}, \\ A_3 &= \frac{p_3}{(\theta_3-\theta_1)(\theta_3-\theta_2)} = \frac{Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{(\theta_3Y_2+\theta_3(-r+\theta_3)Y_1+tY_0)}{r\theta_3^2+2s\theta_3+3t}. \end{aligned}$$

Proof. For the proof, take $r=-1, s=-1, t=1$ in [14], Theorem 3 (a) or $r_1=-1, s_1=-1, t_1=1$ in [14], Theorem 19 (a). □

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} Y_n z^n$ of the sequence Y_n.

Lemma 2.1.
Suppose that $f_{Y_n}(z) = \sum\limits_{n=0}^{\infty} Y_n z^n$ is the ordinary generating function of the generalized co-Tribonacci numbers $\{Y_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} Y_n z^n$ is given by

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{Y_0+(Y_1+Y_0)z+(Y_2+Y_1+Y_0)z^2}{1+z+z^2-z^3}$$

Proof. Set $r=-1, s=-1, t=1$ in [14], Lemma 9. or $r_1=-1, s_1=-1, t_1=1$ in [14], Lemma 24.. □

In this paper, we define and investigate, in detail, two special cases of the generalized co-Tribonacci numbers $\{Y_n\}$ which we call them co-Tribonacci and co-Tribonacci-Lucas numbers. co-Tribonacci numbers $\{U_n\}_{n\geq 0}$ and co-Tribonacci-Lucas numbers $\{S_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$U_n = -U_{n-1} - U_{n-2} + U_{n-3}, \quad U_0 = 0, U_1 = 1, U_2 = -1, \tag{14}$$
$$S_n = -S_{n-1} - S_{n-2} + S_{n-3}, \quad S_0 = 3, S_1 = -1, S_2 = -1, \tag{15}$$

i.e.,

$$U_{n+3} = -U_{n+2} - U_{n+1} + U_n, \quad U_0 = 0, U_1 = 1, U_2 = -1,$$
$$S_{n+3} = -S_{n+2} - S_{n+1} + S_n, \quad S_0 = 3, S_1 = -1, S_2 = -1.$$

The sequences $\{U_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$U_{-n} = U_{-(n-1)} + U_{-(n-2)} + U_{-(n-3)},$$
$$S_{-n} = S_{-(n-1)} + S_{-(n-2)} + S_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences eqs. (14) and (15) hold for all integers n.

Next, we present the first few values of the co-Tribonacci and co-Tribonacci-Lucasnumbers with positive and negative subscripts.

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
U_n	0	1	−1	0	2	−3	1	4	−8	5	7	−20	18	9
U_{-n}	0	0	1	1	2	4	7	13	24	44	81	149	274	504
S_n	3	−1	−1	5	−5	−1	11	−15	3	23	−41	21	43	−105
S_{-n}	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757

For all integers n, Binet's formula of co-Tribonacci and co-Tribonacci-Lucas numbers (using initial conditions eqs. (14) and (15) in theorem 2.1 can be expressed as follows:

Theorem 2.2.
For all integers n, Binet's formulas of co-Tribonacci and co-Tribonacci-Lucas numbers are

$$\begin{aligned} U_n &= \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{\theta_3^{n+1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{\theta_1^{n+2}}{-\theta_1^2-2\theta_1+3} + \frac{\theta_2^{n+2}}{-\theta_2^2-2\theta_2+3} + \frac{\theta_3^{n+2}}{-\theta_3^2-2\theta_3+3}, \end{aligned}$$

and

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n,$$

respectively.

Lemma 2.1 gives the following results as particular examples (generating functions of co-Tribonacci and co-Tribonacci-Lucas numbers).

Corollary 2.1.
Generating functions of co-Tribonacci and co-Tribonacci-Lucas numbers are

$$\sum_{n=0}^{\infty} U_n z^n = \frac{z}{1+z+z^2-z^3},$$
$$\sum_{n=0}^{\infty} S_n z^n = \frac{3+2z+z^2}{1+z+z^2-z^3},$$

respectively.

3. Connections between T_n, K_n and U_n, S_n

S_n can be given as follows.

Lemma 3.1.
For all integers n, we have the following formula for S_n:

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n$$
$$= \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n.$$

Proof. Use [14], Lemma 30. □
We can present the relations between U_n, S_n and T_n, K_n as follows.

Lemma 3.2.
For all integers n, we have the following formulas:

(a) $S_n = \frac{1}{2}(K_n^2 - K_{2n})$.

(b) $U_n = T_{-n-1}$ *and* $U_{-n} = T_{n-1}$.

(c) $S_n = K_{-n}$ *and* $S_{-n} = K_n$.

Proof. Use [14], Lemma 32. □

4. Some Identities of Generalized co-Tribonacci Numbers

In this section, we obtain some identities of co-Tribonacci and co-Tribonacci-Lucas numbers. First, we can give a few basic relations between $\{U_n\}$ and $\{S_n\}$.

Lemma 4.1.
The following equalities are true:

(a) $S_n = 7U_{n+4} + 10U_{n+3} + 11U_{n+2}$

(b) $S_n = 3U_{n+3} + 4U_{n+2} + 7U_{n+1}$

(c) $S_n = U_{n+2} + 4U_{n+1} + 3U_n$

(d) $S_n = 3U_{n+1} + 2U_n + U_{n-1}$

(e) $S_n = -U_n - 2U_{n-1} + 3U_{n-2}$

(f) $44U_n = 6S_{n+4} + 10S_{n+3} + 20S_{n+2}$

(g) $44U_n = 4S_{n+3} + 14S_{n+2} + 6S_{n+1}$

(h) $44U_n = 10S_{n+2} + 2S_{n+1} + 4S_n$

(i) $44U_n = -8S_{n+1} - 6S_n + 10S_{n-1}$

(j) $44U_n = 2S_n + 18S_{n-1} - 8S_{n-2}$

Proof. Set $G_n = U_n$, $H_n = S_n$ and $r = -1, s = -1, t = 1$ in [14], Lemma 36. □ □
Note that all the identities in the above lemma can be proved by induction as well.
Next, we give a few basic relations between $\{U_n\}$ and $\{Y_n\}$.

Lemma 4.2.
The following equalities are true:

(a) $(Y_0^3 - 2Y_0^2 Y_1 - Y_0^2 Y_2 - 2Y_0 Y_1 Y_2 + Y_0 Y_2^2 + 2Y_1^3 + 2Y_1^2 Y_2 + 2Y_1 Y_2^2 + Y_2^3)U_n = -(-Y_0^2 + Y_0 Y_1 + Y_1^2 + Y_2 Y_1)Y_{n+2} + (Y_2^2 - Y_0 Y_1 + Y_0 Y_2 + Y_1 Y_2)Y_{n+1} + (Y_1^2 - Y_0 Y_2)Y_n$

(b) $(Y_0^3-2Y_0^2Y_1-Y_0^2Y_2-2Y_0Y_1Y_2+Y_0Y_2^2+2Y_1^3+2Y_1^2Y_2+2Y_1Y_2^2+Y_2^3)U_n=(-Y_0^2+Y_0Y_2+Y_1^2+2Y_1Y_2+Y_2^2)Y_{n+1}+(-Y_0^2+Y_0Y_1-Y_2Y_0+2Y_1^2+Y_2Y_1)Y_n-(-Y_0^2+Y_0Y_1+Y_1^2+Y_2Y_1)Y_{n-1}$

(c) $(Y_0^3-2Y_0^2Y_1-Y_0^2Y_2-2Y_0Y_1Y_2+Y_0Y_2^2+2Y_1^3+2Y_1^2Y_2+2Y_1Y_2^2+Y_2^3)U_n=-(-Y_1^2+Y_1Y_2-Y_0Y_1+Y_2^2+2Y_0Y_2)Y_n-(-2Y_0^2+Y_0Y_1+Y_0Y_2+2Y_1^2+3Y_1Y_2+Y_2^2)Y_{n-1}+(-Y_0^2+Y_0Y_2+Y_1^2+2Y_1Y_2+Y_2^2)Y_{n-2}$

(d) $Y_n=(Y_0+Y_1+Y_2)U_{n+2}+(2Y_0+Y_1+Y_2)U_{n+1}+(2Y_0+2Y_1+Y_2)U_n$

(e) $Y_n=Y_0U_{n+1}+(Y_0+Y_1)U_n+(Y_0+Y_1+Y_2)U_{n-1}$

(f) $Y_n=Y_1U_n+(Y_1+Y_2)U_{n-1}+Y_0U_{n-2}$

Proof. Set $W_n=Y_n, G_n=U_n$ and $r=-1, s=-1, t=1$ in [14], Lemma 37. □
Now, we present a few basic relations between $\{S_n\}$ and $\{Y_n\}$.

Lemma 4.3.
The following equalities are true:

(a) $\left(Y_0^3-2Y_0^2Y_1-Y_0^2Y_2-2Y_0Y_1Y_2+Y_0Y_2^2+2Y_1^3+2Y_1^2Y_2+2Y_1Y_2^2+Y_2^3\right)S_n=(-Y_0^2-2Y_0Y_1+2Y_0Y_2+2Y_1^2+4Y_1Y_2+3Y_2^2)Y_{n+2}+(-2Y_0^2-2Y_0Y_2+6Y_1^2+4Y_1Y_2+2Y_2^2)Y_{n+1}-(-3Y_0^2+2Y_0Y_2+4Y_1Y_0-Y_2^2+2Y_1Y_2)Y_n$

(b) $\left(Y_0^3-2Y_0^2Y_1-Y_0^2Y_2-2Y_0Y_1Y_2+Y_0Y_2^2+2Y_1^3+2Y_1^2Y_2+2Y_1Y_2^2+Y_2^3\right)S_n=-(Y_0^2-2Y_0Y_1+4Y_0Y_2-4Y_1^2+Y_2^2)Y_{n+1}-(-4Y_0^2+2Y_0Y_1+4Y_0Y_2+2Y_1^2+6Y_1Y_2+2Y_2^2)Y_n+(-Y_0^2-2Y_0Y_1+2Y_0Y_2+2Y_1^2+4Y_1Y_2+3Y_2^2)Y_{n-1}$

(c) $\left(Y_0^3-2Y_0^2Y_1-Y_0^2Y_2-2Y_0Y_1Y_2+Y_0Y_2^2+2Y_1^3+2Y_1^2Y_2+2Y_1Y_2^2+Y_2^3\right)S_n=-(-5Y_0^2+4Y_0Y_1+6Y_1^2+6Y_1Y_2+Y_2^2)Y_n+(-2Y_1^2+4Y_1Y_2-4Y_0Y_1+4Y_2^2+6Y_0Y_2)Y_{n-1}-(Y_0^2-2Y_0Y_1+4Y_0Y_2-4Y_1^2+Y_2^2)Y_{n-2}$

(d) $44Y_n=(6Y_0+14Y_1+4Y_2)S_{n+2}+(10Y_0+16Y_1+14Y_2)S_{n+1}+(20Y_0+10Y_1+6Y_2)S_n$

(e) $44Y_n=(4Y_0+2Y_1+10Y_2)S_{n+1}+(14Y_0-4Y_1+2Y_2)S_n+(6Y_0+14Y_1+4Y_2)S_{n-1}$

(f) $44Y_n=-(6Y_1-10Y_0+8Y_2)S_n+(2Y_0+12Y_1-6Y_2)S_{n-1}+(4Y_0+2Y_1+10Y_2)S_{n-2}$

Proof. Set $W_n=Y_n, H_n=S_n$ and $r=-1, s=-1, t=1$ in [14], Lemma 38.
We can present identities between T_n, K_n and U_n, S_n by using Lemmas given above.

Lemma 4.4.
For all integers n, we have the following formulas:

(a) $22U_{-n}=-4K_{n+2}+9K_{n+1}+K_n$.

(b) $S_{-n}=-T_{n+2}+4T_{n+1}-T_n$.

(c) $S_n=\frac{1}{2}((-T_{n+2}+4T_{n+1}-T_n)^2-(-T_{2n+2}+4T_{2n+1}-T_{2n}))$.

(d) $K_{-n}=U_{n+2}+4U_{n+1}+3U_n$.

(e) $22T_{-n-1}=5S_{n+2}+S_{n+1}+2S_n$.

(f) $22T_{-n}=2S_{n+2}+7S_{n+1}+3S_n$.

Proof. Use lemmas 1.2, 3.2 and 4.1. □
Now, we present some identities of generalized co-Tribonacci numbers and its special cases.

Lemma 4.5.
Suppose that $\{X_n\}_{n\geq 0}=\{X_n(X_0,X_1,X_2)\}_{n\geq 0}$ is also defined by the third-order recurrence relations

$$X_n=-X_{n-1}-X_{n-2}+X_{n-3} \tag{16}$$

i.e.,

$$X_{n+3}=-X_{n+2}-X_{n+1}+X_n$$

with the initial values X_0, X_1, X_2 not all being zero and

$$X_{-n}=X_{-(n-1)}+X_{-(n-2)}+X_{-(n-3)}$$

so that eq. (16) is true for all integer n.
Then the following equalities are true:

(a)
$$(X_0X_3^2+X_1^2X_4+X_2^3-X_0X_2X_4-2X_1X_2X_3)Y_n=q_1X_{n+2}+q_2X_{n+1}+q_3X_n$$

where

$q_1=(X_1^2-X_0X_2)Y_2+(X_0X_3-X_1X_2)Y_1+(X_2^2-X_1X_3)Y_0,$
$q_2=(X_0X_3-X_1X_2)Y_2+(X_2^2-X_0X_4)Y_1+(X_1X_4-X_2X_3)Y_0,$
$q_3=(X_2^2-X_1X_3)Y_2+(X_1X_4-X_2X_3)Y_1+(X_3^2-X_2X_4)Y_0.$

(b)
$$(Y_0Y_3^2+Y_1^2Y_4+Y_2^3-Y_0Y_2Y_4-2Y_1Y_2Y_3)U_n=q_4Y_{n+2}+q_5Y_{n+1}+q_6Y_n$$

where

$q_4=-Y_1^2+Y_0^2-Y_1Y_2-Y_0Y_1,$
$q_5=Y_2^2+Y_1Y_2+Y_0Y_2-Y_0Y_1,$
$q_6=Y_1^2-Y_0Y_2.$

(c)
$$Y_n=q_7U_{n+2}+q_8U_{n+1}+q_9U_n$$

where

$q_7=Y_2+Y_1+Y_0,$
$q_8=Y_2+Y_1+2Y_0,$
$q_9=Y_2+2Y_1+2Y_0.$

(d)
$$(Y_0Y_3^2+Y_1^2Y_4+Y_2^3-Y_0Y_2Y_4-2Y_1Y_2Y_3)S_n=q_{10}Y_{n+2}+q_{11}Y_{n+1}+q_{12}Y_n$$

where

$q_{10}=3Y_2^2+2Y_1^2-Y_0^2+4Y_1Y_2+2Y_0Y_2-2Y_0Y_1,$
$q_{11}=2Y_2^2+6Y_1^2-2Y_0^2+4Y_1Y_2-2Y_0Y_2,$
$q_{12}=Y_2^2+3Y_0^2-2Y_1Y_2-2Y_0Y_2-4Y_0Y_1.$

(e)
$$44Y_n=q_{13}S_{n+2}+q_{14}S_{n+1}+q_{15}S_n$$

where

$q_{13}=4Y_2+14Y_1+6Y_0,$
$q_{14}=14Y_2+16Y_1+10Y_0,$
$q_{15}=6Y_2+10Y_1+20Y_0.$

Proof.

(a) Writing

$$Y_n=q_1\times X_{n+2}+q_2\times X_{n+1}+q_3\times X_n$$

and solving the system of equations

$$Y_0=q_1\times X_2+q_2\times X_1+q_3\times X_0$$
$$Y_1=q_1\times X_3+q_2\times X_2+q_3\times X_1$$
$$Y_2=q_1\times X_4+q_2\times X_3+q_3\times X_2$$

we find the required identity.

(b) Replace Y_n and X_n with U_n and Y_n, respectively in (a).

(c) Replace X_n with U_n in (a).

(d) Replace Y_n and X_n with S_n and Y_n, respectively in (a).

(e) Replace X_n with S_n in (a). □

5. Simson's Formulas of co-Tribonacci Numbers

The following theorem gives Simson's formula of the generalized co-Tribonacci numbers $\{Y_n\}$.

Theorem 5.1 (Simson's Formula of Generalized co-Tribonacci).
For all integers n, we have

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix}$$

$$= \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_0+Y_1+Y_2 \\ Y_0 & Y_0+Y_1+Y_2 & 2Y_0+2Y_1+Y_2 \end{vmatrix}.$$

Proof. Set $W_n = Y_n$ and $r = -1, s = -1, t = 1$ in [14], Theorem 33. □
The previous theorem gives the following results as particular examples.

Corollary 5.1.
For all integers n, Simson's formula of co-Tribonacci and co-Tribonacci-Lucas numbers are given as

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = -1,$$

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = -44,$$

respectively.

Proof. Set $Y_n = U_n$ and $Y_n = S_n$ in theorem 5.1, respectively. □

6. Recurrence Properties of Generalized co-Tribonacci Numbers

The generalized co-Tribonacci numbers W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 6.1.
For $n \in \mathbb{Z}$, we have

$$Y_{-n} = Y_{2n} - S_n Y_n + \frac{1}{2}(S_n^2 - S_{2n}) Y_0.$$

Proof. Set $W_n = Y_n$, $H_n = S_n$ and $r = -1$, $s = -1$, $t = 1$ in [14], Theorem 39. □
As special cases of the above Theorem, we have the following Corollary.

Corollary 6.1.
For $n \in \mathbb{Z}$, we have

(a)
$$U_{-n} = -3U_n^2 + U_{2n} - U_{n+2}U_n - 4U_{n+1}U_n$$

(b)
$$S_{-n} = \frac{1}{2}(S_n^2 - S_{2n})$$

Proof. Take $r = r = -1$, $s = -1$, $t = 1$, and $G_n = U_n$ and $H_n = S_n$,respectively, in [14], Corollary 42 or set $Y_n = U_n$ and $Y_n = S_n$, respectively, in theorem 6.1. □
The last Corollary can be written in the following form by using lemma 3.2.

Corollary 6.2.
For $n \in \mathbb{Z}$, we have

(a)
$$T_{n-1} = -3U_n^2 + U_{2n} - U_{n+2}U_n - 4U_{n+1}U_n$$

(b)
$$K_n = \frac{1}{2}(S_n^2 - S_{2n})$$

Proof. Use lemma 3.2 and corollary 6.1.

7. Sum Formulas $\sum_{k=0}^{n} Y_k$, $\sum_{k=0}^{n} Y_{2k}$, $\sum_{k=0}^{n} Y_{2k+1}$, $\sum_{k=0}^{n} Y_{-k}$, $\sum_{k=0}^{n} Y_{-2k}$, $\sum_{k=0}^{n} Y_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} Y_n z^n$, $\sum_{n=0}^{\infty} Y_{2n} z^n$, $\sum_{n=0}^{\infty} Y_{2n+1} z^n$, $\sum_{n=0}^{\infty} Y_{-n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n+1} z^n$ of Generalized co-Tribonacci Numbers

Next, we present sum formulas of generalized co-Tribonacci numbers.

Theorem 7.1.
For $n \geq 0$, we have the following sum formulas for generalized co-Tribonacci numbers:

(a) $\sum_{k=0}^{n} Y_k = \frac{1}{2}(-Y_{n+2} - 2Y_{n+1} - Y_n + Y_2 + 2Y_1 + 3Y_0)$.

(b) $\sum_{k=0}^{n} Y_{2k} = \frac{1}{2}(-Y_{2n+2} - Y_{2n+1} + Y_2 + Y_1 + 2Y_0)$.

(c) $\sum_{k=0}^{n} Y_{2k+1} = \frac{1}{2}(Y_{2n+1} - Y_{2n} + Y_1 + Y_0)$.

(d) $\sum_{k=0}^{n} Y_{-k} = \frac{1}{2}(Y_{-n+2} + 2Y_{-n+1} + 3Y_{-n} - 2Y_1 - Y_2 - Y_0)$.

(e) $\sum_{k=0}^{n} Y_{-2k} = \frac{1}{2}(Y_{-2n} + Y_{-2n-1} - Y_2 - Y_1)$.

(f) $\sum_{k=0}^{n} Y_{-2k+1} = \frac{1}{2}(-Y_{-2n-1} + Y_{-2n-2} + Y_1 - Y_0)$.

Proof.

(a) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [14], Theorem 62 (a) (i).

(b) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [14], Theorem 62 (b) (i).

(c) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [14], Theorem 62 (c) (i).

(d) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [14], Theorem 62 (d) (i).

(e) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [14], Theorem 62 (e) (i).

(f) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [14], Theorem 62 (f) (i). □

From the last Theorem, we have the following Corollary which gives sum formulas of co-Tribonacci numbers (take $Y_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = -1$).

Corollary 7.1.
For $n \geq 0$, co-Tribonacci numbers have the following properties.

(a) $\sum_{k=0}^{n} U_k = \frac{1}{2}(-U_{n+2} - 2U_{n+1} - U_n + 1)$.

(b) $\sum_{k=0}^{n} U_{2k} = \frac{1}{2}(-U_{2n+2} - U_{2n+1})$.

(c) $\sum_{k=0}^{n} U_{2k+1} = \frac{1}{2}(U_{2n+1} - U_{2n} + 1).$

(d) $\sum_{k=0}^{n} U_{-k} = \frac{1}{2}(U_{-n+2} + 2U_{-n+1} + 3U_{-n} - 1).$

(e) $\sum_{k=0}^{n} U_{-2k} = \frac{1}{2}(U_{-2n} + U_{-2n-1}).$

(f) $\sum_{k=0}^{n} U_{-2k+1} = \frac{1}{2}(-U_{-2n-1} + U_{-2n-2} + 1).$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = -1, S_2 = -1$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Tribonacci-Lucas numbers.

Corollary 7.2.
For $n \geq 0$, co-Tribonacci-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} S_k = \frac{1}{2}(-S_{n+2} - 2S_{n+1} - S_n + 6).$

(b) $\sum_{k=0}^{n} S_{2k} = \frac{1}{2}(-S_{2n+2} - S_{2n+1} + 4).$

(c) $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{2}(S_{2n+1} - S_{2n} + 2).$

(d) $\sum_{k=0}^{n} S_{-k} = \frac{1}{2}(S_{-n+2} + 2S_{-n+1} + 3S_{-n}).$

(e) $\sum_{k=0}^{n} S_{-2k} = \frac{1}{2}(S_{-2n} + S_{-2n-1} + 2).$

(f) $\sum_{k=0}^{n} S_{-2k+1} = \frac{1}{2}(-S_{-2n-1} + S_{-2n-2} - 4).$

Next, we give the ordinary generating function of special cases of the generalized co-Tribonacci numbers $\{Y_{mn+j}\}$.

Corollary 7.3.
The ordinary generating functions of the sequences Y_n, Y_{2n}, Y_{2n+1}, Y_{-n}, Y_{-2n}, Y_{-2n+1} are given as follows:

(a) $(|z| < \min\{|\theta_1|^{-1}, |\theta_2|^{-1}, |\theta_3|^{-1}\} = |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.737352).$

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{(Y_2 + Y_1 + Y_0)z^2 + (Y_1 + Y_0)z + Y_0}{-z^3 + z^2 + z + 1}.$$

(b) $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.543689).$

$$\sum_{n=0}^{\infty} Y_{2n} z^n = \frac{(Y_2 + 2Y_1 + 2Y_0)z^2 + (Y_2 + Y_0)z + Y_0}{-z^3 + 3z^2 + z + 1}.$$

(c) $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.543689).$

$$\sum_{n=0}^{\infty} Y_{2n+1} z^n = \frac{(Y_2 + Y_1 + Y_0)z^2 + (-Y_2 + Y_0)z + Y_1}{-z^3 + 3z^2 + z + 1}.$$

(d) $(|z| < \min\{|\theta_1|, |\theta_2|, |\theta_3|\} = |\theta_1| \simeq 0.543689).$

$$\sum_{n=0}^{\infty} Y_{-n} z^n = \frac{Y_1 z^2 + (Y_2 + Y_1)z + Y_0}{-z^3 - z^2 - z + 1}.$$

(e) $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 0.295597).$

$$\sum_{n=0}^{\infty} Y_{-2n} z^n = \frac{Y_2 z^2 + (Y_2 + 2Y_1 - Y_0)z + Y_0}{-z^3 - z^2 - 3z + 1}$$

(f) $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 0.295597)$.

$$\sum_{n=0}^{\infty} Y_{-2n+1}z^n = \frac{(-Y_2 - Y_1 + Y_0)z^2 + (Y_2 - 2Y_1 + Y_0)z + Y_1}{-z^3 - z^2 - 3z + 1}.$$

Proof. Take $W_n = Y_n$ and $r = -1$, $s = -1$, $t = 1$ in [14], Corollary 67. □
Now, we consider special cases of the last corollary.

Corollary 7.4.
The ordinary generating functions of special cases of the generalized co-Tribonacci numbers are given as follows:

(a) $(|z| < |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.737352)$.

$$\sum_{n=0}^{\infty} U_n z^n = \frac{z}{-z^3 + z^2 + z + 1},$$
$$\sum_{n=0}^{\infty} S_n z^n = \frac{z^2 + 2z + 3}{-z^3 + z^2 + z + 1}.$$

(b) $(|z| < |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.543689)$.

$$\sum_{n=0}^{\infty} U_{2n} z^n = \frac{z^2 - z}{-z^3 + 3z^2 + z + 1},$$
$$\sum_{n=0}^{\infty} S_{2n} z^n = \frac{3z^2 + 2z + 3}{-z^3 + 3z^2 + z + 1}.$$

(c) $(|z| < |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.543689)$.

$$\sum_{n=0}^{\infty} U_{2n+1} z^n = \frac{z + 1}{-z^3 + 3z^2 + z + 1},$$
$$\sum_{n=0}^{\infty} S_{2n+1} z^n = \frac{z^2 + 4z - 1}{-z^3 + 3z^2 + z + 1}.$$

(d) $(|z| < |\theta_1| \simeq 0.543689)$.

$$\sum_{n=0}^{\infty} U_{-n} z^n = \frac{-z^2}{z^3 + z^2 + z - 1},$$
$$\sum_{n=0}^{\infty} S_{-n} z^n = \frac{z^2 + 2z - 3}{z^3 + z^2 + z - 1}.$$

(e) $(|z| < |\theta_1|^2 \simeq 0.295597)$.

$$\sum_{n=0}^{\infty} U_{-2n} z^n = \frac{z^2 - z}{z^3 + z^2 + 3z - 1},$$
$$\sum_{n=0}^{\infty} S_{-2n} z^n = \frac{z^2 + 6z - 3}{z^3 + z^2 + 3z - 1}.$$

(f) $(|z| < |\theta_1|^2 \simeq 0.295597)$.

$$\sum_{n=0}^{\infty} U_{-2n+1} z^n = \frac{3z - 1}{z^3 + z^2 + 3z - 1},$$
$$\sum_{n=0}^{\infty} S_{-2n+1} z^n = \frac{-5z^2 - 4z + 1}{z^3 + z^2 + 3z - 1}.$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 7.5.
We have the following infinite sums .

(a) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{U_n}{2^n} = \frac{4}{13},$$
$$\sum_{n=0}^{\infty} \frac{S_n}{2^n} = \frac{34}{13}.$$

(b) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{U_{2n}}{2^n} = -\frac{2}{17},$$
$$\sum_{n=0}^{\infty} \frac{S_{2n}}{2^n} = \frac{38}{17}.$$

(c) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{U_{2n+1}}{2^n} = \frac{12}{17},$$
$$\sum_{n=0}^{\infty} \frac{S_{2n+1}}{2^n} = \frac{10}{17}$$

(d) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{U_{-n}}{2^n} = 2,$$
$$\sum_{n=0}^{\infty} \frac{S_{-n}}{2^n} = 14.$$

(e) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{U_{-2n}}{4^n} = \frac{12}{11},$$
$$\sum_{n=0}^{\infty} \frac{S_{-2n}}{4^n} = \frac{92}{11}.$$

(f) $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{U_{-2n+1}}{4^n} = \frac{16}{11},$$
$$\sum_{n=0}^{\infty} \frac{S_{-2n+1}}{4^n} = \frac{20}{11}.$$

8. Sum Formulas $\sum_{k=0}^{n} z^k Y_k^2$, $\sum_{k=0}^{n} z^k Y_{k+1} Y_k$, $\sum_{k=0}^{n} z^k Y_{k+2} Y_k$ and Generating Functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1} Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2} Y_n z^n$ of Generalized co-Tribonacci Numbers Numbers

Next, we present sum formulas of generalized co-Tribonacci Numbers numbers.

Theorem 8.1.
For $n \geq 0$, we have the following sum formulas for generalized co-Tribonacci Numbers numbers:

(a) $\sum_{k=0}^{n} Y_k^2 = \frac{1}{4}(Y_{n+3}^2 - 3Y_{n+1}^2 - 2Y_{n+1}Y_{n+3} - 4Y_{n+2}Y_{n+1} - Y_2^2 + 3Y_0^2 + 2Y_0Y_2 + 4Y_1Y_0)$.

(b) $\sum_{k=0}^{n} Y_{k+1}Y_k = \frac{1}{4}(-Y_{n+3}^2 - 2Y_{n+2}^2 - Y_{n+1}^2 - 2Y_{n+2}Y_{n+3} - 2Y_{n+1}Y_{n+2} + Y_2^2 + 2Y_1^2 + Y_0^2 + 2Y_0Y_1 + 2Y_1Y_2)$.

(c) $\sum_{k=0}^{n} Y_{k+2}Y_k = \frac{1}{4}(-Y_{n+3}^2 - Y_{n+1}^2 - 2Y_{n+1}Y_{n+3} + Y_2^2 + Y_0^2 + 2Y_0Y_2)$.

Proof. Note that characteristic equation of the third-order recurrence sequence Y_n is the cubic equation $y^3 + y^2 + y - 1$ whose roots are $\theta_1, \theta_2, \theta_3$ with $\theta_1 \neq \theta_2 \neq \theta_3$. In [15], Theorem 2.1, for $r = -1$, $s = -1$, $t = 1$, we get

$$\Gamma(z) = -(z^3 - 3z^2 - z - 1)(z^3 + z^2 + z - 1)$$

and $\Gamma(1) \neq 0$.

(a) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [15], Theorem 2.1 (a) (i) or in [16], Theorem 2.1 (a) (i).

(b) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [15], Theorem 2.1 (b) (i) or in [16], Theorem 2.1 (b) (i)

(c) Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and $z = 1$ in [15], Theorem 2.1 (c) (i) or in [16], Theorem 2.1 (c) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-Tribonacci numbers (take $Y_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = -1$).

Corollary 8.1.
For $n \geq 0$, co-Tribonacci numbers have the following properties.

(a) $\sum_{k=0}^{n} U_k^2 = \frac{1}{4}(U_{n+3}^2 - 3U_{n+1}^2 - 2U_{n+1}U_{n+3} - 4U_{n+2}U_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} U_{k+1}U_k = \frac{1}{4}(-U_{n+3}^2 - 2U_{n+2}^2 - U_{n+1}^2 - 2U_{n+2}U_{n+3} - 2U_{n+1}U_{n+2} + 1)$.

(c) $\sum_{k=0}^{n} U_{k+2}U_k = \frac{1}{4}(-U_{n+3}^2 - U_{n+1}^2 - 2U_{n+1}U_{n+3} + 1)$.

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = -1, S_2 = -1$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Tribonacci-Lucas numbers.

Corollary 8.2.
For $n \geq 0$, co-Tribonacci-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} S_k^2 = \frac{1}{4}(S_{n+3}^2 - 3S_{n+1}^2 - 2S_{n+1}S_{n+3} - 4S_{n+2}S_{n+1} + 8)$.

(b) $\sum_{k=0}^{n} S_{k+1}S_k = \frac{1}{4}(-S_{n+3}^2 - 2S_{n+2}^2 - S_{n+1}^2 - 2S_{n+2}S_{n+3} - 2S_{n+1}S_{n+2} + 8)$.

(c) $\sum_{k=0}^{n} S_{k+2}S_k = \frac{1}{4}(-S_{n+3}^2 - S_{n+1}^2 - 2S_{n+1}S_{n+3} + 4)$.

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n$ of the sequences $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$.

Theorem 8.2.
Assume that $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2|^{-2} = |\theta_3|^{-2} = |\theta_2\theta_3|^{-1} \simeq 0.543689$. *Then the ordinary generating functions of the sequences* $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$ *are given as follows:*

(a) $\sum_{n=0}^{\infty} Y_n^2 z^n = \frac{1}{-z^6 + 2z^5 + 3z^4 + 6z^3 - z^2 - 1}(z^5(Y_2 + Y_1 + Y_0)^2 + z^4(-Y_2^2 + 2Y_1^2 + 2Y_0^2 + 4Y_0Y_1) + z^3(-Y_2^2 - 2Y_1^2 + 5Y_0^2 - 2Y_1Y_2 + 2Y_0Y_2 + 2Y_0Y_1) - z^2(Y_2^2 + Y_0^2) - zY_1^2 - Y_0^2)$.

(b) $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n = \frac{1}{-z^6 + 2z^5 + 3z^4 + 6z^3 - z^2 - 1}(z^5Y_0(Y_2 + Y_1 + Y_0) + z^4(2Y_1 + Y_2)(Y_1 + Y_2 + Y_0) + z^3(2Y_1^2 + Y_0^2 + Y_1Y_2 - Y_0Y_2 + 3Y_0Y_1) + z^2(Y_2 + Y_1)(Y_2 - Y_0) - zY_1Y_2 - Y_0Y_1)$.

(c) $\sum\limits_{n=0}^{\infty} Y_{n+2}Y_n z^n = \dfrac{1}{-z^6+2z^5+3z^4+6z^3-z^2-1}(z^5 Y_1(Y_2+Y_1+Y_0)+z^4(-2Y_1^2+2Y_0^2-2Y_1Y_2+Y_0Y_2)+2z^3Y_2(Y_2+Y_1+2Y_0)-2z^2Y_1Y_2+zY_1(Y_2+Y_1-Y_0)-Y_0Y_2)$.

Proof. Set $W_n = Y_n$ and $r=-1$, $s=-1$, $t=1$ in [15], Theorem 3.1 or in [16], Theorem 3.1. □
Now, we consider special cases of the last Theorem.

Corollary 8.3.
Assume that $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2|^{-2} = |\theta_3|^{-2} = |\theta_2\theta_3|^{-1} \simeq 0.543689$. *The ordinary generating functions of the sequences* $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *are given as follows:*

(a)

$$\sum_{n=0}^{\infty} U_n^2 z^n = \frac{z^4-z^3-z^2-z}{-z^6+2z^5+3z^4+6z^3-z^2-1},$$
$$\sum_{n=0}^{\infty} S_n^2 z^n = \frac{z^5+7z^4+28z^3-10z^2-z-9}{-z^6+2z^5+3z^4+6z^3-z^2-1}.$$

(b)

$$\sum_{n=0}^{\infty} U_{n+1}U_n z^n = \frac{z^3+z}{-z^6+2z^5+3z^4+6z^3-z^2-1},$$
$$\sum_{n=0}^{\infty} S_{n+1}S_n z^n = \frac{3z^5-3z^4+6z^3+8z^2-z+3}{-z^6+2z^5+3z^4+6z^3-z^2-1}.$$

(c)

$$\sum_{n=0}^{\infty} U_{n+2}U_n z^n = \frac{2z^2}{-z^6+2z^5+3z^4+6z^3-z^2-1},$$
$$\sum_{n=0}^{\infty} S_{n+2}S_n z^n = \frac{-z^5+11z^4-8z^3-2z^2+5z+3}{-z^6+2z^5+3z^4+6z^3-z^2-1}.$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 8.4.
Some infinite sums of $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *are given as follows:*

(a) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{U_n^2}{2^n} = \frac{52}{17},$$
$$\sum_{n=0}^{\infty} \frac{S_n^2}{2^n} = \frac{514}{17}.$$

(b) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{U_{n+1}U_n}{2^n} = -\frac{40}{17},$$
$$\sum_{n=0}^{\infty} \frac{S_{n+1}S_n}{2^n} = -\frac{330}{17}.$$

(c) $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{U_{n+2}U_n}{2^n} = -\frac{32}{17},$$
$$\sum_{n=0}^{\infty} \frac{S_{n+2}S_n}{2^n} = -\frac{298}{17}.$$

9. Generalized co-Tribonacci Numbers by Matrix Methods

In this section, we present matrix representations of the sequences Y_n, U_n and S_n. We also introduce Simson matrix and investigate its properties.

9.1. Matrix Representations of the Sequences Y_n, U_n and S_n

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Some properties of matrix A^n can be given as

$$A^n = rA^{n-1} + sA^{n-2} + tA^{n-3},$$
$$A^{n+m} = A^n A^m = A^m A^n,$$

for all integers m and n. Note that we have the following formulas:

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \\ Y_n \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{n+1} \\ Y_n \\ Y_{n-1} \end{pmatrix},$$

and

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \\ Y_n \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_2 \\ Y_1 \\ Y_0 \end{pmatrix},$$

and

$$\begin{pmatrix} U_{n+2} \\ U_{n+1} \\ U_n \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} U_{n+1} \\ U_n \\ U_{n-1} \end{pmatrix}.$$

We also define

$$B_n = \begin{pmatrix} U_{n+1} & -U_n + U_{n-1} & U_n \\ U_n & -U_{n-1} + U_{n-2} & U_{n-1} \\ U_{n-1} & -U_{n-2} + U_{n-3} & U_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} Y_{n+1} & -Y_n + Y_{n-1} & Y_n \\ Y_n & -Y_{n-1} + Y_{n-2} & Y_{n-1} \\ Y_{n-1} & -Y_{n-2} + Y_{n-3} & Y_{n-2} \end{pmatrix}.$$

Theorem 9.1.
For all integers m, n, we have the following properties:

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} U_{n+1} & -U_n + U_{n-1} & U_n \\ U_n & -U_{n-1} + U_{n-2} & U_{n-1} \\ U_{n-1} & -U_{n-2} + U_{n-3} & U_{n-2} \end{pmatrix}.$$

(b) $D_1 A^n = A^n D_1$.

(c) $D_{n+m} = D_n B_m = B_m D_n$, *i.e.,*

$$\begin{pmatrix} Y_{n+m+1} & -Y_{n+m} + Y_{n+m-1} & Y_{n+m} \\ Y_{n+m} & -Y_{n+m-1} + Y_{n+m-2} & Y_{n+m-1} \\ Y_{n+m-1} & -Y_{n+m-2} + Y_{n+m-3} & Y_{n+m-2} \end{pmatrix}$$
$$= \begin{pmatrix} Y_{n+1} & -Y_n + Y_{n-1} & tY_n \\ Y_n & -Y_{n-1} + Y_{n-2} & tY_{n-1} \\ Y_{n-1} & -Y_{n-2} + Y_{n-3} & tY_{n-2} \end{pmatrix} \begin{pmatrix} U_{m+1} & -U_m + U_{m-1} & U_m \\ U_m & -U_{m-1} + U_{m-2} & U_{m-1} \\ U_{m-1} & -U_{m-2} + U_{m-3} & U_{m-2} \end{pmatrix}$$
$$= \begin{pmatrix} U_{m+1} & -U_m + U_{m-1} & U_m \\ U_m & -U_{m-1} + U_{m-2} & U_{m-1} \\ U_{m-1} & -U_{m-2} + U_{m-3} & U_{m-2} \end{pmatrix} \begin{pmatrix} Y_{n+1} & -Y_n + Y_{n-1} & Y_n \\ Y_n & -Y_{n-1} + Y_{n-2} & Y_{n-1} \\ Y_{n-1} & -Y_{n-2} + Y_{n-3} & Y_{n-2} \end{pmatrix}.$$

(d)

$$A^n = U_{n-1}A^2 + (-U_{n-2} + U_{n-3})A + U_{n-2}I,$$

i.e.,

$A^n = (U_{n+2} + U_{n+1} + U_n)A^2 + (U_{n+2} + U_{n+1} + 2U_n)A + (U_{n+2} + 2U_{n+1} + 2U_n)I,$

that is,

$A^n = U_{n+2}(A^2 + A + I) + U_{n+1}(A^2 + A + 2I) + U_n(A^2 + 2A + 2I),$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Set $W_n = Y_n$, $G_n = U_n$, $H_n = S_n$ and $r = -1, s = -1, t = 1$ in [14], Theorem 51. □

Next, we present matrix formulas for the generalized co-Tribonacci and co-Tribonacci-Lucas numbers..

Corollary 9.1.

For all integers n, we have the following formulas for generalized co-Tribonacci numbers and co-Tribonacci-Lucas numbers.

(a) *Generalized co-Tribonacci numbers.*

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_Y(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$a_{11} = (-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+3} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_{n+2} + (Y_1^2 - Y_0Y_2)Y_{n+1},$

$a_{21} = (-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+2} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n,$

$a_{31} = (-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1},$

$a_{12} = -((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+2} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n) + ((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1}),$

$a_{22} = -((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1}) + ((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_n + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_{n-1} + (Y_1^2 - Y_0Y_2)Y_{n-2}),$

$a_{32} = -((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_n + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_{n-1} + (Y_1^2 - Y_0Y_2)Y_{n-2}) + ((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n-1} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_{n-2} + (Y_1^2 - Y_0Y_2)Y_{n-3}),$

$a_{13} = ((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+2} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n),$

$a_{23} = ((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1}),$

$a_{33} = ((-Y_1^2 + Y_0^2 - Y_1Y_2 - Y_0Y_1)Y_n + (Y_2^2 + Y_1Y_2 + Y_0Y_2 - Y_0Y_1)Y_{n-1} + (Y_1^2 - Y_0Y_2)Y_{n-2}),$

and

$\Lambda_Y(0) = Y_2^3 + 2Y_1^3 + Y_0^3 + 2Y_1Y_2^2 + Y_0Y_2^2 + 2Y_2Y_1^2 - Y_0^2Y_2 - 2Y_0^2Y_1 - 2Y_2Y_1Y_0.$

(b) *co-Tribonacci-Lucas numbers.*

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{44} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$b_{11} = 10S_{n+3} + 2S_{n+2} + 4S_{n+1}$

$b_{21} = 10S_{n+2} + 2S_{n+1} + 4S_n$

$b_{31} = 10S_{n+1} + 2S_n + 4S_{n-1}$

$b_{12} = -10S_{n+2} + 8S_{n+1} - 2S_n + 4S_{n-1}$

$b_{22} = -10S_{n+1} + 8S_n - 2S_{n-1} + 4S_{n-2}$

$b_{32} = -10S_n + 8S_{n-1} - 2S_{n-2} + 4S_{n-3}$

$b_{13} = 10S_{n+2} + 2S_{n+1} + 4S_n$

$b_{23} = 10S_{n+1} + 2S_n + 4S_{n-1}$

$b_{33} = 10S_n + 2S_{n-1} + 4S_{n-2}$

Proof. Set $W_n = Y_n$, $r = -1$, $s = -1$, $t = 1$ and then $H_n = S_n$ in [14], Corollary 52.. □

Note that, a_{12}, a_{22}, a_{32} and b_{12}, b_{22}, b_{32} can be written in the following form:

$a_{12} = (-Y_2^2 - 2Y_1^2 + 2Y_0^2 - 3Y_1Y_2 - Y_0Y_2 - Y_0Y_1)Y_{n+1} + (Y_2^2 - 2Y_1^2 + Y_0^2 + 2Y_0Y_2 - 2Y_0Y_1)Y_n + (2Y_1^2 - Y_0^2 + Y_1Y_2 - Y_0Y_2 + Y_0Y_1)Y_{n-1}$

$a_{22} = (-Y_2^2 - 2Y_1^2 + 2Y_0^2 - 3Y_1Y_2 - Y_0Y_2 - Y_0Y_1)Y_n + (Y_2^2 - 2Y_1^2 + Y_0^2 + 2Y_0Y_2 - 2Y_0Y_1)Y_{n-1} + (2Y_1^2 - Y_0^2 + Y_1Y_2 - Y_0Y_2 + Y_0Y_1)Y_{n-2}$

$a_{32} = (-Y_2^2 - 2Y_1^2 + 2Y_0^2 - 3Y_1Y_2 - Y_0Y_2 - Y_0Y_1)Y_{n-1} + (Y_2^2 - 2Y_1^2 + Y_0^2 + 2Y_0Y_2 - 2Y_0Y_1)Y_{n-2} + (2Y_1^2 - Y_0^2 + Y_1Y_2 - Y_0Y_2 + Y_0Y_1)Y_{n-3}$

and

$b_{12} = 18S_{n+1} + 8S_n - 6S_{n-1}$

$b_{22} = 18S_n + 8S_{n-1} - 6S_{n-2}$

$b_{32} = 18S_{n-1} + 8S_{n-2} - 6S_{n-3}$

Now, we present an identity for Y_{n+m}.

Theorem 9.2.
(Honsberger's Identity) For all integers m and n, we have

$$\begin{aligned} Y_{n+m} &= Y_nU_{m+1} + Y_{n-1}(-U_m + U_{m-1}) + Y_{n-2}U_m \\ &= Y_nU_{m+1} + (-Y_{n-1} + Y_{n-2})U_m + Y_{n-1}U_{m-1}. \end{aligned}$$

Proof. Set $W_n = Y_n$, $G_n = U_n$ and $r = -1$, $s = -1$, $t = 1$ in [14], Theorem 53. □

As special cases of the last Theorem, we have the following corollary.

Corollary 9.2.
For all integers m, n, we have the following properties:

$$\begin{aligned} U_{n+m} &= U_nU_{m+1} + U_{n-1}(-U_m + U_{m-1}) + U_{n-2}U_m, \\ S_{n+m} &= S_nU_{m+1} + S_{n-1}(-U_m + U_{m-1}) + S_{n-2}U_m. \end{aligned}$$

Next, we present identies for Y_{mn+j} and its special cases.

Corollary 9.3.
For all integers m, n, j, we have the following properties:

$$\begin{aligned} Y_{mn+j} &= U_{mn-1}Y_{j+2} + (-U_{mn-2} + U_{mn-3})Y_{j+1} + U_{mn-2}Y_j, \\ U_{mn+j} &= U_{mn-1}U_{j+2} + (-U_{mn-2} + U_{mn-3})U_{j+1} + U_{mn-2}U_j, \\ S_{mn+j} &= U_{mn-1}S_{j+2} + (-U_{mn-2} + U_{mn-3})S_{j+1} + U_{mn-2}S_j. \end{aligned}$$

Proof. Set $r = -1$, $s = -1$, $t = 1$ and $W_n = Y_n$, then take $Y_n = T_n$ and $Y_n = K_n$, respectively, in [14], Corollary 55. □

9.2. Simson Matrix and its Properties

For $n \in \mathbb{Z}$, we define

$$f_Y(n) = \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence Y_n. Similarly, as special cases of Y_n, Simson matrices of the sequences U_n and S_n are

$$f_U(n) = \begin{pmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{pmatrix} \text{ and } f_S(n) = \begin{pmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{pmatrix}$$

,

respectively.

Lemma 9.1.
For all integers n, m and j, the followings hold.

(a) $f_Y(n) = -f_Y(n-1) - f_Y(n-2) + f_Y(n-3)$.

(b) $f_Y(n) = Af_Y(n-1)$ *and* $f_Y(n) = A^n f_Y(0)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \\ Y_{n-1} & Y_{n-2} & Y_{n-3} \end{pmatrix}$$

and

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{pmatrix}.$$

(c) $f_Y(n+m) = A^n f_Y(m)$ *and* $f_Y(n+m) = A^m f_Y(n)$ *i.e.,*

$$\begin{pmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{pmatrix},$$

and

$$\begin{pmatrix} Y_{m+n+2} & Y_{m+n+1} & Y_{m+n} \\ Y_{m+n+1} & Y_{m+n} & Y_{m+n-1} \\ Y_{m+n} & Y_{m+n-1} & Y_{m+n-2} \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix},$$

and $f_Y(n) = A^m f_Y(n-m)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{pmatrix}.$$

Proof. Set $W_n = Y_n$ and $r = -1$, $s = -1$, $t = 1$ in [14], Lemma 56. □

Taking the determinant of both sides of the identities given in the last Lemma, we obtain the following Theorem.

Theorem 9.3.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_Y(n+m)) = \det(f_Y(m)) \text{ and } \det(f_Y(n)) = \det(f_Y(n-m)),$$

i.e.,

$$\begin{vmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{vmatrix} = \begin{vmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = \begin{vmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{vmatrix}.$$

(b) *(see theorem 5.1) Simson's (or Cassini's) Identity:*

$$\det(f_Y(n)) = \det(f_Y(0)),$$

i.e.,

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix}.$$

Proof. Set $W_n = Y_n$ and $r = -1$, $s = -1$, $t = 1$ in [14], Theorem 57.]. □

From the last Theorem, we have the following Corollary which gives determinantal formulas of co-Tribonacci numbers (take $Y_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = -1,$).

Corollary 9.4.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_U(n+m)) = \det(f_U(m)) \ \ and \ \ \det(f_U(n)) = \det(f_U(n-m)),$$

i.e.,

$$\begin{vmatrix} U_{n+m+2} & U_{n+m+1} & U_{n+m} \\ U_{n+m+1} & U_{n+m} & U_{n+m-1} \\ U_{n+m} & U_{n+m-1} & U_{n+m-2} \end{vmatrix} = \begin{vmatrix} U_{m+2} & U_{m+1} & U_m \\ U_{m+1} & U_m & U_{m-1} \\ U_m & U_{m-1} & U_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = \begin{vmatrix} U_{n-m+2} & U_{n-m+1} & U_{n-m} \\ U_{n-m+1} & U_{n-m} & U_{n-m-1} \\ U_{n-m} & U_{n-m-1} & U_{n-m-2} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_U(n)) = \det(f_U(0)),$$

i.e.,

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = -1.$$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = -1, S_2 = -1$ in the last Theorem, we have the following Corollary which gives determinantal formulas of co-Tribonacci-Lucas numbers.

Corollary 9.5.
For all integers n and m, the following identities hold.

(a) *Catalan's Identity:*

$$\det(f_S(n+m)) = \det(f_S(m)) \ \ and \ \ \det(f_S(n)) = \det(f_S(n-m))$$

i.e.,

$$\begin{vmatrix} S_{n+m+2} & S_{n+m+1} & S_{n+m} \\ S_{n+m+1} & S_{n+m} & S_{n+m-1} \\ S_{n+m} & S_{n+m-1} & S_{n+m-2} \end{vmatrix} = \begin{vmatrix} S_{m+2} & S_{m+1} & S_m \\ S_{m+1} & S_m & S_{m-1} \\ S_m & S_{m-1} & S_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = \begin{vmatrix} S_{n-m+2} & S_{n-m+1} & S_{n-m} \\ S_{n-m+1} & S_{n-m} & S_{n-m-1} \\ S_{n-m} & S_{n-m-1} & S_{n-m-2} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_S(n)) = \det(f_S(0)),$$

i.e.,

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = -44.$$

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