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# Sums and Generating Functions of Special Cases of Generalized Tetranacci Polynomials

**Research Article**

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**Abstract:** In this paper, we present special cases of sum formulas $\sum_{k=0}^{n} z^k W_{mk+j}$ and generating functions $\sum_{n=0}^{\infty} W_{mn+j} z^n$ for special cases of generalized Tetranacci polynomials, namely, generalized Tetranacci numbers, generalized fourth order Pell numbers, generalized fourth order Jacobsthal numbers, generalized four primes numbers, generalized Tridovan numbers, generalized Richard numbers, generalized Olivier numbers, generalized Blaise numbers, generalized Friedrich numbers, generalized Pierre numbers, generalized Pandita numbers, generalized Adrien numbers, Moreover, we evaluate the infinite sums of special cases of generalized Tribonacci polynomials.

**MSC:** 11B37 • 11B39 • 11B83

**Keywords:** Tetranacci numbers • Fourth order Pell numbers • Fourth order Jacobsthal numbers • Four primes numbers • Tridovan numbers • Richard numbers • Olivier numbers • Blaise numbers • Friedrich numbers • Pierre numbers • Pandita numbers • Adrien numbers



## 1. Introduction and Preliminaries

The generalized Tetranacci polynomials (or generalized $(r(x), s(x), t(x), u(x))-$Tetranacci polynomials or $x$-Tetranacci polynomials or generalized $(r(x), s(x), t(x), u(x))$-polynomials or 4-step Fibonacci polynomials)

$$\{W_n(W_0(x), W_1(x), W_2(x), W_3(x); r(x), s(x), t(x), u(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} &W_n(x) = r(x)W_{n-1}(x) + s(x)W_{n-2}(x) + t(x)W_{n-3}(x) + u(x)W_{n-4}(x), \\ &W_0(x) = a(x), W_1(x) = b(x), W_2(x) = c(x), W_3(x) = d(x) \quad n \geq 4 \end{aligned} \tag{1}$$

where $W_0(x), W_1(x), W_2(x), W_3(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$, $s(x)$, $t(x)$ and $u(x)$ are polynomials with real coefficients and $u(x) \neq 0$. For more information on generalized Tetranacci polynomials, see Soykan [16]. See also Soykan [4] for generalized Tetranacci numbers.

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**Remark 1.1.**
For the sake of simplicity throughout the rest of the paper, we use

$$W_n, r, s, t, u, W_0, W_1, W_2, W_3$$

instead of

$$W_n(x), r(x), s(x), t(x), u(x), W_0(x), W_1(x), W_2(x), W_3(x)$$

respectively, unless otherwise stated.

In the next two section, we present sums and generating functions of special cases of generalized Tetranacci polynomials. Before this, firstly, in the next two subsections, we present sums and generating functions formulas which we will use them extensively.

## 1.1. Sums

The following theorem presents some linear summing formulas of generalized Tetranacci polynomials with positive subscripts.

**Theorem 1.1.**
*[16], Theorem 7.1. Let $z$ be a real or complex number (in fact $z$ is a real or complex valued function in $x$). For $n \geq 0$ we have the following formulas:*

**(a) (i)** *If $uz^4 + tz^3 + sz^2 + rz - 1 \neq 0$, then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Theta_{1W}(z)}{uz^4 + tz^3 + sz^2 + rz - 1} \tag{2}$$

*where*
$\Theta_{1W}(z) = z^{n+3}W_{n+3} + (z^{n+2} - rz^{n+3})W_{n+2} + (-sz^{n+3} - rz^{n+2} + z^{n+1})W_{n+1} + uz^{n+4}W_n - z^3W_3 + (rz^3 - z^2)W_2 + (sz^3 + rz^2 - z)W_1 + (tz^3 + sz^2 + rz - 1)W_0.$

**(ii)** *If $uz^4 + tz^3 + sz^2 + rz - 1 = (z-a)f(z) = 0$ for some $a \in \mathbb{C}$ (or $\mathbb{R}$, in fact $a$ is a real or complex valued function) and a function (polynomial) $f$ in $z$ with $f(a) \neq 0$ then, for $z = a$, we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\frac{d\Theta_{1W}(z)}{dz}}{\frac{d(uz^4+tz^3+sz^2+rz-1)}{dz}} = \frac{\frac{d\Theta_{1W}(z)}{dz}}{4uz^3 + 3tz^2 + 2sz + r}$$

*where*
$\frac{d\Theta_{1W}(z)}{dz} = (n+3)z^{n+2}W_{n+3} + ((n+2)z^{n+1} - r(n+3)z^{n+2})W_{n+2} + (-s(n+3)z^{n+2} - r(n+2)z^{n+1} + (n+1)z^n)W_{n+1} + u(n+4)z^{n+3}W_n - 3z^2W_3 + (3rz^2 - 2z)W_2 + (3sz^2 + 2rz - 1)W_1 + (3tz^2 + 2sz + r)W_0.$

**(iii)** *If $uz^4 + tz^3 + sz^2 + rz - 1 = (z-a)^2 f(z) = 0$ for some $a \in \mathbb{C}$ and a function (polynomial) $f$ in $z$ with $f(a) \neq 0$ then, for $z = a$, we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\frac{d^2\Theta_{1W}(z)}{dz^2}}{\frac{d^2}{dz^2}(uz^4 + tz^3 + sz^2 + rz - 1)} = \frac{\frac{d^2\Theta_{1W}(z)}{dz^2}}{12uz^2 + 6tz + 2s}$$

*where*
$\frac{d^2\Theta_{1W}(z)}{dz^2} = (n+2)(n+3)z^{n+1}W_{n+3} + ((n+1)(n+2)z^n - r(n+2)(n+3)z^{n+1})W_{n+2} + (-s(n+2)(n+3)z^{n+1} - r(n+1)(n+2)z^n + n(n+1)z^{n-1})W_{n+1} + u(n+3)(n+4)z^{n+2}W_n - 6zW_3 + (6rz - 2)W_2 + (6sz + 2r)W_1 + (6tz + 2s)W_0.$

**(b) (i)** *If $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 \neq 0$ then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Theta_{2W}(z)}{-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1}$$

*where*
$\Theta_{2W}(z) = (-uz^{n+3} - sz^{n+2} + z^{n+1})W_{2n+2} + (ruz^{n+3} + (t + rs)z^{n+2})W_{2n+1} + (-u^2z^{n+4} + (t^2 - su)z^{n+3} + (rt + u)z^{n+2})W_{2n} + u(tz^{n+3} + rz^{n+2})W_{2n-1} - (tz^3 + rz^2)W_3 + ((rt + u)z^3 + (r^2 + s)z^2 - z)W_2 + ((st - ru)z^3 - tz^2)W_1 + ((t^2 - us)z^3 + (2rt + u - s^2)z^2 + (r^2 + 2s)z - 1)W_0.$

**(ii)** *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* $f$ *in* $z$ *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\frac{d\Theta_{2W}(z)}{dz}}{\frac{d(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}{dz}} = \frac{\frac{d\Theta_{2W}(z)}{dz}}{-4u^2z^3+3(t^2-2su)z^2+2(2rt+2u-s^2)z+r^2+2s}$$

*where*

$\frac{d\Theta_{2W}(z)}{dz}=(-u(n+3)z^{n+2}-s(n+2)z^{n+1}+(n+1)z^n)W_{2n+2}+(ru(n+3)z^{n+2}+(n+2)(t+rs)z^{n+1})W_{2n+1}+(-u^2(n+4)z^{n+3}+(t^2-su)(n+3)z^{n+2}+(rt+u)(n+2)z^{n+1})W_{2n}+u(t(n+3)z^{n+2}+r(n+2)z^{n+1})W_{2n-1}-(3tz^2+2rz)W_3+(3(rt+u)z^2+2(r^2+s)z-1)W_2+(3(st-ru)z^2-2tz)W_1+(3(t^2-us)z^2+2(2rt+u-s^2)z+(r^2+2s))W_0$.

**(iii)** *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* $f$ *in* $z$ *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\frac{d^2\Theta_{2W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)} = \frac{\frac{d^2\Theta_{2W}(z)}{dz^2}}{-12u^2z^2+6(t^2-2su)z+4rt+4u-2s^2}$$

*where*

$\frac{d^2\Theta_{2W}(z)}{dz^2}=(-u(n+2)(n+3)z^{n+1}-s(n+1)(n+2)z^n+n(n+1)z^{n-1})W_{2n+2}+(ru(n+2)(n+3)z^{n+1}+(n+1)(n+2)(t+rs)z^n)W_{2n+1}+(-u^2(n+3)(n+4)z^{n+2}+(t^2-su)(n+2)(n+3)z^{n+1}+(rt+u)(n+1)(n+2)z^n)W_{2n}+u(t(n+2)(n+3)z^{n+1}+r(n+1)(n+2)z^n)W_{2n-1}-(6tz+2r)W_3+(6(rt+u)z+2(r^2+s))W_2+(6(st-ru)z-2t)W_1+(6(t^2-us)z+2(2rt+u-s^2))W_0$.

**(c)** **(i)** *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Theta_{3W}(z)}{-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1}$$

*where*

$\Theta_{3W}(z)=(tz^{n+2}+rz^{n+1})W_{2n+2}+(-u^2z^{n+4}+(t^2-2su)z^{n+3}+(rt+u-s^2)z^{n+2}+sz^{n+1})W_{2n+1}+((ru-st)z^{n+2}+tz^{n+1})W_{2n}-u(uz^{n+3}+sz^{n+2}-z^{n+1})W_{2n-1}+(uz^3+sz^2-z)W_3+(-ruz^3-(t+rs)z^2)W_2+(-suz^3+(rt+u-s^2)z^2+(r^2+2s)z-1)W_1-u(tz^3+rz^2)W_0$.

**(ii)** *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function* $f$ *(polynomial) in* $z$ *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\frac{d\Theta_{3W}(z)}{dz}}{\frac{d(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)}{dz}} = \frac{\frac{d\Theta_{3W}(z)}{dz}}{-4u^2z^3+3(t^2-2su)z^2+2(2rt+2u-s^2)z+r^2+2s}$$

*where*

$\frac{d\Theta_{3W}(z)}{dz}=(t(n+2)z^{n+1}+r(n+1)z^n)W_{2n+2}+(-u^2(n+4)z^{n+3}+(n+3)(t^2-2su)z^{n+2}+(n+2)(rt+u-s^2)z^{n+1}+s(n+1)z^n)W_{2n+1}+((n+2)(ru-st)z^{n+1}+t(n+1)z^n)W_{2n}-u(u(n+3)z^{n+2}+s(n+2)z^{n+1}-(n+1)z^n)W_{2n-1}+(3uz^2+2sz-1)W_3+(-3ruz^2-2(t+rs)z)W_2+(-3suz^2+2(rt+u-s^2)z+(r^2+2s))W_1-u(3tz^2+2rz)W_0$.

**(iii)** *If* $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function* $f$ *(polynomial) in* $z$ *with* $f(a)\neq 0$ *then, for* $z=a$*, we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\frac{d^2\Theta_{3W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1)} = \frac{\frac{d^2\Theta_{3W}(z)}{dz^2}}{-12u^2z^2+6(t^2-2su)z+4rt+4u-2s^2}$$

*where*

$\frac{d^2\Theta_{3W}(z)}{dz^2}=(t(n+1)(n+2)z^n+rn(n+1)z^{n-1})W_{2n+2}+(-u^2(n+3)(n+4)z^{n+2}+(n+2)(n+3)(t^2-2su)z^{n+1}+(n+1)(n+2)(rt+u-s^2)z^n+sn(n+1)z^{n-1})W_{2n+1}+((n+1)(n+2)(ru-st)z^n+tn(n+1)z^{n-1})W_{2n}-u(u(n+2)(n+3)z^{n+1}+s(n+1)(n+2)z^n-n(n+1)z^{n-1})W_{2n-1}+(6uz+2s)W_3+(-6ruz-2(t+rs))W_2+(-6suz+2(rt+u-s^2))W_1-u(6tz+2r)W_0$.

The following theorem present some linear summing formulas of generalized Tetranacci polynomials with negative subscripts.

**Theorem 1.2.**
*[16], Theorem 8.1. Let $z$ be a real or complex number (in fact $z$ is a real or complex valued function in $x$). For $n \geq 1$ we have the following formulas:*

**(a)** **(i)** *If $-z^4 + rz^3 + sz^2 + tz + u \neq 0$, then*

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{\Theta_{4W}(z)}{-z^4 + rz^3 + sz^2 + tz + u} \tag{3}$$

*where*
$\Theta_{4W}(z) = -z^{n+1}W_{-n+3} + (-z^{n+2} + rz^{n+1})W_{-n+2} + (-z^{n+3} + rz^{n+2} + sz^{n+1})W_{-n+1} + (-z^{n+4} + rz^{n+3} + sz^{n+2} + tz^{n+1})W_{-n} + zW_3 + (z^2 - rz)W_2 + (z^3 - rz^2 - sz)W_1 + (z^4 - rz^3 - sz^2 - tz)W_0.$

**(ii)** *If $-z^4 + rz^3 + sz^2 + tz + u = (z-a)f(z) = 0$ for some $a \in \mathbb{C}$ (or $\mathbb{R}$, in fact $a$ is a real or complex valued function) and a function (polynomial) $f$ in $z$ with $f(a) \neq 0$ then, for $z = a$, we get*

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{\frac{d\Theta_{4W}(z)}{dz}}{\frac{d(-z^4+rz^3+sz^2+tz+u)}{dz}} = \frac{\frac{d\Theta_{4W}(z)}{dz}}{-4z^3 + 3rz^2 + 2sz + t}$$

*where*
$\frac{d\Theta_{4W}(z)}{dz} = -(n+1)z^n W_{-n+3} + (-(n+2)z^{n+1} + r(n+1)z^n)W_{-n+2} + (-(n+3)z^{n+2} + r(n+2)z^{n+1} + s(n+1)z^n)W_{-n+1} + (-(n+4)z^{n+3} + r(n+3)z^{n+2} + s(n+2)z^{n+1} + t(n+1)z^n)W_{-n} + W_3 + (2z - r)W_2 + (3z^2 - 2rz - s)W_1 + (4z^3 - 3rz^2 - 2sz - t)W_0.$

**(iii)** *If $-z^4 + rz^3 + sz^2 + tz + u = (z-a)^2 f(z) = 0$ for some $a \in \mathbb{C}$ and a function $f$ (polynomial) in $z$ with $f(a) \neq 0$ then, for $z = a$, we get*

$$\sum_{k=1}^{n} z^k W_{-k} = \frac{\frac{d^2\Theta_{4W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-z^4 + rz^3 + sz^2 + tz + u)} = \frac{\frac{d^2\Theta_{4W}(z)}{dz^2}}{-12z^2 + 6rz + 2s}$$

*where*
$\frac{d^2\Theta_{4W}(z)}{dz^2} = -n(n+1)z^{n-1}W_{-n+3} + (-(n+1)(n+2)z^n + rn(n+1)z^{n-1})W_{-n+2} + (-(n+2)(n+3)z^{n+1} + r(n+1)(n+2)z^n + sn(n+1)z^{n-1})W_{-n+1} + (-(n+3)(n+4)z^{n+2} + r(n+2)(n+3)z^{n+1} + s(n+1)(n+2)z^n + tn(n+1)z^{n-1})W_{-n} + 2W_2 + (6z - 2r)W_1 + (12z^2 - 6rz - 2s)W_0.$

**(b)** **(i)** *If $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 \neq 0$ then*

$$\sum_{k=1}^{n} z^k W_{-2k} = \frac{\Theta_{5W}(z)}{-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2}$$

*where*
$\Theta_{5W}(z) = (-z^{n+3} + sz^{n+2} + uz^{n+1})W_{-2n+2} - ((t + rs)z^{n+2} + ruz^{n+1})W_{-2n+1} + (-z^{n+4} + (2s + r^2)z^{n+3} + (rt + u - s^2)z^{n+2} - suz^{n+1})W_{-2n} - u(rz^{n+2} + tz^{n+1})W_{-2n-1} + (rz^2 + tz)W_3 + (z^3 - (s + r^2)z^2 - (u + rt)z)W_2 + (tz^2 + (ru - st)z)W_1 + (z^4 - (2s + r^2)z^3 + (s^2 - 2rt - u)z^2 + (us - t^2)z)W_0.$

**(ii)** *If $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 = (z-a)f(z) = 0$ for some $a \in \mathbb{C}$ (or $\mathbb{R}$, in fact $a$ is a real or complex valued function) and a function (polynomial) $f$ in $z$ with $f(a) \neq 0$ then, for $z = a$, we get*

$$\sum_{k=1}^{n} z^k W_{-2k} = \frac{\frac{d\Theta_{5W}(z)}{dz}}{\frac{d(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)}{dz}} = \frac{\frac{d\Theta_{5W}(z)}{dz}}{-4z^3 + 3(r^2 + 2s)z^2 + 2(2u + 2rt - s^2)z + t^2 - 2us}$$

$\frac{d\Theta_{5W}(z)}{dz} = (-(n+3)z^{n+2} + s(n+2)z^{n+1} + u(n+1)z^n)W_{-2n+2} - ((t+rs)(n+2)z^{n+1} + ru(n+1)z^n)W_{-2n+1} + (-(n+4)z^{n+3} + (2s+r^2)(n+3)z^{n+2} + (rt+u-s^2)(n+2)z^{n+1} - su(n+1)z^n)W_{-2n} - u(r(n+2)z^{n+1} + t(n+1)z^n)W_{-2n-1} + (2rz+t)W_3 + (3z^2 - 2(s+r^2)z - (u+rt))W_2 + (2tz + (ru-st))W_1 + (4z^3 - 3(2s+r^2)z^2 + 2(s^2 - 2rt - u)z + (us - t^2))W_0.$

**(iii)** *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* $f$ *in* $z$ *with* $f(a)\neq 0$ *then, for* $z=a$, *we get*

$$\sum_{k=1}^{n} z^k W_{-2k} = \frac{\frac{d^2\Theta_{5W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)} = \frac{\frac{d^2\Theta_{5W}(z)}{dz^2}}{-12z^2+6(r^2+2s)z-2s^2+4tr+4u}$$

*where*
$\frac{d^2\Theta_{5W}(z)}{dz^2}=(-(n+2)(n+3)z^{n+1}+s(n+1)(n+2)z^n+un(n+1)z^{n-1})W_{-2n+2}-((t+rs)(n+1)(n+2)z^n+run(n+1)z^{n-1})W_{-2n+1}+(-(n+3)(n+4)z^{n+2}+(2s+r^2)(n+2)(n+3)z^{n+1}+(rt+u-s^2)(n+1)(n+2)z^n-sun(n+1)z^{n-1})W_{-2n}-u(r(n+1)(n+2)z^n+tn(n+1)z^{n-1})W_{-2n-1}+2rW_3+(6z-2(s+r^2))W_2+2tW_1+(12z^2-6(2s+r^2)z+2(s^2-2rt-u))W_0.$

**(c)** **(i)** *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2\neq 0$ *then*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\Theta_{6W}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2}$$

*where*
$\Theta_{6W}(z)=-(rz^{n+3}+tz^{n+2})W_{-2n+2}+(-z^{n+4}+(r^2+s)z^{n+3}+(u+rt)z^{n+2})W_{-2n+1}-(tz^{n+3}+(ru-st)z^{n+2})W_{-2n}+u(-z^{n+3}+sz^{n+2}+uz^{n+1})W_{-2n-1}+(z^3-sz^2-uz)W_3+((t+rs)z^2+ruz)W_2+(z^4-(r^2+2s)z^3+(s^2-tr-u)z^2+usz)W_1+u(rz^2+tz)W_0.$

**(ii)** *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)f(z)=0$ *for some* $a\in\mathbb{C}$ *(or* $\mathbb{R}$*, in fact* $a$ *is a real or complex valued function) and a function (polynomial)* $f$ *in* $z$ *with* $f(a)\neq 0$ *then, for* $z=a$, *we get*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\frac{d\Theta_{6W}(z)}{dz}}{\frac{d(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)}{dz}} = \frac{\frac{d\Theta_{6W}(z)}{dz}}{-4z^3+3(r^2+2s)z^2+2(2u+2rt-s^2)z+t^2-2us}$$

$\frac{d\Theta_{6W}(z)}{dz}=-(r(n+3)z^{n+2}+t(n+2)z^{n+1})W_{-2n+2}+(-(n+4)z^{n+3}+(r^2+s)(n+3)z^{n+2}+(u+rt)(n+2)z^{n+1})W_{-2n+1}-(t(n+3)z^{n+2}+(ru-st)(n+2)z^{n+1})W_{-2n}+u(-(n+3)z^{n+2}+s(n+2)z^{n+1}+u(n+1)z^n)W_{-2n-1}+(3z^2-2sz-u)W_3+(2(t+rs)z+ru)W_2+(4z^3-3(r^2+2s)z^2+2(s^2-tr-u)z+us)W_1+u(2rz+t)W_0.$

**(iii)** *If* $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=(z-a)^2f(z)=0$ *for some* $a\in\mathbb{C}$ *and a function (polynomial)* $f$ *in* $z$ *with* $f(a)\neq 0$ *then, for* $z=a$, *we get*

$$\sum_{k=1}^{n} z^k W_{-2k+1} = \frac{\frac{d^2\Theta_{6W}(z)}{dz^2}}{\frac{d^2}{dz^2}(-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2)} = \frac{\frac{d^2\Theta_{6W}(z)}{dz^2}}{-12z^2+6(r^2+2s)z-2s^2+4tr+4u}$$

*where*
$\frac{d^2\Theta_{6W}(z)}{dz^2}=-(r(n+2)(n+3)z^{n+1}+t(n+1)(n+2)z^n)W_{-2n+2}+(-(n+3)(n+4)z^{n+2}+(r^2+s)(n+2)(n+3)z^{n+1}+(u+rt)(n+1)(n+2)z^n)W_{-2n+1}-(t(n+2)(n+3)z^{n+1}+(ru-st)(n+1)(n+2)z^n)W_{-2n}+u(-(n+2)(n+3)z^{n+1}+s(n+1)(n+2)z^n+un(n+1)z^{n-1})W_{-2n-1}+(6z-2s)W_3+2(t+rs)W_2+(12z^2-6(r^2+2s)z+2(s^2-tr-u))W_1+2ruW_0.$

## 1.2. Generating Functions

Next, we give the ordinary generating functions of some special cases of generalized Tetranacci polynomials.

**Lemma 1.1.**
[16], *Lemma 9.1. The ordinary generating functions of the sequences* $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ *are given as follows:*

**(a)** $(|z|<\min\{|\alpha|^{-1},|\beta|^{-1},|\gamma|^{-1},|\delta|^{-1}\})$.

$$\sum_{n=0}^{\infty}W_nz^n=\frac{\Gamma_{1W}(z)}{uz^4+tz^3+sz^2+rz-1}$$

*where*

$\Gamma_{1W}(z)=-z^3W_3+(rz^3-z^2)W_2+(sz^3+rz^2-z)W_1+(tz^3+sz^2+rz-1)W_0.$

**(b)** $(|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2},|\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty}W_{2n}z^n=\frac{\Gamma_{2W}(z)}{-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1}$$

*where*

$\Gamma_{2W}(z)=-(tz^3+rz^2)W_3+((rt+u)z^3+(r^2+s)z^2-z)W_2+((st-ru)z^3-tz^2)W_1+((t^2-us)z^3+(2rt+u-s^2)z^2+(r^2+2s)z-1)W_0.$

**(c)** $(|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2},|\delta|^{-2}\})$.

$$\sum_{n=0}^{\infty}W_{2n+1}z^n=\frac{\Gamma_{3W}(z)}{-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1}$$

*where*

$\Gamma_{3W}(z)=(uz^3+sz^2-z)W_3+(-ruz^3-(t+rs)z^2)W_2+(-suz^3+(rt+u-s^2)z^2+(r^2+2s)z-1)W_1-u(tz^3+rz^2)W_0.$

**(d)** $(|z|<\min\{|\alpha|,|\beta|,|\gamma|,|\delta|\})$.

$$\sum_{n=1}^{\infty}W_{-n}z^n=\frac{\Gamma_{4W}(z)}{-z^4+rz^3+sz^2+tz+u}$$

*where*

$\Gamma_{4W}(z)=zW_3+(z^2-rz)W_2+(z^3-rz^2-sz)W_1+(z^4-rz^3-sz^2-tz)W_0.$

**(e)** $(|z|<\min\{|\alpha|^{2},|\beta|^{2},|\gamma|^{2},|\delta|^{2}\})$.

$$\sum_{n=1}^{\infty}W_{-2n}z^n=\frac{\Gamma_{5W}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2}$$

*where*

$\Gamma_{5W}(z)=(rz^2+tz)W_3+(z^3-(s+r^2)z^2-(u+rt)z)W_2+(tz^2+(ru-st)z)W_1+(z^4-(2s+r^2)z^3+(s^2-2rt-u)z^2+(us-t^2)z)W_0$

**(f)** $(|z|<\min\{|\alpha|^{2},|\beta|^{2},|\gamma|^{2},|\delta|^{2}\})$.

$$\sum_{n=1}^{\infty}W_{-2n+1}z^n=\frac{\Gamma_{6W}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2}$$

*where*

$\Gamma_{6W}(z)=(z^3-sz^2-uz)W_3+((t+rs)z^2+ruz)W_2+(z^4-(r^2+2s)z^3+(s^2-tr-u)z^2+usz)W_1+u(rz^2+tz)W_0.$

(d), (e) and (f) of lemma 1.1 can be given in the standart form as the following Lemma shows.

### Lemma 1.2.

*[16], Lemma 9.2. The ordinary generating functions of the sequences $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ can be given as follows:*

**(i)** $(|z|<\min\{|\alpha|,|\beta|,|\gamma|,|\delta|\})$.

$$\sum_{n=0}^{\infty}W_{-n}z^n=\frac{zW_3+(z^2-rz)W_2+(z^3-rz^2-sz)W_1+uW_0}{-z^4+rz^3+sz^2+tz+u}.$$

**(ii)** $(|z|<\min\{|\alpha|^{2},|\beta|^{2},|\gamma|^{2},|\delta|^{2}\})$.

$$\sum_{n=0}^{\infty}W_{-2n}z^n=\frac{\Gamma_{5aW}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2}$$

*where*

$\Gamma_{5aW}(z)=(rz^2+tz)W_3+(z^3-(s+r^2)z^2-(u+rt)z)W_2+(tz^2+(ru-st)z)W_1+(uz^2-suz-u^2)W_0.$

**(iii)** $(|z|<\min\{|\alpha|^{2},|\beta|^{2},|\gamma|^{2},|\delta|^{2}\})$.

$$\sum_{n=0}^{\infty}W_{-2n+1}z^n=\frac{\Gamma_{6aW}(z)}{-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2}$$

*where*

$\Gamma_{6aW}(z)=(z^3-sz^2-uz)W_3+((t+rs)z^2+ruz)W_2+((rt+u)z^2+(t^2-su)z-u^2)W_1+u(rz^2+tz)W_0.$

## 2. The Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions of Special Cases of Generalized Tetranacci Polynomials/Numbers: First Group

### 2.1. Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} W_n z^n, \sum_{n=0}^{\infty} W_{2n} z^n, \sum_{n=0}^{\infty} W_{2n+1} z^n, \sum_{n=0}^{\infty} W_{-n} z^n, \sum_{n=0}^{\infty} W_{-2n} z^n, \sum_{n=0}^{\infty} W_{-2n+1} z^n$ of Generalized Tetranacci Numbers

In this subsection, we consider the case $r = 1$, $s = 1$, $t = 1$, $u = 1$. A generalized Tetranacci numbers $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = W_{n-1} + W_{n-2} + W_{n-3} + W_{n-3}, \tag{4}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} - W_{-(n-2)} - W_{-(n-3)} + W_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (4) holds for all integer $n$. This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1–3, 17–19].

Characteristic equation of $\{W_n\}$ is

$$z^4 - z^3 - z^2 - z - 1 = 0, \tag{5}$$

whose roots are

$$\begin{aligned}
\alpha &\simeq 1.927561, \\
\beta &\simeq -0.774804, \\
\gamma &\simeq -0.076378 + 0.814703i, \\
\delta &\simeq -0.076378 - 0.814703i.
\end{aligned}$$

Two special cases of the sequence $\{W_n\}$ are Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$. Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \quad M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2 \tag{6}$$

and

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \quad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7 \tag{7}$$

respectively. The sequences $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} - M_{-(n-3)} + M_{-(n-4)}$$

and

$$R_{-n} = -R_{-(n-1)} - R_{-(n-2)} - R_{-(n-3)} + R_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences eqs. (6) and (7) hold for all integer $n$.

Binet's formula of generalized Tetranacci numbers can be given as follows:

**Theorem 2.1.**
*(Binet's formula of generalized Tetranacci numbers):*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
&\quad + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}
\end{aligned}$$

*where*

$$\begin{aligned}
p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\
p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\
p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\
p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0.
\end{aligned}$$

Tetranacci and Tetranacci-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 2.1.**
*Binet's formulas of Tetranacci and Tetranacci-Lucas numbers are*

$$\begin{aligned} M_n &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\ &+ \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \\ R_n &= \alpha^n+\beta^n+\gamma^n+\delta^n. \end{aligned}$$

Next, we present sum formulas of generalized Tetranacci numbers

**Theorem 2.2.**
*For $n \geq 0$, we have the following sum formulas for generalized Tetranacci numbers:*

**(a)** $\sum\limits_{k=0}^{n} W_k = \frac{1}{3}(W_{n+3} - W_{n+1} + W_n - W_3 + W_1 + 2W_0)$.

**(b)** $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{3}(-W_{2n+2} + 3W_{2n+1} + W_{2n} + 2W_{2n-1} - 2W_3 + 3W_2 - W_1 + 4W_0)$.

**(c)** $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{3}(2W_{2n+2} + W_{2n} - W_{2n-1} + W_3 - 3W_2 + 2W_1 - 2W_0)$.

**(d)** $\sum\limits_{k=1}^{n} W_{-k} = \frac{1}{3}(-W_{-n+3} + W_{-n+1} + 2W_{-n} + W_3 - W_1 - 2W_0)$.

**(e)** $\sum\limits_{k=1}^{n} W_{-2k} = \frac{1}{3}(W_{-2n+2} - 3W_{-2n+1} + 2W_{-2n} - 2W_{-2n-1} + 2W_3 - 3W_2 + W_1 - 4W_0)$.

**(f)** $\sum\limits_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-2W_{-2n+2} + 3W_{-2n+1} - W_{-2n} + W_{-2n-1} - W_3 + 3W_2 - 2W_1 + 2W_0)$.

Proof.

**(a)** For $r=1, s=1, t=1, u=1$, we get $uz^4+tz^3+sz^2+rz-1 = z^4+z^3+z^2+z-1$ and then for $z=1$, we get $z^4+z^3+z^2+z-1 \neq 0$ so we use theorem 1.1 (a) (i) with $z=1$.

**(b)** For $r=1, s=1, t=1, u=1$, we get $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1 = -z^4-z^3+3z^2+3z-1$ and then for $z=1$, we get $-z^4-z^3+3z^2+3z-1 \neq 0$ so we use theorem 1.1 (b) (i) with $z=1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (i) with $z=1$.

**(d)** For $r=1, s=1, t=1, u=1$, we get $-z^4+rz^3+sz^2+tz+u = -z^4+z^3+z^2+z+1$ and then for $z=1$, we get $-z^4+z^3+z^2+z+1 \neq 0$ so we use theorem 1.2 (a) (i) with $z=1$.

**(e)** For $r=1, s=1, t=1, u=1$, we get $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2 = -z^4+3z^3+3z^2-z-1$ and then for $z=1$, we get $-z^4+3z^3+3z^2-z-1 \neq 0$ so we use theorem 1.2 (b) (i) with $z=1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (i) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0=0, M_1=1, M_2=1, M_3=2$).

**Corollary 2.2.**
*For $n \geq 0$, Tetranacci numbers have the following properties.*

**(a)** $\sum\limits_{k=0}^{n} M_k = \frac{1}{3}(M_{n+3} - M_{n+1} + M_n - 1)$.

**(b)** $\sum\limits_{k=0}^{n} M_{2k} = \frac{1}{3}(-M_{2n+2} + 3M_{2n+1} + M_{2n} + 2M_{2n-1} - 2)$.

**(c)** $\sum_{k=0}^{n} M_{2k+1} = \frac{1}{3}(2M_{2n+2} + M_{2n} - M_{2n-1} + 1)$.

**(d)** $\sum_{k=1}^{n} M_{-k} = \frac{1}{3}(-M_{-n+3} + M_{-n+1} + 2M_{-n} + 1)$.

**(e)** $\sum_{k=1}^{n} M_{-2k} = \frac{1}{3}(M_{-2n+2} - 3M_{-2n+1} + 2M_{-2n} - 2M_{-2n-1} + 2)$.

**(f)** $\sum_{k=1}^{n} M_{-2k+1} = \frac{1}{3}(-2M_{-2n+2} + 3M_{-2n+1} - M_{-2n} + M_{-2n-1} - 1)$.

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the last Theorem, we have the following Corollary which gives sum formulas of Tetranacci-Lucas numbers.

**Corollary 2.3.**
*For $n \geq 0$, Tetranacci-Lucas numbers have the following properties:*

**(a)** $\sum_{k=0}^{n} R_k = \frac{1}{3}(R_{n+3} - R_{n+1} + R_n + 2)$.

**(b)** $\sum_{k=0}^{n} R_{2k} = \frac{1}{3}(-R_{2n+2} + 3R_{2n+1} + R_{2n} + 2R_{2n-1} + 10)$.

**(c)** $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{3}(2R_{2n+2} + R_{2n} - R_{2n-1} - 8)$.

**(d)** $\sum_{k=1}^{n} R_{-k} = \frac{1}{3}(-R_{-n+3} + R_{-n+1} + 2R_{-n} - 2)$.

**(e)** $\sum_{k=1}^{n} R_{-2k} = \frac{1}{3}(R_{-2n+2} - 3R_{-2n+1} + 2R_{-2n} - 2R_{-2n-1} - 10)$.

**(f)** $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{3}(-2R_{-2n+2} + 3R_{-2n+1} - R_{-2n} + R_{-2n-1} + 8)$.

Next, we give the ordinary generating functions of some special cases of generalized Tetranacci numbers.

**Lemma 2.1.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.51879$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{-z^3 W_3 + (z^3 - z^2) W_2 + (z^3 + z^2 - z) W_1 + (z^3 + z^2 + z - 1) W_0}{z^4 + z^3 + z^2 + z - 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.269143$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{(z^3 + z^2) W_3 - (2z^3 + 2z^2 - z) W_2 + z^2 W_1 - (2z^2 + 3z - 1) W_0}{z^4 + z^3 - 3z^2 - 3z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.269143$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{-(z^3 + z^2 - z) W_3 + (z^3 + 2z^2) W_2 + (z^3 - z^2 - 3z + 1) W_1 + (z^3 + z^2) W_0}{z^4 + z^3 - 3z^2 - 3z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\gamma| \simeq 0.738325$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{z W_3 + (z^2 - z) W_2 + (z^3 - z^2 - z) W_1 + W_0}{-z^4 + z^3 + z^2 + z + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\gamma|^2 \simeq 0.545123$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-(z^2 + z) W_3 + (-z^3 + 2z^2 + 2z) W_2 - z^2 W_1 + (-z^2 + z + 1) W_0}{z^4 - 3z^3 - 3z^2 + z + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\gamma|^2 \simeq 0.545123$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{(-z^3+z^2+z)W_3 - (2z^2+z)W_2 - (2z^2-1)W_1 - (z^2+z)W_0}{z^4-3z^3-3z^2+z+1}.$$

Proof. Use lemmas 1.1 and 1.2. □

Now, we consider special cases of the last Lemma.

**Corollary 2.4.**

*The ordinary generating functions of the sequences $M_n$, $M_{2n}$, $M_{2n+1}$, $M_{-n}$, $M_{-2n}$, $M_{-2n+1}$ and $R_n$, $R_{2n}$, $R_{2n+1}$, $R_{-n}$, $R_{-2n}$, $R_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.51879$

$$\sum_{n=0}^{\infty} M_n z^n = \frac{-z}{z^4+z^3+z^2+z-1},$$

$$\sum_{n=0}^{\infty} R_n z^n = \frac{z^3+2z^2+3z-4}{z^4+z^3+z^2+z-1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.269143$

$$\sum_{n=0}^{\infty} M_{2n} z^n = \frac{z^2+z}{z^4+z^3-3z^2-3z+1},$$

$$\sum_{n=0}^{\infty} R_{2n} z^n = \frac{z^3-6z^2-9z+4}{z^4+z^3-3z^2-3z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.269143$

$$\sum_{n=0}^{\infty} M_{2n+1} z^n = \frac{-z^2-z+1}{z^4+z^3-3z^2-3z+1},$$

$$\sum_{n=0}^{\infty} R_{2n+1} z^n = \frac{z^3+2z^2+4z+1}{z^4+z^3-3z^2-3z+1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\gamma| \simeq 0.738325$

$$\sum_{n=0}^{\infty} M_{-n} z^n = \frac{z^3}{-z^4+z^3+z^2+z+1},$$

$$\sum_{n=0}^{\infty} R_{-n} z^n = \frac{z^3+2z^2+3z+4}{-z^4+z^3+z^2+z+1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\gamma|^2 \simeq 0.545123$

$$\sum_{n=0}^{\infty} M_{-2n} z^n = \frac{-z^3-z^2}{z^4-3z^3-3z^2+z+1},$$

$$\sum_{n=0}^{\infty} R_{-2n} z^n = \frac{-3z^3-6z^2+3z+4}{z^4-3z^3-3z^2+z+1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\gamma|^2 \simeq 0.545123$

$$\sum_{n=0}^{\infty} M_{-2n+1} z^n = \frac{-2z^3-2z^2+z+1}{z^4-3z^3-3z^2+z+1},$$

$$\sum_{n=0}^{\infty} R_{-2n+1} z^n = \frac{-7z^3-5z^2+1}{z^4-3z^3-3z^2+z+1}.$$

From the last corollary, we obtain the following results for Tetranacci and Tetranacci-Lucas numbers.

**Corollary 2.5.**
*Infinite sums of* $M_n, M_{2n}, M_{2n+1}, M_{-n}, M_{-2n}, M_{-2n+1}$ *and* $R_n, R_{2n}, R_{2n+1}, R_{-n}, R_{-2n}, R_{-2n+1}$ *are given as follows:*

**(a)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{M_n}{2^n} = 8$$
$$\sum_{n=0}^{\infty} \frac{R_n}{2^n} = 30$$

**(b)** $z = \frac{1}{8}$

$$\sum_{n=0}^{\infty} \frac{M_{2n}}{8^n} = \frac{576}{2377}$$
$$\sum_{n=0}^{\infty} \frac{R_{2n}}{8^n} = \frac{11400}{2377}$$

**(c)** $z = \frac{1}{8}$

$$\sum_{n=0}^{\infty} \frac{M_{2n+1}}{8^n} = \frac{3520}{2377}$$
$$\sum_{n=0}^{\infty} \frac{R_{2n+1}}{8^n} = \frac{6280}{2377}$$

**(d)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{M_{-n}}{2^n} = \frac{2}{29}$$
$$\sum_{n=0}^{\infty} \frac{R_{-n}}{2^n} = \frac{98}{29}$$

**(e)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{M_{-2n}}{2^n} = -\frac{6}{7}$$
$$\sum_{n=0}^{\infty} \frac{R_{-2n}}{2^n} = \frac{58}{7}$$

**(f)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{M_{-2n+1}}{2^n} = \frac{12}{7}$$
$$\sum_{n=0}^{\infty} \frac{R_{-2n+1}}{2^n} = -\frac{18}{7}$$

### 2.2. Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} W_n z^n$, $\sum_{n=0}^{\infty} W_{2n} z^n$, $\sum_{n=0}^{\infty} W_{2n+1} z^n$, $\sum_{n=0}^{\infty} W_{-n} z^n$, $\sum_{n=0}^{\infty} W_{-2n} z^n$, $\sum_{n=0}^{\infty} W_{-2n+1} z^n$ of Generalized Fourth Order Pell Numbers

In this subsection, we consider the case $r = 2$, $s = 1$, $t = 1$, $u = 1$.A generalized fourth order Pell sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} + W_{n-2} + W_{n-3} + W_{n-4} \tag{8}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$, $W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} - W_{-(n-2)} - 2W_{-(n-3)} + W_{-(n-4)}$$

for $n = 1,2,3,\ldots$. Therefore, recurrence eq. (8) holds for all integer $n$. For more information on generalized fourth order Pell numbers, see Soykan [5].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 2z^3 - z^2 - z - 1 = 0, \tag{9}$$

whose roots are

$$\begin{aligned} \alpha &\simeq 2.592052792, \\ \beta &\simeq -0.6631378984, \\ \gamma &\simeq 0.03554255298 - 0.7619107877i, \\ \delta &\simeq 0.03554255299 + 0.7619107877i. \end{aligned}$$

Two special cases of the sequence $\{W_n\}$ are fourth order Pell sequence $\{P_n\}_{n\geq 0}$ and fourth order Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$. Fourth order Pell and fourth order Pell-Lucas sequences are defined, respectively, by the fourth-order recurrence relations

$$P_{n+4} = 2P_{n+3} + P_{n+2} + P_{n+1} + P_n, \quad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, \tag{10}$$

and

$$Q_{n+4} = 2Q_{n+3} + Q_{n+2} + Q_{n+1} + Q_n, \quad Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17, \tag{11}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} - P_{-(n-2)} - 2P_{-(n-3)} + P_{-(n-4)}$$

and

$$Q_{-n} = -Q_{-(n-1)} - Q_{-(n-2)} - 2Q_{-(n-3)} + Q_{-(n-4)}$$

for $n = 1,2,3,\ldots$ respectively. Therefore, recurrences eqs. (10) and (11)) hold for all integer $n$.

Binet's formula of generalized fourth order Pell numbers can be given as follows:

**Theorem 2.3.**
*(Binet's formula of generalized fourth order Pell numbers):*

$$\begin{aligned} W_n = {}& \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\ &+ \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \end{aligned}$$

*where*

$$\begin{aligned} p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\ p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0. \end{aligned}$$

Fourth order Pell and fourth order Pell-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 2.6.**
*Binet's formulas of fourth order Pell and fourth order Pell-Lucas numbers are*

$$P_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},$$

*and*

$$Q_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

*respectively.*

Next, we present sum formulas of generalized fourth order Pell numbers

**Theorem 2.4.**
*For $n \geq 0$, we have the following sum formulas for generalized fourth order Pell numbers:*

**(a)** $\sum_{k=0}^{n} W_k = \frac{1}{4}(W_{n+3} - W_{n+2} - 2W_{n+1} + W_n - W_3 + W_2 + 2W_1 + 3W_0).$

**(b)** $\sum_{k=0}^{n} W_{2k} = \frac{1}{8}(-W_{2n+2} + 5W_{2n+1} + 2W_{2n} + 3W_{2n-1} - 3W_3 + 7W_2 - 2W_1 + 9W_0).$

**(c)** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{8}(3W_{2n+2} + W_{2n+1} + 2W_{2n} - W_{2n-1} + W_3 - 5W_2 + 6W_1 - 3W_0).$

**(d)** $\sum_{k=1}^{n} W_{-k} = \frac{1}{4}(-W_{-n+3} + W_{-n+2} + 2W_{-n+1} + 3W_{-n} + W_3 - W_2 - 2W_1 - 3W_0).$

**(e)** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{8}(W_{-2n+2} - 5W_{-2n+1} - 3W_{-2n-1} + 6W_{-2n} + 3W_3 - 7W_2 + 2W_1 - 9W_0).$

**(f)** $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{8}(-3W_{-2n+2} + 7W_{-2n+1} + W_{-2n-1} - 2W_{-2n} - W_3 + 5W_2 - 6W_1 + 3W_0).$

Proof.

**(a)** For $r = 2, s = 1, t = 1, u = 1$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = z^4 + z^3 + z^2 + 2z - 1$ and then for $z = 1$, we get $z^4 + z^3 + z^2 + 2z - 1 \neq 0$ so we use theorem 1.1 (a) (i) with $z = 1$.

**(b)** For $r = 2, s = 1, t = 1, u = 1$, we get $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = -z^4 - z^3 + 5z^2 + 6z - 1$ and then for $z = 1$, we get $-z^4 - z^3 + 5z^2 + 6z - 1 \neq 0$ so we use theorem 1.1 (b) (i) with $z = 1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (i) with $z = 1$.

**(d)** For $r = 2, s = 1, t = 1, u = 1$, we get $-z^4 + rz^3 + sz^2 + tz + u = -z^4 + 2z^3 + z^2 + z + 1$ and then for $z = 1$, we get $-z^4 + 2z^3 + z^2 + z + 1 \neq 0$ so we use theorem 1.2 (a) (i) with $z = 1$.

**(e)** For $r = 2, s = 1, t = 1, u = 1$, we get $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 = -z^4 + 6z^3 + 5z^2 - z - 1$ and then for $z = 1$, we get $-z^4 + 6z^3 + 5z^2 - z - 1 \neq 0$ so we use theorem 1.2 (b) (i) with $z = 1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (i) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of fourth order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

**Corollary 2.7.**
*For $n \geq 0$, fourth order Pell numbers have the following properties.*

**(a)** $\sum_{k=0}^{n} P_k = \frac{1}{4}(P_{n+3} - P_{n+2} - 2P_{n+1} + P_n - 1).$

**(b)** $\sum_{k=0}^{n} P_{2k} = \frac{1}{8}(-P_{2n+2} + 5P_{2n+1} + 2P_{2n} + 3P_{2n-1} - 3).$

**(c)** $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{8}(3P_{2n+2} + P_{2n+1} + 2P_{2n} - P_{2n-1} + 1).$

**(d)** $\sum_{k=1}^{n} P_{-k} = \frac{1}{4}(-P_{-n+3} + P_{-n+2} + 2P_{-n+1} + 3P_{-n} + 1).$

**(e)** $\sum_{k=1}^{n} P_{-2k} = \frac{1}{8}(P_{-2n+2} - 5P_{-2n+1} - 3P_{-2n-1} + 6P_{-2n} + 3).$

**(f)** $\sum_{k=1}^{n} P_{-2k+1} = \frac{1}{8}(-3P_{-2n+2} + 7P_{-2n+1} + P_{-2n-1} - 2P_{-2n} - 1).$

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last Theorem, we have the following Corollary which gives sum formulas of fourth order Pell-Lucas numbers.

**Corollary 2.8.**
*For $n \geq 0$, fourth order Pell-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+3} - Q_{n+2} - 2Q_{n+1} + Q_n + 5)$.

**(b)** $\sum\limits_{k=0}^{n} Q_{2k} = \frac{1}{8}(-Q_{2n+2} + 5Q_{2n+1} + 2Q_{2n} + 3Q_{2n-1} + 23)$.

**(c)** $\sum\limits_{k=0}^{n} Q_{2k+1} = \frac{1}{8}(3Q_{2n+2} + Q_{2n+1} + 2Q_{2n} - Q_{2n-1} - 13)$.

**(d)** $\sum\limits_{k=1}^{n} Q_{-k} = \frac{1}{4}(-Q_{-n+3} + Q_{-n+2} + 2Q_{-n+1} + 3Q_{-n} - 5)$.

**(e)** $\sum\limits_{k=1}^{n} Q_{-2k} = \frac{1}{8}(Q_{-2n+2} - 5Q_{-2n+1} - 3Q_{-2n-1} + 6Q_{-2n} - 23)$.

**(f)** $\sum\limits_{k=1}^{n} Q_{-2k+1} = \frac{1}{8}(-3Q_{-2n+2} + 7Q_{-2n+1} + Q_{-2n-1} - 2Q_{-2n} + 13)$.

Next, we give the ordinary generating functions of some special cases of generalized fourth order Pell numbers.

**Lemma 2.2.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.385794$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{-z^3 W_3 + (2z^3 - z^2)W_2 + (z^3 + 2z^2 - z)W_1 + (z^3 + z^2 + 2z - 1)W_0}{z^4 + z^3 + z^2 + 2z - 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.148837$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{(z^3 + 2z^2)W_3 - (3z^3 + 5z^2 - z)W_2 + (z^3 + z^2)W_1 - (4z^2 + 6z - 1)W_0}{z^4 + z^3 - 5z^2 - 6z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.148837$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{-(z^3 + z^2 - z)W_3 + (2z^3 + 3z^2)W_2 + (z^3 - 2z^2 - 6z + 1)W_1 + (z^3 + 2z^2)W_0}{z^4 + z^3 - 5z^2 - 6z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| \simeq 0.663137$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{zW_3 + (z^2 - 2z)W_2 - \left(-z^3 + 2z^2 + z\right)W_1 + W_0}{-z^4 + 2z^3 + z^2 + z + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 \simeq 0.439751$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-(2z^2 + z)W_3 + (-z^3 + 5z^2 + 3z)W_2 - (z^2 + z)W_1 + (-z^2 + z + 1)W_0}{z^4 - 6z^3 - 5z^2 + z + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 \simeq 0.439751$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{(-z^3 + z^2 + z)W_3 - (3z^2 + 2z)W_2 - (3z^2 - 1)W_1 - (2z^2 + z)W_0}{z^4 - 6z^3 - 5z^2 + z + 1}.$$

Proof. Use lemmas 1.1 and 1.2. □
Now, we consider special cases of the last Lemma.

**Corollary 2.9.**
*The ordinary generating functions of the sequences $P_n$, $P_{2n}$, $P_{2n+1}$, $P_{-n}$, $P_{-2n}$, $P_{-2n+1}$ $Q_n$, $Q_{2n}$, $Q_{2n+1}$, $Q_{-n}$, $Q_{-2n}$, $Q_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.385794$

$$\sum_{n=0}^{\infty} P_n z^n = \frac{-z}{z^4+z^3+z^2+2z-1},$$
$$\sum_{n=0}^{\infty} Q_n z^n = \frac{z^3+2z^2+6z-4}{z^4+z^3+z^2+2z-1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.148837$

$$\sum_{n=0}^{\infty} P_{2n} z^n = \frac{z^2+2z}{z^4+z^3-5z^2-6z+1},$$
$$\sum_{n=0}^{\infty} Q_{2n} z^n = \frac{z^3-10z^2-18z+4}{z^4+z^3-5z^2-6z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.148837$

$$\sum_{n=0}^{\infty} P_{2n+1} z^n = \frac{-z^2-z+1}{z^4+z^3-5z^2-6z+1},$$
$$\sum_{n=0}^{\infty} Q_{2n+1} z^n = \frac{z^3+5z^2+5z+2}{z^4+z^3-5z^2-6z+1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| \simeq 0.663137$

$$\sum_{n=0}^{\infty} P_{-n} z^n = \frac{z^3}{-z^4+2z^3+z^2+z+1},$$
$$\sum_{n=0}^{\infty} Q_{-n} z^n = \frac{2z^3+2z^2+3z+4}{-z^4+2z^3+z^2+z+1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 \simeq 0.439751$

$$\sum_{n=0}^{\infty} P_{-2n} z^n = \frac{-2z^3-z^2}{z^4-6z^3-5z^2+z+1},$$
$$\sum_{n=0}^{\infty} Q_{-2n} z^n = \frac{-6z^3-10z^2+3z+4}{z^4-6z^3-5z^2+z+1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 \simeq 0.439751$

$$\sum_{n=0}^{\infty} P_{-2n+1} z^n = \frac{-5z^3-4z^2+z+1}{z^4-6z^3-5z^2+z+1},$$
$$\sum_{n=0}^{\infty} Q_{-2n+1} z^n = \frac{-17z^3-15z^2+z+2}{z^4-6z^3-5z^2+z+1}.$$

From the last corollary, we obtain the following results for fourth order Pell and fourth order Pell-Lucas numbers.

**Corollary 2.10.**
*Infinite sums of* $P_n, P_{2n}, P_{2n+1}, P_{-n}, P_{-2n}, P_{-2n+1}$ *and* $Q_n, Q_{2n}, Q_{2n+1}, Q_{-n}, Q_{-2n}, Q_{-2n+1}$ *are given as follows:*

**(a)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{P_n}{4^n} = \frac{64}{107},$$
$$\sum_{n=0}^{\infty} \frac{Q_n}{4^n} = \frac{604}{107}.$$

**(b)** $z = \frac{1}{8}$

$$\sum_{n=0}^{\infty} \frac{P_{2n}}{8^n} = \frac{1088}{713},$$
$$\sum_{n=0}^{\infty} \frac{Q_{2n}}{8^n} = \frac{6536}{713}.$$

**(c)** $z = \frac{1}{8}$

$$\sum_{n=0}^{\infty} \frac{P_{2n+1}}{8^n} = \frac{3520}{713},$$
$$\sum_{n=0}^{\infty} \frac{Q_{2n+1}}{8^n} = \frac{11080}{713}.$$

**(d)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_{-n}}{2^n} = \frac{2}{31},$$
$$\sum_{n=0}^{\infty} \frac{Q_{-n}}{2^n} = \frac{100}{31}.$$

**(e)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{P_{-2n}}{4^n} = -\frac{24}{217},$$
$$\sum_{n=0}^{\infty} \frac{Q_{-2n}}{4^n} = \frac{1032}{217}.$$

**(f)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{P_{-2n+1}}{4^n} = \frac{236}{217},$$
$$\sum_{n=0}^{\infty} \frac{Q_{-2n+1}}{4^n} = \frac{268}{217}.$$

### 2.3. Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} W_n z^n$, $\sum_{n=0}^{\infty} W_{2n} z^n$, $\sum_{n=0}^{\infty} W_{2n+1} z^n$, $\sum_{n=0}^{\infty} W_{-n} z^n$, $\sum_{n=0}^{\infty} W_{-2n} z^n$, $\sum_{n=0}^{\infty} W_{-2n+1} z^n$ of Generalized Fourth Order Jacobsthal Numbers

In this subsection, we consider the case $r = 1$, $s = 1$, $t = 1$, $u = 2$. A generalized fourth order Jacobsthal sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth order recurrence relations

$$W_n = W_{n-1} + W_{n-2} + W_{n-3} + 2W_{n-4} \tag{12}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2} W_{-(n-1)} - \frac{1}{2} W_{-(n-2)} - \frac{1}{2} W_{-(n-3)} + \frac{1}{2} W_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (12) holds for all integer $n$. For more information on generalized fourth order Jacobsthal numbers, see [6].

Characteristic equation of $\{W_n\}$ is

$$z^4 - z^3 - z^2 - z - 2 = 0. \tag{13}$$

whose roots are

$$\alpha = -1,$$
$$\beta = 2,$$
$$\gamma = i,$$
$$\delta = -i.$$

Four special cases of the sequence $\{W_n\}$ are adjusted fourth order Jacobsthal sequence $\{S_n\}_{n\geq 0}$, modified fourth order Jacobsthal-Lucas sequence $\{R_n\}_{n\geq 0}$, fourth order Jacobsthal sequence $\{J_n\}_{n\geq 0}$ and fourth order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$. Adjusted fourth orderJacobsthal, modified fourth order Jacobsthal-Lucas, fourth order Jacobsthal and fourth order Jacobsthal-Lucas sequences are defined, respectively, by the fourth order recurrence relations

$$S_{n+4} = S_{n+3} + S_{n+2} + S_{n+1} + 2S_n, \quad S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, \tag{14}$$
$$R_{n+4} = R_{n+3} + R_{n+2} + R_{n+1} + 2R_n, \quad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7, \tag{15}$$
$$J_{n+4} = J_{n+3} + J_{n+2} + J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, \tag{16}$$
$$j_{n+4} = j_{n+3} + j_{n+2} + j_{n+1} + 2j_n, \quad j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, \tag{17}$$

The sequences $\{S_n\}_{n\geq 0}, \{R_n\}_{n\geq 0}, \{J_n\}_{n\geq 0}$, and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$S_{-n} = -\frac{1}{2}S_{-(n-1)} - \frac{1}{2}S_{-(n-2)} - \frac{1}{2}S_{-(n-3)} + \frac{1}{2}S_{-(n-4)}, \tag{18}$$
$$R_{-n} = -\frac{1}{2}R_{-(n-1)} - \frac{1}{2}R_{-(n-2)} - \frac{1}{2}R_{-(n-3)} + \frac{1}{2}R_{-(n-4)} \tag{19}$$
$$J_{-n} = -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} - \frac{1}{2}J_{-(n-3)} + \frac{1}{2}J_{-(n-4)}, \tag{20}$$
$$j_{-n} = -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} - \frac{1}{2}j_{-(n-3)} + \frac{1}{2}j_{-(n-4)}, \tag{21}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences eq. (14)–eq. (17) hold for all integer $n$.

Binet's formula of generalized fourth order Jacobsthal numbers can be given as follows:

**Theorem 2.5.**
*For all integers $n$, Binet's formula of generalized fourth order Jacobsthal numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

*where*

$$p_1 = W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0,$$
$$p_2 = W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0,$$
$$p_3 = W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0,$$
$$p_4 = W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0.$$

Adjusted fourth order Jacobsthal and modified fourth order Jacobsthal-Lucas, fourth order Jacobsthal and fourth order Jacobsthal-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 2.11.**
*Binet formulas of adjusted fourth order Jacobsthal and modified fourth order Jacobsthal-Lucas, fourth order Jacobsthal*

*and fourth order Jacobsthal-Lucas numbers are*

$$\begin{aligned} S_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^{n+2} + \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^{n+2} \\ &+ \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^{n+2} + \frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^{n+2}, \\ R_n &= \alpha^n+\beta^n+\gamma^n+\delta^n \\ J_n &= \frac{(\alpha^2-1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(\beta^2-1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &+ \frac{(\gamma^2-1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(\delta^2-1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n, \\ j_n &= \frac{(2\alpha^3-\alpha^2+2\alpha+2)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(2\beta^3-\beta^2+2\beta+2)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &+ \frac{(2\gamma^3-\gamma^2+2\gamma+2)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(2\delta^3-\delta^2+2\delta+2)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n, \end{aligned}$$

*respectively.*

The above formulas can be written as follows

$$\begin{aligned} S_n &= -\frac{1}{6}\alpha^n + \frac{4}{15}\beta^n + (-\frac{1}{20}-\frac{3}{20}i)\gamma^n + (-\frac{1}{20}+\frac{3}{20}i)\delta^n \\ R_n &= \alpha^n+\beta^n+\gamma^n+\delta^n, \\ J_n &= \frac{1}{5}\beta^n + (-\frac{1}{10}-\frac{3}{10}i)\gamma^n + (-\frac{1}{10}+\frac{3}{10}i)\delta^n, \\ j_n &= \frac{1}{2}\alpha^n + \frac{6}{5}\beta^n + (\frac{3}{20}+\frac{9}{20}i)\gamma^n + (\frac{3}{20}-\frac{9}{20}i)\delta^n. \end{aligned}$$

Next, we present sum formulas of generalized fourth order Jacobsthal numbers

**Theorem 2.6.**
*For $n \geq 0$, we have the following sum formulas for generalized fourth order Jacobsthal numbers:*

**(a)** $\sum_{k=0}^{n} W_k = \frac{1}{4}(W_{n+3}-W_{n+1}+2W_n-W_3+W_1+2W_0)$.

**(b)** $\sum_{k=0}^{n} W_{2k} = \frac{1}{12}((2n+7)W_{2n+2}-2(2n+5)W_{2n+1}+(2n+13)W_{2n}-2(2n+5)W_{2n-1}+5W_3-12W_2+5W_1-6W_0)$.

**(c)** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{12}(-(2n+3)W_{2n+2}+4(n+5)W_{2n+1}-(2n+3)W_{2n}+2(2n+7)W_{2n-1}-7W_3+10W_2-W_1+10W_0)$.

**(d)** $\sum_{k=1}^{n} W_{-k} = \frac{1}{4}(-W_{-n+3}+W_{-n+1}+2W_{-n}+W_3-W_1-2W_0)$.

**(e)** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{12}((2n+1)W_{-2n+2}-2(2n+3)W_{-2n+1}+(2n+7)W_{-2n}-2(2n+3)W_{-2n-1}+3W_3-4W_2+3W_1-10W_0)$.

**(f)** $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{12}(-(2n+5)W_{-2n+2}+4(n+2)W_{-2n+1}-(2n+5)W_{-2n}+2(2n+1)W_{-2n-1}-W_3+6W_2-7W_1+6W_0)$.

Proof.

**(a)** For $r=1, s=1, t=1, u=2$, we get $uz^4+tz^3+sz^2+rz-1=(2z-1)(z+1)(z^2+1)$ and then for $z=1$, we get $(2z-1)(z+1)(z^2+1)\neq 0$ so we use theorem 1.1 (a) (i) with $z=1$.

**(b)** For $r=1, s=1, t=1, u=2$, we get $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=-(z-1)(4z-1)(z+1)^2$ and then for $z=1$, we get $-(z-1)(4z-1)(z+1)^2=0$ with multiplicity 1 so we use theorem 1.1 (b) (ii) with $z=1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (ii) with $z=1$.

**(d)** For $r=1, s=1, t=1, u=2$, we get $-z^4+rz^3+sz^2+tz+u=-(z-2)(z+1)(z^2+1)$ and then for $z=1$, we get $-(z-2)(z+1)(z^2+1)\neq 0$ so we use theorem 1.2 (a) (i) with $z=1$.

**(e)** For $r=1, s=1, t=1, u=2$, we get $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2=-(z-1)(z-4)(z+1)^2$ and then for $z=1$, we get $-(z-1)(z-4)(z+1)^2=0$ with multiplicity 1 so we use theorem 1.2 (b) (ii) with $z=1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (ii) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of adjusted fourth order Jacobsthal numbers (take $W_n=S_n$ with $S_0=0, S_1=1, S_2=1, S_3=2$).

**Corollary 2.12.**
*For $n\geq 0$, adjusted fourth order Jacobsthal numbers have the following properties.*

**(a)** $\sum\limits_{k=0}^{n} S_k = \frac{1}{4}(S_{n+3}-S_{n+1}+2S_n-1)$.

**(b)** $\sum\limits_{k=0}^{n} S_{2k} = \frac{1}{12}((2n+7)S_{2n+2}-2(2n+5)S_{2n+1}+(2n+13)S_{2n}-2(2n+5)S_{2n-1}+3)$.

**(c)** $\sum\limits_{k=0}^{n} S_{2k+1} = \frac{1}{12}(-(2n+3)S_{2n+2}+4(n+5)S_{2n+1}-(2n+3)S_{2n}+2(2n+7)S_{2n-1}-5)$.

**(d)** $\sum\limits_{k=1}^{n} S_{-k} = \frac{1}{4}(-S_{-n+3}+S_{-n+1}+2S_{-n}+1)$.

**(e)** $\sum\limits_{k=1}^{n} S_{-2k} = \frac{1}{12}((2n+1)S_{-2n+2}-2(2n+3)S_{-2n+1}+(2n+7)S_{-2n}-2(2n+3)S_{-2n-1}+5)$.

**(f)** $\sum\limits_{k=1}^{n} S_{-2k+1} = \frac{1}{12}(-(2n+5)S_{-2n+2}+4(n+2)S_{-2n+1}-(2n+5)S_{-2n}+2(2n+1)S_{-2n-1}-3)$.

Taking $W_n=R_n$ with $R_0=4, R_1=1, R_2=3, R_3=7$ in the last Theorem, we have the following Corollary which gives sum formulas of modified fourth order Jacobsthal-Lucas numbers.

**Corollary 2.13.**
*For $n\geq 0$, modified fourth order Jacobsthal-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} R_k = \frac{1}{4}(R_{n+3}-R_{n+1}+2R_n+2)$.

**(b)** $\sum\limits_{k=0}^{n} R_{2k} = \frac{1}{12}((2n+7)R_{2n+2}-2(2n+5)R_{2n+1}+(2n+13)R_{2n}-2(2n+5)R_{2n-1}-20)$.

**(c)** $\sum\limits_{k=0}^{n} R_{2k+1} = \frac{1}{12}(-(2n+3)R_{2n+2}+4(n+5)R_{2n+1}-(2n+3)R_{2n}+2(2n+7)R_{2n-1}+20)$.

**(d)** $\sum\limits_{k=1}^{n} R_{-k} = \frac{1}{4}(-R_{-n+3}+R_{-n+1}+2R_{-n}-2)$.

**(e)** $\sum\limits_{k=1}^{n} R_{-2k} = \frac{1}{12}((2n+1)R_{-2n+2}-2(2n+3)R_{-2n+1}+(2n+7)R_{-2n}-2(2n+3)R_{-2n-1}-28)$.

**(f)** $\sum\limits_{k=1}^{n} R_{-2k+1} = \frac{1}{12}(-(2n+5)R_{-2n+2}+4(n+2)R_{-2n+1}-(2n+5)R_{-2n}+2(2n+1)R_{-2n-1}+28)$.

From the last Theorem, we have the following Corollary which gives sum formulas of fourth order Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1$).

**Corollary 2.14.**
*For $n\geq 0$, fourth order Jacobsthal numbers have the following properties.*

**(a)** $\sum\limits_{k=0}^{n} J_k = \frac{1}{4}(J_{n+3}-J_{n+1}+2J_n)$.

**(b)** $\sum\limits_{k=0}^{n} J_{2k} = \frac{1}{12}((2n+7)J_{2n+2}-2(2n+5)J_{2n+1}+(2n+13)J_{2n}-2(2n+5)J_{2n-1}-2)$.

**(c)** $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{12}(-(2n+3)J_{2n+2} + 4(n+5)J_{2n+1} - (2n+3)J_{2n} + 2(2n+7)J_{2n-1} + 2).$

**(d)** $\sum_{k=1}^{n} J_{-k} = \frac{1}{4}(-J_{-n+3} + J_{-n+1} + 2J_{-n}).$

**(e)** $\sum_{k=1}^{n} J_{-2k} = \frac{1}{12}((2n+1)J_{-2n+2} - 2(2n+3)J_{-2n+1} + (2n+7)J_{-2n} - 2(2n+3)J_{-2n-1} + 2).$

**(f)** $\sum_{k=1}^{n} J_{-2k+1} = \frac{1}{12}(-(2n+5)J_{-2n+2} + 4(n+2)J_{-2n+1} - (2n+5)J_{-2n} + 2(2n+1)J_{-2n-1} - 2).$

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$ in the last Theorem, we have the following Corollary which gives sum formulas of fourth order Jacobsthal-Lucas numbers.

**Corollary 2.15.**
*For $n \geq 0$, fourth order Jacobsthal-Lucas numbers have the following properties:*

**(a)** $\sum_{k=0}^{n} j_k = \frac{1}{4}(j_{n+3} - j_{n+1} + 2j_n - 5).$

**(b)** $\sum_{k=0}^{n} j_{2k} = \frac{1}{12}((2n+7)j_{2n+2} - 2(2n+5)j_{2n+1} + (2n+13)j_{2n} - 2(2n+5)j_{2n-1} - 17).$

**(c)** $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{12}(-(2n+3)j_{2n+2} + 4(n+5)j_{2n+1} - (2n+3)j_{2n} + 2(2n+7)j_{2n-1} - 1).$

**(d)** $\sum_{k=1}^{n} j_{-k} = \frac{1}{4}(-j_{-n+3} + j_{-n+1} + 2j_{-n} + 5)$

**(e)** $\sum_{k=1}^{n} j_{-2k} = \frac{1}{12}((2n+1)j_{-2n+2} - 2(2n+3)j_{-2n+1} + (2n+7)j_{-2n} - 2(2n+3)j_{-2n-1} - 7).$

**(f)** $\sum_{k=1}^{n} j_{-2k+1} = \frac{1}{12}(-(2n+5)j_{-2n+2} + 4(n+2)j_{-2n+1} - (2n+5)j_{-2n} + 2(2n+1)j_{-2n-1} + 25).$

Next, we give the ordinary generating functions of some special cases of generalized fourth order Jacobsthal numbers.

**Lemma 2.3.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\beta|^{-1} = 0.5$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{-z^3W_3 + (z^3 - z^2)W_2 + (z^3 + z^2 - z)W_1 + (z^3 + z^2 + z - 1)W_0}{2z^4 + z^3 + z^2 + z - 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\beta|^{-2} = 0.25$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{((z^3 + z^2)W_3 - (3z^3 + 2z^2 - z)W_2 + (z^3 + z^2)W_1 + (z^3 - 3z^2 - 3z + 1)W_0}{4z^4 + 3z^3 - 5z^2 - 3z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\beta|^{-2} = 0.25$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{-(2z^3 + z^2 - z)W_3 + 2(z^3 + z^2)W_2 + (2z^3 - 2z^2 - 3z + 1)W_1 + 2(z^3 + z^2)W_0}{4z^4 + 3z^3 - 5z^2 - 3z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\alpha| = |\gamma| = |\delta| = 1$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{zW_3 + (z^2 - z)W_2 + (z^3 - z^2 - z)W_1 + 2W_0}{-z^4 + z^3 + z^2 + z + 2}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\alpha|^2 = |\beta|^2 = |\delta|^2 = 1$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-(z^2+z)W_3 + (-z^3+2z^2+3z)W_2 - (z^2+z)W_1 + (-2z^2+2z+4)W_0}{z^4-3z^3-5z^2+3z+4}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\alpha|^2 = |\beta|^2 = |\delta|^2 = 1$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{(-z^3+z^2+2z)W_3 + (-3z^2+z+4)W_1 - (2z^2+2z)W_2 - 2(z^2+z)W_0}{z^4-3z^3-5z^2+3z+4}.$$

Proof. Use lemmas 1.1 and 1.2. □

Now, we consider special cases of the last Lemma.

**Corollary 2.16.**

*The ordinary generating functions of the sequences $S_n$, $S_{2n}$, $S_{2n+1}$, $S_{-n}$, $S_{-2n}$, $S_{-2n+1}$ and $R_n$, $R_{2n}$, $R_{2n+1}$, $R_{-n}$, $R_{-2n}$, $R_{-2n+1}$ and $J_n$, $J_{2n}$, $J_{2n+1}$, $J_{-n}$, $J_{-2n}$, $J_{-2n+1}$ and $j_n$, $j_{2n}$, $j_{2n+1}$, $j_{-n}$, $j_{-2n}$, $j_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\beta|^{-1} = 0.5$

$$\sum_{n=0}^{\infty} S_n z^n = \frac{-z}{2z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} R_n z^n = \frac{z^3+2z^2+3z-4}{2z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} J_n z^n = \frac{-z+z^3}{2z^4+z^3+z^2+z-1},$$
$$\sum_{n=0}^{\infty} j_n z^n = \frac{-2z^3-2z^2+z-2}{2z^4+z^3+z^2+z-1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\beta|^{-2} = 0.25$

$$\sum_{n=0}^{\infty} S_{2n} z^n = \frac{z^2+z}{4z^4+3z^3-5z^2-3z+1},$$
$$\sum_{n=0}^{\infty} R_{2n} z^n = \frac{3z^3-10z^2-9z+4}{4z^4+3z^3-5z^2-3z+1},$$
$$\sum_{n=0}^{\infty} J_{2n} z^n = \frac{z-z^3}{4z^4+3z^3-5z^2-3z+1},$$
$$\sum_{n=0}^{\infty} j_{2n} z^n = \frac{-2z^3-5z^2-z+2}{4z^4+3z^3-5z^2-3z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\beta|^{-2} = 0.25$

$$\sum_{n=0}^{\infty} S_{2n+1} z^n = \frac{-2z^2-z+1}{4z^4+3z^3-5z^2-3z+1},$$
$$\sum_{n=0}^{\infty} R_{2n+1} z^n = \frac{2z^3+5z^2+4z+1}{4z^4+3z^3-5z^2-3z+1},$$
$$\sum_{n=0}^{\infty} J_{2n+1} z^n = \frac{2z^3-z^2-2z+1}{4z^4+3z^3-5z^2-3z+1},$$
$$\sum_{n=0}^{\infty} j_{2n+1} z^n = \frac{-4z^3+2z^2+7z+1}{4z^4+3z^3-5z^2-3z+1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\alpha| = |\gamma| = |\delta| = 1$

$$\sum_{n=0}^{\infty} S_{-n} z^n = \frac{z^3}{-z^4+z^3+z^2+z+2},$$
$$\sum_{n=0}^{\infty} R_{-n} z^n = \frac{z^3+2z^2+3z+8}{-z^4+z^3+z^2+z+2},$$
$$\sum_{n=0}^{\infty} J_{-n} z^n = \frac{-z+z^3}{-z^4+z^3+z^2+z+2},$$
$$\sum_{n=0}^{\infty} j_{-n} z^n = \frac{z^3+4z^2+4z+4}{-z^4+z^3+z^2+z+2}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\alpha|^2 = |\beta|^2 = |\delta|^2 = 1$

$$\sum_{n=0}^{\infty} S_{-2n} z^n = \frac{-z^3 - z^2}{z^4 - 3z^3 - 5z^2 + 3z + 4},$$
$$\sum_{n=0}^{\infty} R_{-2n} z^n = \frac{-3z^3 - 10z^2 + 9z + 16}{z^4 - 3z^3 - 5z^2 + 3z + 4},$$
$$\sum_{n=0}^{\infty} J_{-2n} z^n = \frac{z - z^3}{z^4 - 3z^3 - 5z^2 + 3z + 4},$$
$$\sum_{n=0}^{\infty} j_{-2n} z^n = \frac{-5z^3 - 5z^2 + 8z + 8}{z^4 - 3z^3 - 5z^2 + 3z + 4}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\alpha|^2 = |\beta|^2 = |\delta|^2 = 1$

$$\sum_{n=0}^{\infty} S_{-2n+1} z^n = \frac{-2z^3 - 3z^2 + 3z + 4}{z^4 - 3z^3 - 5z^2 + 3z + 4},$$
$$\sum_{n=0}^{\infty} R_{-2n+1} z^n = \frac{-7z^3 - 10z^2 + z + 4}{z^4 - 3z^3 - 5z^2 + 3z + 4},$$
$$\sum_{n=0}^{\infty} J_{-2n+1} z^n = \frac{-z^3 - 4z^2 + z + 4}{z^4 - 3z^3 - 5z^2 + 3z + 4},$$
$$\sum_{n=0}^{\infty} j_{-2n+1} z^n = \frac{-10z^3 - 7z^2 + 7z + 4}{z^4 - 3z^3 - 5z^2 + 3z + 4}.$$

From the last corollary, we obtain the following results for adjusted fourth order Jacobsthal, modified fourth order Jacobsthal-Lucas, fourth order Jacobsthal and fourth order Jacobsthal-Lucas numbers.

**Corollary 2.17.**

*Infinite sums of* $S_n$, $S_{2n}$, $S_{2n+1}$, $S_{-n}$, $S_{-2n}$, $S_{-2n+1}$ *and* $R_n$, $R_{2n}$, $R_{2n+1}$, $R_{-n}$, $R_{-2n}$, $R_{-2n+1}$ *and* $J_n$, $J_{2n}$, $J_{2n+1}$, $J_{-n}$, $J_{-2n}$, $J_{-2n+1}$ *and* $j_n$, $j_{2n}$, $j_{2n+1}$, $j_{-n}$, $j_{-2n}$, $j_{-2n+1}$ *are given as follows:*

**(a)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{S_n}{4^n} = \frac{32}{85},$$
$$\sum_{n=0}^{\infty} \frac{R_n}{4^n} = \frac{398}{85},$$
$$\sum_{n=0}^{\infty} \frac{J_n}{4^n} = \frac{6}{17},$$
$$\sum_{n=0}^{\infty} \frac{j_n}{4^n} = \frac{244}{85}.$$

**(b)** $z = \frac{1}{8}$

$$\sum_{n=0}^{\infty} \frac{S_{2n}}{8^n} = \frac{16}{63},$$
$$\sum_{n=0}^{\infty} \frac{R_{2n}}{8^n} = \frac{310}{63},$$
$$\sum_{n=0}^{\infty} \frac{J_{2n}}{8^n} = \frac{2}{9},$$
$$\sum_{n=0}^{\infty} \frac{j_{2n}}{8^n} = \frac{68}{21}.$$

**(c)** $z=\frac{1}{8}$

$$\sum_{n=0}^{\infty}\frac{S_{2n+1}}{8^n}=\frac{32}{21},$$
$$\sum_{n=0}^{\infty}\frac{R_{2n+1}}{8^n}=\frac{20}{7},$$
$$\sum_{n=0}^{\infty}\frac{J_{2n+1}}{8^n}=\frac{4}{3},$$
$$\sum_{n=0}^{\infty}\frac{j_{2n+1}}{8^n}=\frac{24}{7}.$$

**(d)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{S_{-n}}{2^n}=\frac{2}{45},$$
$$\sum_{n=0}^{\infty}\frac{R_{-n}}{2^n}=\frac{18}{5},$$
$$\sum_{n=0}^{\infty}\frac{J_{-n}}{2^n}=-\frac{2}{15},$$
$$\sum_{n=0}^{\infty}\frac{j_{-n}}{2^n}=\frac{38}{15}.$$

**(e)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{S_{-2n}}{2^n}=-\frac{2}{21},$$
$$\sum_{n=0}^{\infty}\frac{R_{-2n}}{2^n}=\frac{94}{21},$$
$$\sum_{n=0}^{\infty}\frac{J_{-2n}}{2^n}=\frac{2}{21},$$
$$\sum_{n=0}^{\infty}\frac{j_{-2n}}{2^n}=\frac{18}{7}.$$

**(f)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{S_{-2n+1}}{2^n}=\frac{8}{7},$$
$$\sum_{n=0}^{\infty}\frac{R_{-2n+1}}{2^n}=\frac{2}{7},$$
$$\sum_{n=0}^{\infty}\frac{J_{-2n+1}}{2^n}=\frac{6}{7},$$
$$\sum_{n=0}^{\infty}\frac{j_{-2n+1}}{2^n}=\frac{8}{7}.$$

### 2.4. Sum Formulas $\sum_{k=0}^{n}W_k$, $\sum_{k=0}^{n}W_{2k}$, $\sum_{k=0}^{n}W_{2k+1}$, $\sum_{k=0}^{n}W_{-k}$, $\sum_{k=0}^{n}W_{-2k}$, $\sum_{k=0}^{n}W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty}W_nz^n$, $\sum_{n=0}^{\infty}W_{2n}z^n$, $\sum_{n=0}^{\infty}W_{2n+1}z^n$, $\sum_{n=0}^{\infty}W_{-n}z^n$, $\sum_{n=0}^{\infty}W_{-2n}z^n$, $\sum_{n=0}^{\infty}W_{-2n+1}z^n$ of Generalized Four Primes Numbers

In this subsection, we consider the case $r=2$, $s=3$, $t=5$, $u=7$. A generalized four primes sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2,W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n=2W_{n-1}+3W_{n-2}+5W_{n-3}+7W_{n-4} \tag{22}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{5}{7}W_{-(n-1)} - \frac{3}{7}W_{-(n-2)} - \frac{2}{7}W_{-(n-3)} + \frac{1}{7}W_{-(n-4)}$$

for $n = 1,2,3,\ldots$. Therefore, recurrence eq. (22) holds for all integer $n$. For more information on generalized four primes numbers, see Soykan [7].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 2z^3 - 3z^2 - 5z - 7 = 0,$$

whose roots are

$$\begin{aligned}
\alpha &\simeq 3.456157801461113 \\
\beta &\simeq -1.184059685093579 \\
\gamma &\simeq -0.1360490581837671 + 1.300777225148450i \\
\delta &\simeq -0.1360490581837671 - 1.300777225148450i
\end{aligned}$$

Two special cases of the sequence $\{W_n\}$ are four primes sequence $\{G_n\}_{n\geq 0}$ and four primes-Lucas sequence $\{H_n\}_{n\geq 0}$. Four primes sequence $\{G_n\}_{n\geq 0}$ and four primes-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$G_{n+4} = 2G_{n+3} + 3G_{n+2} + 5G_{n+1} + 7G_n, \quad G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2,$$

and

$$H_{n+4} = 2H_{n+3} + 3H_{n+2} + 5H_{n+1} + 7H_n, \quad H_0 = 4, H_1 = 2, H_2 = 10, H_3 = 41.$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{5}{7}G_{-(n-1)} - \frac{3}{7}G_{-(n-2)} - \frac{2}{7}G_{-(n-3)} + \frac{1}{7}G_{-(n-4)}, \tag{23}$$

$$H_{-n} = -\frac{5}{7}H_{-(n-1)} - \frac{3}{7}H_{-(n-2)} - \frac{2}{7}H_{-(n-3)} + \frac{1}{7}H_{-(n-4)} \tag{24}$$

and for $n = 1,2,3,\ldots$ respectively. Therefore, recurrences eqs. (23) and (24) hold for all integer $n$.

Binet's formula of generalized four primes numbers can be given as follows:

**Theorem 2.7.**
*For all integers n, Binet's formula of generalized four primes numbers is*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
&\quad + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}
\end{aligned}$$

*where*

$$\begin{aligned}
p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\
p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\
p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\
p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0.
\end{aligned}$$

Four primes and four primes-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 2.18.**
*For all integers n, Binet's formulas of four primes and four primes-Lucas numbers are*

$$\begin{aligned}
G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
&\quad + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \\
H_n &= \alpha^n + \beta^n + \gamma^n + \delta^n,
\end{aligned}$$

*respectively.*

Next, we present sum formulas of generalized four primes numbers

**Theorem 2.8.**
*For $n \geq 0$, we have the following sum formulas for generalized four primes numbers:*

**(a)** $\sum_{k=0}^{n} W_k = \frac{1}{16}(W_{n+3} - W_{n+2} - 4W_{n+1} + 7W_n - W_3 + W_2 + 4W_1 + 9W_0).$

**(b)** $\sum_{k=0}^{n} W_{2k} = \frac{1}{32}(9W_{2n+2} - 25W_{2n+1} + 28W_{2n} - 49W_{2n-1} + 7W_3 - 23W_2 + 4W_1 - 31W_0).$

**(c)** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{32}(-7W_{2n+2} + 55W_{2n+1} - 4W_{2n} + 63W_{2n-1} - 9W_3 + 25W_2 + 4W_1 + 49W_0).$

**(d)** $\sum_{k=1}^{n} W_{-k} = \frac{1}{16}(-W_{-n+3} + W_{-n+2} + 4W_{-n+1} + 9W_{-n} + W_3 - W_2 - 4W_1 - 9W_0).$

**(e)** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{32}(-9W_{-2n+2} + 25W_{-2n+1} + 4W_{-2n} + 49W_{-2n-1} - 7W_3 + 23W_2 - 4W_1 + 31W_0).$

**(f)** $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{32}(7W_{-2n+2} - 23W_{-2n+1} + 4W_{-2n} - 63W_{-2n-1} + 9W_3 - 25W_2 - 4W_1 - 49W_0).$

Proof.

**(a)** For $r = 2, s = 3, t = 5, u = 7$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = 7z^4 + 5z^3 + 3z^2 + 2z - 1$ and then for $z = 1$, we get $7z^4 + 5z^3 + 3z^2 + 2z - 1 \neq 0$ so we use theorem 1.1 (a) (i) with $z = 1$.

**(b)** For $r = 2, s = 3, t = 5, u = 7$, we get $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = -49z^4 - 17z^3 + 25z^2 + 10z - 1$ and then for $z = 1$, we get $-49z^4 - 17z^3 + 25z^2 + 10z - 1 \neq 0$ so we use theorem 1.1 (b) (i) with $z = 1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (i) with $z = 1$.

**(d)** For $r = 2, s = 3, t = 5, u = 7$, we get $-z^4 + rz^3 + sz^2 + tz + u = -z^4 + 2z^3 + 3z^2 + 5z + 7$ and then for $z = 1$, we get $-z^4 + 2z^3 + 3z^2 + 5z + 7 \neq 0$ so we use theorem 1.2 (a) (i) with $z = 1$.

**(e)** For $r = 2, s = 3, t = 5, u = 7$, we get $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 = -z^4 + 10z^3 + 25z^2 - 17z - 49$ and then for $z = 1$, we get $-z^4 + 10z^3 + 25z^2 - 17z - 49 \neq 0$ so we use theorem 1.2 (b) (i) with $z = 1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (i) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of four primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

**Corollary 2.19.**
*For $n \geq 0$, four primes numbers have the following properties.*

**(a)** $\sum_{k=0}^{n} G_k = \frac{1}{16}(G_{n+3} - G_{n+2} - 4G_{n+1} + 7G_n - 1).$

**(b)** $\sum_{k=0}^{n} G_{2k} = \frac{1}{32}(9G_{2n+2} - 25G_{2n+1} + 28G_{2n} - 49G_{2n-1} - 9).$

**(c)** $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{32}(-7G_{2n+2} + 55G_{2n+1} - 4G_{2n} + 63G_{2n-1} + 7).$

**(d)** $\sum_{k=1}^{n} G_{-k} = \frac{1}{16}(-G_{-n+3} + G_{-n+2} + 4G_{-n+1} + 9G_{-n} + 1).$

**(e)** $\sum_{k=1}^{n} G_{-2k} = \frac{1}{32}(-9G_{-2n+2} + 25G_{-2n+1} + 4G_{-2n} + 49G_{-2n-1} + 9).$

**(f)** $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{32}(7G_{-2n+2} - 23G_{-2n+1} + 4G_{-2n} - 63G_{-2n-1} - 7).$

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 10, H_3 = 41$ in the last Theorem, we have the following Corollary which gives sum formulas of four primes-Lucas numbers.

**Corollary 2.20.**
*For $n \geq 0$, four primes-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} H_k = \frac{1}{16}(H_{n+3} - H_{n+2} - 4H_{n+1} + 7H_n + 13).$

**(b)** $\sum\limits_{k=0}^{n} H_{2k} = \frac{1}{32}(9H_{2n+2} - 25H_{2n+1} + 28H_{2n} - 49H_{2n-1} - 59).$

**(c)** $\sum\limits_{k=0}^{n} H_{2k+1} = \frac{1}{32}(-7H_{2n+2} + 55H_{2n+1} - 4H_{2n} + 63H_{2n-1} + 85).$

**(d)** $\sum\limits_{k=1}^{n} H_{-k} = \frac{1}{16}(-H_{-n+3} + H_{-n+2} + 4H_{-n+1} + 9H_{-n} - 13).$

**(e)** $\sum\limits_{k=1}^{n} H_{-2k} = \frac{1}{32}(-9H_{-2n+2} + 25H_{-2n+1} + 4H_{-2n} + 49H_{-2n-1} + 59).$

**(f)** $\sum\limits_{k=1}^{n} H_{-2k+1} = \frac{1}{32}(7H_{-2n+2} - 23H_{-2n+1} + 4H_{-2n} - 63H_{-2n-1} - 85).$

Next, we give the ordinary generating functions of some special cases of generalized numbers.

**Lemma 2.4.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.289338$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{-z^3W_3 + (2z^3 - z^2)W_2 + (3z^3 + 2z^2 - z)W_1 + (5z^3 + 3z^2 + 2z - 1)W_0}{7z^4 + 5z^3 + 3z^2 + 2z - 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.083716$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{(5z^3 + 2z^2)W_3 - (17z^3 + 7z^2 - z)W_2 + (5z^2 - z^3)W_1 - (4z^3 + 18z^2 + 10z - 1)W_0}{49z^4 + 17z^3 - 25z^2 - 10z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.083716$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{-(7z^3 + 3z^2 - z)W_3 + (14z^3 + 11z^2)W_2 + 7(5z^3 + 2z^2)W_0 + (21z^3 - 8z^2 - 10z + 1)W_1}{49z^4 + 17z^3 - 25z^2 - 10z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| \simeq 1.184059$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{zW_3 + (z^2 - 2z)W_2 + (z^3 - 2z^2 - 3z)W_1 + 7W_0}{-z^4 + 2z^3 + 3z^2 + 5z + 7}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\gamma|^2 = |\delta|^2 \simeq 1.71053$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-(2z^2 + 5z)W_3 + (-z^3 + 7z^2 + 17z)W_2 + (z - 5z^2)W_1 + (-7z^2 + 21z + 49)W_0}{z^4 - 10z^3 - 25z^2 + 17z + 49}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\gamma|^2 = |\delta|^2 \simeq 1.71053$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{(-z^3 + 3z^2 + 7z)W_3 - (11z^2 + 14z)W_2 - (17z^2 + 4z - 49)W_1 - 7(2z^2 + 5z)W_0}{z^4 - 10z^3 - 25z^2 + 17z + 49}.$$

Proof. Use Lemma lemmas 1.1 and 1.2. □
Now, we consider special cases of the last Lemma.

**Corollary 2.21.**
*The ordinary generating functions of the sequences $G_n$, $G_{2n}$, $G_{2n+1}$, $G_{-n}$, $G_{-2n}$, $G_{-2n+1}$ and $H_n$, $H_{2n}$, $H_{2n+1}$, $H_{-n}$, $H_{-2n}$, $H_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.289338$

$$\sum_{n=0}^{\infty} G_n z^n = \frac{-z^2}{7z^4+5z^3+3z^2+2z-1},$$
$$\sum_{n=0}^{\infty} H_n z^n = \frac{5z^3+6z^2+6z-4}{7z^4+5z^3+3z^2+2z-1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.083716$

$$\sum_{n=0}^{\infty} G_{2n} z^n = \frac{-7z^3-3z^2+z}{49z^4+17z^3-25z^2-10z+1},$$
$$\sum_{n=0}^{\infty} H_{2n} z^n = \frac{17z^3-50z^2-30z+4}{49z^4+17z^3-25z^2-10z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.083716$

$$\sum_{n=0}^{\infty} G_{2n+1} z^n = \frac{5z^2+2z}{49z^4+17z^3-25z^2-10z+1},$$
$$\sum_{n=0}^{\infty} H_{2n+1} z^n = \frac{35z^3+27z^2+21z+2}{49z^4+17z^3-25z^2-10z+1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| \simeq 1.184059$

$$\sum_{n=0}^{\infty} G_{-n} z^n = \frac{z^2}{-z^4+2z^3+3z^2+5z+7},$$
$$\sum_{n=0}^{\infty} H_{-n} z^n = \frac{2z^3+6z^2+15z+28}{-z^4+2z^3+3z^2+5z+7}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\gamma|^2 = |\delta|^2 \simeq 1.71053$

$$\sum_{n=0}^{\infty} G_{-2n} z^n = \frac{-z^3+3z^2+7z}{z^4-10z^3-25z^2+17z+49},$$
$$\sum_{n=0}^{\infty} H_{-2n} z^n = \frac{-10z^3-50z^2+51z+196}{z^4-10z^3-25z^2+17z+49}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\gamma|^2 = |\delta|^2 \simeq 1.71053$

$$\sum_{n=0}^{\infty} G_{-2n+1} z^n = \frac{-2z^3-5z^2}{z^4-10z^3-25z^2+17z+49},$$
$$\sum_{n=0}^{\infty} H_{-2n+1} z^n = \frac{-41z^3-77z^2-z+98}{z^4-10z^3-25z^2+17z+49}.$$

From the last corollary, we obtain the following results for four primes and four primes-Lucas numbers.

**Corollary 2.22.**
*Infinite sums of $G_n$, $G_{2n}$, $G_{2n+1}$, $G_{-n}$, $G_{-2n}$, $G_{-2n+1}$ and $H_n$, $H_{2n}$, $H_{2n+1}$, $H_{-n}$, $H_{-2n}$, $H_{-2n+1}$ are given as follows:*

**(a)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{G_n}{4^n} = \frac{16}{53},$$
$$\sum_{n=0}^{\infty} \frac{H_n}{4^n} = \frac{524}{53}.$$

**(b)** $z = \frac{1}{16}$

$$\sum_{n=0}^{\infty} \frac{G_{2n}}{16^n} = \frac{3216}{18497},$$
$$\sum_{n=0}^{\infty} \frac{H_{2n}}{16^n} = \frac{126736}{18497}.$$

**(c)** $z = \frac{1}{16}$

$$\sum_{n=0}^{\infty} \frac{G_{2n+1}}{16^n} = \frac{9472}{18497},$$
$$\sum_{n=0}^{\infty} \frac{H_{2n+1}}{16^n} = \frac{224560}{18497}.$$

**(d)** $z = 1$

$$\sum_{n=0}^{\infty} G_{-n} = \frac{1}{16},$$
$$\sum_{n=0}^{\infty} H_{-n} = \frac{51}{16}.$$

**(e)** $z = 1$

$$\sum_{n=0}^{\infty} G_{-2n} = \frac{9}{32},$$
$$\sum_{n=0}^{\infty} H_{-2n} = \frac{187}{32}.$$

**(f)** $z = 1$

$$\sum_{n=0}^{\infty} G_{-2n+1} = -\frac{7}{32},$$
$$\sum_{n=0}^{\infty} H_{-2n+1} = -\frac{21}{32}.$$

### 2.5. Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} W_n z^n$, $\sum_{n=0}^{\infty} W_{2n} z^n$, $\sum_{n=0}^{\infty} W_{2n+1} z^n$, $\sum_{n=0}^{\infty} W_{-n} z^n$, $\sum_{n=0}^{\infty} W_{-2n} z^n$, $\sum_{n=0}^{\infty} W_{-2n+1} z^n$ of Generalized Tridovan Numbers

In this subsection, we consider the case $r = 0$, $s = 1$, $t = 1$, $u = 1$.A generalized Tridovan sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth order recurrence relation

$$W_n = W_{n-2} + W_{n-3} + W_{n-4} \tag{25}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} - W_{-(n-2)} + W_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (25) holds for all integers $n$. For more information on generalized Tridovan numbers, see [8].

Characteristic equation of $\{W_n\}$ is

$$z^4 - z^2 - z - 1 = (z^3 - z^2 - 1)(z + 1) = 0, \tag{26}$$

whose roots are

$$\alpha = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$
$$\beta = \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$
$$\gamma = \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$
$$\delta = -1,$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Two special cases of the sequence $\{W_n\}$ are Tridovan sequence $\{T_n\}_{n\geq 0}$ and Tridovan-Lucas sequence $\{H_n\}_{n\geq 0}$. Tridovan sequence $\{T_n\}_{n\geq 0}$ and Tridovan-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$T_n = T_{n-2} + T_{n-3} + T_{n-4}, \quad T_0 = 0, T_1 = 1, T_2 = 0, T_3 = 1, \tag{27}$$
$$H_n = H_{n-2} + H_{n-3} + H_{n-4}, \quad H_0 = 4, H_1 = 0, H_2 = 2, H_3 = 3. \tag{28}$$

The sequences $\{T_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-4)},$$
$$H_{-n} = -H_{-(n-1)} - H_{-(n-2)} + H_{-(n-4)},$$

for $n = 1, 2, 3, \ldots$ respectively. So, recurrences eq. (27)–eq. (28) hold for all integer $n$.

Binet's formula of generalized Tridovan numbers can be given as follows:

**Theorem 2.9.**
*For all integers n, Binet's formula of generalized Tridovan numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

*where*

$$p_1 = W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0,$$
$$p_2 = W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0,$$
$$p_3 = W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0,$$
$$p_4 = W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0.$$

Tridovan and Tridovan-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 2.23.**
*For all integers n, Binet's formulas of Tridovan and Tridovan-Lucas numbers are*

$$T_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},$$
$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

*respectively.*

Next, we present sum formulas of generalized Tridovan numbers

**Theorem 2.10.**
*For $n \geq 0$, we have the following sum formulas for generalized Tridovan numbers:*

**(a)** $\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+3} + W_{n+2} + W_n - W_3 - W_2 + W_0)$.

**(b)** $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}((n+4)W_{2n+2} - (n+2)W_{2n+1} + 2W_{2n} - (n+3)W_{2n-1} + 3W_3 - W_1 - 4W_2 - 2W_0)$.

**(c)** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(n+2)W_{2n+2} + (n+6)W_{2n+1} + W_{2n} + (n+4)W_{2n-1} - 4W_3 + 2W_2 + W_1 + 3W_0)$.

**(d)** $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-W_{-n+3} - W_{-n+2} + W_{-n} + W_3 + W_2 - W_0)$.

**(e)** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(nW_{-2n+2} - (n+2)W_{-2n+1} + W_{-2n} - (n+1)W_{-2n-1} + W_3 + W_1 - 2W_0)$.

**(f)** $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(n+2)W_{-2n+2} + (n+1)W_{-2n+1} - W_{-2n} + nW_{-2n-1} + 2W_2 - W_1 + W_0)$.

Proof.

**(a)** For $r = 0, s = 1, t = 1, u = 1$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = (z+1)(z^3 + z - 1)$ and then for $z = 1$, we get $(z+1)(z^3 + z - 1) \neq 0$ so we use theorem 1.1 (a) (i) with $z = 1$.

**(b)** For $r = 0, s = 1, t = 1, u = 1$, we get $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = -(z-1)(z^3 + 2z^2 + z - 1)$ and then for $z = 1$, we get $-(z-1)(z^3 + 2z^2 + z - 1) = 0$ with multiplicity 1 so we use theorem 1.1 (b) (ii) with $z = 1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (ii) with $z = 1$.

**(d)** For $r = 0, s = 1, t = 1, u = 1$, we get $-z^4 + rz^3 + sz^2 + tz + u = (z+1)(-z^3 + z^2 + 1)$ and then for $z = 1$, we get $(z+1)(-z^3 + z^2 + 1) \neq 0$ so we use theorem 1.2 (a) (i) with $z = 1$.

**(e)** For $r = 0, s = 1, t = 1, u = 1$, we get $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 = (z-1)(-z^3 + z^2 + 2z + 1)$ and then for $z = 1$, we get $(z-1)(-z^3 + z^2 + 2z + 1) = 0$ with multiplicity 1 so we use theorem 1.2 (b) (ii) with $z = 1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Tridovan numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 0, T_3 = 1$).

**Corollary 2.24.**
*For $n \geq 0$, Tridovan numbers have the following properties.*

**(a)** $\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+3} + T_{n+2} + T_n - 1)$.

**(b)** $\sum_{k=0}^{n} T_{2k} = \frac{1}{3}((n+4)T_{2n+2} - (n+2)T_{2n+1} + 2T_{2n} - (n+3)T_{2n-1} + 2)$.

**(c)** $\sum_{k=0}^{n} T_{2k+1} = \frac{1}{3}(-(n+2)T_{2n+2} + (n+6)T_{2n+1} + T_{2n} + (n+4)T_{2n-1} - 3)$.

**(d)** $\sum_{k=1}^{n} T_{-k} = \frac{1}{2}(-T_{-n+3} - T_{-n+2} + T_{-n} + 1)$.

**(e)** $\sum_{k=1}^{n} T_{-2k} = \frac{1}{3}(nT_{-2n+2} - (n+2)T_{-2n+1} + T_{-2n} - (n+1)T_{-2n-1} + 2)$.

**(f)** $\sum_{k=1}^{n} T_{-2k+1} = \frac{1}{3}(-(n+2)T_{-2n+2} + (n+1)T_{-2n+1} - T_{-2n} + nT_{-2n-1} - 1)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 0, H_2 = 2, H_3 = 3$ in the last Theorem, we have the following Corollary which gives sum formulas of Tridovan-Lucas numbers.

**Corollary 2.25.**
*For $n \geq 0$, Tridovan-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} H_k = \frac{1}{2}(H_{n+3} + H_{n+2} + H_n - 1)$.

**(b)** $\sum\limits_{k=0}^{n} H_{2k} = \frac{1}{3}((n+4)H_{2n+2} - (n+2)H_{2n+1} + 2H_{2n} - (n+3)H_{2n-1} - 7)$.

**(c)** $\sum\limits_{k=0}^{n} H_{2k+1} = \frac{1}{3}(-(n+2)H_{2n+2} + (n+6)H_{2n+1} + H_{2n} + (n+4)H_{2n-1} + 4)$.

**(d)** $\sum\limits_{k=1}^{n} H_{-k} = \frac{1}{2}(-H_{-n+3} - H_{-n+2} + H_{-n} + 1)$.

**(e)** $\sum\limits_{k=1}^{n} H_{-2k} = \frac{1}{3}(nH_{-2n+2} - (n+2)H_{-2n+1} + H_{-2n} - (n+1)H_{-2n-1} - 5)$.

**(f)** $\sum\limits_{k=1}^{n} H_{-2k+1} = \frac{1}{3}(-(n+2)H_{-2n+2} + (n+1)H_{-2n+1} - H_{-2n} + nH_{-2n-1} + 8)$.

Next, we give the ordinary generating functions of some special cases of generalized Tridovan numbers.

**Lemma 2.5.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.682327$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{-z^3W_3 - z^2W_2 + (z^3 - z)W_1 + (z^3 + z^2 - 1)W_0}{z^4 + z^3 + z^2 - 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.465571$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{z^3W_3 - (z^3 + z^2 - z)W_2 + (z^2 - z^3)W_1 - (2z - 1)W_0}{z^4 + z^3 - z^2 - 2z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.465571$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{-(z^3 + z^2 - z)W_3 + z^2W_2 + (z^3 - 2z + 1)W_1 + z^3W_0}{z^4 + z^3 - z^2 - 2z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.826031$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{zW_3 + z^2W_2 + (z^3 - z)W_1 + W_0}{-z^4 + z^2 + z + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.682327$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-zW_3 + (-z^3 + z^2 + z)W_2 + (z - z^2)W_1 + (-z^2 + z + 1)W_0}{z^4 - 2z^3 - z^2 + z + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.682327$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{(-z^3 + z^2 + z)W_3 - z^2W_2 - (z^2 - 1)W_1 - zW_0}{z^4 - 2z^3 - z^2 + z + 1}.$$

Proof. Use lemmas 1.1 and 1.2. □
Now, we consider special cases of the last Lemma.

**Corollary 2.26.**
*The ordinary generating functions of the sequences $T_n$, $T_{2n}$, $T_{2n+1}$, $T_{-n}$, $T_{-2n}$, $T_{-2n+1}$ and $H_n$, $H_{2n}$, $H_{2n+1}$, $H_{-n}$, $H_{-2n}$, $H_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.682327$

$$\sum_{n=0}^{\infty} T_n z^n = \frac{-z}{z^4+z^3+z^2-1},$$
$$\sum_{n=0}^{\infty} H_n z^n = \frac{z^3+2z^2-4}{z^4+z^3+z^2-1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.465571$

$$\sum_{n=0}^{\infty} T_{2n} z^n = \frac{z^2}{z^4+z^3-z^2-2z+1},$$
$$\sum_{n=0}^{\infty} H_{2n} z^n = \frac{z^3-2z^2-6z+4}{z^4+z^3-z^2-2z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.465571$

$$\sum_{n=0}^{\infty} T_{2n+1} z^n = \frac{-z^2-z+1}{z^4+z^3-z^2-2z+1},$$
$$\sum_{n=0}^{\infty} H_{2n+1} z^n = \frac{z^3-z^2+3z}{z^4+z^3-z^2-2z+1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.826031$

$$\sum_{n=0}^{\infty} T_{-n} z^n = \frac{z^3}{-z^4+z^2+z+1},$$
$$\sum_{n=0}^{\infty} H_{-n} z^n = \frac{2z^2+3z+4}{-z^4+z^2+z+1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.682327$

$$\sum_{n=0}^{\infty} T_{-2n} z^n = \frac{-z^2}{z^4-2z^3-z^2+z+1},$$
$$\sum_{n=0}^{\infty} H_{-2n} z^n = \frac{-2z^3-2z^2+3z+4}{z^4-2z^3-z^2+z+1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.682327$

$$\sum_{n=0}^{\infty} T_{-2n+1} z^n = \frac{-z^3+z+1}{z^4-2z^3-z^2+z+1},$$
$$\sum_{n=0}^{\infty} H_{-2n+1} z^n = \frac{-3z^3+z^2-z}{z^4-2z^3-z^2+z+1}.$$

From the last corollary, we obtain the following results for Tridovan and Tridovan-Lucas numbers.

**Corollary 2.27.**
*Infinite sums of* $T_n, T_{2n}, T_{2n+1}, T_{-n}, T_{-2n}, T_{-2n+1}$ *and* $H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1}$ *are given as follows:*

**(a)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{T_n}{2^n} = \frac{8}{9},$$
$$\sum_{n=0}^{\infty} \frac{H_n}{2^n} = 6.$$

**(b)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{T_{2n}}{4^n} = \frac{16}{117},$$
$$\sum_{n=0}^{\infty} \frac{H_{2n}}{4^n} = \frac{68}{13}.$$

**(c)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{T_{2n+1}}{4^n} = \frac{176}{117},$$
$$\sum_{n=0}^{\infty} \frac{H_{2n+1}}{4^n} = \frac{20}{13}.$$

**(d)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{T_{-n}}{2^n} = \frac{2}{27},$$
$$\sum_{n=0}^{\infty} \frac{H_{-n}}{2^n} = \frac{32}{9}.$$

**(e)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{T_{-2n}}{2^n} = -\frac{4}{17},$$
$$\sum_{n=0}^{\infty} \frac{H_{-2n}}{2^n} = \frac{76}{17}.$$

**(f)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{T_{-2n+1}}{2^n} = \frac{22}{17},$$
$$\sum_{n=0}^{\infty} \frac{H_{-2n+1}}{2^n} = -\frac{10}{17}.$$

## 3. The Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions of Special Cases of Generalized Tetranacci Polynomials/Numbers: Second Group

In this section, we present special cases of sum formulas of generalized Tetranacci numbers.

### 3.1. Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} W_n z^n$, $\sum_{n=0}^{\infty} W_{2n} z^n$, $\sum_{n=0}^{\infty} W_{2n+1} z^n$, $\sum_{n=0}^{\infty} W_{-n} z^n$, $\sum_{n=0}^{\infty} W_{-2n} z^n$, $\sum_{n=0}^{\infty} W_{-2n+1} z^n$ of Generalized Richard Numbers

In this subsection, we consider the case $r = 1, s = 1, t = 0, u = -1$. A generalized Richard sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = W_{n-1} + W_{n-2} - W_{n-4} \tag{29}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (29) holds for all integers $n$. For more information on generalized Richard numbers, see Soykan [9].

Characteristic equation of $\{W_n\}$ is

$$z^4 - z^3 - z^2 + 1 = (z^3 - z - 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned}
\alpha &= \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} = 1.32471795724, \\
\beta &= \omega\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}, \\
\gamma &= \omega^2\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Two special cases of the sequence $\{W_n\}$ are Richard sequence $\{R_n\}_{n\geq 0}$ and Richard-Lucas sequence $\{Q_n\}_{n\geq 0}$. Richard sequence$\{R_n\}_{n\geq 0}$ and Richard-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}
R_n &= R_{n-1} + R_{n-2} - R_{n-4}, \quad R_0 = 0, R_1 = 1, R_2 = 1, R_3 = 2, \quad n \geq 4, \\
Q_n &= Q_{n-1} + Q_{n-2} - Q_{n-4}, \quad Q_0 = 4, Q_1 = 1, Q_2 = 3, Q_3 = 4, \quad n \geq 4.
\end{aligned}$$

The sequences $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
R_{-n} &= R_{-(n-2)} + R_{-(n-3)} - R_{-(n-4)} \\
Q_{-n} &= Q_{-(n-2)} + Q_{-(n-3)} - Q_{-(n-4)}
\end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively.

Binet's formula of generalized Richard numbers can be given as follows:

**Theorem 3.1.**
*For all integers n, Binet's formula of generalized Richard numbers is*

$$\begin{aligned}
W_n &= \frac{(\alpha W_3 - \alpha(1-\alpha)W_2 + (-\alpha^2 + 1)W_1 - W_0)\alpha^n}{2\alpha^2 + \alpha - 3} \\
&\quad + \frac{(\beta W_3 - \beta(1-\beta)W_2 + (-\beta^2 + 1)W_1 - W_0)\beta^n}{2\beta^2 + \beta - 3} \\
&\quad + \frac{(\gamma W_3 - \gamma(1-\gamma)W_2 + (-\gamma^2 + 1)W_1 - W_0)\gamma^n}{2\gamma^2 + \gamma - 3} - W_3 + W_1 + W_0.
\end{aligned}$$

Richard and Richard-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 3.1.**
*For all integers n, Binet's formulas of Richard and Richard-Lucas numbers are*

$$\begin{aligned}
R_n &= \frac{(\alpha+1)\alpha^n}{2\alpha^2 + \alpha - 3} + \frac{(\beta+1)\beta^n}{2\beta^2 + \beta - 3} + \frac{(\gamma+1)\gamma^n}{2\gamma^2 + \gamma - 3} - 1 \\
&= \frac{\alpha^{n+3}}{2\alpha^2 + \alpha - 3} + \frac{\beta^{n+3}}{2\beta^2 + \beta - 3} + \frac{\gamma^{n+3}}{2\gamma^2 + \gamma - 3} - 1, \\
Q_n &= \alpha^n + \beta^n + \gamma^n + 1,
\end{aligned}$$

*respectively.*

Next, we present sum formulas of generalized Richard numbers

**Theorem 3.2.**
*For $n \geq 0$, we have the following sum formulas for generalized Richard numbers:*

**(a)** $\sum_{k=0}^{n} W_k = -(n+3)W_{n+3} + W_{n+2} + (n+4)W_{n+1} + 4W_n + nW_n + 3W_3 - W_2 - 4W_1 - 3W_0.$

**(b)** $\sum_{k=0}^{n} W_{2k} = -(n+2)W_{2n+2} + W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 3W_1 - 2W_0.$

**(c)** $\sum_{k=0}^{n} W_{2k+1} = -(n+1)W_{2n+2} + W_{2n+1} + (n+2)W_{2n} + nW_{2n-1} + 2W_{2n-1} + 2W_3 - W_2 - 2W_1 - 2W_0.$

**(d)** $\sum_{k=1}^{n} W_{-k} = -(n+1)W_{-n+3} - W_{-n+2} + nW_{-n+1} + (n+1)W_{-n} + W_3 + W_2 - W_0.$

**(e)** $\sum_{k=1}^{n} W_{-2k} = -(n+2)W_{-2n+2} - W_{-2n+1} + (n+2)W_{-2n} + (n+2)W_{-2n-1} + 2W_3 - W_1 - 2W_0.$

**(f)** $\sum_{k=1}^{n} W_{-2k+1} = -(n+3)W_{-2n+2} + (n+2)W_{-2n} + (n+2)W_{-2n-1} + 2W_3 + W_2 - 2W_1 - 2W_0.$

Proof.

**(a)** For $r = 1, s = 1, t = 0, u = -1$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = -(z-1)(z^3 + z^2 - 1)$ and then for $z = 1$, we get $-(z-1)(z^3 + z^2 - 1) = 0$ with multiplicity 1 so we use theorem 1.1 (a) (ii) with $z = 1$.

**(b)** For $r = 1, s = 1, t = 0, u = -1$, we get $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = (z-1)(-z^3 + z^2 - 2z + 1)$ and then for $z = 1$, we get $(z-1)(-z^3 + z^2 - 2z + 1) = 0$ with multiplicity 1 so we use theorem 1.1 (b) (ii) with $z = 1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (ii) with $z = 1$.

**(d)** For $r = 1, s = 1, t = 0, u = -1$, we get $-z^4 + rz^3 + sz^2 + tz + u = -(z-1)(z^3 - z - 1)$ and then for $z = 1$, we get $-(z-1)(z^3 - z - 1) = 0$ with multiplicity 1 so we use theorem 1.2 (a) (ii) with $z = 1$.

**(e)** For $r = 1, s = 1, t = 0, u = -1$, we get $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 = (z-1)(-z^3 + 2z^2 - z + 1)$ and then for $z = 1$, we get $(z-1)(-z^3 + 2z^2 - z + 1) = 0$ with multiplicity 1 so we use theorem 1.2 (b) (ii) with $z = 1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Richard numbers (take $W_n = R_n$ with $R_0 = 0, R_1 = 1, R_2 = 1, R_3 = 2$).

**Corollary 3.2.**
*For $n \geq 0$, Richard numbers have the following properties.*

**(a)** $\sum_{k=0}^{n} R_k = -(n+3)R_{n+3} + R_{n+2} + (n+4)R_{n+1} + 4R_n + nR_n + 1.$

**(b)** $\sum_{k=0}^{n} R_{2k} = -(n+2)R_{2n+2} + R_{2n+1} + (n+3)R_{2n} + (n+2)R_{2n-1} + 1.$

**(c)** $\sum_{k=0}^{n} R_{2k+1} = -(n+1)R_{2n+2} + R_{2n+1} + (n+2)R_{2n} + nR_{2n-1} + 2R_{2n-1} + 1.$

**(d)** $\sum_{k=1}^{n} R_{-k} = -(n+1)R_{-n+3} - R_{-n+2} + nR_{-n+1} + (n+1)R_{-n} + 3.$

**(e)** $\sum_{k=1}^{n} R_{-2k} = -(n+2)R_{-2n+2} - R_{-2n+1} + (n+2)R_{-2n} + (n+2)R_{-2n-1} + 3.$

**(f)** $\sum_{k=1}^{n} R_{-2k+1} = -(n+3)R_{-2n+2} + (n+2)R_{-2n} + (n+2)R_{-2n-1} + 3.$

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 1, Q_2 = 3, Q_3 = 4$ in the last Theorem, we have the following Corollary which gives sum formulas of Richard-Lucas numbers.

**Corollary 3.3.**
*For $n \geq 0$, Richard-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} Q_k = -(n+3)Q_{n+3} + Q_{n+2} + (n+4)Q_{n+1} + 4Q_n + nQ_n - 7.$

**(b)** $\sum\limits_{k=0}^{n} Q_{2k} = -(n+2)Q_{2n+2} + Q_{2n+1} + (n+3)Q_{2n} + (n+2)Q_{2n-1} - 3.$

**(c)** $\sum\limits_{k=0}^{n} Q_{2k+1} = -(n+1)Q_{2n+2} + Q_{2n+1} + (n+2)Q_{2n} + nQ_{2n-1} + 2Q_{2n-1} - 5.$

**(d)** $\sum\limits_{k=1}^{n} Q_{-k} = -(n+1)Q_{-n+3} - Q_{-n+2} + nQ_{-n+1} + (n+1)Q_{-n} + 3.$

**(e)** $\sum\limits_{k=1}^{n} Q_{-2k} = -(n+2)Q_{-2n+2} - Q_{-2n+1} + (n+2)Q_{-2n} + (n+2)Q_{-2n-1} - 1.$

**(f)** $\sum\limits_{k=1}^{n} Q_{-2k+1} = -(n+3)Q_{-2n+2} + (n+2)Q_{-2n} + (n+2)Q_{-2n-1} + 1.$

Next, we give the ordinary generating functions of some special cases of generalized Richard numbers.

**Lemma 3.1.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.754877.$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{-z^3W_3 + (z^3-z^2)W_2 + (z^3+z^2-z)W_1 + (z^2+z-1)W_0}{-z^4+z^2+z-1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.56984.$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{z^2W_3 + (z^3-2z^2+z)W_2 - z^3W_1 + (-z^3+2z^2-3z+1)W_0}{z^4-2z^3+3z^2-3z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.56984.$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{(z^3-z^2+z)W_3 + (z^2-z^3)W_2 + (-z^3+2z^2-3z+1)W_1 - z^2W_0}{z^4-2z^3+3z^2-3z+1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.868836.$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{zW_3 + (z^2-rz)W_2 + (z^3-rz^2-sz)W_1 + uW_0}{-z^4+rz^3+sz^2+tz+u}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.754877.$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-z^2W_3 - (z^3-2z^2+z)W_2 + zW_1 + (z^2-z+1)W_0}{z^4-3z^3+3z^2-2z+1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.754877.$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{-(z^3-z^2+z)W_3 + \left(z-z^2\right)W_2 + (z^2-z+1)W_1 + z^2W_0}{z^4-3z^3+3z^2-2z+1}.$$

Proof. Use lemmas 1.1 and 1.2. □
Now, we consider special cases of the last Lemma.

**Corollary 3.4.**
*The ordinary generating functions of the sequences $R_n$, $R_{2n}$, $R_{2n+1}$, $R_{-n}$, $R_{-2n}$, $R_{-2n+1}$ and $Q_n$, $Q_{2n}$, $Q_{2n+1}$, $Q_{-n}$, $Q_{-2n}$, $Q_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.754877.$

$$\sum_{n=0}^{\infty} R_n z^n = \frac{-z}{-z^4+z^2+z-1},$$
$$\sum_{n=0}^{\infty} Q_n z^n = \frac{2z^2+3z-4}{-z^4+z^2+z-1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.56984.$

$$\sum_{n=0}^{\infty} R_{2n} z^n = \frac{z}{z^4-2z^3+3z^2-3z+1},$$
$$\sum_{n=0}^{\infty} Q_{2n} z^n = \frac{-2z^3+6z^2-9z+4}{z^4-2z^3+3z^2-3z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.56984.$

$$\sum_{n=0}^{\infty} R_{2n+1} z^n = \frac{z^2-z+1}{z^4-2z^3+3z^2-3z+1},$$
$$\sum_{n=0}^{\infty} Q_{2n+1} z^n = \frac{-3z^2+z+1}{z^4-2z^3+3z^2-3z+1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.868836.$

$$\sum_{n=0}^{\infty} R_{-n} z^n = \frac{z^3}{-z^4+z^3+z^2-1},$$
$$\sum_{n=0}^{\infty} Q_{-n} z^n = \frac{z^3+2z^2-4}{-z^4+z^3+z^2-1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.754877.$

$$\sum_{n=0}^{\infty} R_{-2n} z^n = \frac{-z^3}{z^4-3z^3+3z^2-2z+1},$$
$$\sum_{n=0}^{\infty} Q_{-2n} z^n = \frac{-3z^3+6z^2-6z+4}{z^4-3z^3+3z^2-2z+1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.754877.$

$$\sum_{n=0}^{\infty} R_{-2n+1} z^n = \frac{-2z^3+2z^2-2z+1}{z^4-3z^3+3z^2-2z+1},$$
$$\sum_{n=0}^{\infty} Q_{-2n+1} z^n = \frac{-4z^3+6z^2-2z+1}{z^4-3z^3+3z^2-2z+1}.$$

From the last corollary, we obtain the following results for Richard and Richard-Lucas numbers.

**Corollary 3.5.**
*Infinite sums of $R_n$, $R_{2n}$, $R_{2n+1}$, $R_{-n}$, $R_{-2n}$, $R_{-2n+1}$ and $Q_n$, $Q_{2n}$, $Q_{2n+1}$, $Q_{-n}$, $Q_{-2n}$, $Q_{-2n+1}$ are given as follows:*

**(a)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{R_n}{2^n} = \frac{8}{5},$$
$$\sum_{n=0}^{\infty} \frac{Q_n}{2^n} = \frac{32}{5}.$$

**(b)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{R_{2n}}{2^n}=8,$$
$$\sum_{n=0}^{\infty}\frac{Q_{2n}}{2^n}=12.$$

**(c)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{R_{2n+1}}{2^n}=12,$$
$$\sum_{n=0}^{\infty}\frac{Q_{2n+1}}{2^n}=12.$$

**(d)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{R_{-n}}{2^n}=-\frac{2}{11},$$
$$\sum_{n=0}^{\infty}\frac{Q_{-n}}{2^n}=\frac{54}{11}.$$

**(e)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{R_{-2n}}{2^n}=-\frac{2}{7},$$
$$\sum_{n=0}^{\infty}\frac{Q_{-2n}}{2^n}=\frac{34}{7}.$$

**(f)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{R_{-2n+1}}{2^n}=\frac{4}{7},$$
$$\sum_{n=0}^{\infty}\frac{Q_{-2n+1}}{2^n}=\frac{16}{7}.$$

### 3.2. Sum Formulas $\sum_{k=0}^{n}W_k$, $\sum_{k=0}^{n}W_{2k}$, $\sum_{k=0}^{n}W_{2k+1}$, $\sum_{k=0}^{n}W_{-k}$, $\sum_{k=0}^{n}W_{-2k}$, $\sum_{k=0}^{n}W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty}W_nz^n$, $\sum_{n=0}^{\infty}W_{2n}z^n$, $\sum_{n=0}^{\infty}W_{2n+1}z^n$, $\sum_{n=0}^{\infty}W_{-n}z^n$, $\sum_{n=0}^{\infty}W_{-2n}z^n$, $\sum_{n=0}^{\infty}W_{-2n+1}z^n$ of Generalized Olivier Numbers

In this subsection, we consider the case $r=1, s=2, t=-1, u=-1$. A generalized Olivier sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2,W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n=W_{n-1}+2W_{n-2}-W_{n-3}-W_{n-4} \tag{30}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2, W_3=c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=-W_{-(n-1)}+2W_{-(n-2)}+W_{-(n-3)}-W_{-(n-4)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence eq. (30) holds for all integers $n$. For more information on generalized Olivier numbers, see Soykan [10].

Characteristic equation of $\{W_n\}$ is

$$z^4-z^3-2z^2+z+1=(z^3-2z-1)(z-1)=(z^2-z-1)(z+1)(z-1)=0$$

whose roots are

$$\alpha = \frac{1+\sqrt{5}}{2},$$
$$\beta = \frac{1-\sqrt{5}}{2},$$
$$\gamma = -1,$$
$$\delta = 1.$$

Two special cases of the sequence $\{W_n\}$ are Olivier sequence $\{O_n\}_{n\geq 0}$ and Olivier-Lucas sequence $\{K_n\}_{n\geq 0}$. Olivier sequence $\{O_n\}_{n\geq 0}$ and Olivier-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$O_n = O_{n-1} + 2O_{n-2} - O_{n-3} - O_{n-4}, \quad O_0 = 0, O_1 = 1, O_2 = 1, O_3 = 3, \quad n \geq 4,$$
$$K_n = K_{n-1} + 2K_{n-2} - K_{n-3} - K_{n-4}, \quad K_0 = 4, K_1 = 1, K_2 = 5, K_3 = 4, \quad n \geq 4,$$

The sequences $\{O_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$O_{-n} = -O_{-(n-1)} + 2O_{-(n-2)} + O_{-(n-3)} - O_{-(n-4)}$$
$$K_{-n} = -K_{-(n-1)} + 2K_{-(n-2)} + K_{-(n-3)} - K_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$ respectively.

Binet's formula of generalized Olivier numbers can be given as follows:

**Theorem 3.3.**
*For all integers n, Binet's formula of generalized Olivier numbers is*

$$W_n = \frac{(\alpha W_3 - \alpha(1-\alpha)W_2 + (-\alpha^2+1)W_1 - W_0)\alpha^n}{4\alpha^2 - \alpha - 3} + \frac{(\beta W_3 - \beta(1-\beta)W_2 + (-\beta^2+1)W_1 - W_0)\beta^n}{4\beta^2 - \beta - 3} + \frac{(-W_3 + 2W_2 - W_0)(-1)^n}{2} - \frac{W_3 - 2W_1 - W_0}{2}.$$

Olivier and Olivier-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 3.6.**
*For all integers n, Binet's formulas of Olivier and Olivier-Lucas numbers are*

$$O_n = \frac{(2\alpha+1)\alpha^n}{4\alpha^2-\alpha-3} + \frac{(2\beta+1)\beta^n}{4\beta^2-\beta-3} - \frac{1}{2}\gamma^n - \frac{1}{2}$$
$$= \frac{1}{10}\left((5-\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^n + (5+\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^n - 5(-1)^n - 5\right)$$

*and*

$$K_n = \alpha^n + \beta^n + \gamma^n + 1 = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n + (-1)^n + 1$$

*respectively.*

Next, we present sum formulas of generalized Olivier numbers

**Theorem 3.4.**
*For $n \geq 0$, we have the following sum formulas for generalized Olivier numbers:*

**(a)** $\sum_{k=0}^{n} W_k = \frac{1}{2}(-(n+3)W_{n+3} + W_{n+2} + (2n+7)W_{n+1} + (n+4)W_n + 3W_3 - W_2 - 7W_1 - 2W_0).$

**(b)** $\sum_{k=0}^{n} W_{2k} = W_{2n+2} - (n+2)W_{2n+1} + (n+1)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 3W_2 - 2W_1 + 2W_0.$

**(c)** $\sum\limits_{k=0}^{n} W_{2k+1} = -(n+1)W_{2n+2} + (n+3)W_{2n+1} + (n+1)W_{2n} - W_{2n-1} - W_3 + 2W_2 - 2W_0.$

**(d)** $\sum\limits_{k=1}^{n} W_{-k} = \frac{1}{2}(-(n+1)W_{-n+3} - W_{-n+2} + (2n+1)W_{-n+1} + (n+2)W_{-n} + W_3 + W_2 - W_1 - 2W_0).$

**(e)** $\sum\limits_{k=1}^{n} W_{-2k} = -W_{-2n+2} - (n+1)W_{-2n+1} + (n+3)W_{-2n} + (n+1)W_{-2n-1} + W_3 - W_1 - 2W_0.$

**(f)** $\sum\limits_{k=1}^{n} W_{-2k+1} = -(n+2)W_{-2n+2} + (n+1)W_{-2n+1} + (n+2)W_{-2n} + W_{-2n-1} + W_3 + W_2 - 3W_1 - W_0.$

Proof.

**(a)** For $r=1, s=2, t=-1, u=-1$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = -(z-1)(z+1)(z^2+z-1)$ and then for $z=1$, we get $-(z-1)(z+1)(z^2+z-1) = 0$ with multiplicity 1 so we use theorem 1.1 (a) (ii) with $z=1$.

**(b)** For $r=1, s=2, t=-1, u=-1$, we get $-u^2z^4 + (t^2-2su)z^3 + (2u+2rt-s^2)z^2 + (2s+r^2)z - 1 = -(z-1)^2(z^2-3z+1)$ and then for $z=1$, we get $-(z-1)^2(z^2-3z+1) = 0$ with multiplicity 2 so we use theorem 1.1 (b) (iii) with $z=1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (iii) with $z=1$.

**(d)** For $r=1, s=2, t=-1, u=-1$, we get $-z^4 + rz^3 + sz^2 + tz + u = -(z-1)(z+1)(z^2-z-1)$ and then for $z=1$, we get $-(z-1)(z+1)(z^2-z-1) = 0$ with multiplicity 1 so we use theorem 1.2 (a) (ii) with $z=1$.

**(e)** For $r=1, s=2, t=-1, u=-1$, we get $-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2 = -(z-1)^2(z^2-3z+1)$ and then for $z=1$, we get $-(z-1)^2(z^2-3z+1) = 0$ with multiplicity 2 so we use theorem 1.2 (b) (iii) with $z=1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (iii) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Olivier numbers (take $W_n = O_n$ with $O_0=0, O_1=1, O_2=1, O_3=3$).

**Corollary 3.7.**
*For $n \geq 0$, Olivier numbers have the following properties.*

**(a)** $\sum\limits_{k=0}^{n} O_k = \frac{1}{2}(-(n+3)O_{n+3} + O_{n+2} + (2n+7)O_{n+1} + (n+4)O_n + 1).$

**(b)** $\sum\limits_{k=0}^{n} O_{2k} = O_{2n+2} - (n+2)O_{2n+1} + (n+1)O_{2n} + (n+2)O_{2n-1} + 1.$

**(c)** $\sum\limits_{k=0}^{n} O_{2k+1} = -(n+1)O_{2n+2} + (n+3)O_{2n+1} + (n+1)O_{2n} - O_{2n-1} - 1.$

**(d)** $\sum\limits_{k=1}^{n} O_{-k} = \frac{1}{2}(-(n+1)O_{-n+3} - O_{-n+2} + (2n+1)O_{-n+1} + (n+2)O_{-n} + 3).$

**(e)** $\sum\limits_{k=1}^{n} O_{-2k} = -O_{-2n+2} - (n+1)O_{-2n+1} + (n+3)O_{-2n} + (n+1)O_{-2n-1} + 2.$

**(f)** $\sum\limits_{k=1}^{n} O_{-2k+1} = -(n+2)O_{-2n+2} + (n+1)O_{-2n+1} + (n+2)O_{-2n} + O_{-2n-1} + 1.$

Taking $W_n = K_n$ with $K_0=4, K_1=1, K_2=5, K_3=4$ in the last Theorem, we have the following Corollary which gives sum formulas of Olivier-Lucas numbers.

**Corollary 3.8.**
*For $n \geq 0$, Olivier-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} K_k = \frac{1}{2}(-(n+3)K_{n+3} + K_{n+2} + (2n+7)K_{n+1} + (n+4)K_n - 8).$

**(b)** $\sum\limits_{k=0}^{n} K_{2k} = K_{2n+2} - (n+2)K_{2n+1} + (n+1)K_{2n} + (n+2)K_{2n-1} - 1.$

**(c)** $\sum_{k=0}^{n} K_{2k+1} = -(n+1)K_{2n+2} + (n+3)K_{2n+1} + (n+1)K_{2n} - K_{2n-1} - 2.$

**(d)** $\sum_{k=1}^{n} K_{-k} = \frac{1}{2}(-(n+1)K_{-n+3} - K_{-n+2} + (2n+1)K_{-n+1} + (n+2)K_{-n}).$

**(e)** $\sum_{k=1}^{n} K_{-2k} = -K_{-2n+2} - (n+1)K_{-2n+1} + (n+3)K_{-2n} + (n+1)K_{-2n-1} - 5.$

**(f)** $\sum_{k=1}^{n} K_{-2k+1} = -(n+2)K_{-2n+2} + (n+1)K_{-2n+1} + (n+2)K_{-2n} + K_{-2n-1} + 2.$

Next, we give the ordinary generating functions of some special cases of generalized Olivier numbers.

**Lemma 3.2.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.618033.$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^3 W_3 + (z^2 - z^3)W_2 + (z^3 - 2z^2 - z + 1)W_0 - \left(2z^3 + z^2 - z\right)W_1}{z^4 + z^3 - 2z^2 - z + 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.381966.$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{(z^2 - z^3)W_3 + (2z^3 - 3z^2 + z)W_2 + (z^3 - z^2)W_1 - \left(3z^3 - 7z^2 + 5z - 1\right)W_0}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.381966.$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{\left(z^3 - 2z^2 + z\right)W_3 + \left(z^2 - z^3\right)W_2 - \left(2z^3 - 6z^2 + 5z - 1\right)W_1 + (z^3 - z^2)W_0}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| \simeq 0.618033.$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{-zW_3 + (z - z^2)W_2 + (-z^3 + z^2 + 2z)W_1 + W_0}{z^4 - z^3 - 2z^2 + z + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 \simeq 0.381966.$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{\left(z - z^2\right)W_3 - \left(z^3 - 3z^2 + 2z\right)W_2 + (z^2 - z)W_1 + \left(z^2 - 2z + 1\right)W_0}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 \simeq 0.381966.$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{-\left(z^3 - 2z^2 + z\right)W_3 + \left(z - z^2\right)W_2 + \left(2z^2 - 3z + 1\right)W_1 + (z^2 - z)W_0}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

Proof. Use lemmas 1.1 and 1.2. □

Now, we consider special cases of the last Lemma.

**Corollary 3.9.**
*The ordinary generating functions of the sequences $O_n$, $O_{2n}$, $O_{2n+1}$, $O_{-n}$, $O_{-2n}$, $O_{-2n+1}$ and $K_n$, $K_{2n}$, $K_{2n+1}$, $K_{-n}$, $K_{-2n}$, $K_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.618033.$

$$\sum_{n=0}^{\infty} O_n z^n = \frac{z}{z^4 + z^3 - 2z^2 - z + 1},$$

$$\sum_{n=0}^{\infty} K_n z^n = \frac{z^3 - 4z^2 - 3z + 4}{z^4 + z^3 - 2z^2 - z + 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.381966$.

$$\sum_{n=0}^{\infty} O_{2n} z^n = \frac{z - z^2}{z^4 - 5z^3 + 8z^2 - 5z + 1},$$
$$\sum_{n=0}^{\infty} K_{2n} z^n = \frac{-5z^3 + 16z^2 - 15z + 4}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.381966$.

$$\sum_{n=0}^{\infty} O_{2n+1} z^n = \frac{z^2 - 2z + 1}{z^4 - 5z^3 + 8z^2 - 5z + 1},$$
$$\sum_{n=0}^{\infty} K_{2n+1} z^n = \frac{z^3 - z^2 - z + 1}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| \simeq 0.618033$.

$$\sum_{n=0}^{\infty} O_{-n} z^n = \frac{-z^3}{z^4 - z^3 - 2z^2 + z + 1},$$
$$\sum_{n=0}^{\infty} K_{-n} z^n = \frac{-z^3 - 4z^2 + 3z + 4}{z^4 - z^3 - 2z^2 + z + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 \simeq 0.381966$.

$$\sum_{n=0}^{\infty} O_{-2n} z^n = \frac{z^2 - z^3}{z^4 - 5z^3 + 8z^2 - 5z + 1},$$
$$\sum_{n=0}^{\infty} K_{-2n} z^n = \frac{-5z^3 + 16z^2 - 15z + 4}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 \simeq 0.381966$.

$$\sum_{n=0}^{\infty} O_{-2n+1} z^n = \frac{-3z^3 + 7z^2 - 5z + 1}{z^4 - 5z^3 + 8z^2 - 5z + 1},$$
$$\sum_{n=0}^{\infty} K_{-2n+1} z^n = \frac{-4z^3 + 9z^2 - 6z + 1}{z^4 - 5z^3 + 8z^2 - 5z + 1}.$$

From the last corollary, we obtain the following results for Olivier and Olivier-Lucas numbers.

**Corollary 3.10.**
*Infinite sums of* $O_n, O_{2n}, O_{2n+1}, O_{-n}, O_{-2n}, O_{-2n+1}$ *and* $K_n, K_{2n}, K_{2n+1}, K_{-n}, K_{-2n}, K_{-2n+1}$ *are given as follows:*

**(a)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{O_n}{2^n} = \frac{8}{3},$$
$$\sum_{n=0}^{\infty} \frac{K_n}{2^n} = \frac{26}{3}.$$

**(b)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{O_{2n}}{4^n} = \frac{16}{15},$$
$$\sum_{n=0}^{\infty} \frac{K_{2n}}{4^n} = \frac{20}{3}.$$

**(c)** $z=\frac{1}{4}$

$$\sum_{n=0}^{\infty}\frac{O_{2n+1}}{4^n}=\frac{16}{5},$$
$$\sum_{n=0}^{\infty}\frac{K_{2n+1}}{4^n}=4.$$

**(d)** $z=\frac{1}{2}$

$$\sum_{n=0}^{\infty}\frac{O_{-n}}{2^n}=-\frac{2}{15},$$
$$\sum_{n=0}^{\infty}\frac{K_{-n}}{2^n}=\frac{14}{3}.$$

**(e)** $z=\frac{1}{4}$

$$\sum_{n=0}^{\infty}\frac{O_{-2n}}{4^n}=\frac{4}{15},$$
$$\sum_{n=0}^{\infty}\frac{K_{-2n}}{4^n}=\frac{20}{3}.$$

**(f)** $z=\frac{1}{4}$

$$\sum_{n=0}^{\infty}\frac{O_{-2n+1}}{4^n}=\frac{4}{5},$$
$$\sum_{n=0}^{\infty}\frac{K_{-2n+1}}{4^n}=0.$$

### 3.3. Sum Formulas $\sum_{k=0}^{n}W_k$, $\sum_{k=0}^{n}W_{2k}$, $\sum_{k=0}^{n}W_{2k+1}$, $\sum_{k=0}^{n}W_{-k}$, $\sum_{k=0}^{n}W_{-2k}$, $\sum_{k=0}^{n}W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty}W_nz^n$, $\sum_{n=0}^{\infty}W_{2n}z^n$, $\sum_{n=0}^{\infty}W_{2n+1}z^n$, $\sum_{n=0}^{\infty}W_{-n}z^n$, $\sum_{n=0}^{\infty}W_{-2n}z^n$, $\sum_{n=0}^{\infty}W_{-2n+1}z^n$ of Generalized Blaise Numbers

In this subsection, we consider the case $r=1, s=1, t=1, u=-2$. A generalized Blaise sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2,W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n=W_{n-1}+W_{n-2}+W_{n-3}-2W_{n-4} \tag{31}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2, W_3=c_3$ not all being zero. For more information on generalized Blaise numbers, see Soykan [11].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=\frac{1}{2}W_{-(n-1)}+\frac{1}{2}W_{-(n-2)}+\frac{1}{2}W_{-(n-3)}-\frac{1}{2}W_{-(n-4)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence eq. (31) holds for all integers $n$.

Characteristic equation of $\{W_n\}$ is

$$z^4-z^3-z^2-z+2=(z^3-z-2)(z-1)=0$$

whose roots are

$$\alpha=\sqrt[3]{1+\frac{\sqrt{78}}{9}}+\sqrt[3]{1-\frac{\sqrt{78}}{9}}\simeq 1.521379706804568,$$
$$\beta=\omega\sqrt[3]{1+\frac{\sqrt{78}}{9}}+\omega^2\sqrt[3]{1-\frac{\sqrt{78}}{9}},$$
$$\gamma=\omega^2\sqrt[3]{1+\frac{\sqrt{78}}{9}}+\omega\sqrt[3]{1-\frac{\sqrt{78}}{9}},$$
$$\delta=1,$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Two special cases of the sequence $\{W_n\}$ are Blaise sequence $\{B_n\}_{n\geq 0}$ and Blaise-Lucas sequence $\{C_n\}_{n\geq 0}$. Blaise sequence $\{B_n\}_{n\geq 0}$ and Blaise-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$B_n = B_{n-1}+B_{n-2}+B_{n-3}-2B_{n-4}, \quad B_0=0, B_1=1, B_2=1, B_3=2, \qquad n\geq 4,$$
$$C_n = C_{n-1}+C_{n-2}+C_{n-3}-2C_{n-4}, \quad C_0=4, C_1=1, C_2=3, C_3=7, \qquad n\geq 4.$$

The sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$B_{-n} = \frac{1}{2}B_{-(n-1)} + \frac{1}{2}B_{-(n-2)} + \frac{1}{2}B_{-(n-3)} - \frac{1}{2}B_{-(n-4)}$$
$$C_{-n} = \frac{1}{2}C_{-(n-1)} + \frac{1}{2}C_{-(n-2)} + \frac{1}{2}C_{-(n-3)} - \frac{1}{2}C_{-(n-4)}$$

for $n=1,2,3,...$ respectively.

Binet's formula of generalized Blaise numbers can be given as follows:

**Theorem 3.5.**
*For all integers n, Binet's formula of generalized Blaise numbers is*

$$W_n = \frac{(\alpha W_3-\alpha(1-\alpha)W_2+(-\alpha^2+2)W_1-2W_0)\alpha^n}{2\alpha^2+4\alpha-6} + \frac{(\beta W_3-\beta(1-\beta)W_2+(-\beta^2+2)W_1-2W_0)\beta^n}{2\beta^2+4\beta-6}$$
$$+\frac{(\gamma W_3-\gamma(1-\gamma)W_2+(-\gamma^2+2)W_1-2W_0)\gamma^n}{2\gamma^2+4\gamma-6} - \frac{W_3-W_1-2W_0}{2}$$

Blaise and Blaise-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 3.11.**
*For all integers n, Binet's formulas of Blaise and Blaise-Lucas numbers are*

$$B_n = \frac{\alpha^{n+3}}{2\alpha^2+4\alpha-6} + \frac{\beta^{n+3}}{2\beta^2+4\beta-6} + \frac{\gamma^{n+3}}{2\gamma^2+4\gamma-6} - \frac{1}{2}$$

*and*

$$C_n = \alpha^n+\beta^n+\gamma^n+1$$

*respectively.*

Next, we present sum formulas of generalized Blaise numbers

**Theorem 3.6.**
*For $n\geq 0$, we have the following sum formulas for generalized Blaise numbers:*

**(a)** $\sum_{k=0}^{n} W_k = \frac{1}{2}(-(n+3)W_{n+3}+W_{n+2}+(n+4)W_{n+1}+2(n+4)W_n+3W_3-W_2-4W_1-6W_0).$

**(b)** $\sum_{k=0}^{n} W_{2k} = \frac{1}{4}(-(2n+5)W_{2n+2}+2W_{2n+1}+(2n+9)W_{2n}+2(2n+5)W_{2n-1}+5W_3-7W_1-10W_0).$

**(c)** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{4}(-(2n+3)W_{2n+2}+4W_{2n+1}+(2n+5)W_{2n}+2(2n+5)W_{2n-1}+5W_3-2W_2-5W_1-10W_0).$

**(d)** $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-(n+1)W_{-n+3}-W_{-n+2}+nW_{-n+1}+2(n+1)W_{-n}+W_3+W_2-2W_0).$

**(e)** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{4}(-(2n+3)W_{-2n+2}-2W_{-2n+1}+(2n+3)W_{-2n}+2(2n+3)W_{-2n-1}+3W_3-W_1-6W_0).$

**(f)** $\sum\limits_{k=1}^{n} W_{-2k+1} = \frac{1}{4}(-(2n+5)W_{-2n+2} + (2n+3)W_{-2n} + 2(2n+3)W_{-2n-1} + 3W_3 + 2W_2 - 3W_1 - 6W_0)$.

Proof.

**(a)** For $r = 1, s = 1, t = 1, u = -2$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = -(z-1)(2z^3 + z^2 - 1)$ and then for $z = 1$, we get $-(z-1)(2z^3 + z^2 - 1) = 0$ with multiplicity 1 so we use theorem 1.1 bunu ve aşağıdaki leri de uygun şekilde ayarla% (a) (ii) with $z = 1$.

**(b)** For $r = 1, s = 1, t = 1, u = -2$, we get $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = (z-1)(-4z^3 + z^2 - 2z + 1)$ and then for $z = 1$, we get $(z-1)(-4z^3 + z^2 - 2z + 1) = 0$ with multiplicity 1 so we use theorem 1.1 (b) (ii) with $z = 1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (ii) with $z = 1$.

**(d)** For $r = 1, s = 1, t = 1, u = -2$, we get $-z^4 + rz^3 + sz^2 + tz + u = -(z-1)(z^3 - z - 2)$ and then for $z = 1$, we get $-(z-1)(z^3 - z - 2) = 0$ with multiplicity 1 so we use theorem 1.2 (a) (ii) with $z = 1$.

**(e)** For $r = 1, s = 1, t = 1, u = -2$, we get $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 = (z-1)(-z^3 + 2z^2 - z + 4)$ and then for $z = 1$, we get $(z-1)(-z^3 + 2z^2 - z + 4) = 0$ with multiplicity 1 so we use theorem 1.2 (b) (ii) with $z = 1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Blaise numbers (take $W_n = B_n$ with $B_0 = 0, B_1 = 1, B_2 = 1, B_3 = 2$).

**Corollary 3.12.**
*For $n \geq 0$, Blaise numbers have the following properties.*

**(a)** $\sum\limits_{k=0}^{n} B_k = \frac{1}{2}(-(n+3)B_{n+3} + B_{n+2} + (n+4)B_{n+1} + 2(n+4)B_n + 1)$.

**(b)** $\sum\limits_{k=0}^{n} B_{2k} = \frac{1}{4}(-(2n+5)B_{2n+2} + 2B_{2n+1} + (2n+9)B_{2n} + 2(2n+5)B_{2n-1} + 3)$.

**(c)** $\sum\limits_{k=0}^{n} B_{2k+1} = \frac{1}{4}(-(2n+3)B_{2n+2} + 4B_{2n+1} + (2n+5)B_{2n} + 2(2n+5)B_{2n-1} + 3)$.

**(d)** $\sum\limits_{k=1}^{n} B_{-k} = \frac{1}{2}(-(n+1)B_{-n+3} - B_{-n+2} + nB_{-n+1} + 2(n+1)B_{-n} + 3)$.

**(e)** $\sum\limits_{k=1}^{n} B_{-2k} = \frac{1}{4}(-(2n+3)B_{-2n+2} - 2B_{-2n+1} + (2n+3)B_{-2n} + 2(2n+3)B_{-2n-1} + 5)$.

**(f)** $\sum\limits_{k=1}^{n} B_{-2k+1} = \frac{1}{4}(-(2n+5)B_{-2n+2} + (2n+3)B_{-2n} + 2(2n+3)B_{-2n-1} + 5)$.

Taking $W_n = C_n$ with $C_0 = 4, C_1 = 1, C_2 = 3, C_3 = 7$ in the last Theorem, we have the following Corollary which gives sum formulas of Blaise-Lucas numbers.

**Corollary 3.13.**
*For $n \geq 0$, Blaise-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} C_k = \frac{1}{2}(-(n+3)C_{n+3} + C_{n+2} + (n+4)C_{n+1} + 2(n+4)C_n - 10)$.

**(b)** $\sum\limits_{k=0}^{n} C_{2k} = \frac{1}{4}(-(2n+5)C_{2n+2} + 2C_{2n+1} + (2n+9)C_{2n} + 2(2n+5)C_{2n-1} - 12)$.

**(c)** $\sum\limits_{k=0}^{n} C_{2k+1} = \frac{1}{4}(-(2n+3)C_{2n+2} + 4C_{2n+1} + (2n+5)C_{2n} + 2(2n+5)C_{2n-1} - 16)$.

**(d)** $\sum\limits_{k=1}^{n} C_{-k} = \frac{1}{2}(-(n+1)C_{-n+3} - C_{-n+2} + nC_{-n+1} + 2(n+1)C_{-n} + 2)$.

**(e)** $\sum\limits_{k=1}^{n} C_{-2k} = \frac{1}{4}(-(2n+3)C_{-2n+2} - 2C_{-2n+1} + (2n+3)C_{-2n} + 2(2n+3)C_{-2n-1} - 4)$.

**(f)** $\sum_{k=1}^{n} C_{-2k+1} = \frac{1}{4}(-(2n+5)C_{-2n+2} + (2n+3)C_{-2n} + 2(2n+3)C_{-2n-1})$.

Next, we give the ordinary generating functions of some special cases of generalized Blaise numbers.

**Lemma 3.3.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.657298$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{-z^3W_3 + (z^3 - z^2)W_2 + (z^3 + z^2 - z)W_1 + (z^3 + z^2 + z - 1)W_0}{-2z^4 + z^3 + z^2 + z - 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.43204$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{(z^3 + z^2)W_3 + (z^3 - 2z^2 + z)W_2 - (3z^3 - z^2)W_1 - (3z^3 - z^2 + 3z - 1)W_0}{4z^4 - 5z^3 + 3z^2 - 3z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.43204$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{(2z^3 - z^2 + z)W_3 + (2z^2 - 2z^3)W_2 + (-2z^3 + 2z^2 - 3z + 1)W_1 - 2(z^3 + z^2)W_0}{4z^4 - 5z^3 + 3z^2 - 3z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\delta| = 1$.

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{zW_3 + (z^2 - rz)W_2 + (z^3 - rz^2 - sz)W_1 + uW_0}{-z^4 + rz^3 + sz^2 + tz + u} = -\frac{2W_0 - zW_3 + W_2(z - z^2) + W_1(-z^3 + z^2 + z)}{-z^4 + z^3 + z^2 + z - 2}.$$

:

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\delta|^2 \simeq 1$.

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-(z^2 + z)W_3 - (z^3 - 2z^2 + z)W_2 + (3z - z^2)W_1 + (2z^2 - 2z + 4)W_0}{z^4 - 3z^3 + 3z^2 - 5z + 4}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\delta|^2 \simeq 1$.

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{-(z^3 - z^2 + 2z)W_3 - (2z^2 - 2z)W_2 + (z^2 - 3z + 4)W_1 + 2(z^2 + z)W_0}{z^4 - 3z^3 + 3z^2 - 5z + 4}.$$

Proof. Use lemmas 1.1 and 1.2. □
Now, we consider special cases of the last Lemma.

**Corollary 3.14.**
*The ordinary generating functions of the sequences $B_n$, $B_{2n}$, $B_{2n+1}$, $B_{-n}$, $B_{-2n}$, $B_{-2n+1}$ and $C_n$, $C_{2n}$, $C_{2n+1}$, $C_{-n}$, $C_{-2n}$, $C_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.657298$.

$$\sum_{n=0}^{\infty} B_n z^n = \frac{-z}{-2z^4 + z^3 + z^2 + z - 1},$$

$$\sum_{n=0}^{\infty} C_n z^n = \frac{z^3 + 2z^2 + 3z - 4}{-2z^4 + z^3 + z^2 + z - 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.43204$.

$$\sum_{n=0}^{\infty} B_{2n} z^n = \frac{z^2 + z}{4z^4 - 5z^3 + 3z^2 - 3z + 1},$$

$$\sum_{n=0}^{\infty} C_{2n} z^n = \frac{-5z^3 + 6z^2 - 9z + 4}{4z^4 - 5z^3 + 3z^2 - 3z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.43204.$

$$\sum_{n=0}^{\infty} B_{2n+1} z^n = \frac{2z^2 - z + 1}{4z^4 - 5z^3 + 3z^2 - 3z + 1},$$
$$\sum_{n=0}^{\infty} C_{2n+1} z^n = \frac{-2z^3 - 7z^2 + 4z + 1}{4z^4 - 5z^3 + 3z^2 - 3z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\delta| = 1.$

$$\sum_{n=0}^{\infty} B_{-n} z^n = \frac{z^3}{-z^4 + z^3 + z^2 + z - 2},$$
$$\sum_{n=0}^{\infty} C_{-n} z^n = \frac{z^3 + 2z^2 + 3z - 8}{-z^4 + z^3 + z^2 + z - 2}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\delta|^2 \simeq 1.$

$$\sum_{n=0}^{\infty} B_{-2n} z^n = \frac{-z^3 - z^2}{z^4 - 3z^3 + 3z^2 - 5z + 4},$$
$$\sum_{n=0}^{\infty} C_{-2n} z^n = \frac{-3z^3 + 6z^2 - 15z + 16}{z^4 - 3z^3 + 3z^2 - 5z + 4}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\delta|^2 \simeq 1.$

$$\sum_{n=0}^{\infty} B_{-2n+1} z^n = \frac{-2z^3 + z^2 - 5z + 4}{z^4 - 3z^3 + 3z^2 - 5z + 4},$$
$$\sum_{n=0}^{\infty} C_{-2n+1} z^n = \frac{-7z^3 + 10z^2 - 3z + 4}{z^4 - 3z^3 + 3z^2 - 5z + 4}.$$

From the last corollary, we obtain the following results for Blaise and Blaise-Lucas numbers.

**Corollary 3.15.**
*Infinite sums of $B_n$, $B_{2n}$, $B_{2n+1}$, $B_{-n}$, $B_{-2n}$, $B_{-2n+1}$ and $C_n$, $C_{2n}$, $C_{2n+1}$, $C_{-n}$, $C_{-2n}$, $C_{-2n+1}$ are given as follows:*

**(a)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{B_n}{4^n} = \frac{32}{87},$$
$$\sum_{n=0}^{\infty} \frac{C_n}{4^n} = \frac{398}{87}.$$

**(b)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{4^n} = \frac{5}{6},$$
$$\sum_{n=0}^{\infty} \frac{C_{2n}}{4^n} = \frac{131}{24}.$$

**(c)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{B_{2n+1}}{4^n} = \frac{7}{3},$$
$$\sum_{n=0}^{\infty} \frac{C_{2n+1}}{4^n} = \frac{49}{12}.$$

**(d)** $z=\frac{1}{4}$

$$\sum_{n=0}^{\infty}\frac{B_{-n}}{4^n}=-\frac{4}{429},$$
$$\sum_{n=0}^{\infty}\frac{C_{-n}}{4^n}=\frac{140}{33}.$$

**(e)** $z=\frac{1}{4}$

$$\sum_{n=0}^{\infty}\frac{B_{-2n}}{4^n}=-\frac{20}{741},$$
$$\sum_{n=0}^{\infty}\frac{C_{-2n}}{4^n}=\frac{3220}{741}.$$

**(f)** $z=\frac{1}{4}$

$$\sum_{n=0}^{\infty}\frac{B_{-2n+1}}{4^n}=\frac{712}{741},$$
$$\sum_{n=0}^{\infty}\frac{C_{-2n+1}}{4^n}=\frac{964}{741}.$$

### 3.4. Sum Formulas $\sum_{k=0}^{n}W_k$, $\sum_{k=0}^{n}W_{2k}$, $\sum_{k=0}^{n}W_{2k+1}$, $\sum_{k=0}^{n}W_{-k}$, $\sum_{k=0}^{n}W_{-2k}$, $\sum_{k=0}^{n}W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty}W_nz^n$, $\sum_{n=0}^{\infty}W_{2n}z^n$, $\sum_{n=0}^{\infty}W_{2n+1}z^n$, $\sum_{n=0}^{\infty}W_{-n}z^n$, $\sum_{n=0}^{\infty}W_{-2n}z^n$, $\sum_{n=0}^{\infty}W_{-2n+1}z^n$ of Generalized Friedrich Numbers

In this subsection, we consider the case $r=2, s=0, t=1, u=-2$. A generalized Friedrich sequence $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2,W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n=2W_{n-1}+W_{n-3}-2W_{n-4} \tag{32}$$

with the initial values $W_0=c_0, W_1=c_1, W_2=c_2, W_3=c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=\frac{1}{2}W_{-(n-1)}+W_{-(n-3)}-\frac{1}{2}W_{-(n-4)}$$

for $n=1,2,3,....$ Therefore, recurrence eq. (32) holds for all integers $n$. For more information on generalized Friedrich numbers, see Soykan [12].

Characteristic equation of $\{W_n\}$ is

$$z^4-2z^3-z+2=(z^3-z^2-z-2)(z-1)=(z^2+z+1)(z-2)(z-1)=0$$

whose roots are

$$\alpha=2,$$
$$\beta=\frac{-1+i\sqrt{3}}{2},$$
$$\gamma=\frac{-1-i\sqrt{3}}{2},$$
$$\delta=1.$$

Two special cases of the sequence $\{W_n\}$ are Friedrich sequence $\{F_n\}_{n\geq 0}$ and Friedrich-Lucas sequence $\{C_n\}_{n\geq 0}$. Friedrich sequence $\{F_n\}_{n\geq 0}$ and Friedrich-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$F_n=2F_{n-1}+F_{n-3}-2F_{n-4}, \quad F_0=0, F_1=1, F_2=2, F_3=4, \quad n\geq 4, \tag{33}$$
$$C_n=2C_{n-1}+C_{n-3}-2C_{n-4}, \quad C_0=4, C_1=2, C_2=4, C_3=11, \quad n\geq 4. \tag{34}$$

The sequences $\{F_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$F_{-n}=\frac{1}{2}F_{-(n-1)}+F_{-(n-3)}-\frac{1}{2}F_{-(n-4)},$$
$$C_{-n}=\frac{1}{2}C_{-(n-1)}+C_{-(n-3)}-\frac{1}{2}C_{-(n-4)},$$

for $n=1,2,3,...$ respectively.

Binet's formula of generalized Friedrich numbers can be given as follows:

**Theorem 3.7.**
*For all integers n, Binet's formula of generalized Friedrich numbers is*

$$\begin{aligned} W_n =& \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+2)W_1-2W_0)\alpha^n}{2\alpha^2+5\alpha-4} \\ &+\frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+2)W_1-2W_0)\beta^n}{2\beta^2+5\beta-4} \\ &+\frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+2)W_1-2W_0)\gamma^n}{2\gamma^2+5\gamma-4} \\ &+\frac{W_3-W_2-W_1-2W_0}{-3}. \end{aligned}$$

Friedrich and Friedrich-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 3.16.**
*For all integers n, Binet's formulas of Friedrich and Friedrich-Lucas numbers are*

$$\begin{aligned} F_n &= \frac{(\alpha^2+\alpha+2)\alpha^n}{2\alpha^2+5\alpha-4}+\frac{(\beta^2+\beta+2)\beta^n}{2\beta^2+5\beta-4}+\frac{(\gamma^2+\gamma+2)\gamma^n}{2\gamma^2+5\gamma-4}-\frac{1}{3} \\ &= \frac{1}{7}\times 2^{n+2}-\frac{1}{42}(5+i\sqrt{3})\left(\frac{-1+i\sqrt{3}}{2}\right)^n-\frac{1}{42}(5-i\sqrt{3})\left(\frac{-1-i\sqrt{3}}{2}\right)^n-\frac{1}{3}, \end{aligned} \tag{35}$$

*and*

$$C_n = \alpha^n+\beta^n+\gamma^n+1 = 2^n+\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n+1, \tag{36}$$

*respectively.*

Next, we present sum formulas of generalized Friedrich numbers

**Theorem 3.8.**
*For $n \geq 0$, we have the following sum formulas for generalized Friedrich numbers:*

**(a)** $\sum_{k=0}^{n} W_k = \frac{1}{3}(-(n+3)W_{n+3}+(n+4)W_{n+2}+(n+3)W_{n+1}+2(n+4)W_n+3W_3-4W_2-3W_1-5W_0).$

**(b)** $\sum_{k=0}^{n} W_{2k} = \frac{1}{9}(-(3n+7)W_{2n+2}+(3n+10)W_{2n+1}+(3n+13)W_{2n}+2(3n+7)W_{2n-1}+7W_3-7W_2-10W_1-11W_0).$

**(c)** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{9}(-(3n+4)W_{2n+2}+(3n+13)W_{2n+1}+(3n+7)W_{2n}+2(3n+7)W_{2n-1}+7W_3-10W_2-4W_1-14W_0).$

**(d)** $\sum_{k=1}^{n} W_{-k} = \frac{1}{3}(-(n+1)W_{-n+3}+nW_{-n+2}+(n+1)W_{-n+1}+(2n+3)W_{-n}+W_3-W_1-3W_0).$

**(e)** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{9}(-(3n+5)W_{-2n+2}+(3n+2)W_{-2n+1}+(3n+8)W_{-2n}+2(3n+5)W_{-2n-1}+5W_3-5W_2-2W_1-13W_0).$

**(f)** $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{9}(-(3n+8)W_{-2n+2}+(3n+8)W_{-2n+1}+(3n+5)W_{-2n}+(6n+10)W_{-2n-1}+5W_3-2W_2-8W_1-10W_0).$

Proof.

**(a)** For $r=2, s=0, t=1, u=-2$, we get $uz^4+tz^3+sz^2+rz-1=-(z-1)(2z-1)(z^2+z+1)$ and then for $z=1$, we get $-(z-1)(2z-1)(z^2+z+1)=0$ with multiplicity 1 so we use theorem 1.1 (a) (ii) with $z=1$.

**(b)** For $r=2, s=0, t=1, u=-2$, we get $-u^2z^4+(t^2-2su)z^3+(2u+2rt-s^2)z^2+(2s+r^2)z-1=-(z-1)(4z-1)(z^2+z+1)$ and then for $z=1$, we get $-(z-1)(4z-1)(z^2+z+1)=0$ with multiplicity 1 so we use theorem 1.1 (b) (ii) with $z=1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (ii) with $z=1$.

**(d)** For $r = 2, s = 0, t = 1, u = -2$, we get $-z^4 + rz^3 + sz^2 + tz + u = -(z-2)(z-1)(z^2+z+1)$ and then for $z = 1$, we get $-(z-2)(z-1)(z^2+z+1) = 0$ with multiplicity 1 so we use theorem 1.2 (a) (ii) with $z = 1$.

**(e)** For $r = 2, s = 0, t = 1, u = -2$, we get $-z^4+(r^2+2s)z^3+(2u+2rt-s^2)z^2+(t^2-2su)z-u^2 = -(z-4)(z-1)(z^2+z+1)$ and then for $z = 1$,we get $-(z-4)(z-1)(z^2+z+1) = 0$ with multiplicity 1 so we use theorem 1.2 (b) (ii) with $z = 1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Friedrich numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1, F_2 = 2, F_3 = 4$).

**Corollary 3.17.**
*For $n \geq 0$, Friedrich numbers have the following properties.*

**(a)** $\sum\limits_{k=0}^{n} F_k = \frac{1}{3}(-(n+3)F_{n+3} + (n+4)F_{n+2} + (n+3)F_{n+1} + 2(n+4)F_n + 1).$

**(b)** $\sum\limits_{k=0}^{n} F_{2k} = \frac{1}{9}(-(3n+7)F_{2n+2} + (3n+10)F_{2n+1} + (3n+13)F_{2n} + 2(3n+7)F_{2n-1} + 4).$

**(c)** $\sum\limits_{k=0}^{n} F_{2k+1} = \frac{1}{9}(-(3n+4)F_{2n+2} + (3n+13)F_{2n+1} + (3n+7)F_{2n} + 2(3n+7)F_{2n-1} + 4).$

**(d)** $\sum\limits_{k=1}^{n} F_{-k} = \frac{1}{3}(-(n+1)F_{-n+3} + nF_{-n+2} + (n+1)F_{-n+1} + (2n+3)F_{-n} + 3).$

**(e)** $\sum\limits_{k=1}^{n} F_{-2k} = \frac{1}{9}(-(3n+5)F_{-2n+2} + (3n+2)F_{-2n+1} + (3n+8)F_{-2n} + 2(3n+5)F_{-2n-1} + 8).$

**(f)** $\sum\limits_{k=1}^{n} F_{-2k+1} = \frac{1}{9}(-(3n+8)F_{-2n+2} + (3n+8)F_{-2n+1} + (3n+5)F_{-2n} + (6n+10)F_{-2n-1} + 8).$

Taking $W_n = C_n$ with $C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 11$ in the last Theorem, we have the following Corollary which gives sum formulas of Friedrich-Lucas numbers.

**Corollary 3.18.**
*For $n \geq 0$, Friedrich-Lucas numbers have the following properties:*

**(a)** $\sum\limits_{k=0}^{n} C_k = \frac{1}{3}(-(n+3)C_{n+3} + (n+4)C_{n+2} + (n+3)C_{n+1} + 2(n+4)C_n - 9).$

**(b)** $\sum\limits_{k=0}^{n} C_{2k} = \frac{1}{9}(-(3n+7)C_{2n+2} + (3n+10)C_{2n+1} + (3n+13)C_{2n} + 2(3n+7)C_{2n-1} - 15).$

**(c)** $\sum\limits_{k=0}^{n} C_{2k+1} = \frac{1}{9}(-(3n+4)C_{2n+2} + (3n+13)C_{2n+1} + (3n+7)C_{2n} + 2(3n+7)C_{2n-1} - 27).$

**(d)** $\sum\limits_{k=1}^{n} C_{-k} = \frac{1}{3}(-(n+1)C_{-n+3} + nC_{-n+2} + (n+1)C_{-n+1} + (2n+3)C_{-n} - 3).$

**(e)** $\sum\limits_{k=1}^{n} C_{-2k} = \frac{1}{9}(-(3n+5)C_{-2n+2} + (3n+2)C_{-2n+1} + (3n+8)C_{-2n} + 2(3n+5)C_{-2n-1} - 21).$

**(f)** $\sum\limits_{k=1}^{n} C_{-2k+1} = \frac{1}{9}(-(3n+8)C_{-2n+2} + (3n+8)C_{-2n+1} + (3n+5)C_{-2n} + (6n+10)C_{-2n-1} - 9).$

Next, we give the ordinary generating functions of some special cases of generalized Friedrich numbers.

**Lemma 3.4.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} = 0.5$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^3 W_3 - (2z^3 - z^2)W_2 + (z - 2z^2)W_1 - (z^3 + 2z - 1)W_0}{2z^4 - z^3 - 2z + 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} = 0.25$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{(z^3+2z^2)W_3 + (z-4z^2)W_2 - (4z^3-z^2)W_1 - (z^3+2z^2+4z-1)W_0}{4z^4-z^3-4z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} = 0.25$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{(2z^3+z)W_3 - (4z^3-z^2)W_2 - (4z-1)W_1 - 2(z^3+2z^2)W_0}{4z^4-z^3-4z+1}.$$

**(d)** $\min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| = |\delta| = 1$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{zW_3 + (z^2-2z)W_2 + (z^3-2z^2)W_1 - 2W_0}{-z^4+2z^3+z-2}.$$

**(e)** $|z| < \min\{|\alpha|^{2}, |\beta|^{2}, |\gamma|^{2}, |\delta|^{2}\} = |\beta|^{2} = |\gamma|^{2} = |\delta|^{2} = 1$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{(2z^2+z)W_3 + (z^3-4z^2)W_2 + (z^2-4z)W_1 + (-2z^2-4)W_0}{-z^4+4z^3+z-4}.$$

**(f)** $|z| < \min\{|\alpha|^{2}, |\beta|^{2}, |\gamma|^{2}, |\delta|^{2}\} = |\beta|^{2} = |\gamma|^{2} = |\delta|^{2} = 1$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{(z^3+2z)W_3 + (z^2-4z)W_2 + (z-4)W_1 - 2(2z^2+z)W_0}{-z^4+4z^3+z-4}.$$

Proof. Use lemmas 1.1 and 1.2. □

Now, we consider special cases of the last Lemma.

**Corollary 3.19.**

*The ordinary generating functions of the sequences $F_n$, $F_{2n}$, $F_{2n+1}$, $F_{-n}$, $F_{-2n}$, $F_{-2n+1}$ and $C_n$, $C_{2n}$, $C_{2n+1}$, $C_{-n}$, $C_{-2n}$, $C_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} = 0.5$

$$\sum_{n=0}^{\infty} F_n z^n = \frac{z}{2z^4-z^3-2z+1},$$
$$\sum_{n=0}^{\infty} C_n z^n = \frac{-z^3-6z+4}{2z^4-z^3-2z+1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} = 0.25$

$$\sum_{n=0}^{\infty} F_{2n} z^n = \frac{z^2+2z}{4z^4-z^3-4z+1},$$
$$\sum_{n=0}^{\infty} C_{2n} z^n = \frac{-z^3-12z+4}{4z^4-z^3-4z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} = 0.25$

$$\sum_{n=0}^{\infty} F_{2n+1} z^n = \frac{2z^2+1}{4z^4-z^3-4z+1},$$
$$\sum_{n=0}^{\infty} C_{2n+1} z^n = \frac{-2z^3-12z^2+3z+2}{4z^4-z^3-4z+1}.$$

**(d)** $\min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| = |\delta| = 1$

$$\sum_{n=0}^{\infty} F_{-n} z^n = \frac{z^3}{-z^4+2z^3+z-2},$$
$$\sum_{n=0}^{\infty} C_{-n} z^n = \frac{2z^3+3z-8}{-z^4+2z^3+z-2}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 = |\delta|^2 = 1$

$$\sum_{n=0}^{\infty} F_{-2n} z^n = \frac{2z^3 + z^2}{-z^4 + 4z^3 + z - 4},$$

$$\sum_{n=0}^{\infty} C_{-2n} z^n = \frac{4z^3 + 3z - 16}{-z^4 + 4z^3 + z - 4}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 = |\delta|^2 = 1$

$$\sum_{n=0}^{\infty} F_{-2n+1} z^n = \frac{4z^3 + 2z^2 + z - 4}{-z^4 + 4z^3 + z - 4},$$

$$\sum_{n=0}^{\infty} C_{-2n+1} z^n = \frac{11z^3 - 12z^2 - 8}{-z^4 + 4z^3 + z - 4}.$$

From the last corollary, we obtain the following results for Friedrich and Friedrich-Lucas numbers.

**Corollary 3.20.**
*Infinite sums of* $F_n, F_{2n}, F_{2n+1}, F_{-n}, F_{-2n}, F_{-2n+1}$ *and* $C_n, C_{2n}, C_{2n+1}, C_{-n}, C_{-2n}, C_{-2n+1}$ *are given as follows:*

**(a)** $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{F_n}{4^n} = \frac{32}{63},$$

$$\sum_{n=0}^{\infty} \frac{C_n}{4^n} = \frac{106}{21}.$$

**(b)** $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{F_{2n}}{8^n} = \frac{272}{511},$$

$$\sum_{n=0}^{\infty} \frac{C_{2n}}{8^n} = \frac{2558}{511}.$$

**(c)** $z = \frac{1}{8}$.

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}}{8^n} = \frac{1056}{511},$$

$$\sum_{n=0}^{\infty} \frac{C_{2n+1}}{8^n} = \frac{2236}{511}.$$

**(d)** $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{F_{-n}}{2^n} = -\frac{2}{21},$$

$$\sum_{n=0}^{\infty} \frac{C_{-n}}{2^n} = \frac{100}{21}.$$

**(e)** $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{F_{-2n}}{2^n} = -\frac{8}{49},$$

$$\sum_{n=0}^{\infty} \frac{C_{-2n}}{2^n} = \frac{32}{7}.$$

**(f)** $z = \frac{1}{2}$.

$$\sum_{n=0}^{\infty} \frac{F_{-2n+1}}{2^n} = \frac{40}{49},$$

$$\sum_{n=0}^{\infty} \frac{C_{-2n+1}}{2^n} = \frac{22}{7}.$$

### 3.5. Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} W_n z^n$, $\sum_{n=0}^{\infty} W_{2n} z^n$, $\sum_{n=0}^{\infty} W_{2n+1} z^n$, $\sum_{n=0}^{\infty} W_{-n} z^n$, $\sum_{n=0}^{\infty} W_{-2n} z^n$, $\sum_{n=0}^{\infty} W_{-2n+1} z^n$ of Generalized Pierre Numbers

In this subsection, we consider the case $r = 2, s = 0, t = 0, u = -1$. A generalized Pierre sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-4} \tag{37}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (37) holds for all integers $n$. For more information on generalized Pierre numbers, see Soykan [13].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 2z^3 + 1 = (z^3 - z^2 - z - 1)(z - 1) = 0$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Two special cases of the sequence $\{W_n\}$ are Pierre sequence $\{P_n\}_{n\geq 0}$ and Pierre-Lucas sequence $\{C_n\}_{n\geq 0}$. Pierre sequence $\{P_n\}_{n\geq 0}$ and Pierre-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}
P_n &= 2P_{n-1} - P_{n-4}, \quad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4, \quad n \geq 4, \\
C_n &= 2C_{n-1} - C_{n-4}, \quad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8, \quad n \geq 4.
\end{aligned}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
P_{-n} &= 2P_{-(n-3)} - P_{-(n-4)}, \\
C_{-n} &= 2C_{-(n-3)} - C_{-(n-4)},
\end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively.

Binet's formula of generalized Pierre numbers can be given as follows:

**Theorem 3.9.**
*For all integers n, Binet's formula of generalized Pierre numbers is*

$$\begin{aligned}
W_n &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0)\alpha^n}{2\alpha^2 + 2\alpha - 2} \\
&+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0)\beta^n}{2\beta^2 + 2\beta - 2} \\
&+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0)\gamma^n}{2\gamma^2 + 2\gamma - 2} \\
&- \frac{W_3 - W_2 - W_1 - W_0}{2}.
\end{aligned}$$

Pierre and Pierre-Lucas numbers can be expressed using Binet's formulas as follows.

**Corollary 3.21.**
*For all integers n, Binet's formula of Pierre and Pierre-Lucas numbers are*

$$P_n = \frac{(\alpha^2+\alpha+1)\alpha^n}{2(\alpha^2+\alpha-1)} + \frac{(\beta^2+\beta+1)\beta^n}{2(\beta^2+\beta-1)} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)} - \frac{1}{2},$$

*and*

$$C_n = \alpha^n + \beta^n + \gamma^n + 1,$$

*respectively.*

Next, we present sum formulas of generalized Pierre numbers

**Theorem 3.10.**
*For $n \geq 0$, we have the following sum formulas for generalized Pierre numbers:*

**(a)** $\sum_{k=0}^{n} W_k = \frac{1}{2}(-(n+3)W_{n+3} + (n+4)W_{n+2} + (n+3)W_{n+1} + (n+4)W_n + 3W_3 - 4W_2 - 3W_1 - 2W_0).$

**(b)** $\sum_{k=0}^{n} W_{2k} = \frac{1}{2}(-(n+2)W_{2n+2} + (n+3)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 2W_2 - 3W_1 - W_0).$

**(c)** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(-(n+1)W_{2n+2} + (n+3)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 3W_2 - W_1 - 2W_0).$

**(d)** $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-(n+1)W_{-n+3} + nW_{-n+2} + (n+1)W_{-n+1} + (n+2)W_{-n} + W_3 - W_1 - 2W_0).$

**(e)** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{2}(-(n+2)W_{-2n+2} + (n+1)W_{-2n+1} + (n+3)W_{-2n} + (n+2)W_{-2n-1} + 2W_3 - 2W_2 - W_1 - 3W_0).$

**(f)** $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{2}(-nW_{-2n+2} - 3W_{-2n+2} + nW_{-2n+1} + 3W_{-2n+1} + nW_{-2n} + 2W_{-2n} + nW_{-2n-1} + 2W_{-2n-1} + 2W_3 - W_2 - 3W_1 - 2W_0).$

Proof.

**(a)** For $r = 2,\ s = 0,\ t = 0,\ u = -1$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = -(z-1)(z^3+z^2+z-1)$ and then for $z = 1$, we get $-(z-1)(z^3+z^2+z-1) = 0$ with multiplicity 1 so we use theorem 1.1 (a) (ii) with $z = 1$.

**(b)** For $r = 2,\ s = 0,\ t = 0,\ u = -1$, we get $-u^2z^4 + (t^2-2su)z^3 + (2u+2rt-s^2)z^2 + (2s+r^2)z - 1 = -(z-1)(z^3+z^2+3z-1)$ and then for $z = 1$, we get $-(z-1)(z^3+z^2+3z-1) = 0$ with multiplicity 1 so we use theorem 1.1 (b) (ii) with $z = 1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (ii) with $z = 1$.

**(d)** For $r = 2,\ s = 0,\ t = 0,\ u = -1$, we get $-z^4 + rz^3 + sz^2 + tz + u = (z-1)(-z^3+z^2+z+1)$ and then for $z = 1$, we get $(z-1)(-z^3+z^2+z+1) = 0$ with multiplicity 1 so we use theorem 1.2 (a) (ii) with $z = 1$.

**(e)** For $r = 2,\ s = 0,\ t = 0,\ u = -1$, we get $-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2 = -(z-1)(z^3-3z^2-z-1)$ and then for $z = 1$, we get $-(z-1)(z^3-3z^2-z-1) = 0$ with multiplicity 1 so we use theorem 1.2 (b) (ii) with $z = 1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Pierre numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4$).

**Corollary 3.22.**
*For $n \geq 0$, Pierre numbers have the following properties.*

**(a)** $\sum_{k=0}^{n} P_k = \frac{1}{2}(-(n+3)P_{n+3} + (n+4)P_{n+2} + (n+3)P_{n+1} + (n+4)P_n + 1).$

**(b)** $\sum_{k=0}^{n} P_{2k} = \frac{1}{2}(-(n+2)P_{2n+2} + (n+3)P_{2n+1} + (n+3)P_{2n} + (n+2)P_{2n-1} + 1).$

**(c)** $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{2}(-(n+1)P_{2n+2} + (n+3)P_{2n+1} + (n+2)P_{2n} + (n+2)P_{2n-1} + 1).$

**(d)** $\sum_{k=1}^{n} P_{-k} = \frac{1}{2}(-(n+1)P_{-n+3} + nP_{-n+2} + (n+1)P_{-n+1} + (n+2)P_{-n} + 3).$

**(e)** $\sum_{k=1}^{n} P_{-2k} = \frac{1}{2}(-(n+2)P_{-2n+2} + (n+1)P_{-2n+1} + (n+3)P_{-2n} + (n+2)P_{-2n-1} + 3).$

**(f)** $\sum_{k=1}^{n} P_{-2k+1} = \frac{1}{2}(-nP_{-2n+2} - 3P_{-2n+2} + nP_{-2n+1} + 3P_{-2n+1} + nP_{-2n} + 2P_{-2n} + nP_{-2n-1} + 2P_{-2n-1} + 3).$

Taking $W_n = C_n$ with $C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8$ in the last Theorem, we have the following Corollary which gives sum formulas of Pierre-Lucas numbers.

**Corollary 3.23.**
*For $n \geq 0$, Pierre-Lucas numbers have the following properties:*

**(a)** $\sum_{k=0}^{n} C_k = \frac{1}{2}(-(n+3)C_{n+3} + (n+4)C_{n+2} + (n+3)C_{n+1} + (n+4)C_n - 6).$

**(b)** $\sum_{k=0}^{n} C_{2k} = \frac{1}{2}(-(n+2)C_{2n+2} + (n+3)C_{2n+1} + (n+3)C_{2n} + (n+2)C_{2n-1} - 2).$

**(c)** $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{2}(-(n+1)C_{2n+2} + (n+3)C_{2n+1} + (n+2)C_{2n} + (n+2)C_{2n-1} - 6).$

**(d)** $\sum_{k=1}^{n} C_{-k} = \frac{1}{2}(-(n+1)C_{-n+3} + nC_{-n+2} + (n+1)C_{-n+1} + (n+2)C_{-n} - 2).$

**(e)** $\sum_{k=1}^{n} C_{-2k} = \frac{1}{2}(-(n+2)C_{-2n+2} + (n+1)C_{-2n+1} + (n+3)C_{-2n} + (n+2)C_{-2n-1} - 6).$

**(f)** $\sum_{k=1}^{n} C_{-2k+1} = \frac{1}{2}(-nC_{-2n+2} - 3C_{-2n+2} + nC_{-2n+1} + 3C_{-2n+1} + nC_{-2n} + 2C_{-2n} + nC_{-2n-1} + 2C_{-2n-1} - 2).$

Next, we give the ordinary generating functions of some special cases of generalized Pierre numbers.

**Lemma 3.5.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.543689$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^3 W_3 - (2z^3 - z^2)W_2 - (2z^2 - z)W_1 - (2z-1)W_0}{z^4 - 2z + 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.295597$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{2z^2 W_3 + (z^3 - 4z^2 + z)W_2 - 2z^3 W_1 + (z^2 - 4z + 1)W_0}{z^4 + 2z^2 - 4z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.295597$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{(z^3 + z)W_3 - 2z^3 W_2 + (z^2 - 4z + 1)W_1 - 2z^2 W_0}{z^4 + 2z^2 - 4z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.737352$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{-zW_3 + \left(2z - z^2\right)W_2 + \left(2z^2 - z^3\right)W_1 + W_0}{z^4 - 2z^3 + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.543689$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{(z^2+1)W_0 + 2zW_1 - 2z^2 W_3 - (z^3 - 4z^2 + z)W_2}{z^4 - 4z^3 + 2z^2 + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.543689$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{-(z^3+z)W_3 + 2zW_2 + (z^2+1)W_1 + 2z^2 W_0}{z^4 - 4z^3 + 2z^2 + 1}.$$

Proof. Use lemmas 1.1 and 1.2. □

Now, we consider special cases of the last Lemma.

**Corollary 3.24.**
*The ordinary generating functions of the sequences $P_n$, $P_{2n}$, $P_{2n+1}$, $P_{-n}$, $P_{-2n}$, $P_{-2n+1}$ and $C_n$, $C_{2n}$, $C_{2n+1}$, $C_{-n}$, $C_{-2n}$, $C_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.543689$

$$\sum_{n=0}^{\infty} P_n z^n = \frac{z}{z^4 - 2z + 1},$$
$$\sum_{n=0}^{\infty} C_n z^n = \frac{-6z + 4}{z^4 - 2z + 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.295597$

$$\sum_{n=0}^{\infty} P_{2n} z^n = \frac{2z}{z^4 + 2z^2 - 4z + 1},$$
$$\sum_{n=0}^{\infty} C_{2n} z^n = \frac{4z^2 - 12z + 4}{z^4 + 2z^2 - 4z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.295597$

$$\sum_{n=0}^{\infty} P_{2n+1} z^n = \frac{z^2 + 1}{z^4 + 2z^2 - 4z + 1},$$
$$\sum_{n=0}^{\infty} C_{2n+1} z^n = \frac{-6z^2 + 2}{z^4 + 2z^2 - 4z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.737352$

$$\sum_{n=0}^{\infty} P_{-n} z^n = \frac{-z^3}{z^4 - 2z^3 + 1},$$
$$\sum_{n=0}^{\infty} C_{-n} z^n = \frac{-2z^3 + 4}{z^4 - 2z^3 + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.543689$

$$\sum_{n=0}^{\infty} P_{-2n} z^n = \frac{-2z^3}{z^4 - 4z^3 + 2z^2 + 1},$$
$$\sum_{n=0}^{\infty} C_{-2n} z^n = \frac{-4z^3 + 4z^2 + 4}{z^4 - 4z^3 + 2z^2 + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.543689$

$$\sum_{n=0}^{\infty} P_{-2n+1} z^n = \frac{-4z^3 + z^2 + 1}{z^4 - 4z^3 + 2z^2 + 1},$$
$$\sum_{n=0}^{\infty} C_{-2n+1} z^n = \frac{-8z^3 + 10z^2 + 2}{z^4 - 4z^3 + 2z^2 + 1}.$$

From the last corollary, we obtain the following results for Pierre and Pierre-Lucas numbers.

**Corollary 3.25.**
*Infinite sums of* $P_n, P_{2n}, P_{2n+1}, P_{-n}, P_{-2n}, P_{-2n+1}$ *and* $C_n, C_{2n}, C_{2n+1}, C_{-n}, C_{-2n}, C_{-2n+1}$ *are given as follows:*

**(a)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_n}{2^n} = 8,$$
$$\sum_{n=0}^{\infty} \frac{C_n}{2^n} = 16.$$

**(b)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{P_{2n}}{4^n} = \frac{128}{33},$$
$$\sum_{n=0}^{\infty} \frac{C_{2n}}{4^n} = \frac{320}{33}.$$

**(c)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{P_{2n+1}}{4^n} = \frac{272}{33},$$
$$\sum_{n=0}^{\infty} \frac{C_{2n+1}}{4^n} = \frac{416}{33}.$$

**(d)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_{-n}}{2^n} = -\frac{2}{13},$$
$$\sum_{n=0}^{\infty} \frac{C_{-n}}{2^n} = \frac{60}{13}.$$

**(e)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_{-2n}}{2^n} = -\frac{4}{17},$$
$$\sum_{n=0}^{\infty} \frac{C_{-2n}}{2^n} = \frac{72}{17}.$$

**(f)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_{-2n+1}}{2^n} = \frac{12}{17},$$
$$\sum_{n=0}^{\infty} \frac{C_{-2n+1}}{2^n} = \frac{56}{17}.$$

### 3.6. Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} W_n z^n$, $\sum_{n=0}^{\infty} W_{2n} z^n$, $\sum_{n=0}^{\infty} W_{2n+1} z^n$, $\sum_{n=0}^{\infty} W_{-n} z^n$, $\sum_{n=0}^{\infty} W_{-2n} z^n$, $\sum_{n=0}^{\infty} W_{-2n+1} z^n$ of Generalized Pandita Numbers

In this subsection, we consider the case $r = 2, s = -1, t = 1, u = -1$. A generalized Pandita sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-2} + W_{n-3} - W_{n-4} \tag{38}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = W_{-(n-1)} - W_{-(n-2)} + 2W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1,2,3,\ldots$. Therefore, recurrence eq. (38) holds for all integers $n$. For more information on generalized Pandita numbers, see Soykan [14].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 2z^3 + z^2 - z + 1 = (z^3 - z^2 - 1)(z-1) = 0$$

whose roots are

$$\alpha = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$
$$\beta = \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$
$$\gamma = \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3},$$
$$\delta = 1,$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Two special cases of the sequence $\{W_n\}$ are Pandita sequence $\{P_n\}_{n\geq 0}$ and Pandita-Lucas sequence $\{S_n\}_{n\geq 0}$. Pandita sequence $\{P_n\}_{n\geq 0}$ and Pandita-Lucas sequence $\{S_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, \quad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \quad n \geq 4,$$
$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}, \quad S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5, \quad n \geq 4.$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)},$$
$$S_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)},$$

for $n = 1,2,3,\ldots$ respectively.

Binet's formula of generalized Pandita numbers can be given as follows:

**Theorem 3.11.**
*For all integers $n$, Binet's formula of generalized Pandita numbers is*

$$W_n = \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0)\alpha^n}{3\alpha - 2} + \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0)\beta^n}{3\beta - 2} + \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0)\gamma^n}{3\gamma - 2} - W_3 + W_2 + W_0.$$

Pandita and Pandita-Lucas numbers can be expressed using Binet's formulas as follows.

**Corollary 3.26.**
*For all integers $n$, Binet's formula of Pandita and Pandita-Lucas numbers are*

$$P_n = \frac{\alpha^{n+3}}{3\alpha - 2} + \frac{\beta^{n+3}}{3\beta - 2} + \frac{\gamma^{n+3}}{3\gamma - 2} - 1,$$

*and*

$$S_n = \alpha^n + \beta^n + \gamma^n + 1,$$

*respectively.*

Next, we present sum formulas of generalized Pandita numbers

**Theorem 3.12.**
*For $n \geq 0$, we have the following sum formulas for generalized Pandita numbers:*

**(a)** $\sum_{k=0}^{n} W_k = -(n+3)W_{n+3} + (n+4)W_{n+2} + (n+4)W_n + 3W_3 - 4W_2 - 3W_0.$

**(b)** $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(-3(n+2)W_{2n+2} + (3n+8)W_{2n+1} + 2W_{2n} + (3n+7)W_{2n-1} + 7W_3 - 8W_2 - W_1 - 6W_0).$

**(c)** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(3n+4)W_{2n+2} + (3n+8)W_{2n+1} + W_{2n} + 3(n+2)W_{2n-1} + 6W_3 - 8W_2 + W_1 - 7W_0).$

**(d)** $\sum_{k=1}^{n} W_{-k} = -(n+1)W_{-n+3} + nW_{-n+2} + (n+1)W_{-n} + W_3 - W_0.$

**(e)** $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(-3(n+2)W_{-2n+2} + (3n+4)W_{-2n+1} + W_{-2n} + (3n+5)W_{-2n-1} + 5W_3 - 4W_2 + W_1 - 6W_0).$

**(f)** $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(3n+8)W_{-2n+2} + (3n+7)W_{-2n+1} - W_{-2n} + 3(n+2)W_{-2n-1} + 6W_3 - 4W_2 - W_1 - 5W_0).$

Proof.

**(a)** For $r = 2, s = -1, t = 1, u = -1$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = -(z-1)(z^3 + z - 1)$ and then for $z = 1$, we get $-(z-1)(z^3 + z - 1) = 0$ with multiplicity 1 so we use theorem 1.1 (a) (ii) with $z = 1$.

**(b)** For $r = 2, s = -1, t = 1, u = -1$, we get $-u^2z^4 + (t^2 - 2su)z^3 + (2u + 2rt - s^2)z^2 + (2s + r^2)z - 1 = -(z-1)(z^3 + 2z^2 + z - 1)$ and then for $z = 1$, we get $-(z-1)(z^3 + 2z^2 + z - 1) = 0$ so we use theorem 1.1 (b) (ii) with $z = 1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (ii) with $z = 1$.

**(d)** For $r = 2, s = -1, t = 1, u = -1$, we get $-z^4 + rz^3 + sz^2 + tz + u = (z-1)(-z^3 + z^2 + 1)$ and then for $z = 1$, we get $(z-1)(-z^3 + z^2 + 1) = 0$ with multiplicity 1 so we use theorem 1.2 (a) (ii) with $z = 1$.

**(e)** For $r = 2, s = -1, t = 1, u = -1$, we get $-z^4 + (r^2 + 2s)z^3 + (2u + 2rt - s^2)z^2 + (t^2 - 2su)z - u^2 = (z-1)(-z^3 + z^2 + 2z + 1)$ and then for $z = 1$, we get $(z-1)(-z^3 + z^2 + 2z + 1) = 0$ with multiplicity 1 so we use theorem 1.2 (b) (ii) with $z = 1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (ii) with $z = 1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Pandita numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3$).

**Corollary 3.27.**
*For $n \geq 0$, Pandita numbers have the following properties.*

**(a)** $\sum_{k=0}^{n} P_k = -(n+3)P_{n+3} + (n+4)P_{n+2} + (n+4)P_n + 1.$

**(b)** $\sum_{k=0}^{n} P_{2k} = \frac{1}{3}(-3(n+2)P_{2n+2} + (3n+8)P_{2n+1} + 2P_{2n} + (3n+7)P_{2n-1} + 4).$

**(c)** $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{3}(-(3n+4)P_{2n+2} + (3n+8)P_{2n+1} + P_{2n} + 3(n+2)P_{2n-1} + 3).$

**(d)** $\sum_{k=1}^{n} P_{-k} = -(n+1)P_{-n+3} + nP_{-n+2} + (n+1)P_{-n} + 3.$

**(e)** $\sum_{k=1}^{n} P_{-2k} = \frac{1}{3}(-3(n+2)P_{-2n+2} + (3n+4)P_{-2n+1} + P_{-2n} + (3n+5)P_{-2n-1} + 8).$

**(f)** $\sum_{k=1}^{n} P_{-2k+1} = \frac{1}{3}(-(3n+8)P_{-2n+2} + (3n+7)P_{-2n+1} - P_{-2n} + 3(n+2)P_{-2n-1} + 9).$

Taking $W_n = S_n$ with $S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5$ in the last Theorem, we have the following Corollary which gives sum formulas of Pandita-Lucas numbers.

**Corollary 3.28.**
*For $n \geq 0$, Pandita-Lucas numbers have the following properties:*

**(a)** $\sum_{k=0}^{n} S_k = -(n+3)S_{n+3} + (n+4)S_{n+2} + (n+4)S_n - 5.$

**(b)** $\sum_{k=0}^{n} S_{2k} = \frac{1}{3}(-3(n+2)S_{2n+2} + (3n+8)S_{2n+1} + 2S_{2n} + (3n+7)S_{2n-1} - 7).$

**(c)** $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{3}(-(3n+4)S_{2n+2} + (3n+8)S_{2n+1} + S_{2n} + 3(n+2)S_{2n-1} - 12).$

**(d)** $\sum_{k=1}^{n} S_{-k} = -(n+1)S_{-n+3} + nS_{-n+2} + (n+1)S_{-n} + 1.$

**(e)** $\sum_{k=1}^{n} S_{-2k} = \frac{1}{3}(-3(n+2)S_{-2n+2} + (3n+4)S_{-2n+1} + S_{-2n} + (3n+5)S_{-2n-1} - 5).$

**(f)** $\sum_{k=1}^{n} S_{-2k+1} = \frac{1}{3}(-(3n+8)S_{-2n+2} + (3n+7)S_{-2n+1} - S_{-2n} + 3(n+2)S_{-2n-1}).$

Next, we give the ordinary generating functions of some special cases of generalized Pandita numbers.

**Lemma 3.6.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.682327$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^3W_3 - (2z^3 - z^2)W_2 + (z^3 - 2z^2 + z)W_1 - (z^3 - z^2 + 2z - 1)W_0}{z^4 - z^3 + z^2 - 2z + 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.465571$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{(z^3 + 2z^2)W_3 - (z^3 + 3z^2 - z)W_2 + (z^2 - z^3)W_1 - (2z^2 + 2z - 1)W_0}{z^4 + z^3 - z^2 - 2z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.465571$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{(z^3 + z^2 + z)W_3 - (2z^3 + z^2)W_2 + (z^3 - 2z + 1)W_1 - (z^3 + 2z^2)W_0}{z^4 + z^3 - z^2 - 2z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.826031$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{-zW_3 + (2z - z^2)W_2 - (z^3 - 2z^2 + z)W_1 + W_0}{z^4 - 2z^3 + z^2 - z + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.682327$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-(2z^2 + z)W_3 + (-z^3 + 3z^2 + z)W_2 + (z - z^2)W_1 + (z^2 + z + 1)W_0}{z^4 - 2z^3 - z^2 + z + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.682327$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{-(z^3 + z^2 + z)W_3 + (z^2 + 2z)W_2 - (z^2 - 1)W_1 + (2z^2 + z)W_0}{z^4 - 2z^3 - z^2 + z + 1}.$$

Proof. Use lemmas 1.1 and 1.2. □
Now, we consider special cases of the last Lemma.

**Corollary 3.29.**
*The ordinary generating functions of the sequences $P_n$, $P_{2n}$, $P_{2n+1}$, $P_{-n}$, $P_{-2n}$, $P_{-2n+1}$ and $S_n$, $S_{2n}$, $S_{2n+1}$, $S_{-n}$, $S_{-2n}$, $S_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.682327$

$$\sum_{n=0}^{\infty} P_n z^n = \frac{z}{z^4 - z^3 + z^2 - 2z + 1},$$
$$\sum_{n=0}^{\infty} S_n z^n = \frac{-z^3 + 2z^2 - 6z + 4}{z^4 - z^3 + z^2 - 2z + 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.465571$

$$\sum_{n=0}^{\infty} P_{2n} z^n = \frac{z^2 + 2z}{z^4 + z^3 - z^2 - 2z + 1},$$
$$\sum_{n=0}^{\infty} S_{2n} z^n = \frac{z^3 - 2z^2 - 6z + 4}{z^4 + z^3 - z^2 - 2z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.465571$

$$\sum_{n=0}^{\infty} P_{2n+1} z^n = \frac{z^2 + z + 1}{z^4 + z^3 - z^2 - 2z + 1},$$
$$\sum_{n=0}^{\infty} S_{2n+1} z^n = \frac{-z^3 - 5z^2 + z + 2}{z^4 + z^3 - z^2 - 2z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.826031$

$$\sum_{n=0}^{\infty} P_{-n} z^n = \frac{-z^3}{z^4 - 2z^3 + z^2 - z + 1},$$
$$\sum_{n=0}^{\infty} S_{-n} z^n = \frac{-2z^3 + 2z^2 - 3z + 4}{z^4 - 2z^3 + z^2 - z + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.682327$

$$\sum_{n=0}^{\infty} P_{-2n} z^n = \frac{-2z^3 - z^2}{z^4 - 2z^3 - z^2 + z + 1},$$
$$\sum_{n=0}^{\infty} S_{-2n} z^n = \frac{-2z^3 - 2z^2 + 3z + 4}{z^4 - 2z^3 - z^2 + z + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.682327$

$$\sum_{n=0}^{\infty} P_{-2n+1} z^n = \frac{-3z^3 - 2z^2 + z + 1}{z^4 - 2z^3 - z^2 + z + 1},$$
$$\sum_{n=0}^{\infty} S_{-2n+1} z^n = \frac{-5z^3 + 3z^2 + 3z + 2}{z^4 - 2z^3 - z^2 + z + 1}.$$

From the last corollary, we obtain the following results for Pandita and Pandita-Lucas numbers.

**Corollary 3.30.**
*Infinite sums of $P_n$, $P_{2n}$, $P_{2n+1}$, $P_{-n}$, $P_{-2n}$, $P_{-2n+1}$ and $S_n$, $S_{2n}$, $S_{2n+1}$, $S_{-n}$, $S_{-2n}$, $S_{-2n+1}$ are given as follows:*

**(a)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_n}{2^n} = \frac{8}{3},$$
$$\sum_{n=0}^{\infty} \frac{S_n}{2^n} = \frac{22}{3}.$$

**(b)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{P_{2n}}{4^n} = \frac{16}{13},$$
$$\sum_{n=0}^{\infty} \frac{S_{2n}}{4^n} = \frac{68}{13}.$$

**(c)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{P_{2n+1}}{4^n} = \frac{112}{39},$$
$$\sum_{n=0}^{\infty} \frac{S_{2n+1}}{4^n} = \frac{164}{39}.$$

**(d)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_{-n}}{2^n} = -\frac{2}{9},$$
$$\sum_{n=0}^{\infty} \frac{S_{-n}}{2^n} = \frac{44}{9}.$$

**(e)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_{-2n}}{2^n} = -\frac{8}{17},$$
$$\sum_{n=0}^{\infty} \frac{S_{-2n}}{2^n} = \frac{76}{17}.$$

**(f)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{P_{-2n+1}}{2^n} = \frac{10}{17},$$
$$\sum_{n=0}^{\infty} \frac{S_{-2n+1}}{2^n} = \frac{58}{17}.$$

### 3.7. Sum Formulas $\sum_{k=0}^{n} W_k$, $\sum_{k=0}^{n} W_{2k}$, $\sum_{k=0}^{n} W_{2k+1}$, $\sum_{k=0}^{n} W_{-k}$, $\sum_{k=0}^{n} W_{-2k}$, $\sum_{k=0}^{n} W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} W_n z^n$, $\sum_{n=0}^{\infty} W_{2n} z^n$, $\sum_{n=0}^{\infty} W_{2n+1} z^n$, $\sum_{n=0}^{\infty} W_{-n} z^n$, $\sum_{n=0}^{\infty} W_{-2n} z^n$, $\sum_{n=0}^{\infty} W_{-2n+1} z^n$ of Generalized Adrien Numbers

In this subsection, we consider the case $r = 3, s = -1, t = 0, u = -1$. A generalized Adrien sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4} \tag{39}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence eq. (39) holds for all integers $n$. For more information on generalized Adrien numbers, see Soykan [15].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 3z^3 + z^2 + 1 = (z^3 - 2z^2 - z - 1)(z - 1) = 0$$

whose roots are

$$\alpha = \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}$$
$$\beta = \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}$$
$$\gamma = \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}$$
$$\delta = 1$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Two special cases of the sequence $\{W_n\}$ are Adrien sequence $\{A_n\}_{n\geq 0}$ and Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$. Adrien sequence $\{A_n\}_{n\geq 0}$ and Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}, \quad A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8, \quad n \geq 4,$$
$$B_n = 3B_{n-1} - B_{n-2} - B_{n-4}, \quad B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18, \quad n \geq 4.$$

The sequences $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$A_{-n} = -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)},$$
$$B_{-n} = -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively.

Binet's formula of generalized Adrien numbers can be given as follows:

**Theorem 3.13.**
*For all integers n, Binet's formula of generalized Adrien numbers is*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - 1W_0)\alpha^n}{4\alpha^2 + 3\alpha - 1} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - 1W_0)\beta^n}{4\beta^2 + 3\beta - 1} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - 1W_0)\gamma^n}{4\gamma^2 + 3\gamma - 1} \\ &+ \frac{W_3 - 2W_2 - W_1 - W_0}{-3}. \end{aligned}$$

Adrien and Adrien-Lucas numbers can be expressed using Binet's formulas as follows.

**Corollary 3.31.**
*For all integers n, Binet's formula of Adrien and Adrien-Lucas numbers are*

$$A_n = \frac{(2\alpha^2 + \alpha + 1)\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{(2\beta^2 + \beta + 1)\beta^n}{4\beta^2 + 3\beta - 1} + \frac{(2\gamma^2 + \gamma + 1)\gamma^n}{4\gamma^2 + 3\gamma - 1} - \frac{1}{3},$$

*and*

$$B_n = \alpha^n + \beta^n + \gamma^n + 1,$$

*respectively.*

Next, we present sum formulas of generalized Adrien numbers

**Theorem 3.14.**
*For $n \geq 0$, we have the following sum formulas for generalized Adrien numbers:*

**(a)** $\sum\limits_{k=0}^{n} W_k = \frac{1}{3}(-(n+3)W_{n+3} + (2n+7)W_{n+2} + (n+2)W_{n+1} + (n+4)W_n + 3W_3 - 7W_2 - 2W_1 - W_0).$

**(b)** $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{3}(-(n+2)W_{2n+2} + (2n+5)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 4W_2 - 3W_1).$

**(c)** $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(n+1)W_{2n+2} + (2n+5)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 5W_2 - 2W_0).$

**(d)** $\sum\limits_{k=1}^{n} W_{-k} = \frac{1}{3}(-(n+1)W_{-n+3} + (2n+1)W_{-n+2} + (n+2)W_{-n+1} + (n+3)W_{-n} + W_3 - W_2 - 2W_1 - 3W_0).$

**(e)** $\sum\limits_{k=1}^{n} W_{-2k} = \frac{1}{3}(-(n+2)W_{-2n+2} + (2n+3)W_{-2n+1} + (n+4)W_{-2n} + (n+2)W_{-2n-1} + 2W_3 - 4W_2 - W_1 - 4W_0).$

**(f)** $\sum\limits_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(n+3)W_{-2n+2} + 2(n+3)W_{-2n+1} + (n+2)W_{-2n} + (n+2)W_{-2n-1} + 2W_3 - 3W_2 - 4W_1 - 2W_0).$

Proof.

**(a)** For $r=3, s=-1, t=0, u=-1$, we get $uz^4 + tz^3 + sz^2 + rz - 1 = -(z-1)(z^3+z^2+2z-1)$ and then for $z=1$, we get $-(z-1)(z^3+z^2+2z-1) = 0$ with multiplicity 1 so we use theorem 1.1 (a) (ii) with $z=1$.

**(b)** For $r=3, s=-1, t=0, u=-1$, we get $-u^2z^4 + (t^2-2su)z^3 + (2u+2rt-s^2)z^2 + (2s+r^2)z - 1 = -(z-1)(z^3+3z^2+6z-1)$ and then for $z=1$, we get $-(z-1)(z^3+3z^2+6z-1) = 0$ with multiplicity 1 so we use theorem 1.1 (b) (ii) with $z=1$.

**(c)** Similarly as in (b), we use theorem 1.1 (c) (ii) with $z=1$.

**(d)** For $r=3, s=-1, t=0, u=-1$, we get $-z^4 + rz^3 + sz^2 + tz + u = (z-1)(-z^3+2z^2+z+1)$ and then for $z=1$, we get $(z-1)(-z^3+2z^2+z+1) = 0$ with multiplicity 1 so we use theorem 1.2 (a) (ii) with $z=1$.

**(e)** For $r=3, s=-1, t=0, u=-1$, we get $-z^4 + (r^2+2s)z^3 + (2u+2rt-s^2)z^2 + (t^2-2su)z - u^2 = -(z-1)(z^3-6z^2-3z-1)$ and then for $z=1$, we get $-(z-1)(z^3-6z^2-3z-1) = 0$ with multiplicity 1 so we use theorem 1.2 (b) (ii) with $z=1$.

**(f)** Similarly as in (e), we use theorem 1.2 (c) (ii) with $z=1$. □

From the last Theorem, we have the following Corollary which gives sum formulas of Adrien numbers (take $W_n = A_n$ with $A_0=0, A_1=1, A_2=3, A_3=8$).

**Corollary 3.32.**
*For $n \geq 0$, Adrien numbers have the following properties.*

**(a)** $\sum\limits_{k=0}^{n} A_k = \frac{1}{3}(-(n+3)A_{n+3} + (2n+7)A_{n+2} + (n+2)A_{n+1} + (n+4)A_n + 1).$

**(b)** $\sum\limits_{k=0}^{n} A_{2k} = \frac{1}{3}(-(n+2)A_{2n+2} + (2n+5)A_{2n+1} + (n+3)A_{2n} + (n+2)A_{2n-1} + 1).$

**(c)** $\sum\limits_{k=0}^{n} A_{2k+1} = \frac{1}{3}(-(n+1)A_{2n+2} + (2n+5)A_{2n+1} + (n+2)A_{2n} + (n+2)A_{2n-1} + 1).$

**(d)** $\sum\limits_{k=1}^{n} A_{-k} = \frac{1}{3}(-(n+1)A_{-n+3} + (2n+1)A_{-n+2} + (n+2)A_{-n+1} + (n+3)A_{-n} + 3).$

**(e)** $\sum\limits_{k=1}^{n} A_{-2k} = \frac{1}{3}(-(n+2)A_{-2n+2} + (2n+3)A_{-2n+1} + (n+4)A_{-2n} + (n+2)A_{-2n-1} + 3).$

**(f)** $\sum\limits_{k=1}^{n} A_{-2k+1} = \frac{1}{3}(-(n+3)A_{-2n+2} + 2(n+3)A_{-2n+1} + (n+2)A_{-2n} + (n+2)A_{-2n-1} + 3).$

Taking $W_n = B_n$ with $B_0=4, B_1=3, B_2=7, B_3=18$ in the last Theorem, we have the following Corollary which gives sum formulas of Adrien-Lucas numbers.

**Corollary 3.33.**
*For $n \geq 0$, Adrien-Lucas numbers have the following properties:*

**(a)** $\sum_{k=0}^{n} B_k = \frac{1}{3}(-(n+3)B_{n+3} + (2n+7)B_{n+2} + (n+2)B_{n+1} + (n+4)B_n - 5)$.

**(b)** $\sum_{k=0}^{n} B_{2k} = \frac{1}{3}(-(n+2)B_{2n+2} + (2n+5)B_{2n+1} + (n+3)B_{2n} + (n+2)B_{2n-1} - 1)$.

**(c)** $\sum_{k=0}^{n} B_{2k+1} = \frac{1}{3}(-(n+1)B_{2n+2} + (2n+5)B_{2n+1} + (n+2)B_{2n} + (n+2)B_{2n-1} - 7)$.

**(d)** $\sum_{k=1}^{n} B_{-k} = \frac{1}{3}(-(n+1)B_{-n+3} + (2n+1)B_{-n+2} + (n+2)B_{-n+1} + (n+3)B_{-n} - 7)$.

**(e)** $\sum_{k=1}^{n} B_{-2k} = \frac{1}{3}(-(n+2)B_{-2n+2} + (2n+3)B_{-2n+1} + (n+4)B_{-2n} + (n+2)B_{-2n-1} - 11)$.

**(f)** $\sum_{k=1}^{n} B_{-2k+1} = \frac{1}{3}(-(n+3)B_{-2n+2} + 2(n+3)B_{-2n+1} + (n+2)B_{-2n} + (n+2)B_{-2n-1} - 5)$.

Next, we give the ordinary generating functions of some special cases of generalized Adrien numbers.

**Lemma 3.7.**
*The ordinary generating functions of the sequences $W_n$, $W_{2n}$, $W_{2n+1}$, $W_{-n}$, $W_{-2n}$, $W_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.392646$

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^3W_3 - (3z^3 - z^2)W_2 + (z^3 - 3z^2 + z)W_1 + (z^2 - 3z + 1)W_0}{z^4 + z^2 - 3z + 1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.154171$

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{3z^2W_3 + (z^3 - 8z^2 + z)W_2 - 3z^3W_1 + (z^3 + 2z^2 - 7z + 1)W_0}{z^4 + 2z^3 + 3z^2 - 7z + 1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.154171$

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{(z^3 + z^2 + z)W_3 - (3z^3 + 3z^2)W_2 + (z^3 + 2z^2 - 7z + 1)W_1 - 3z^2W_0}{z^4 + 2z^3 + 3z^2 - 7z + 1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.626615$

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{-zW_3 + (3z - z^2)W_2 - (z^3 - 3z^2 + z)W_1 + W_0}{z^4 - 3z^3 + z^2 + 1}.$$

**(e)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.392646$

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{-3z^2W_3 - (z^3 - 8z^2 + z)W_2 + 3zW_1 + (z^2 + z + 1)W_0}{z^4 - 7z^3 + 3z^2 + 2z + 1}.$$

**(f)** $|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2, |\delta|^2\} = |\beta|^2 = |\gamma|^2 \simeq 0.392646$

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{-(z^3 + z^2 + z)W_3 + (3z^2 + 3z)W_2 + (z^2 + z + 1)W_1 + 3z^2W_0}{z^4 - 7z^3 + 3z^2 + 2z + 1}.$$

Proof. Use lemmas 1.1 and 1.2. □
Now, we consider special cases of the last Lemma.

**Corollary 3.34.**
*The ordinary generating functions of the sequences $A_n$, $A_{2n}$, $A_{2n+1}$, $A_{-n}$, $A_{-2n}$, $A_{-2n+1}$ and $B_n$, $B_{2n}$, $B_{2n+1}$, $B_{-n}$, $B_{-2n}$, $B_{-2n+1}$ are given as follows:*

**(a)** $|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}, |\delta|^{-1}\} = |\alpha|^{-1} \simeq 0.392646$

$$\sum_{n=0}^{\infty} A_n z^n = \frac{z}{z^4+z^2-3z+1},$$
$$\sum_{n=0}^{\infty} B_n z^n = \frac{2z^2-9z+4}{z^4+z^2-3z+1}.$$

**(b)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.154171$

$$\sum_{n=0}^{\infty} A_{2n} z^n = \frac{3z}{z^4+2z^3+3z^2-7z+1},$$
$$\sum_{n=0}^{\infty} B_{2n} z^n = \frac{2z^3+6z^2-21z+4}{z^4+2z^3+3z^2-7z+1}.$$

**(c)** $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\delta|^{-2}\} = |\alpha|^{-2} \simeq 0.154171$

$$\sum_{n=0}^{\infty} A_{2n+1} z^n = \frac{z^2+z+1}{z^4+2z^3+3z^2-7z+1},$$
$$\sum_{n=0}^{\infty} B_{2n+1} z^n = \frac{-9z^2-3z+3}{z^4+2z^3+3z^2-7z+1}.$$

**(d)** $|z| < \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\} = |\beta| = |\gamma| \simeq 0.626615$

$$\sum_{n=0}^{\infty} A_{-n} z^n = \frac{-z^3}{z^4-3z^3+z^2+1},$$
$$\sum_{n=0}^{\infty} B_{-n} z^n = \frac{-3z^3+2z^2+4}{z^4-3z^3+z^2+1}.$$

**(e)** $|z| < \min\{|\alpha|^{2}, |\beta|^{2}, |\gamma|^{2}, |\delta|^{2}\} = |\beta|^{2} = |\gamma|^{2} \simeq 0.392646$

$$\sum_{n=0}^{\infty} A_{-2n} z^n = \frac{-3z^3}{z^4-7z^3+3z^2+2z+1},$$
$$\sum_{n=0}^{\infty} B_{-2n} z^n = \frac{-7z^3+6z^2+6z+4}{z^4-7z^3+3z^2+2z+1}.$$

**(f)** $|z| < \min\{|\alpha|^{2}, |\beta|^{2}, |\gamma|^{2}, |\delta|^{2}\} = |\beta|^{2} = |\gamma|^{2} \simeq 0.392646$

$$\sum_{n=0}^{\infty} A_{-2n+1} z^n = \frac{-8z^3+2z^2+2z+1}{z^4-7z^3+3z^2+2z+1},$$
$$\sum_{n=0}^{\infty} B_{-2n+1} z^n = \frac{-18z^3+18z^2+6z+3}{z^4-7z^3+3z^2+2z+1}.$$

From the last corollary, we obtain the following results for Adrien and Adrien-Lucas numbers.

**Corollary 3.35.**
*Infinite sums of $A_n$, $A_{2n}$, $A_{2n+1}$, $A_{-n}$, $A_{-2n}$, $A_{-2n+1}$ and $B_n$, $B_{2n}$, $B_{2n+1}$, $B_{-n}$, $B_{-2n}$, $B_{-2n+1}$ are given as follows:*

**(a)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{A_n}{4^n} = \frac{64}{81},$$
$$\sum_{n=0}^{\infty} \frac{B_n}{4^n} = \frac{160}{27}.$$

**(b)** $z = \frac{1}{8}$

$$\sum_{n=0}^{\infty} \frac{A_{2n}}{8^n} = \frac{1536}{721},$$
$$\sum_{n=0}^{\infty} \frac{B_{2n}}{8^n} = \frac{6032}{721}.$$

**(c)** $z = \frac{1}{8}$

$$\sum_{n=0}^{\infty} \frac{A_{2n+1}}{8^n} = \frac{4672}{721},$$
$$\sum_{n=0}^{\infty} \frac{B_{2n+1}}{8^n} = \frac{10176}{721}.$$

**(d)** $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{A_{-n}}{2^n} = -\frac{2}{15},$$
$$\sum_{n=0}^{\infty} \frac{B_{-n}}{2^n} = \frac{22}{5}.$$

**(e)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{A_{-2n}}{4^n} = -\frac{4}{135},$$
$$\sum_{n=0}^{\infty} \frac{B_{-2n}}{4^n} = \frac{164}{45}.$$

**(f)** $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{A_{-2n+1}}{4^n} = \frac{128}{135},$$
$$\sum_{n=0}^{\infty} \frac{B_{-2n+1}}{4^n} = \frac{152}{45}.$$

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