Int. J. Adv. Appl. Math. and Mech. 1(3) (2014) 56 - 75
ISSN: 2347-2529
Available online at www.ijaamm.com

International Journal of Advances in **Applied Mathematics and Mechanics**

Fully discrete formulation of Galerkin-Partial artificial diffusion finite element method for coupled Burgers' problem

Research Article

Najat Jaleel Noon[1]*

[1]*Department of Mathematics, Basrah University, College of Education for Pure Science, Basrah-Iraq*

Received 03 October 2013; accepted (in revised version) 26 February 2014

Abstract: In this paper, the fully discrete formulation of Galerkin partial artificial diffusion finite element method for solving coupled Burgers' problem in 2-D is studied . The Crank-Nicholson method for the time variable is considered. The theoretical evidence proved the stability and proved that the error estimate is of order $O(h^{2r}+k^{\frac{5}{2}})$. The discretization with respect to space and time variables is applied. This leads to a large system of algebraic equations which are solved through implementation in MATLAB. The numerical results are compared with the exact solution.

MSC: 65NXX • 65N30

Keywords: Galerkin-Partial artificial diffusion • Error estimate • Burgers' equation

1. Introduction

Nonlinear partial differential equations arise in many fields of science, particularly in physics, engineering, chemistry and finance such as fluid mechanics, plasma physics, optical Fibers and quantum mechanics. One of such important equations is known as Burgers' equation. Some attention has been given to the convection-dominated case. Several methods have been intensively studied to remove such a drawback for this problem and relevant problem, a popular idea is to add stabilization terms to the formulation of the problem. In actuality, this is mainly a achieved by stabilized methods, such as the artificial diffusion, In this direction, Heitmann[2] applied subgrid scale eddy viscosity for convection dominated diffusive transport. The method consists of adding artificial viscosity term $\alpha(P^{\perp}_{L_H}\nabla u_h, P^{\perp}_{L_H}\nabla v_h)$ of orthogonal projection acting only on the fine scales, he give a comprehensive analysis of this method, in [3] he applied this method in a finite difference

* *Corresponding author. E-mail: fini.burg@gmail.com*

method by using an appropriate interpretation of the term $\alpha(P_{L_H}^{\perp}\nabla u_h,\nabla v_h)\equiv\alpha(\nabla u_h,\nabla v_h)-\alpha(\nabla\overline{u}_h,\nabla v_h)$, where $\overline{u}_h$ is an average over itself and its five nearest discrete neighbors. Kashkool and Noon [4] used the classical artificial diffusion for Galerkin and Galerkin-Conservation finite element methods for coupled Burgers' problem in 2-D, the fully discrete formulation with the backward Euler -Galerkin and Galerkin-Conservation methods were considered, the error estimate of these methods were of order $O(h,k)$ and the numerical results were compared with the exact solution. Noon [6] presented a stabilized finite element method for solving coupled Burgers' equation in 2-D, the method consists of adding artificial viscosity acting only the fine scales to a variational formulation of the problem and consider semi-discrete approximation, we proved stability and convergence for this approximation, the error estimate of this method was of order $O(h^{2r})$, the numerical solution was compared with the exact solution. In this paper, we consider a fully-discrete approximation with a Crank-Nicholson scheme for the time variable for this method, we prove stability and convergence for the approximation, the numerical solution is compared with the exact solution. This paper is organized as follows. In section 2, definitions and some important theorems are given. In Section 3, we present the time-dependent modeling problem and a weak form of 2-D Burgers' problem. The discrete problem, fully-discrete approximation, stability and error estimate are presented in section 4. In section 5 the finite element approximation, test problem and numerical results are introduced. The conclusions is shown in section 6.

2. Definitions and some important theorems

It is beneficial to mention the definitions and some important theorems which will be used frequently in the equal.

Definition 2.1.
[2] For Ω an open, connected subset of $\mathbb{R}^n$, we define norm over discrete time $(t_n=nk, n=0,1,...,N=T/k)$,

$$\|u\|_{l^2(L^2)}=\left(k\sum_{n=0}^{N}\|u^n\|^2\right)^{\frac{1}{2}}.$$

Definition 2.2.
[2] For $\Omega\subset\mathbb{R}^m$, the (a,b) weighted norm of a function $u:\Omega\to\mathbb{R}$ is defined by,

$$\|u\|_{a,b}^2=a\|u\|^2+b\|\nabla u\|^2.$$

Definition 2.3.
[2] For $\Omega\subset\mathbb{R}^m$, the (a,b,α) weighted norm of a function $u:\Omega\to\mathbb{R}$ is defined by,

$$\|u\|_{a,b,\alpha}^2=\|u\|_{a,b}^2+\alpha\|P_{L_H}^{\perp}\nabla u\|^2.$$

Definition 2.4.
[2] For $\Omega\subset\mathbb{R}^m$, the l^2 in time, $0\le nk\le T$, (a,b,α) weighted norm of a function $u:\Omega\to\mathbb{R}$ is defined by,

$$|||u|||^2=k\sum_{n=0}^{N-1}\|u_{nhalf}\|_{1,\epsilon,\alpha}^2.$$

Definition 2.5.
[2] For $T > 0$, the weighted $L^2(0, T, H^1(\Omega))$ norm $||\mu||_{(c:(0,T))}$ is defined by,

$$||\mu||^2_{(c:(0,T))} = \int_0^T e^{t-T}\ [\epsilon||\nabla\mu||^2 + \alpha||P^{\perp}_{L_H}\nabla\mu||^2]dt.$$

Theorem 2.1 (Poincare' Inequality [5]).
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then, there is constant $C = C(\Omega)$, such that for any $u \in H^1_0$,

$$||u||_{L^2(\Omega)} \le C||\nabla u||_{L^2(\Omega)}. \tag{1}$$

Theorem 2.2 (Inverse Estimate [5]).
On a quasi-uniform mesh any $u \in V_h$ satisfies the inverse estimate,

$$||\nabla u||_{L^2(\Omega)} \le Ch^{-1}||u||_{L^2(\Omega)}. \tag{2}$$

Abstract Results[1]

Let H be a separable Hilbert space , let a be a bounded symmetric bilinear form on H with the property that for some $\delta > 0$,

$$a(u,u) \ge \delta||u||^2,\ \ \forall u \in H, \tag{3}$$

and let B be a trilinear form on H such that there exists a constant $\beta > 0$ such that

$$|B(u,v,w)| \le \beta||u||||v||||w||,\ \ \forall u,v,w \in H, \tag{4}$$

Remark 2.1.
[2] We shall state the following inequality which will be used frequently in this paper,

$$||u - Iu|| \le Ch^r||u||_r,\ \ 1 \le r \le s,\ \ s \ge 2. \tag{5}$$

3. Time - Dependent Modeling Problem

Consider the nonlinear time-dependent for the two dimensional coupled Burgers' problem [8].

$$u_t - \epsilon\Delta u + uu_x + vu_y = f, \tag{6}$$

$$v_t - \epsilon\Delta v + uv_x + vv_y = g \tag{7}$$

with boundary conditions

$$u(x,y,t) = 0,\quad v(x,y,t) = 0,\qquad \text{on}\quad \partial\Omega \times (0,T],$$

and initial conditions

$$u(x,y,0)=u^0(x,y), \quad v(x,y,0)=v^0(x,y),$$

where $\epsilon>0$ is a viscosity constant, $\Omega\subset\mathbb{R}^2$ with boundary $\partial\Omega$, $u=u(x,y,t)$, $v=v(x,y,t)$, f and $g\in L^2(\Omega)$.

The weak formulation of (6,7) is : find $u,v\in H_0^1(\Omega)$ such that:

$$(u_t,\varphi)+a(u,\varphi)+B(u,u,\varphi)+B(v,u,\varphi)=(f,\varphi) \tag{8}$$

$$(v_t,\varphi)+a(v,\varphi)+B(u,v,\varphi)+B(v,v,\varphi)=(g,\varphi), \ \ \forall\varphi\in H_0^1(\Omega) \tag{9}$$

$$(u(x,y,0),\varphi)=(u^0,\varphi), \qquad (v(x,y,0),\varphi)=(v^0,\varphi),$$

where,

$a(u,\varphi)=(\epsilon\nabla u,\nabla\varphi)$, $a(v,\varphi)=(\epsilon\nabla v,\nabla\varphi)$, $B(u,u,\varphi)=(u u_x,\varphi)$,

$B(v,u,\varphi)=(v u_y,\varphi)$, $B(u,v,\varphi)=(u v_x,\varphi)$ and $B(v,v,\varphi)=(v v_y,\varphi)$.

The weak formulation (8,9) with artificial viscosity term is : find $u,v\in H_0^1(\Omega)$ such that:

$$(u_t,\varphi)+A(u,\varphi)=(f,\varphi) \tag{10}$$

$$(v_t,\varphi)+A(v,\varphi)=(g,\varphi), \ \ \forall\varphi\in H_0^1(\Omega) \tag{11}$$

where,

$$A(u,\varphi)=a(u,\varphi)+\alpha(P_{L_H}^{\perp}\nabla u,P_{L_H}^{\perp}\nabla\varphi)+B(u,u,\varphi)+B(v,u,\varphi),$$

$$A(v,\varphi)=a(v,\varphi)+\alpha(P_{L_H}^{\perp}\nabla v,P_{L_H}^{\perp}\nabla\varphi)+B(u,v,\varphi)+B(v,v,\varphi),$$

Assumption 3.1.
We assume that the solution u and v of problem (6,7) satisfied the following condition

(A1) $u,v\in L^\infty(0,T,H_0^1(\Omega))\bigcap L^\infty(0,T,H^2(\Omega))$,
$u_t,u_{tt},u_{ttt},v_t,v_{tt},v_{ttt}\in L^\infty(0,T,L^\infty(\Omega))$.

Lemma 3.1.
[6] $A(u,\varphi)$ and $A(v,\varphi)$ given by (10,11) are continuous and V-elliptic.

4. The discrete problem

Given finite dimensional spaces $V_h\subset H_0^1(\Omega)$ then the approximate solution u_h,v_h to u,v respectively is the solution of the following problem:

$$(u_{h,t},\varphi_h)+a(u_h,\varphi_h)+B(u_h,u_h,\varphi_h)+B(v_h,u_h,\varphi_h)=(f,\varphi_h),$$

$$(v_{h,t},\varphi_h)+a(v_h,\varphi_h)+B(u_h,v_h,\varphi_h)+B(v_h,v_h,\varphi_h)=(g,\varphi_h), \ \forall\varphi_h\in V_h.$$

4.1. Mathematical Formulation of an Artificial Viscosity Term

It is well known that when $\epsilon < h$, where h is mesh size, the convection term dominates over diffusion and the standard Galerkin finite element method produce an oscillating solution which is not close to exact solution. In the following we analyze an approach stabilizing the approximation through the introduction of an artificial viscosity term which acts only on the fine scales of the finite element mesh. We add and subtract $\alpha(\nabla u, \nabla\varphi)$and $\alpha(\nabla v, \nabla\varphi)$ to (8) and (9) respectively where $\alpha = \alpha(h)$ is a positive constant. This gives,

$$(u_t, \varphi) + (\epsilon + \alpha)(\nabla u, \nabla\varphi) - \alpha(\nabla u, \nabla\varphi) + B(u, u, \varphi) + B(v, u, \varphi) = (f, \varphi),$$

$$(v_t, \varphi) + (\epsilon + \alpha)(\nabla v, \nabla\varphi) - \alpha(\nabla v, \nabla\varphi) + B(u, v, \varphi) + B(v, v, \varphi) = (g, \varphi), \forall\varphi \in H_0^1(\Omega),$$

This suggests a mixed methods formulation wherein we define $q_1 \equiv \nabla u$ and $q_2 \equiv \nabla v \in (L^2(\Omega))^2$ [2]. We obtain the system, find $((u, v), (q_1, q_2)) \in (H_0^1, (L^2(\Omega))^2)$ satisfying

$$(u_t, \varphi) + (\epsilon + \alpha)(\nabla u, \nabla\varphi) - \alpha(q_1, \nabla\varphi) + B(u, u, \varphi) + B(v, u, \varphi) = (f, \varphi),$$

$$(v_t, \varphi) + (\epsilon + \alpha)(\nabla v, \nabla\varphi) - \alpha(q_2, \nabla\varphi) + B(u, v, \varphi) + B(v, v, \varphi) = (g, \varphi), \forall\varphi \in H_0^1(\Omega),$$

$$(q_1 - \nabla u, l) = 0, \ (q_2 - \nabla v, l) = 0, \ \forall\varphi \in H_0^1(\Omega), \ l \in (L^2(\Omega))^2.$$

In the discretized problem, let h and H represent two mesh widths (with $h < H$). Let $L_H \subset (L^2(\Omega))^2$ and $V_h \subset H_0^1$ be finite element spaces. The problem then is to find $((u_h, v_h), (q_{1H}, q_{2H})) \in (V_h, L_H)$ satisfying

$$(u_{h,t}, \varphi_h) + (\epsilon + \alpha)(\nabla u_h, \nabla\varphi_h) - \alpha(q_{1H}, \nabla\varphi_h) + B(u_h, u_h, \varphi_h) + B(v_h, u_h, \varphi_h) = (f, \varphi_h), \tag{12}$$

$$(v_{h,t}, \varphi_h) + (\epsilon + \alpha)(\nabla v_h, \nabla\varphi_h) - \alpha(q_{2H}, \nabla\varphi_h) + B(u_h, v_h, \varphi_h) + B(v_h, v_h, \varphi_h) = (g, \varphi_h) \tag{13}$$

$$(q_{1H} - \nabla u_h, l_H) = 0, \ (q_{2H} - \nabla v_h, l_H) = 0, \quad \forall\varphi_h \in V_h, \ l_H \in L_H. \tag{14}$$

We note that, If $L_H = \{0\}$, L_H is small, then $q_{1H}, q_{2H} = 0$, and we have a straight artificial viscosity formulation, The key then will be to select L_H, guided by precise and general error analysis, in such a way as to achieve a beneficial balance, we set $q_{1H} = P_{L_H}\nabla u_h$ and $q_{2H} = P_{L_H}\nabla v_h$ [2], where P_{L_H} is the orthogonal projection of L^2 onto L_H and $P_{L_H}^{\perp} = (I - P_{L_H})$ is orthogonal projection of L^2 on $L_H^{\perp}$, where $L_H^{\perp} = \{w \in L^2(\Omega), (w, s) = 0, \forall s \in L_H\}$. Then we have $\nabla u_h = P_{L_H}^{\perp}\nabla u_h + P_{L_H}\nabla u_h$ and $\nabla v_h = P_{L_H}^{\perp}\nabla v_h + P_{L_H}\nabla v_h$.

Lemma 4.1.
[6] If $q_{1H} = P_{L_H}\nabla u_h$ and $q_{2H} = P_{L_H}\nabla v_h$ then the system (12,13, 14) is equivalent to:

$$(u_{h,t}, \varphi_h) + A(u_h, \varphi_h) = (f, \varphi_h) \tag{15}$$

$$(v_{h,t}, \varphi_h) + A(v_h, \varphi_h) = (g, \varphi_h), \forall\varphi_h \in V_h \tag{16}$$

where, $A(.,.)$ is defined previously.

4.2. The Fully-Discrete Approximation

We consider a fully discrete formulation of (15,16), in particular, we will turn our attention to the Crank-Nicholson method. We use the subscript $n+\frac{1}{2}$ to represent the average of a quantity over the two discrete times for example $f^{n+\frac{1}{2}} = \frac{f^{n+1}+f^n}{2}$.

$$\frac{1}{k}(u_h^{n+1}-u_h^n,\varphi_h)+A(u_h^{n+\frac{1}{2}},\varphi_h)=(f^{n+\frac{1}{2}},\varphi_h) \tag{17}$$

$$\frac{1}{k}(v_h^{n+1}-v_h^n,\varphi_h)+A(v_h^{n+\frac{1}{2}},\varphi_h)=(g^{n+\frac{1}{2}},\varphi_h) \tag{18}$$

Lemma 4.2.
The method described by (17,18) is stable over finite time, specifically, for any $N>0$,

$$\|u_h^N\| \le \|u_h^0\| + k\sum_{n=0}^{N-1}\|f^{n+\frac{1}{2}}\|,$$

$$\|v_h^N\| \le \|v_h^0\| + k\sum_{n=0}^{N-1}\|g^{n+\frac{1}{2}}\|.$$

Proof: By choosing $\varphi_h = u_h^{n+1}+u_h^n$ in (17), $\varphi_h = v_h^{n+1}+v_h^n$ in (18) and using the Cauchy-Schwrtiz inequality on the right side we obtain,

$$\frac{1}{k}(u_h^{n+1}-u_h^n,u_h^{n+1}+u_h^n)+A(u_h^{n+\frac{1}{2}},u_h^{n+1}+u_h^n) \le \|f^{n+\frac{1}{2}}\|\|u_h^{n+1}+u_h^n\|,$$
$$\frac{1}{k}(v_h^{n+1}-v_h^n,v_h^{n+1}+v_h^n)+A(v_h^{n+\frac{1}{2}},v_h^{n+1}+v_h^n) \le \|g^{n+\frac{1}{2}}\|\|v_h^{n+1}+v_h^n\|.$$

Consider the first terms on the left hand sides

$$\begin{aligned}(u_h^{n+1}-u_h^n,u_h^{n+1}+u_h^n) &= \|u_h^{n+1}\|^2-\|u_h^n\|^2\\ &= (\|u_h^{n+1}\|-\|u_h^n\|)(\|u_h^{n+1}\|+\|u_h^n\|),\end{aligned}$$

$$\begin{aligned}(v_h^{n+1}-v_h^n,v_h^{n+1}+v_h^n) &= \|v_h^{n+1}\|^2-\|v_h^n\|^2\\ &= (\|v_h^{n+1}\|-\|v_h^n\|)(\|v_h^{n+1}\|+\|v_h^n\|),\end{aligned}$$

The second terms on the left hand side is non-negative by lemma 3.1, thus we can eliminate this terms, multiplying by k and applying the triangle inequality on the right hand side gives,

$$(\|u_h^{n+1}\|-\|u_h^n\|)(\|u_h^{n+1}\|+\|u_h^n\|) \le k\|f^{n+\frac{1}{2}}\|(\|u_h^{n+1}\|+\|u_h^n\|),$$

$$(\|v_h^{n+1}\|-\|v_h^n\|)(\|v_h^{n+1}\|+\|v_h^n\|) \le k\|g^{n+\frac{1}{2}}\|(\|v_h^{n+1}\|+\|v_h^n\|),$$

So, $\|u_h^{n+1}\| \le \|u_h^n\| + k\|f^{n+\frac{1}{2}}\|, \qquad \|v_h^{n+1}\| \le \|v_h^n\| + k\|g^{n+\frac{1}{2}}\|,$

Summing both sides from $n=0$ to $n=N-1$ gives,

$$\|u_h^N\| \le \|u_h^0\| + k\sum_{n=0}^{N-1}\|f^{n+\frac{1}{2}}\|,$$

$$\|v_h^N\| \le \|v_h^0\| + k\sum_{n=0}^{N-1}\|g^{n+\frac{1}{2}}\|.$$

With these result the proof is complete.

For the error analysis we first need to establish the existence of the equilibrium projection $pu_h, pv_h \in V_h$ of u and v respectively which is given by,

$$\left.\begin{aligned}
&A(u^{n+\frac{1}{2}} - pu_h^{n+\frac{1}{2}}, \varphi_h)\\
&= a(u^{n+\frac{1}{2}} - pu_h^{n+\frac{1}{2}}, \varphi_h) + \alpha(P_{L_H}^{\perp}\nabla(u^{n+\frac{1}{2}} - pu_h^{n+\frac{1}{2}}), P_{L_H}^{\perp}\nabla\varphi_h)\\
&+ B(u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}, \varphi_h) - B(pu_h^{n+\frac{1}{2}}, pu_h^{n+\frac{1}{2}}, \varphi_h)\\
&+ B(v^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}, \varphi_h) - B(pv_h^{n+\frac{1}{2}}, pu_h^{n+\frac{1}{2}}, \varphi_h) = 0.
\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}
&A(v^{n+\frac{1}{2}} - pv_h^{n+\frac{1}{2}}, \varphi_h)\\
&= a(v^{n+\frac{1}{2}} - pv_h^{n+\frac{1}{2}}, \varphi_h) + \alpha(P_{L_H}^{\perp}\nabla(v^{n+\frac{1}{2}} - pv_h^{n+\frac{1}{2}}), P_{L_H}^{\perp}\nabla\varphi_h)\\
&+ B(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, \varphi_h) - B(pu_h^{n+\frac{1}{2}}, pv_h^{n+\frac{1}{2}}, \varphi_h)\\
&+ B(v^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, \varphi_h) - B(pv_h^{n+\frac{1}{2}}, pv_h^{n+\frac{1}{2}}, \varphi_h) = 0, \quad \forall\varphi_h \in V_h.
\end{aligned}\right\} \tag{20}$$

Lemma 4.3.
[6] Let $u, v \in H_0^1(\Omega)$, the equilibrium projection pu_h, pv_h of u, v respectively, given by (19,20) exist uniquely.

Lemma 4.4.
[6] Let $u, v \in H_0^1(\Omega)$, let $pu_h, pv_h \in V_h$ be the equilibrium projection the assumptions of the finite element space there exists a constant C_3 and C_4 independent of ϵ, α, h and H such that

$$\|u - pu\|_{L^\infty(L^2)} \le C_3(h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}),$$

$$\|v - pv\|_{L^\infty(L^2)} \le C_4(h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}).$$

Lemma 4.5.
Let $u^{n+1}, v^{n+1} \in H_0^1(\Omega)$, let $pu_h^{n+1}, pv_h^{n+1} \in V_h$ be the equilibrium projection given by (19,20). Under the assumptions of lemma (4.4) there exists a constant C_3 and C_4 independent of ϵ, α, h and H such that

$$\max_{0\le n\le N}\|u^n - pu_h^n\| \le C_3(h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}),$$

$$\max_{0\le n\le N}\|v^n - pv_h^n\| \le C_4(h^{2r} + h^r + \epsilon h^{r-2} + h^{r-1}).$$

Proof: Let $\eta_1^{n+1} = u^{n+1} - I u^{n+1}$, $\eta_2^{n+1} = v^{n+1} - I v^{n+1}$, $\theta_1^{n+1} = p u_h^{n+1} - I u^{n+1}$ and $\theta_2^{n+1} = p v_h^{n+1} - I v^{n+1}$, by the triangle inequality, we have,

$$\max_{0\le n\le N} \|u^n - p u_h^n\| \le \max_{0\le n\le N} \|\eta_1^n\| + \max_{0\le n\le N} \|\theta_1^n\|,$$

$$\max_{0\le n\le N} \|v^n - p v_h^n\| \le \max_{0\le n\le N} \|\eta_2^n\| + \max_{0\le n\le N} \|\theta_2^n\|,$$

For the first terms of the right hand sides, we have,

$$\max_{0\le n\le N} \|\eta_1^n\| = \max_{0\le n\le N} \|u^n - I u^n\|,$$

From (2.5) we have,

$$\max_{0\le n\le N} \|\eta_1^n\| \le C h^r \max_{0\le n\le N} \|u^n\|_r,$$

From $(A1)$ we have,

$$\max_{0\le n\le N} \|\eta_1^n\| \le C_1 h^r \tag{21}$$

Also,

$$\max_{0\le n\le N} \|\eta_2^n\| \le C h^r \max_{0\le n\le N} \|v^n\|_r \le C_2 h^r. \tag{22}$$

To bound the $\max_{0\le n\le N} \|\theta_1^n\|$ and $\max_{0\le n\le N} \|\theta_2^n\|$ terms we rewrite equation (19,20)

$$A(p u_h^{n+\frac{1}{2}}, \varphi_h) = A(u^{n+\frac{1}{2}}, \varphi_h) \tag{23}$$

$$A(p v_h^{n+\frac{1}{2}}, \varphi_h) = A(v^{n+\frac{1}{2}}, \varphi_h), \quad \forall \varphi_h \in V_h \tag{24}$$

subtracting $A(I u^{n+\frac{1}{2}}, \varphi_h)$ from (23) and $A(I v^{n+\frac{1}{2}}, \varphi_h)$ from (24), choosing $\varphi_h = \theta_1^n$ in (23) and $\varphi_h = \theta_2^n$ in (24) give,

$$\left.\begin{aligned} &a(\theta_1^{n+\frac{1}{2}}, \theta_1^n) + \alpha(P_{L_H}^{\perp} \nabla \theta_1^{n+\frac{1}{2}}, P_{L_H}^{\perp} \nabla \theta_1^n) + B(p u_h^{n+\frac{1}{2}}, p u_h^{n+\frac{1}{2}}, \theta_1^n) - B(I u^{n+\frac{1}{2}}, I u^{n+\frac{1}{2}}, \theta_1^n) \\ &+ B(p v_h^{n+\frac{1}{2}}, p u_h^{n+\frac{1}{2}}, \theta_1^n) - B(I v^{n+\frac{1}{2}}, I u^{n+\frac{1}{2}}, \theta_1^n) \\ &= a(\eta_1^{n+\frac{1}{2}}, \theta_1^n) + \alpha(P_{L_H}^{\perp} \nabla \eta_1^{n+\frac{1}{2}}, P_{L_H}^{\perp} \nabla \theta_1^n) + B(u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}, \theta_1^n) - B(I u^{n+\frac{1}{2}}, I u^{n+\frac{1}{2}}, \theta_1^n) \\ &+ B(v^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}, \theta_1^n) - B(I v^{n+\frac{1}{2}}, I u^{n+\frac{1}{2}}, \theta_1^n) \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} &a(\theta_2^{n+\frac{1}{2}}, \theta_2^n) + \alpha(P_{L_H}^{\perp} \nabla \theta_2^{n+\frac{1}{2}}, P_{L_H}^{\perp} \nabla \theta_2^n) + B(p u_h^{n+\frac{1}{2}}, p v_h^{n+\frac{1}{2}}, \theta_2^n) - B(I u^{n+\frac{1}{2}}, I v^{n+\frac{1}{2}}, \theta_2^n) \\ &+ B(p v_h^{n+\frac{1}{2}}, p v_h^{n+\frac{1}{2}}, \theta_2^n) - B(I v^{n+\frac{1}{2}}, I v^{n+\frac{1}{2}}, \theta_2^n) \\ &= a(\eta_2^{n+\frac{1}{2}}, \theta_2^n) + \alpha(P_{L_H}^{\perp} \nabla \eta_2^{n+\frac{1}{2}}, P_{L_H}^{\perp} \nabla \theta_2^n) + B(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, \theta_2^n) - B(I u^{n+\frac{1}{2}}, I v^{n+\frac{1}{2}}, \theta_2^n) \\ &+ B(v^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}, \theta_2^n) - B(I v^{n+\frac{1}{2}}, I v^{n+\frac{1}{2}}, \theta_2^n) \end{aligned}\right\} \tag{26}$$

Consider the left hand side of (25),

$$a(\theta_1^{n+\frac{1}{2}}, \theta_1^n) = \frac{1}{2}[a(\theta_1^{n+1}, \theta_1^n) + a(\theta_1^n, \theta_1^n)],$$

using the Cauchy-Schwrtiz inequality and Young's inequality for the first term gives,

$$a(\theta_1^{n+\frac{1}{2}},\theta_1^n)\le\|\theta_1^{n+1}\|\|\theta_1^n\|\le\frac{1}{4\delta}\|\theta_1^{n+1}\|^2+\delta\|\theta_1^n\|^2,$$

and from (3) we have,

$$a(\theta_1^n,\theta_1^n)\ge\delta\|\theta_1^n\|^2.$$

$$\alpha(P_{L_H}^{\perp}\nabla\theta_1^{n+\frac{1}{2}},P_{L_H}^{\perp}\nabla\theta_1^n)=\frac{1}{2}[\alpha(P_{L_H}^{\perp}\nabla\theta_1^{n+1},P_{L_H}^{\perp}\nabla\theta_1^n)+\alpha(P_{L_H}^{\perp}\nabla\theta_1^n,P_{L_H}^{\perp}\nabla\theta_1^n)]$$
$$\le\frac{1}{2}[\alpha\|P_{L_H}^{\perp}\nabla\theta_1^{n+1}\|\|P_{L_H}^{\perp}\nabla\theta_1^n\|+\alpha\|P_{L_H}^{\perp}\nabla\theta_1^n\|^2].$$

From (4) we have,

$$B(pu_h^{n+\frac{1}{2}},pu_h^{n+\frac{1}{2}},\theta_1^n)-B(Iu^{n+\frac{1}{2}},Iu^{n+\frac{1}{2}},\theta_1^n)\le\beta(\|pu_h^{n+\frac{1}{2}}\|^2-\|Iu_h^{n+\frac{1}{2}}\|^2)\|\theta_1^n\|$$

$$\le\beta\|pu_h^{n+\frac{1}{2}}-Iu_h^{n+\frac{1}{2}}\|^2\|\theta_1^n\|=\beta\|\theta_1^{n+\frac{1}{2}}\|^2\|\theta_1^n\|,$$

$$B(pv_h^{n+\frac{1}{2}},pu_h^{n+\frac{1}{2}},\theta_1^n)-B(Iv^{n+\frac{1}{2}},Iu^{n+\frac{1}{2}},\theta_1^n)$$
$$\le\beta(\|pv_h^{n+\frac{1}{2}}\|\|pu_h^{n+\frac{1}{2}}\|-\|Iv_h^{n+\frac{1}{2}}\|\|Iu_h^{n+\frac{1}{2}}\|)\|\theta_1^n\|.$$

Similarly for (26) we get,

$$a(\theta_2^{n+\frac{1}{2}},\theta_2^n)=\frac{1}{2}[a(\theta_2^{n+1},\theta_2^n)+a(\theta_2^n,\theta_2^n)],$$

$$a(\theta_2^{n+1},\theta_2^n)\le\|\theta_2^{n+1}\|\|\theta_2^n\|\le\frac{1}{4\delta}\|\theta_2^{n+1}\|^2+\delta\|\theta_2^n\|^2,$$

and

$$a(\theta_2^n,\theta_2^n)\ge\delta\|\theta_2^n\|^2,$$

$$\alpha(P_{L_H}^{\perp}\nabla\theta_2^{n+\frac{1}{2}},P_{L_H}^{\perp}\nabla\theta_2^n)=\frac{1}{2}[\alpha(P_{L_H}^{\perp}\nabla\theta_2^{n+1},P_{L_H}^{\perp}\nabla\theta_2^n)+\alpha(P_{L_H}^{\perp}\nabla\theta_2^n,P_{L_H}^{\perp}\nabla\theta_2^n)]$$
$$\le\frac{1}{2}[\alpha\|P_{L_H}^{\perp}\nabla\theta_2^{n+1}\|\|P_{L_H}^{\perp}\nabla\theta_2^n\|+\alpha\|P_{L_H}^{\perp}\nabla\theta_2^n\|^2],$$

$$B(pu_h^{n+\frac{1}{2}},pv_h^{n+\frac{1}{2}},\theta_2^n)-B(Iu^{n+\frac{1}{2}},Iv^{n+\frac{1}{2}},\theta_2^n)\le$$
$$\beta(\|pu_h^{n+\frac{1}{2}}\|\|pv_h^{n+\frac{1}{2}}\|-\|Iu^{n+\frac{1}{2}}\|\|Iv^{n+\frac{1}{2}}\|)\|\theta_2^n\|,$$

$$B(pv_h^{n+\frac{1}{2}},pv_h^{n+\frac{1}{2}},\theta_2^n)-B(Iv^{n+\frac{1}{2}},Iv^{n+\frac{1}{2}},\theta_2^n)\le\beta\|\theta_2^{n+\frac{1}{2}}\|^2\|\theta_2^n\|.$$

For the right hand side of (25), by using Cauchy-Schwartz inequality we have,

$$a(\eta_1^{n+\frac{1}{2}},\theta_1^n)\le\epsilon\|\nabla\eta_1^{n+\frac{1}{2}}\|\|\nabla\theta_1^n\|,$$

using (2) on $\|\nabla\theta_1^n\|$ gets,

$$a(\eta_1^{n+\frac{1}{2}},\theta_1^n)\leq Ch^{-1}\epsilon\|\nabla\eta_1^{n+\frac{1}{2}}\|\|\theta_1^n\|,$$

using Young's inequality gives,

$$a(\eta_1^{n+\frac{1}{2}},\theta_1^n)\leq\frac{3C^2h^{-2}\epsilon^2}{4\delta}\|\nabla\eta_1^{n+\frac{1}{2}}\|^2+\frac{\delta}{3}\|\theta_1^n\|^2.$$

From (4) we have,

$$B(u^{n+\frac{1}{2}},u^{n+\frac{1}{2}},\theta_1^n)-B(Iu^{n+\frac{1}{2}},Iu^{n+\frac{1}{2}},\theta_1^n)\leq\beta(\|u^{n+\frac{1}{2}}\|^2-\|Iu^{n+\frac{1}{2}}\|^2)\|\theta_1^n\|$$

$$\leq\beta\|u^{n+\frac{1}{2}}-Iu^{n+\frac{1}{2}}\|^2\|\theta_1^n\|=\beta\|\eta_1^{n+\frac{1}{2}}\|^2\|\theta_1^n\|,$$

$$\leq\frac{3\beta^2}{4\delta}\|\eta_1^{n+\frac{1}{2}}\|^4+\frac{\delta}{3}\|\theta_1^n\|^2.$$

$$B(v^{n+\frac{1}{2}},u^{n+\frac{1}{2}},\theta_1^n)-B(Iv^{n+\frac{1}{2}},Iu^{n+\frac{1}{2}},\theta_1^n)\leq\beta(\|v^{n+\frac{1}{2}}\|\|u^{n+\frac{1}{2}}\|-\|Iv^{n+\frac{1}{2}}\|\|Iu^{n+\frac{1}{2}}\|)\|\theta_1^n\|,$$

using Young's inequality for the first term gives,

$$B(v^{n+\frac{1}{2}},u^{n+\frac{1}{2}},\theta_1^n)-B(Iv^{n+\frac{1}{2}},Iu^{n+\frac{1}{2}},\theta_1^n)\leq$$

$$\frac{3\beta^2}{4\delta}\|v^{n+\frac{1}{2}}\|^2\|u^{n+\frac{1}{2}}\|^2+\frac{\delta}{3}\|\theta_1^n\|^2-\beta\|Iv^{n+\frac{1}{2}}\|\|Iu^{n+\frac{1}{2}}\|\|\theta_1^n\|.$$

Similarly for the right hand side of (26) we get,

$$a(\eta_2^{n+\frac{1}{2}},\theta_2^n)\leq\frac{3C^2h^{-2}\epsilon^2}{4\delta}\|\nabla\eta_2^{n+\frac{1}{2}}\|^2+\frac{\delta}{3}\|\theta_2^n\|^2.$$

$$B(u^{n+\frac{1}{2}},v^{n+\frac{1}{2}},\theta_2^n)-B(Iu^{n+\frac{1}{2}},Iv^{n+\frac{1}{2}},\theta_2^n)\leq$$

$$\frac{3\beta^2}{4\delta}\|u^{n+\frac{1}{2}}\|^2\|v^{n+\frac{1}{2}}\|^2+\frac{\delta}{3}\|\theta_2^n\|^2-\beta\|Iu^{n+\frac{1}{2}}\|\|Iv^{n+\frac{1}{2}}\|\|\theta_2^n\|,$$

$$B(v^{n+\frac{1}{2}},v^{n+\frac{1}{2}},\theta_2^n)-B(Iv^{n+\frac{1}{2}},Iv^{n+\frac{1}{2}},\theta_2^n)\leq\frac{3\beta^2}{4\delta}\|\eta_2^{n+\frac{1}{2}}\|^4+\frac{\delta}{3}\|\theta_2^n\|^2.$$

Inserting these inequalities into (25,26) with using Cauchy-Schwartz inequality and Young's inequality to $\alpha(P_{L_H}^{\perp}\nabla\eta_1^{n+\frac{1}{2}},P_{L_H}^{\perp}\nabla\theta_1^n)$ and $\alpha(P_{L_H}^{\perp}\nabla\eta_2^{n+\frac{1}{2}},P_{L_H}^{\perp}\nabla\theta_2^n)$ we have,

$$\frac{1}{2}[\frac{1}{4\delta}\|\theta_1^{n+1}\|^2+\delta\|\theta_1^n\|^2+\delta\|\theta_1^n\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_1^{n+1}\|\|P_{L_H}^{\perp}\nabla\theta_1^n\|+\alpha\|P_{L_H}^{\perp}\nabla\theta_1^n\|^2]+\beta$$

$$\|\theta_1^{n+\frac{1}{2}}\|^2\|\theta_1^n\|+\beta(\|pv_h^{n+\frac{1}{2}}\|\|pu_h^{n+\frac{1}{2}}\|-\|Iv^{n+\frac{1}{2}}\|\|Iu^{n+\frac{1}{2}}\|)\|\theta_1^n\|\leq\frac{3C^2h^{-2}\epsilon^2}{4\delta}\|\nabla\eta_1^{n+\frac{1}{2}}\|^2$$

$$+\frac{\delta}{3}\|\theta_1^n\|^2+\frac{\alpha}{2}\|P_{L_H}^{\perp}\nabla\eta_1^{n+\frac{1}{2}}\|+\frac{\alpha}{2}\|P_{L_H}^{\perp}\nabla\theta_1^n\|+\frac{3\beta^2}{4\delta}\|\eta_1^{n+\frac{1}{2}}\|^4+\frac{\delta}{3}\|\theta_1^n\|^2+\frac{3\beta^2}{4\delta}\|v^{n+\frac{1}{2}}\|^2\|u^{n+\frac{1}{2}}\|^2$$

$$+\frac{\delta}{3}\|\theta_1^n\|^2-\beta\|Iv^{n+\frac{1}{2}}\|\|Iu^{n+\frac{1}{2}}\|\|\theta_1^n\|,$$

$$\frac{1}{2}[\frac{1}{4\delta}\|\theta_2^{n+1}\|^2+\delta\|\theta_2^n\|^2+\delta\|\theta_2^n\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_2^{n+1}\|\|P_{L_H}^{\perp}\nabla\theta_2^n\|+\alpha\|P_{L_H}^{\perp}\nabla\theta_2^n\|^2]+\beta$$

$$(\|pu_h^{n+\frac{1}{2}}\|\|pv_h^{n+\frac{1}{2}}\|-\|Iu^{n+\frac{1}{2}}\|\|Iv^{n+\frac{1}{2}}\|)\|\theta_2^n\|+\beta\|\theta_2^{n+\frac{1}{2}}\|^2\|\theta_2^n\|\leq\frac{3C^2h^{-2}\epsilon^2}{4\delta}\|\nabla\eta_2^{n+\frac{1}{2}}\|^2$$
$$+\frac{\delta}{3}\|\theta_2^n\|^2+\frac{\alpha}{2}\|P_{L_H}^{\perp}\nabla\eta_2^{n+\frac{1}{2}}\|+\frac{\alpha}{2}\|P_{L_H}^{\perp}\nabla\theta_2^n\|+\frac{3\beta^2}{4\delta}\|u^{n+\frac{1}{2}}\|^2\|v^{n+\frac{1}{2}}\|^2+\frac{\delta}{3}\|\theta_2^n\|^2-$$
$$\beta\|Iu^{n+\frac{1}{2}}\|\|Iv^{n+\frac{1}{2}}\|\|\theta_2^n\|+\frac{3\beta^2}{4\delta}\|\eta_2^{n+\frac{1}{2}}\|^4+\frac{\delta}{3}\|\theta_2^n\|^2,$$

multiplying by two and rearranging gives,

$$\frac{1}{4\delta}\|\theta_1^{n+1}\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_1^{n+1}\|\|P_{L_H}^{\perp}\nabla\theta_1^n\|+2\beta\|\theta_1^{n+\frac{1}{2}}\|^2\|\theta_1^n\|+2\beta\|pv_h^{n+\frac{1}{2}}\|\|pu_h^{n+\frac{1}{2}}\|\|\theta_1^n\|$$
$$\leq\frac{3C^2h^{-2}\epsilon^2}{2\delta}\|\nabla\eta_1^{n+\frac{1}{2}}\|^2+\alpha\|P_{L_H}^{\perp}\nabla\eta_1^{n+\frac{1}{2}}\|+\frac{3\beta^2}{2\delta}\|\eta_1^{n+\frac{1}{2}}\|^4+\frac{3\beta^2}{2\delta}\|v^{n+\frac{1}{2}}\|^2\|u^{n+\frac{1}{2}}\|^2,$$
$$\frac{1}{4\delta}\|\theta_2^{n+1}\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_2^{n+1}\|\|P_{L_H}^{\perp}\nabla\theta_2^n\|+2\beta\|pu_h^{n+\frac{1}{2}}\|\|pv_h^{n+\frac{1}{2}}\|\|\theta_2^n\|+2\beta\|\theta_2^{n+\frac{1}{2}}\|^2\|\theta_2^n\|$$
$$\leq\frac{3C^2h^{-2}\epsilon^2}{2\delta}\|\nabla\eta_2^{n+\frac{1}{2}}\|^2+\alpha\|P_{L_H}^{\perp}\nabla\eta_2^{n+\frac{1}{2}}\|+\frac{3\beta^2}{2\delta}\|u^{n+\frac{1}{2}}\|^2\|v^{n+\frac{1}{2}}\|^2+\frac{3\beta^2}{2\delta}\|\eta_2^{n+\frac{1}{2}}\|^4,$$

since $\alpha\|P_{L_H}^{\perp}\nabla\theta_1^{n+1}\|\|P_{L_H}^{\perp}\nabla\theta_1^n\|,\beta\|\theta_1^{n+\frac{1}{2}}\|^2,\|\theta_1^n\|,\beta\|pv_h^{n+\frac{1}{2}}\|\|pu_h^{n+\frac{1}{2}}\|,\alpha\|P_{L_H}^{\perp}\nabla\theta_2^{n+1}\|$
$\|P_{L_H}^{\perp}\nabla\theta_2^n\|,$ and $\beta\|\theta_2^{n+\frac{1}{2}}\|^2,\|\theta_2^n\|$ are nonnegative, we have,

$$\|\theta_1^{n+1}\|^2\leq C_1[h^{-2}\epsilon^2\|\nabla\eta_1^{n+\frac{1}{2}}\|^2+\|P_{L_H}^{\perp}\nabla\eta_1^{n+\frac{1}{2}}\|+\|\eta_1^{n+\frac{1}{2}}\|^4+\|v^{n+\frac{1}{2}}\|^2\|u^{n+\frac{1}{2}}\|^2],$$
$$\|\theta_2^{n+1}\|^2\leq C_2[h^{-2}\epsilon^2\|\nabla\eta_2^{n+\frac{1}{2}}\|^2+\|P_{L_H}^{\perp}\nabla\eta_2^{n+\frac{1}{2}}\|+\|\eta_2^{n+\frac{1}{2}}\|^4+\|u^{n+\frac{1}{2}}\|^2\|v^{n+\frac{1}{2}}\|^2],$$

from (5) we have,

$$\|\theta_1^{n+1}\|^2\leq C_1[(\epsilon^2h^{2(r-2)}+h^{2(r-1)})\|u^{n+\frac{1}{2}}\|_r^2+h^{4r}\|u^{n+\frac{1}{2}}\|_r^4+\|v^{n+\frac{1}{2}}\|_r^2\|u^{n+\frac{1}{2}}\|_r^2],$$
$$\|\theta_2^{n+1}\|^2\leq C_2[(\epsilon^2h^{2(r-2)}+h^{2(r-1)})\|v^{n+\frac{1}{2}}\|_r^2+h^{4r}\|v^{n+\frac{1}{2}}\|_r^4+\|u^{n+\frac{1}{2}}\|_r^2\|v^{n+\frac{1}{2}}\|_r^2],$$

which implies,

$$\max_{0\leq n\leq N}\|\theta_1^n\|\leq$$
$$C_1\{(h^{2r}+\epsilon h^{r-2}+h^{r-1})\|u^{n+\frac{1}{2}}\|_{L\infty(H^r)}+\|v^{n+\frac{1}{2}}\|_{L\infty(H^r)}\|u^{n+\frac{1}{2}}\|_{L\infty(H^r)}\},$$

$$\max_{0\leq n\leq N}\|\theta_2^n\|\leq$$
$$C_2\{(h^{2r}+\epsilon h^{r-2}+h^{r-1})\|v^{n+\frac{1}{2}}\|_{L\infty(H^r)}+\|u^{n+\frac{1}{2}}\|_{L\infty(H^r)}\|v^{n+\frac{1}{2}}\|_{L\infty(H^r)}\},$$

from $(A1)$ we have,

$$\max_{0\leq n\leq N}\|\theta_1^n\|\leq C_3(h^{2r}+\epsilon h^{r-2}+h^{r-1}),$$
$$\max_{0\leq n\leq N}\|\theta_2^n\|\leq C_4(h^{2r}+\epsilon h^{r-2}+h^{r-1}).$$

By these results and (21,22) the proof is complete.

Theorem 4.1.
Let u^{n+1}, v^{n+1}, u_h^{n+1}, and v_h^{n+1} be the solutions of (8,9) and (17,18) respectively, then there exists constants C_1, C_2 independent of ϵ, α, h and H such that,

$$\max_{0\le n\le N}\|u^n-u_h^n\|\le C_1\{h^{2r}+h^r+\epsilon h^{r-2}+h^{r-1}+k^{\frac{5}{2}}+\sqrt{\alpha}\},$$

$$\max_{0\le n\le N}\|v^n-v_h^n\|\le C_2\{h^{2r}+h^r+\epsilon h^{r-2}+h^{r-1}+k^{\frac{5}{2}}+\sqrt{\alpha}\}.$$

Proof: we write the errors in terms of equilibrium projection pu_h^{n+1} and pv_h^{n+1} of u^{n+1} and v^{n+1}

$$u^{n+1}-u_h^{n+1}=(u^{n+1}-pu_h^{n+1})-(u_h^{n+1}-pu_h^{n+1})=\rho_1^{n+1}-\theta_1^{n+1}$$

$$v^{n+1}-v_h^{n+1}=(v^{n+1}-pv_h^{n+1})-(v_h^{n+1}-pv_h^{n+1})=\rho_2^{n+1}-\theta_2^{n+1}$$

then,

$$\max_{0\le n\le N}\|u^n-u_h^n\|\le\max_{0\le n\le N}\|\rho_1^n\|+\max_{0\le n\le N}\|\theta_1^n\|$$

$$\max_{0\le n\le N}\|v^n-v_h^n\|\le\max_{0\le n\le N}\|\rho_2^n\|+\max_{0\le n\le N}\|\theta_2^n\|$$

we have bounds on the $\max_{0\le n\le N}\|\rho_1^n\|$ and $\max_{0\le n\le N}\|\rho_2^n\|$ from lemma (4.5). For having bounds on the $\max_{0\le n\le N}\|\theta_1^n\|$ and $\max_{0\le n\le N}\|\theta_2^n\|$, note that,

$$\frac{1}{k}(\theta_1^{n+1}-\theta_1^n,\varphi_h)+A(\theta_1^{n+\frac{1}{2}},\varphi_h)=$$
$$\frac{1}{k}(u_h^{n+1}-u_h^n,\varphi_h)+A(u_h^{n+\frac{1}{2}},\varphi_h)-\frac{1}{k}(pu_h^{n+1}-pu_h^n,\varphi_h)-A(pu_h^{n+\frac{1}{2}},\varphi_h),$$
$$\frac{1}{k}(\theta_2^{n+1}-\theta_2^n,\varphi_h)+A(\theta_2^{n+\frac{1}{2}},\varphi_h)=$$
$$\frac{1}{k}(v_h^{n+1}-v_h^n,\varphi_h)+A(v_h^{n+\frac{1}{2}},\varphi_h)-\frac{1}{k}(pv_h^{n+1}-pv_h^n,\varphi_h)-A(pv_h^{n+\frac{1}{2}},\varphi_h),$$
$$\frac{1}{k}(\theta_1^{n+1}-\theta_1^n,\varphi_h)+A(\theta_1^{n+\frac{1}{2}},\varphi_h)=(f^{n+\frac{1}{2}},\varphi_h)-\frac{1}{k}(pu_h^{n+1}-pu_h^n,\varphi_h)-A(pu_h^{n+\frac{1}{2}},\varphi_h),$$
$$\frac{1}{k}(\theta_2^{n+1}-\theta_2^n,\varphi_h)+A(\theta_2^{n+\frac{1}{2}},\varphi_h)=(g^{n+\frac{1}{2}},\varphi_h)-\frac{1}{k}(pv_h^{n+1}-pv_h^n,\varphi_h)-A(pv_h^{n+\frac{1}{2}},\varphi_h),$$

from (19,20) we have,

$$\left.\begin{aligned}&\tfrac{1}{k}(\theta_1^{n+1}-\theta_1^n,\varphi_h)+A(\theta_1^{n+\frac{1}{2}},\varphi_h)=(f^{n+\frac{1}{2}},\varphi_h)-\tfrac{1}{k}(pu_h^{n+1}-pu_h^n,\varphi_h)-A(u^{n+\frac{1}{2}},\varphi_h)\\&=(u_t^{n+\frac{1}{2}},\varphi_h)+a(u^{n+\frac{1}{2}},\varphi_h)+B(u^{n+\frac{1}{2}},u^{n+\frac{1}{2}},\varphi_h)+B(v^{n+\frac{1}{2}},u^{n+\frac{1}{2}},\varphi_h)-\tfrac{1}{k}(pu_h^{n+1}-pu_h^n,\varphi_h)-A(u^{n+\frac{1}{2}},\varphi_h)\\&=(u_t^{n+\frac{1}{2}},\varphi_h)-\tfrac{1}{k}(pu_h^{n+1}-pu_h^n,\varphi_h)-\alpha(P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}},P_{L_H}^{\perp}\varphi_h)\end{aligned}\right\}\quad(27)$$

$$\left.\begin{aligned}&\tfrac{1}{k}(\theta_2^{n+1}-\theta_2^n,\varphi_h)+A(\theta_2^{n+\frac{1}{2}},\varphi_h)\\&=(g^{n+\frac{1}{2}},\varphi_h)-\tfrac{1}{k}(pv_h^{n+1}-pv_h^n,\varphi_h)-A(v^{n+\frac{1}{2}},\varphi_h)\\&=(v_t^{n+\frac{1}{2}},\varphi_h)+a(v^{n+\frac{1}{2}},\varphi_h)+B(u^{n+\frac{1}{2}},v^{n+\frac{1}{2}},\varphi_h)+B(v^{n+\frac{1}{2}},v^{n+\frac{1}{2}},\varphi_h)-\tfrac{1}{k}(pv_h^{n+1}-pv_h^n,\varphi_h)-A(v^{n+\frac{1}{2}},\varphi_h)\end{aligned}\right\}\quad(28)$$

adding and subtracting $\frac{1}{k}(u^{n+1}-u^n,\varphi_h)$, $\frac{1}{k}(v^{n+1}-v^n,\varphi_h)$ to right hand sides, choosing $\varphi_h=\theta_1^{n+\frac{1}{2}}$ and $\varphi_h=\theta_2^{n+\frac{1}{2}}$ to (27) and (28) respectively gives ,

$$\frac{1}{2}(\theta_1^{n+1}-\theta_1^n,\theta_1^{n+1}+\theta_1^n)+kA(\theta_1^{n+\frac{1}{2}},\theta_1^{n+\frac{1}{2}})=$$

$$k(\frac{\rho_1^{n+1}-\rho_1^n}{k},\theta_1^{n+\frac{1}{2}})+k(\xi_1^n,\theta_1^{n+\frac{1}{2}})-k\alpha(P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}},P_{L_H}^{\perp}\theta_1^{n+\frac{1}{2}})$$

$$\frac{1}{2}(\theta_2^{n+1}-\theta_2^n,\theta_2^{n+1}+\theta_2^n)+kA(\theta_2^{n+\frac{1}{2}},\theta_2^{n+\frac{1}{2}})=$$

$$k(\frac{\rho_2^{n+1}-\rho_2^n}{k},\theta_2^{n+\frac{1}{2}})+k(\xi_2^n,\theta_2^{n+\frac{1}{2}})-k\alpha(P_{L_H}^{\perp}\nabla v^{n+\frac{1}{2}},P_{L_H}^{\perp}\theta_2^{n+\frac{1}{2}})$$

where, $\xi_1^n=u_t^{n+\frac{1}{2}}-\frac{1}{k}(u^{n+1}-u^n)$ and $\xi_2^n=v_t^{n+\frac{1}{2}}-\frac{1}{k}(v^{n+1}-v^n)$.

As noted in the proof of lemma (3.1) for V-elliptic property,

$$A(\theta_1^{n+\frac{1}{2}},\theta_1^{n+\frac{1}{2}})\geq C[\|\nabla\theta_1^{n+\frac{1}{2}}\|^2+\|\theta_1^{n+\frac{1}{2}}\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_1^{n+\frac{1}{2}}\|^2],$$

$$A(\theta_2^{n+\frac{1}{2}},\theta_2^{n+\frac{1}{2}})\geq C[\|\nabla\theta_2^{n+\frac{1}{2}}\|^2+\|\theta_2^{n+\frac{1}{2}}\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_2^{n+\frac{1}{2}}\|^2].$$

By using Cauchy-Schwartz inequality and applying Young's inequality to the right hand sides , we have,

$$\frac{1}{2}[\|\theta_1^{n+1}\|^2+\|\theta_1^n\|^2]+Ck[\|\nabla\theta_1^{n+\frac{1}{2}}\|^2+\|\theta_1^{n+\frac{1}{2}}\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_1^{n+\frac{1}{2}}\|^2]\leq\frac{k}{2C}\|\frac{\rho_1^{n+1}-\rho_1^n}{k}\|^2$$

$$+\frac{kC}{2}\|\theta_1^{n+\frac{1}{2}}\|^2+\frac{k}{2C}\|\xi_1^n\|^2+\frac{kC}{2}\|\theta_1^{n+\frac{1}{2}}\|^2+\frac{k\alpha}{4C}\|P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}}\|^2+k\alpha C\|P_{L_H}^{\perp}\theta_1^{n+\frac{1}{2}}\|^2$$

$$\frac{1}{2}[\|\theta_2^{n+1}\|^2+\|\theta_2^n\|^2]+Ck[\|\nabla\theta_2^{n+\frac{1}{2}}\|^2+\|\theta_2^{n+\frac{1}{2}}\|^2+\alpha\|P_{L_H}^{\perp}\nabla\theta_2^{n+\frac{1}{2}}\|^2]\leq\frac{k}{2C}\|\frac{\rho_2^{n+1}-\rho_2^n}{k}\|^2$$

$$+\frac{kC}{2}\|\theta_2^{n+\frac{1}{2}}\|^2+\frac{k}{2C}\|\xi_2^n\|^2+\frac{kC}{2}\|\theta_2^{n+\frac{1}{2}}\|^2+\frac{k\alpha}{4C}\|P_{L_H}^{\perp}\nabla v^{n+\frac{1}{2}}\|^2+k\alpha C\|P_{L_H}^{\perp}\theta_2^{n+\frac{1}{2}}\|^2$$

rearranging and multiplying both sides by 2, we have,

$$\|\theta_1^{n+1}\|^2-\|\theta_1^n\|^2+2Ck\|\nabla\theta_1^{n+\frac{1}{2}}\|^2\leq\frac{k}{C}\|\frac{\rho_1^{n+1}-\rho_1^n}{k}\|^2+\frac{k}{C}\|\xi_1^n\|^2+\frac{k\alpha}{2C}\|P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}}\|^2,$$

$$\|\theta_2^{n+1}\|^2-\|\theta_2^n\|^2+2Ck\|\nabla\theta_2^{n+\frac{1}{2}}\|^2\leq\frac{k}{C}\|\frac{\rho_2^{n+1}-\rho_2^n}{k}\|^2+\frac{k}{C}\|\xi_2^n\|^2+\frac{k\alpha}{2C}\|P_{L_H}^{\perp}\nabla v^{n+\frac{1}{2}}\|^2,$$

since, $Ck\|\nabla\theta_1^{n+\frac{1}{2}}\|^2$ and $Ck\|\nabla\theta_2^{n+\frac{1}{2}}\|^2$ are nonnegative we have,

$$\|\theta_1^{n+1}\|^2-\|\theta_1^n\|^2\leq\frac{1}{C}[\frac{1}{k}\|\rho_1^{n+1}-\rho_1^n\|^2+k\|\xi_1^n\|^2+k\alpha\|P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}}\|^2],$$

$$\|\theta_2^{n+1}\|^2-\|\theta_2^n\|^2\leq\frac{1}{C}[\frac{1}{k}\|\rho_2^{n+1}-\rho_2^n\|^2+k\|\xi_2^n\|^2+k\alpha\|P_{L_H}^{\perp}\nabla v^{n+\frac{1}{2}}\|^2],$$

summing both sides from $n=0$ to $n=N-1$ gives,

$$\|\theta_1^N\|^2 \le C_1[\|\theta_1^0\|^2 + \frac{1}{k}\sum_{n=0}^{N-1}\|\rho_1^{n+1}-\rho_1^n\|^2 + k\sum_{n=0}^{N-1}\|\xi_1^n\|^2 + k\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}}\|^2],$$

$$\|\theta_2^N\|^2 \le C_2[\|\theta_2^0\|^2 + \frac{1}{k}\sum_{n=0}^{N-1}\|\rho_2^{n+1}-\rho_2^n\|^2 + k\sum_{n=0}^{N-1}\|\xi_2^n\|^2 + k\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla v^{n+\frac{1}{2}}\|^2],$$

also, note that, there exist $0 \le n^* \le N$ such that $\|\theta_1^{n^*}\| = \max_{0\le n\le N}\|\theta_1^n\|$ and $\|\theta_2^{n^*}\| = \max_{0\le n\le N}\|\theta_2^n\|$, The analysis performed in the last step can be repeated with summing both sides from $n = 0$ to $n = n^*$, and the bound still holds consequently, this implies ,

$$\max_{0\le n\le N}\|\theta_1^n\|^2 \le C_1\{\|\theta_1^0\|^2 + \frac{1}{k}\sum_{n=0}^{N-1}\|\rho_1^{n+1}-\rho_1^n\|^2 + k\sum_{n=0}^{N-1}\|\xi_1^n\|^2 + k\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}}\|^2\}, \tag{29}$$

$$\max_{0\le n\le N}\|\theta_2^n\|^2 \le C_2\{\|\theta_2^0\|^2 + \frac{1}{k}\sum_{n=0}^{N-1}\|\rho_2^{n+1}-\rho_2^n\|^2 + k\sum_{n=0}^{N-1}\|\xi_2^n\|^2 + k\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla v^{n+\frac{1}{2}}\|^2\} \tag{30}$$

the first terms on right hand sides were bounded by (4.11, [6])

$$\|\theta_1^0\|^2 \le C_1 h^{2r} \text{ and } \|\theta_2^0\|^2 \le C_2 h^{2r}$$

For the second terms consider ,

$$\rho_1^{n+1}-\rho_1^n = \int_{t_n}^{t_{n+1}} \rho_{1,t}\ dt, \qquad \rho_2^{n+1}-\rho_2^n = \int_{t_n}^{t_{n+1}} \rho_{2,t}\ dt,$$

this implies ,

$$\|\rho_1^{n+1}-\rho_1^n\| = \int_{t_n}^{t_{n+1}} \|\rho_{1,t}\|dt, \qquad \|\rho_2^{n+1}-\rho_2^n\| = \int_{t_n}^{t_{n+1}} \|\rho_{2,t}\|dt,$$

so,

$$\|\rho_1^{n+1}-\rho_1^n\|^2 = k^2(\int_{t_n}^{t_{n+1}} \|\rho_{1,t}\|\frac{dt}{k})^2, \qquad \|\rho_2^{n+1}-\rho_2^n\|^2 = k^2(\int_{t_n}^{t_{n+1}} \|\rho_{2,t}\|\frac{dt}{k})^2,$$

applying Jensen's inequality (see[2]) to the right hand sides,

$$\|\rho_1^{n+1}-\rho_1^n\|^2 \le k\int_{t_n}^{t_{n+1}} \|\rho_{1,t}\|^2 dt, \qquad \|\rho_2^{n+1}-\rho_2^n\|^2 \le k\int_{t_n}^{t_{n+1}} \|\rho_{2,t}\|^2 dt,$$

then,

$$\frac{1}{k}\sum_{n=0}^{N-1}\|\rho_1^{n+1}-\rho_1^n\|^2 \le \int_0^T \|\rho_{1,t}\|^2 dt = \|\rho_{1,t}\|^2_{L^2(L^2)},$$

$$\frac{1}{k}\sum_{n=0}^{N-1}\|\rho_2^{n+1}-\rho_2^n\|^2 \le \int_0^T \|\rho_{2,t}\|^2 dt = \|\rho_{2,t}\|^2_{L^2(L^2)},$$

which has been bounded by (4.12, [6]), thus,

$$\frac{1}{k}\sum_{n=0}^{N-1}\|\rho_1^{n+1}-\rho_1^n\|^2 \le C_1\{h^{4r}+\epsilon^2 h^{2(r-2)}+h^{2(r-1)}\},$$

$$\frac{1}{k}\sum_{n=0}^{N-1}\|\rho_2^{n+1}-\rho_2^n\|^2 \le C_2\{h^{4r}+\epsilon^2 h^{2(r-2)}+h^{2(r-1)}\}.$$

For the third term of (29) (see[2]),

$$k\sum_{n=0}^{N-1}\|\xi_1^n\|^2 \le Ck^5\int_0^T \|u_{ttt}\|^2 dt = Ck^5\|u_{ttt}\|^2_{L^2(L^2)},$$

from $(A1)$,

$$k\sum_{n=0}^{N-1}\|\xi_1^n\|^2 \le C_1 k^5.$$

Similarly for the third term of (30),

$$k\sum_{n=0}^{N-1}\|\xi_2^n\|^2 \le Ck^5\int_0^T \|v_{ttt}\|^2 dt = Ck^5\|v_{ttt}\|^2_{L^2(L^2)}$$

$$\le C_2 k^5.$$

For the final term of (29), we note,

$$k\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}}\|^2 \le Ck\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla u^{n+1}+P_{L_H}^{\perp}\nabla u^n\|^2 \le$$

$$Ck\alpha\sum_{n=0}^{N-1}(\|P_{L_H}^{\perp}\nabla u^{n+1}\|^2+\|P_{L_H}^{\perp}\nabla u^n\|^2) \le Ck\alpha\sum_{n=0}^{N}\|P_{L_H}^{\perp}\nabla u^n\|^2,$$

from definition 2.1 we have,

$$k\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}}\|^2 \le C\alpha\|P_{L_H}^{\perp}\nabla u^n\|^2_{l^2(L^2)}$$

from$(A1)$ we have,

$$k\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla u^{n+\frac{1}{2}}\|^2 \le C_1\alpha.$$

Similarly for the final term of (30),

$$k\alpha\sum_{n=0}^{N-1}\|P_{L_H}^{\perp}\nabla v^{n+\frac{1}{2}}\|^2 \le C\alpha\|P_{L_H}^{\perp}\nabla v^n\|^2_{l^2(L^2)} \le C_2\alpha.$$

Then,

$$\max_{0\le n\le N}\|\theta_1^n\|^2 \le C_1\{h^{2r}+h^{4r}+\epsilon^2 h^{2(r-2)}+h^{2(r-1)}+k^5+\alpha\},$$

$$\max_{0\le n\le N}\|\theta_2^n\|^2 \le C_2\{h^{2r}+h^{4r}+\epsilon^2 h^{2(r-2)}+h^{2(r-1)}+k^5+\alpha\},$$

this implies,

$$\max_{0\le n\le N}\|\theta_1^n\| \le C_1\{h^{2r}+h^r+\epsilon h^{r-2}+h^{r-1}+k^{\frac{5}{2}}+\sqrt{\alpha}\},$$

$$\max_{0\le n\le N}\|\theta_2^n\| \le C_2\{h^{2r}+h^r+\epsilon h^{r-2}+h^{r-1}+k^{\frac{5}{2}}+\sqrt{\alpha}\}.$$

Combining these bounds with the bounds of $\max_{0\le n\le N}\|\rho_1^n\|$ and $\max_{0\le n\le N}\|\rho_2^n\|$ the proof is completed.

5. The Finite Element Approximation

To approximate the artificial viscosity terms in the equation (17,18), note that,

$$\nabla\varphi_h = P_{L_H}\nabla\varphi_h + P_{L_H}^{\perp}\nabla\varphi_h$$

$$P_{L_H}^{\perp}\nabla\varphi_h = \nabla\varphi_h - P_{L_H}\nabla\varphi_h$$

then,

$$\alpha(P_{L_H}^{\perp}\nabla u_h^{n+\frac{1}{2}}, P_{L_H}^{\perp}\nabla\varphi_h) = \alpha(P_{L_H}^{\perp}\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h) - \alpha(P_{L_H}^{\perp}\nabla u_h^{n+\frac{1}{2}}, P_{L_H}\nabla\varphi_h)$$

from the definition of $P_{L_H}^{\perp}$ the second term equal to zero, this implies,

$$\alpha(P_{L_H}^{\perp}\nabla u_h^{n+\frac{1}{2}}, P_{L_H}^{\perp}\nabla\varphi_h) \equiv \alpha(P_{L_H}^{\perp}\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h),$$

similarly,

$$\alpha(P_{L_H}^{\perp}\nabla v_h^{n+\frac{1}{2}}, P_{L_H}^{\perp}\nabla\varphi_h) \equiv \alpha(P_{L_H}^{\perp}\nabla v_h^{n+\frac{1}{2}}, \nabla\varphi_h).$$

The main point comes in finding an appropriate interpretation of the $\alpha(P_{L_H}^{\perp}\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h)$ and $\alpha(P_{L_H}^{\perp}\nabla v_h^{n+\frac{1}{2}}, \nabla\varphi_h)$ terms, since ,

$$\nabla u_h^{n+\frac{1}{2}} = P_{L_H}\nabla u_h^{n+\frac{1}{2}} + P_{L_H}^{\perp}\nabla u_h^{n+\frac{1}{2}}, \quad \nabla v_h^{n+\frac{1}{2}} = P_{L_H}\nabla v_h^{n+\frac{1}{2}} + P_{L_H}^{\perp}\nabla v_h^{n+\frac{1}{2}},$$

we rewrite,

$$\alpha(P_{L_H}^{\perp}\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h) = \alpha(\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h) - \alpha(P_{L_H}\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h),$$

$$\alpha(P_{L_H}^{\perp}\nabla v_h^{n+\frac{1}{2}}, \nabla\varphi_h) = \alpha(\nabla v_h^{n+\frac{1}{2}}, \nabla\varphi_h) - \alpha(P_{L_H}\nabla v_h^{n+\frac{1}{2}}, \nabla\varphi_h).$$

As noted in section four, L_H is chosen such that $P_{L_H}^{\perp}$ is a projection onto fine scales and P_{L_H} is a projection onto the large scales, we can think of the large scale as representing average values, this implies[3],

$$\alpha(\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h) - \alpha(P_{L_H}\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h) \approx \alpha(\nabla u_h^{n+\frac{1}{2}}, \nabla\varphi_h) - \alpha(\nabla\overline{u}_h^{n+\frac{1}{2}}, \nabla\varphi_h),$$

$$\alpha(\nabla v_h^{n+\frac{1}{2}}, \nabla\varphi_h) - \alpha(P_{L_H}\nabla v_h^{n+\frac{1}{2}}, \nabla\varphi_h) \approx \alpha(\nabla v_h^{n+\frac{1}{2}}, \nabla\varphi_h) - \alpha(\nabla\overline{v}_h^{n+\frac{1}{2}}, \nabla\varphi_h),$$

where, $\overline{u}_h^{n+\frac{1}{2}}$ and $\overline{v}_h^{n+\frac{1}{2}}$ is an average over itself and its five nearest discrete neighbors.

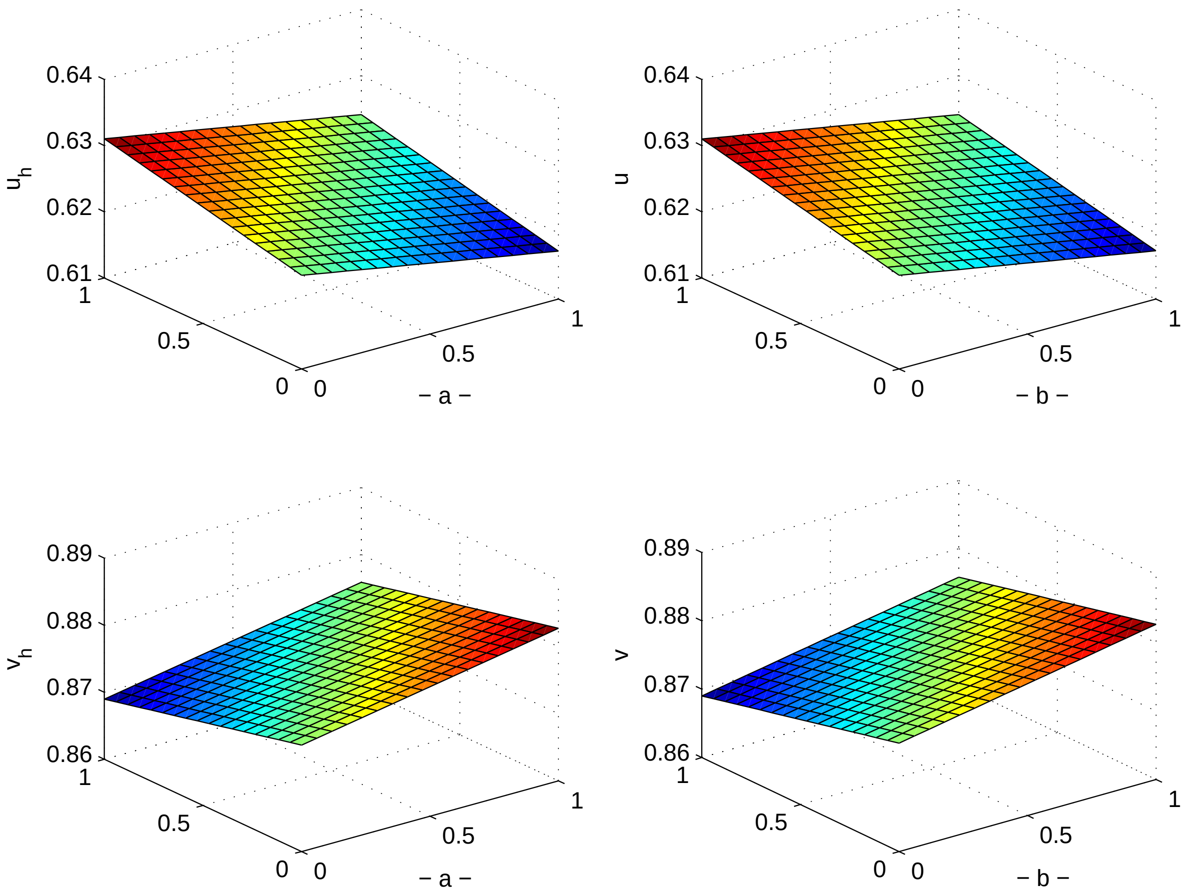

Figure 1. a-Numerical solution of G. method of u and v, b-Exact solution of u and v, at $N = 18$, $t = .5$, $k = .01$ and $\epsilon = 1.14$.

5.1. Test problem

In this subsection, we present the test problem to illustrate G.P.A.D. with Cranck-Nicholson scheme for the time variable to Burgers' equation (15,16). The exact solution of Burgers' equation(6,7) can be generated by using the Hopf-Cole transformation (see [7]) which are :

$$u(x,y,t) = \frac{3}{4} - \frac{1}{4[1+\exp^{\frac{(-4x+4y-t)}{32\epsilon}}]} \quad , \quad v(x,y,t) = \frac{3}{4} + \frac{1}{4[1+\exp^{\frac{(-4x+4y-t)}{32\epsilon}}]} \quad .$$

In this problem ϵ can take on various values and $f = g = 0$. The domain Ω where the problem is to be solved is the unit square domain $\overline{\Omega} = [0,1] \times [0,1]$. We are discretized it using a uniform triangular mesh with mesh width parameter $h = \frac{1}{N-1}$ where we take $N = 18$.

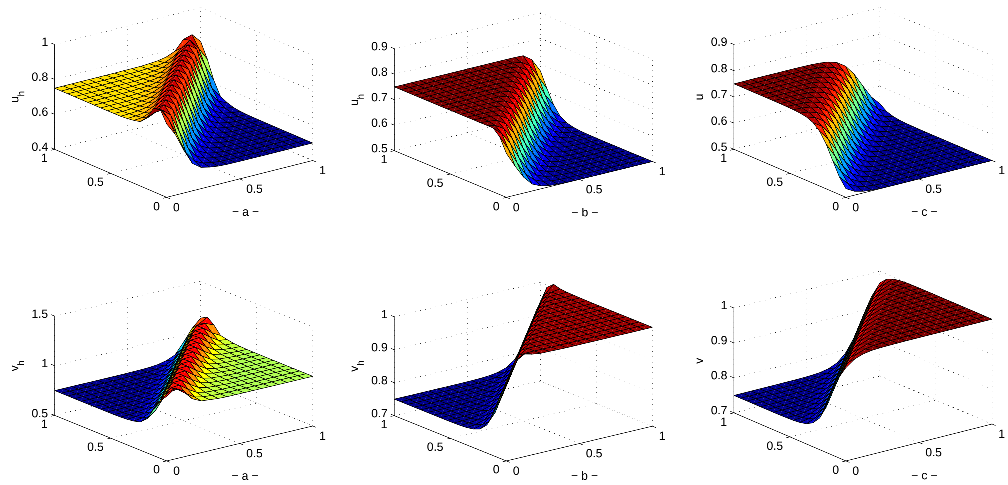

Figure 2. a-Numerical solution of G.method without P.A.D. of u and v, b-Numerical solution of G.P.A.D. of u and v, c-Exact solution of u and v, at $N = 18$, $t = .5$, $k = .01$ and $\epsilon = \frac{1}{120}$.

5.2. Numerical Results

This subsection consists of two case is discussed as follows:

Case 1: In this case the problem was run with $\epsilon = 1.14$ at $t = 0.5$ and $k = .01$, we note that $\epsilon > h$, there is no need to run this problem with P.A.D. (i.e. $\alpha = 0$), where the numerical solution of the standard Galerkin(G.) finite element method are convergent to the exact solution see figure 1

Case 2: In this case we take $\epsilon = \frac{1}{120}$ and $\epsilon = \frac{1}{240}$ respectively at $t = 0.5$ and $k = .01$, where $\epsilon < h$. In Figure 2(a) and 3(a) the problem run without P.A.D. (i.e. $\alpha = 0$),we see that the standard G. finite element method produce an oscillating solution which is not close to the exact solution especially when ϵ decreasing with respect to h. In Figure 2(b) and 3(b) the problem run with G.P.A.D. finite element method by using Cranck-Nicholson scheme for the time variable. where, $\alpha = 0.25 * h$, where the numerical solution became more convergent to the exact solution. In comparing with Case 2 [6], although we see that the standard G. finite element method produce slightly more oscillated solution with respect to the exact solution than Case 2 (Fig. 5.2.2-a, Fig. 5.2.3-a) [6], the G.P.A.D. finite element method is convergent to the exact solution as in Case 2 (Fig. 5.2.2-b, Fig. 5.2.3-b) [6].

6. Conclusions

From the theoretical analysis and the numerical results in the fully discrete case, we can conclude the following :

1. The stability condition of G.P.A.D. finite element method is satisfied .

2. Theoretical analysis shows that G.P.A.D. finite element method are convergent with $O(h^{2r} + k^{\frac{5}{2}})$.

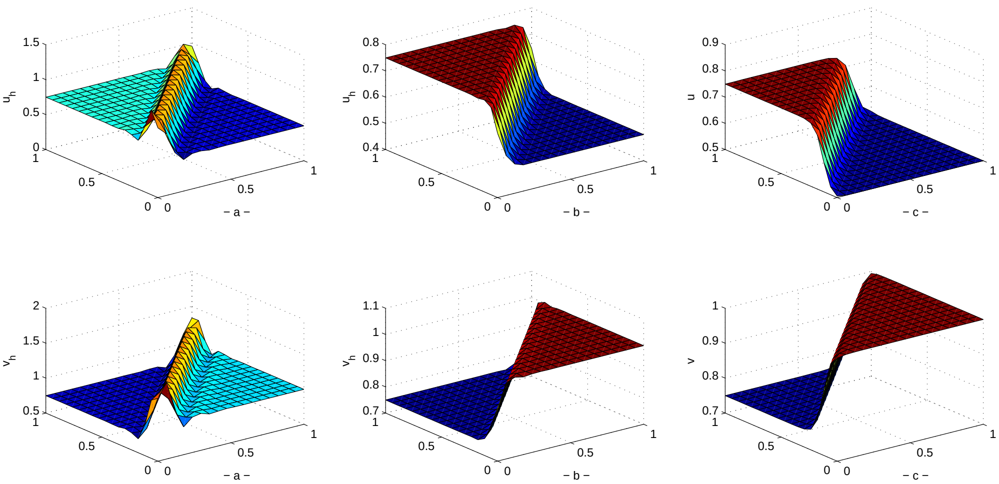

Figure 3. a-Numerical solution of G.method without P.A.D. of u and v, b-Numerical solution of G.P.A.D. of u and v, c-Exact solution of u and v, at $N = 18$, $t = .5$, $k = .01$ and $\epsilon = \frac{1}{240}$.

3. The G.P.A.D. method removed all oscillations occur when we use the standard Galerkin in the convection-dominated case, and the numerical solutions obtained from this method are consistent with the exact solution .

4. In comparing with [6] in the convection-dominated case, although we see that the standard G. finite element method produce slightly more oscillated solution with respect to the exact solution than [6], the G.P.A.D. finite element method is convergent to the exact solution as in [6] .

References

[1] A. N. Boules, On the existence of the solution of Burgers' equation for $n \leq 4$, Department of Mathematical Sciences, University of North Florida, Int. J. Math. and Math. Sci. 13(4) (1998)645-650.

[2] N.F. Heitmann, Subgridscale eddy viscosity for convection dominated diffusive transport, Department of Mathematics, Pittsburgh University, 2002.

[3] N.F. Heitmann, A stabilization scheme for convection dominated diffusive transport, Department of Mathematics , Pittsburgh University, 2002.

[4] H.A. Kashkool, N.J. Noon, The modification of Galerkin and Galerkin-Conservation finite element methods for solving coupled Burgers' problem, in Misan Journal of Academic Studies 9 (18) (2012).

[5] M.G. Larson, F. Bengzon, The finite element method: theory, implementation, and applications, Texts in Computational Science and Engineering, ISSN 1611-0994, Verlag Berlin Heidelberg, 2013.

[6] N.J. Noon, Semi discrete formulation of Galerkin -partial artificial diffusion finite element method for coupled Burgers' problem, International Journal of Pure and Applied Research in Engineering and Technology 2 (4) (2013) 1-24.

[7] V.K. Srivastava, M. Tamsir, U. Bhardwaj, Y. Sanyasiraju, Crank Nicolson finite difference method for two dimensional coupled nonlinear viscous Burgers' equations, Int. J. of Scientific and Engineering Research 2 (5) (2011).

[8] Q. Yang, The upwind finite volume element method for two-dimensional Burgers equation, Hindawi Publishing Corporation, Abstract and Applied Analysis, Article ID 351619, 11 pages (2013).