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Restrictive Taylor Approximation for Gardner and KdV Equations

Research Article

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Abstract: The Restrictive Taylor Approximation is implemented to find numerical solution of linear and nonlinear partial differential equation.
In this paper we introduce the numerical solution of Gardner equation and General KdV equation, also we study the stability of both equations. The scheme is based on evaluation the restrictive term at the first solution level and then applies the formula for the next levels.

MSC: 65M06 • 65F60 • 65N06

Keywords: Gardner Equation • Restrictive Taylor Approximation • General KdV • Mixed KdV-mKdV equation

1. Introduction

The nonlinear partial differential equations are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves and chemical physics. The nonlinear wave phenomena was observed in the above mentioned scientific fields, are often modeled by the bell-shaped $sech$ solutions and the kink-shaped $tanh$ solutions. The availability of these exact solutions, for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, new exact solutions may help to find new phenomena. Also, the explicit formulas may provide physical information and help us to understand the mechanism of the related physical models. [1].

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Recently, there have been a multitude of methods presented for solving nonlinear partial differential equations for instance, the tanh method, the homogeneous balance method , the homotopy analysis method, Wazwaz [2] applied the sine-cosine method, the F-expansion method , exp-function method [3], three-wave method , extended homoclinic test approach and the (G'/G)-expansion method. [4]

In this paper we used the techniques of finite difference methods that developed in [5]. This technique is based to find the restrictive term by using the exact solution or by using accurate method to find some solution at some points to calculate the restrictive term. This new method was applied for Parabolic Partial Differential Equations by Ismail and Elbarbary [6], Schrodinger Equation [7], Generalized Burger's Equation [8], General Korteweg and de Vries (KdV) Equation [9] and Convection-Diffusion Equation [10].

The Gardner equation is known as the mixed KdV-mKdV equation, it's very widely studied in various areas of Physics that includes Plasma Physics, Fluid Dynamics, Quantum Field Theory, Solid State Physics and others [11].

2. Restrictive Taylor's approximation for Gardner equation

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\mu u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0 \tag{1}$$

We use the restrictive Taylor approximation to solve the Gardner equation. We define the first derivative to x and t and the third derivative of x to the form.

$$\begin{aligned} D_t u_{i,j} &= \tfrac{u_{i,j+1}-u_{i,j}}{k}\\ D_x u_{i,j} &= \tfrac{u_{i+1,j}-u_{i-1,j}}{2h}\\ D_x^3 u_{i,j} &= \tfrac{u_{i+2,j}-2u_{i+1,j}+2u_{i-1,j}-u_{i-2,j}}{2h^3}\end{aligned} \tag{2}$$

where $h=\Delta x$ and $k=\Delta t$.

$$u(x,t+k)=u(x,t)+k\tfrac{\partial u(x,t)}{\partial t}+\tfrac{k^2}{2!}\tfrac{\partial^2 u(x,t)}{\partial t^2}+\ldots=\left(1+\tfrac{k}{1!}\tfrac{\partial}{\partial t}+\tfrac{k^2}{2!}\tfrac{\partial^2}{\partial t^2}+\ldots\right)u(x,t)$$

$$u_{i,j+1}=EXP\left[k\frac{\partial}{\partial t}\right]u_{i,j} \tag{3}$$

using equation (1) then

$$u_{i,j+1}=Exp\left[-k\left((\alpha u+\mu u^2)\frac{\partial}{\partial x}+\beta\frac{\partial^3}{\partial x^3}\right)\right]u_{i,j} \tag{4}$$

substituted from Eq. (2) in(4)

$$\begin{aligned} u_{i,j+1}=Exp(-k)[&\left(\alpha\frac{u_{i+1,j}+u_{i,j}+u_{i-1,j}}{3}+\mu(\frac{u_{i+1,j}+u_{i,j}+u_{i-1,j}}{3})^2\right)\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h}\right)\\ &+\beta\frac{u_{i+2,j}-2u_{i+1,j}+2u_{i-1,j}-u_{i-2,j}}{2h^3}]\end{aligned} \tag{5}$$

But restrictive Taylor's approximation of the first order $RT_{1,exp(xA)}(xA)$ of the exponential matrix function $exp(xA)$.

$$RT_{1,exp(xA)}(xA)=I+r\Gamma A=I+x\epsilon_i A \tag{6}$$

where (A) is a $N \times N$ matrix.I is the identity matrix and $\in_{L_1} = [\in_{i,L_1}]$ is the diagonal matrix of the restrictive term.

Then the equivalent scalar approximation of restrictive Taylor's approximation for the Gardner equation is

$$u_{i,j+1} = \frac{-k}{2h^3} \in_{i,L_1} \Big[h^2 \Big(\alpha \frac{u_{i+1,j} + u_{i,j} + u_{i-1,j}}{3} + \mu \big(\frac{u_{i+1,j} + u_{i,j} + u_{i-1,j}}{3} \big)^2 \Big) \big(u_{i+1,j} - u_{i-1,j} \big) + \beta \big(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j} \big) \Big] + u_{i,j} \tag{7}$$

3. Numerical Example for Restrictive Taylor (RT) Approximation for Gardner Equation

Wazwaz [13] solve the Gardner equation on the form

$$\frac{\partial u}{\partial t} + 2au\frac{\partial u}{\partial x} - 3bu^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{8}$$

which have the exact kink solution on the form:

$$u(x,t) = \frac{a}{3b}\left(1 \pm tanh\left(\frac{a}{3\sqrt{2b}}\left(x - \frac{2a^2}{9b}t\right)\right)\right) \; where \; a,b > 0 \tag{9}$$

For $a = b = 1$

$u(x,t) = \frac{1}{3}\left(1 + tanh\left(\frac{1}{3\sqrt{2}}\left(x - \frac{2}{9}t\right)\right)\right)$

To find the numerical solution we choose the step lengths to be $x = h = 0.1$ and $t = k = 0.0001$ and using equation (7), the absolute error in Table 1 at various values of time and distance.

Table 1. The absolute error of the solution of (1) using RT at $k = 0.0001$ and $h = 0.1$ for various values of (t,x)

t	$x = 0.1$	$x = 0.5$	$x = 0.9$
0.0001	6.661338147750939E-16	2.664535259100375E-15	3.885780586188048E-15
0.001	7.477352070850429E-14	3.029798634202052E-13	3.816946758661288E-13
0.009	5.041522754822836E-12	3.379413415771637E-11	1.266903248975381E-11
0.01	6.878275726762695E-12	4.184824708985957E-11	1.430205953667496E-11
0.02	6.566991395118293E-11	1.667365134849774E-10	5.750133702520088E-11
0.05	5.024072269321778E-10	1.015013628702377E-9	4.87995754916426E-10
0.1	2.093099193967162E-9	4.019774901831141E-9	2.059257764308597E-9
0.2	8.509731141970178E-9	1.582620018369951E-8	8.430403541925813E-9
0.5	4.897918975954596E-8	8.567726123009933E-8	5.685116194475981E-8

The executive time of calculating 10000 steps by restrictive Taylor method is 4.29 Second, this is relatively very small.

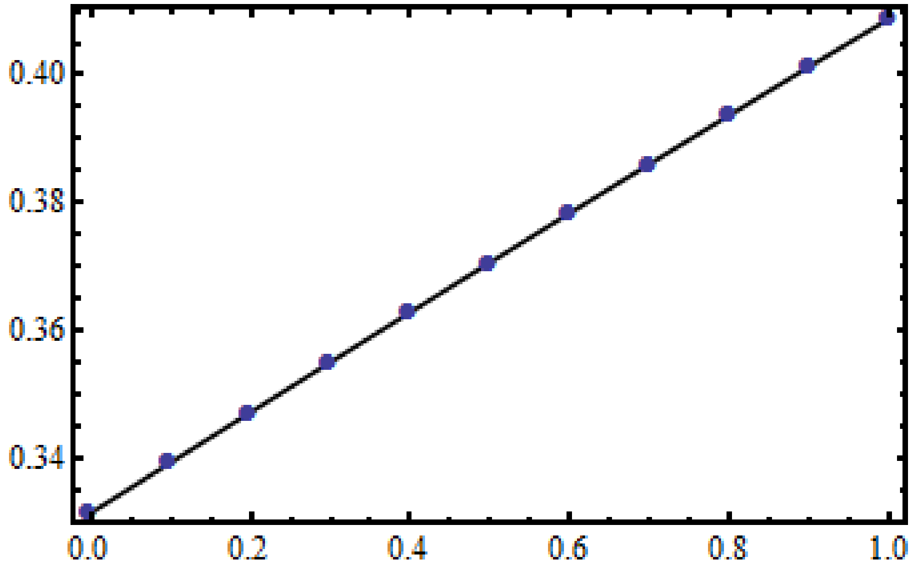

Figure 1. Exact (Solid)and numerical solution(dote) of (1) at $t = 0.1$.

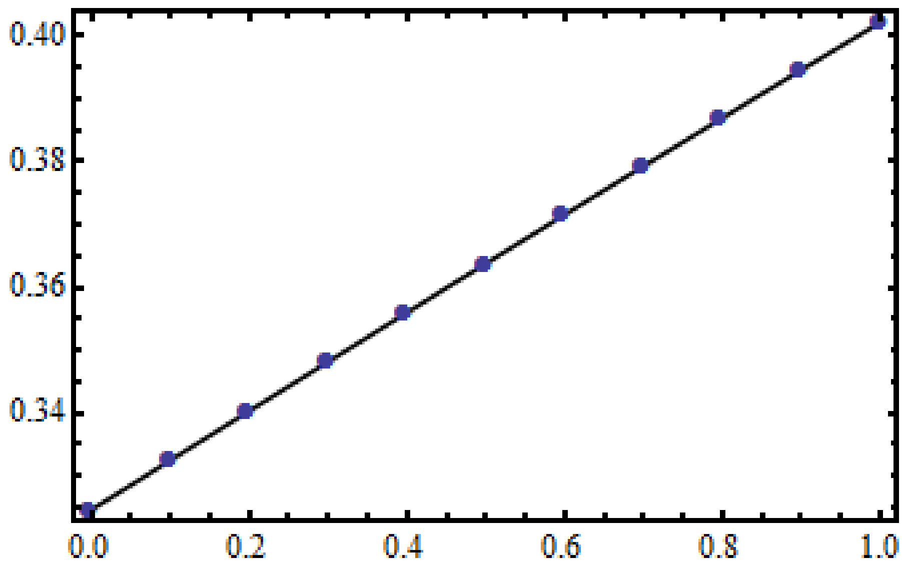

Figure 2. exact (Solid)and numerical solution(dote) of (1) at $t = 0.5$.

4. Practical Trials of Stability Analysis

From equation (7)

$$\begin{aligned} u_{i,j+1} = [\, & u_{i,j} - \frac{k}{2h^3} \in_{i,L_1} \left(h^2 \alpha M + \mu M^2 - 2\beta\right) u_{i+1,j} + \quad \frac{k}{2h^3} \in_{i,L_1} \left(h^2 \alpha M + \mu M^2 - 2\beta\right) u_{i-1,j} \\ & - \frac{k}{2h^3} \in_{i,L_1} \beta\, u_{i+2,j} + \frac{k}{2h^3} \in_{i,L_1} \beta\, u_{i-2,j} \,] \end{aligned} \tag{10}$$

This scalar form can be written in the general vector form:

$$u_{i,j+1} = A\underline{u}_{i,j} + \underline{b} \tag{11}$$

where

$$A = \begin{bmatrix} a & -b & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & a & -b & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & b & a & -b & -c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & b & a & -b & -c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & b & a & -b & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & b & a & -b & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & b & a & -b & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & b & a & -b & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & b & a & -b & -c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & b & a & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & b & a \end{bmatrix} \tag{12}$$

$a = 1, b = \frac{k}{2h^3} \in_{i,L_1} \left(h^2 \propto M + \mu M^2 - 2\beta\right), c = \frac{k}{2h^3} \in_{i,L_1} \beta$

The necessary and sufficient condition of stability $\lambda_A < 1$

Table 2 we assume values of λ_A that satisfy the stability condition and find the constraints on h and k

$$0 < h < 0.7245,\ R = \frac{k}{2h^3} \tag{13}$$

Table 2. different values of h and k for sufficient condition of stability

$\lambda_A = 0.1$		$\lambda_A = 0.5$		$\lambda_A = 0.9$	
h	k	h	k	h	k
0.1	$0 < k < 0.0107$	0.1	$0 < k < 0.0059$	0.1	$0 < k < 0.0011$
0.2	$0 < k < 0.0904$	0.2	$0 < k < 0.0523$	0.2	$0 < k < 0.0104$
0.3	$0 < k < 0.374$	0.3	$0 < k < 0.207$	0.3	$0 < k < 0.04157$

5. Restrictive Approximation for KdV

In 1895, Korteweg and de Vries derived KdV equation to model Russell's phenomenon of solitons . Solitons are localized waves that propagate without change of its shape and velocity properties and stable against mutual collision.

The traditional methods available for the numerical solution of partial differential equations are finite difference, finite element, finite volume and spectral methods. The difficulty of mesh generation, especially in two or three dimensions, makes these methods hard to implement.

5.1. Restrictive Taylor Approximation for GKdV Equation

5.1.1. When $\mu = 0$ equation (1) become the GKdV equation [9]

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0 \tag{14}$$

the restrictive algorism of equation (14) will be

$$\begin{aligned} u_{i,j+1} = \frac{-k}{2h^3} \in_{i,L_2} \Big[h^2 \Big(\alpha \frac{u_{i+1,j} + u_{i,j} + u_{i-1,j}}{3} \Big) (u_{i+1,j} - u_{i-1,j}) \\ + \beta \left(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j} \right) \Big] + u_{i,j} \end{aligned} \tag{15}$$

The exact solution of equation (14) $\beta = 4.84 \times 10^{-4}$ and $\alpha = 1$ is

$u(x,t) = 3\ c\ Sech^2[a_1 x - b_1 t + d_1] \quad 0 \le x \le 2 \quad t \ge 0$

where $a_1 = \frac{1}{2}\sqrt{\frac{\alpha c}{\beta}} \quad b_1 = a c a_1 \quad d_1 = -6$

The initial and boundary conditions are defined as to agree with the exact solution. The absolute errors of restrictive Taylor (RT) method in Eq. (15) for various values of time t are given in Table 3 along $x = 0.5, 1$ and 1.5, where $h = 0.1$ and $k = 0.0001$.

Table 3. The absolute error of the solution of (14) using RT at $k = 0.0001$ and $h = 0.1$ for various values of (t, x)

t	$x = 0.5$		$x = 1$		$x = 1.5$	
	RT inclosed	RT (Ismail et.al 2005)	RT inclosed	RT (Ismail et.al 2005)	RT inclosed	RT (Ismail et.al 2005)
0.0005	3.5362E-7	2.688E-7	3.3520E-14	4.358E-14	2.4557E-25	3.619E-25
0.001	1.5991E-6	1.206E-6	1.7187E-13	2.320E-13	2.4686E-24	3.490E-24
0.002	6.8174E-6	5.061E-6	9.0607E-13	1.286E-12	1.6111E-23	2.635E-23
0.003	1.5758E-5	1.152E-5	2.4960E-12	3.651E-12	5.6707E-23	1.061E-22
0.004	2.8525E-5	2.053E-5	5.2472E-12	7.831E-12	1.5404E-22	3.241E-22
0.005	4.5219E-5	3.203E-5	9.4772E-12	1.434E-11	3.6180E-22	8.360E-22
0.01	1.9112E-4	1.253E-4	6.4799E-11	1.006E-10	8.0638E-21	2.349E-20
0.1	2.9911E-2	—————	1.1253E-7	—————	3.6886E-15	—————

The restrictive term $\in_{i,L_2}$ calculated from Eqs. (15) when the approximated solution at the first level equals the exact solution.

The executive time of calculating 1000 steps by restrictive Taylor method is 2.4 Second. which is relatives very small comparing with [9]. More over in (Ismail et. al 2005) we find the solution at $t = 0.1$ and the absolute error for the second and third columns are relatives small the numerical and exact solutions are shown in figure 3

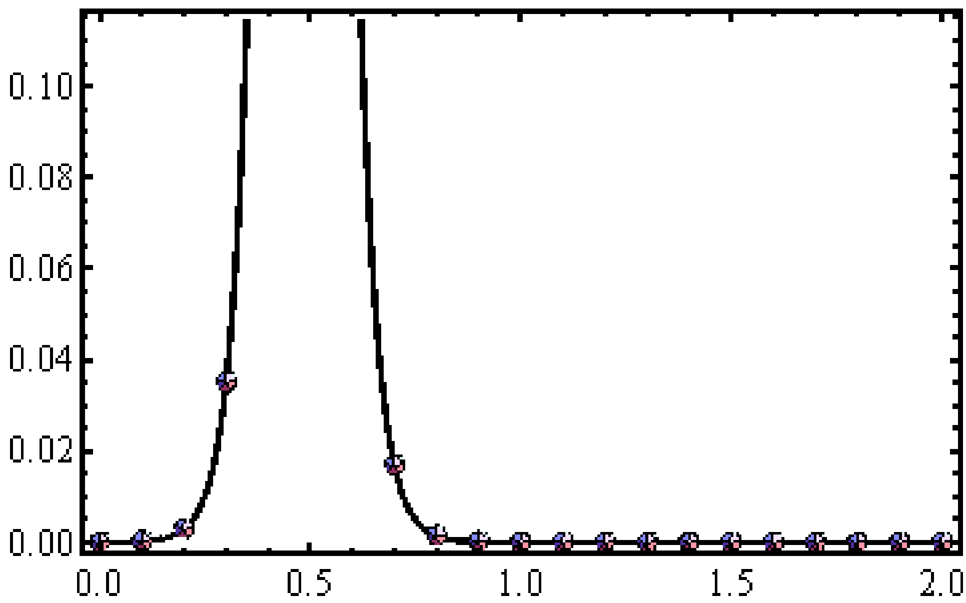

Figure 3. exact (Solid)and numerical solution(dote) of (14) at $t = 0.01$.

5.1.2. When $\alpha = 0$ equation (1) become the GKdV equation [9]

$$\frac{\partial u}{\partial t} + \mu u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0 \tag{16}$$

and the restrictive algorism of equation (16) will be

$$\begin{aligned} u_{i,j+1} = \frac{-k}{2h^3} \in_{i,L_3} \Big[h^2 \Big(\mu \big(\frac{u_{i+1,j} + u_{i,j} + u_{i-1,j}}{3} \big)^2 \Big) \big(u_{i+1,j} - u_{i-1,j} \big) \\ + \beta \big(u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j} \big) \Big] + u_{i,j} \end{aligned} \tag{17}$$

The exact solution of equation (16) $\beta = 4.84 \times 10^{-4}$ and $\mu = 1$ is

$$u(x,t) = \sqrt{3\,c}\; Sech[a_1 x - b_1 t + d_1],\; 0 \le x \le 2,\; t \ge 0 \tag{18}$$

where $a_1 = \frac{1}{2}\sqrt{\frac{\alpha c}{\beta}}$ $b_1 = \mu\ c\ a_1,\ d_1 = -6$

and the initial and boundary conditions are defined as to agree with the exact solution.

The absolute errors of restrictive Taylor (RT) method in Eq. (17) for various values of time t are given in Table 4 along $x = 0.6,\ 1.2\ and\ 1.8$, where $h = 0.1$ and $k = 0.0001$ and it is clear that the absolute errors are better than ( ).

The restrictive term $\in_{i,L_3}$ calculated from Eqs. (17) when the approximated solution at the first level equals the exact solution. The executive time of calculating 1000 steps by restrictive Taylor method is 2 Second. which is relatives very

Table 4. The absolute error of the solution of (16) using RT at $k = 0.0001$ and $h = 0.1$ for various values of (t, x)

t	$x = 0.6$		$x = 1.2$		$x = 1.8$	
	RT inclosed	RT (Ismail et.al 2005)	RT inclosed	RT (Ismail et.al 2005)	RT inclosed	RT (Ismail et.al 2005)
0.0005	3.0399E-7	2.688E-7	1.6362E-18	2.526E-15	1.6408E-28	7.745E-13
0.001	1.3659E-6	1.209E-6	8.6817E-18	1.376E-14	1.8932E-28	7.745E-13
0.003	1.3124E-5	1.165E-5	1.5903E-16	2.701E-13	7.1313E-28	7.445E-13
0.005	3.6741E-5	3.272E-5	7.7427E-16	1.341E-12	1.1360E-27	7.744E-13
0.01	1.4633E-4	1.312E-4	8.6762E-15	1.442E-11	2.9377E-27	7.743E-13
0.03	1.3124E-5	1.145E-3	1.5903E-16	7.578E-10	7.1313E-28	7.734E-13
0.05	3.2822E-3	3.03E-3	5.8251E-12	4.514E-9	2.9172E-22	7.721E-13
0.1	1.0466E-2	1.008E-2	1.3274E-10	4.312E-8	4.9686E-20	8.085E-13

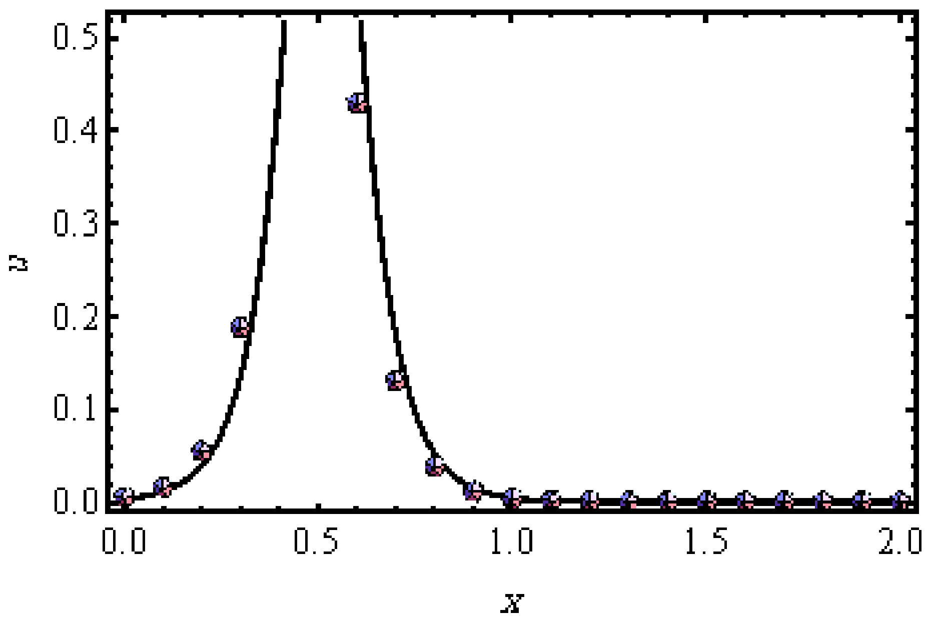

Figure 4. Exact (Solid)and numerical solution(dote) of (16) at $t = 0.1$.

small comparing with [9].

More over in (Ismail et. al 2005) the absolute errors for the second and third columns are relatives small,the numerical and exact solutions are shown in figure 4

5.2. Stability Analysis

The explicit difference scheme equation (15), the truncation error approaches to zero, as defined (Ismail 2005), and is also consistent with equation (17). A stability analysis of the nonlinear numerical scheme of equation (15) using the Von Neuman method is not easy to handle unless it is assumed that u, in the nonlinear term can be considered as locally constant; this is equivalent to replacing the term $(u_{i+1,j} + u_{i,j} + u_{i-1,j})/3$ in equation (15) by $\overline{u}$ as done in [9].

$u^n_{i,j} = \xi^n e^{I\theta i} e^{I\varphi j}$ Then the stability condition of the finite difference equation (15) is $|\xi| \leq 1$.

where

$$\in_{i,L_2} \frac{k}{h}\left(\alpha\left(\bar{u}\right)+\frac{4\beta}{h^2}\right) \leq 1 L_2 = 0(1)m, m \geq 2, i = 1(1)n \tag{19}$$

Similarly the stability condition of the finite difference equation (17) is $|\xi| \leq 1$.

where

$$\in_{i,L_3} \frac{k}{h}\left(\alpha\left(\bar{u}^2\right)+\frac{4\beta}{h^2}\right) \leq 1 L_3 = 0(1)m, m \geq 2, i = 1(1)n \tag{20}$$

6. Conclusion

The advantages of the restrictive Taylor approximation can be summarized as follows:

- The executive time of calculating by restrictive Taylor method is relatively very small.
- The high accuracy of this approach, this appears in the absolute error table.
- The method gives the exact solution if it is known at one level of time, for example at k , i.e $u(x,t) = u(ih,k)$ $i = 1(1)N$.
- Without knowing the exact solution at one level, we try to use an approximate, fast efficient and accurate method with suitable very small step sizes h and k, to get the needed almost exact solution at specific level, after which we continue the usual restrictive Taylor process.
- The needed exact solution at the first level need not be in closed form, i.e., we need only a table of the exact solution at some points.

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