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Existence result of fractional functional integro-differential equation with not instantaneous impulse

Research Article

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Abstract: In this investigation, we have established the existence and uniqueness results of solutions for class of an abstract fractional functional integro-differential equations with state dependent delay subject to not instantaneous impulse by using fixed point theorems. One example is presented to illustrate the main results of the paper.

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Keywords: Fractional order differential equations • Functional differential equations • Impulsive conditions • Fixed point theorems

1. Introduction

Due to the memory and hereditary property, fractional derivatives are more applicable than the ordinary derivative and with the different conditions such as initial, impulsive and nonlocal so these derivatives became more realistic. The fractional differential equations with impulsive effects have been appeared as in natural description evolution processes for many real problems. The impulsive effects can be shown in many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated system etc. See [1, 2, 4, 5, 13, 18] for current updates of the theory.

Fractional order functional differential equations originate in several field of applied mathematics and science. Recently, fractional functional differential equations with state dependent delay seems frequently in modeling of equations, panorama of natural phenomena and porous media for details developments of the theory one can see the papers [6–12, 14, 19].

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Byszewski [16] have established the existence, uniqueness of mild and classical solutions for a nonlocal Cauchy problem. As remarked by the author the nonlocal condition can be more realistic than the standard initial conditions to describe some physical phenomena. See [16, 17] for more detail and reference therein.

Feckan et al. [2] have presented a counterexample to show an essence error in the the formula of the solutions to the impulsive Cauchy problems used by several authors in literature [5, 13]. Further, Wang et al. [18] established sufficient conditions for existence of the solutions for the following fractional differential equations

$$\begin{aligned}
&{}^{c}D_t^q u(t)=f(t,u(t)), \quad t\in J', q\in(1,2),\\
&\Delta u(t_k)=y_k, \Delta u'(t_k)=\bar{y}_k \quad k=1,2,\cdots,m,\\
&u(0)=u_0,\ \bar{u}(0)=\bar{u}_0,
\end{aligned}$$

and by applying fixed point theorems. In their subsequent study authors [18] consider the following problem with nonlinear impulsive condition of the form:

$$\begin{aligned}
&{}^{c}D_t^q u(t)=f(t,u(t)), \quad t\in J', q\in(1,2),\\
&\Delta u(t_k)=I_k(u(t_k^-)), \Delta u'(t_k)=J_k(u(t_k^-)) \quad k=1,2,\cdots,m,\\
&u(0)=u_0,\ \bar{u}(0)=\bar{u}_0.
\end{aligned}$$

Nonlinear impulsive conditions arises where abrupt change occur in any state at certain time of moments for example in phenomena involving thresholds, bursting rhythm models in medicine and biology etc. Recently, not instantaneous impulsive condition first time used by author's Hernandez and O'Regan [4] for the following problem of the form:

$$\begin{aligned}
&u'(t)=Au(t)+f(t,u(t)),\ t\in(s_i,t_{i+1}],\ i=0,1,\cdots,N,\\
&u(t)=g_i(t,u(t)),\ t\in(t_i,s_i],\ i=1,2,\cdots,N,\ u(0)=x_0,
\end{aligned}$$

The dynamical system where impulses start abruptly at the certain time and their action continues on the finite time interval, modeled as not instantaneous differential equations.

Our present work motivated by the papers [2, 4, 18], we study not instantaneous impulsive fractional functional integro-differential equation of the form:

$$D_t^\alpha y(t)=J_t^{2-\alpha}f(t,y_{\rho(t,y_t)},B(y_{\rho(t,y_t)}),\ t\in(s_i,t_{i+1}],\ i=0,1,\cdots,N, \tag{1}$$

$$y(t)=g_i(t,y(t)),\ y'(t)=q_i(t,y(t)),\ t\in(t_i,s_i],\ i=1,2,\cdots,N, \tag{2}$$

$$y(t)+u(y)=\phi(t),\ y'(t)+v(y)=\varphi(t),\ t\in(-\infty,0], \tag{3}$$

where D_t^α is Caputo's derivative of order $\alpha\in(1,2]$ and $J^{2-\alpha}$ is Riemann-Liouville fractional integral. y' denotes the derivative of y with respect to t and operational interval $J=[0,T], 0<T<\infty$. $f: J\times\mathfrak{B}_h\times\mathfrak{B}_h\to X$, $u,v: X\to X$ are given functions. $\mathfrak{B}_h$ is a abstract phase space and y_t the element of $\mathfrak{B}_h$ defined by $y_t(\theta)=y(t+\theta)$, $\theta\in(-\infty,0]$. The term $B(y_{\rho(t,y_t)})$ is given by $B(y_{\rho(t,y_t)}) = \int_0^t K(t,s)(y_{\rho(s,y_s)})ds$, where $K \in C(D, \mathbb{R}^+)$, is the set of all positive functions which are continuous on $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t < T\}$ and $B^* = \sup_{t\in[0,t]} \int_0^t K(t,s)ds < \infty$. Here $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_N \le s_N \le t_{N+1} = T$, are pre-fixed numbers, $g_i, q_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, 2, \cdots, N$.

To the best of our knowledge, our results are new for the fractional functional integro-differential equation with state dependent delay subject to not instantaneous impulsive condition. This work is divided in to four sections. In section two we include the basic definitions and assumptions, next in third section contains the main results and in last section an example is presented.

2. Preliminary

Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|y\|_X = \sup_{t\in J}\{|y(t)| : y \in X\}$. For infinite delay we use abstract phase space $\mathfrak{B}_h$ as defined in [15] details are as follow:

Assume that $h : (-\infty, 0] \to (0, \infty)$ is a continuous functions with $l = \int_{-\infty}^0 h(s)ds < \infty, t \in (-\infty, 0]$. For any $a > 0$, we define

$$\mathfrak{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\},$$

and equipped the space $\mathfrak{B}$ with the norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|_X, \forall\, \psi \in \mathfrak{B}$. Let us define

$$\mathfrak{B}_h = \{\psi : (-\infty, 0] \to X, \text{ s.t. for any } c > 0, \psi\,|_{[-c,0]} \in \mathfrak{B} \;\&\; \int_{-\infty}^0 h(s)\|\psi\|_{[s,0]}ds < \infty\}.$$

If $\mathfrak{B}_h$ is endowed with the norm $\|\psi\|_{\mathfrak{B}_h} = \int_{-\infty}^0 h(s)\|\psi\|_{[s,0]}ds,\ \forall\ \psi \in \mathfrak{B}_h$, then it is clear that $(\mathfrak{B}_h, \|.\|_{\mathfrak{B}_h})$ is a complete Banach space.

To treat the impulsive conditions, we consider the following setting

$$\mathfrak{B}'_h := PC((-\infty, T]; X),\ T < \infty,$$

be a Banach space of all such functions $y : (-\infty, T] \to X$, which are continuous every where except for a finite number of points $t_i \in (0, T),\ i = 1, 2, \ldots, N$, at which $y(t_i^+)$ and $y(t_i^-)$ exists and endowed with the norm

$$\|y\|_{\mathfrak{B}'_h} = \sup\{\|y(s)\|_X : s \in J\} + \|\phi\|_{\mathfrak{B}_h}, y \in \mathfrak{B}'_h,$$

where $\|\cdot\|_{\mathfrak{B}'_h}$ to be a semi-norm in $\mathfrak{B}'_h$.

For a function $y \in \mathfrak{B}'_h$ and $i \in \{0, 1, ..., N\}$, we introduce the function $\bar{y}_i \in C([t_i, t_{i+1}]; X)$ given by

$$\bar{y}_i(t) = \begin{cases} y(t), \text{ for } t \in (t_i, t_{i+1}], \\ y(t_i^+), \text{ for } t = t_i, \end{cases}$$

and setting

$$\mathfrak{B}''_h := PC^1((-\infty, T]; X),\ T < \infty,$$

be a Banach space of all such functions $y : (-\infty, T] \to X$, which are continuously differentiable every where except for a finite number of points $t_i \in (0, T),\ i = 1, 2, \ldots, N$, at which $y'(t_i^+)$ and $y'(t_i^-)$ exists and endowed with the semi-norm

$$\|y\|_{\mathfrak{B}''_h} = \sup_{t\in[0,T]} \{\|y(s)\|_X, \|y'(s)\|_X\} + \|\phi\|_{\mathfrak{B}_h},\ y \in \mathfrak{B}''_h.$$

For a function $y \in \mathfrak{B}''_h$ and $i \in \{0, 1, ..., N\}$, we introduce the function $\bar{y}_i \in C^1([t_i, t_{i+1}]; X)$ given by

$$\bar{y}_i(t) = \begin{cases} y'(t), \text{ for } t \in (t_i, t_{i+1}], \\ y'(t_i^+), \text{ for } t = t_i. \end{cases}$$

If function $y : (-\infty, T] \to X$ such that $y \in \mathfrak{B}''_h$ then for all $t \in [0, T]$, the following conditions hold:

(C_1) $y_t \in \mathfrak{B}_h$.

(C_2) $\|y(t)\|_X \le H\|y_t\|_{\mathfrak{B}_h}$.

(C_3) $\|y_t\|_{\mathfrak{B}_h} \le K(t)\sup\{\|y(s)\|_X : 0 \le s \le t\} + M(t)\|\phi\|_{\mathfrak{B}_h}$, where $H > 0$ is constant; $K, M : [0,\infty) \to [0,\infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and K, M are independent of $y(t)$.

(C_{4_φ}) The function t$\to \phi_t$ is well defined and continuous from the set

$$\mathfrak{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in [0, T] \times \mathfrak{B}_h\},$$

into $\mathfrak{B}_h$ and there exists a continuous and bounded function $J^\phi : \mathfrak{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_{\mathfrak{B}_h} \le J^\phi(t)\|\phi\|_{\mathfrak{B}_h}$ for every $t \in \mathfrak{R}(\rho^-)$.

Lemma 2.1 ([5], lemma 3.6).
Let $y : (-\infty, T] \to X$ be a function such that $y \in \mathfrak{B}''_h$ with $y_0 = \phi$, $y|_{J_k} \in C^1(J_k, X)$ and if (C_{4_φ}) hold, then

$$\|y_s\|_{\mathfrak{B}_h} \le (M_b + J^\phi)\|\phi\|_{\mathfrak{B}_h} + K_b \sup\{\|y(\theta)\|_X;\ \theta \in [0, \max\{0, s\}]\},\ s \in \mathfrak{R}(\rho^-) \cup J,$$

where

$$J^\phi = \sup_{t\in\mathfrak{R}(\rho^-)} J^\phi(t),\ M_b = \sup_{s\in[0,T]} M(s) \text{ and } K_b = \sup_{s\in[0,T]} K(s).$$

Definition 2.1.
Caputo's derivative of order $\alpha > 0$ for a function $f : [a, \infty) \to \mathbb{R}$ is defined as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = {}_a J_t^{n-\alpha} f^{(n)}(t),$$

where $a \ge 0,\ n \in N$. It is clear that derivative of constant function is zero.

Definition 2.2.
The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a function $f \in L^1(\mathbb{R}^+, X)$ is defined by

$$ {}_aJ_t^0 f(t) = f(t),\ {}_aJ_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,\ t > 0, $$

where $a \geq 0,\ n \in N$ and $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 2.2 ([5], lemma 3.3).
For $\alpha > 0$, solution of fractional differential equations with lower limit not zero ${}_aJ_t^\alpha D_t^\alpha y(t) = y(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3 + \cdots + c_{n-1}(t-a)^{n-1}$ where $c_i \in \mathbb{R},\ i = 0, 1, \cdots, n-1,\ n = [\alpha]+1$ and $[\alpha]$ represent the integral part of the real number α.

Our following result is based definition 2.1 in [4].

Lemma 2.3.
A function $y : (-\infty, T] \to X$ s.t. $y \in \mathfrak{B}_h''$ is called a solution of the problem (1)-(3) if $y(0) = \phi(0), y'(0) = \varphi(0),\ y(t) = g_j(t, y(t)), y'(t) = q_j(t, y(t))$ for $t \in (t_j, s_j],\ j = 1, 2, \cdots, N$, and satisfying the following integral equation

$$ y(t) = \begin{cases} \phi(0) - u(y) + (\varphi(0) - v(y))t + \int_0^t (t-s) f(s, y_{\rho(s,y_s)}, B y_{\rho(s,y_s)}) ds, & t \in [0, t_1], \\ g_i(s_i, y(s_i)) + q_i(s_i, y(s_i))t + \int_{s_i}^t (t-s) f(s, y_{\rho(s,y_s)}, B y_{\rho(s,y_s)}) ds, & t \in [s_i, t_{i+1}], \end{cases} $$

for every $i = 1, 2, \cdots, N$.

3. Main results

To prove our results we shall assume the function $\rho : [0, T] \times \mathfrak{B}_h \to (-\infty, T]$ is continuous and $\phi, \varphi \in \mathfrak{B}_h$. If $y \in \mathfrak{B}_h$ we defined $\bar{y} : (-\infty, T) \to X$ as the extension of y to $(-\infty, T]$ such that $y\bar{(}t) = \phi$. We defined $\tilde{y} : (-\infty, T) \to X$ such that $\tilde{y} = y + x$ where $x : (-\infty, T) \to X$ is the extension of $\phi \in \mathfrak{B}_h$ such that $x(t) = \phi(0)$ for $t \in [0, T]$. In additional if $y' \in \mathfrak{B}_h$ we defined $\bar{y}' : (-\infty, T) \to X$ as the extension of y' to $(-\infty, T]$ such that $y'\bar{(}t) = \varphi$. We defined $\tilde{y}' : (-\infty, T) \to X$ such that $\tilde{y}' = y' + x'$ where $x' : (-\infty, T) \to X$ is the extension of $\varphi \in \mathfrak{B}_h$ such that $x'(t) = \varphi(0)$ for $t \in [0, T]$. Now we introduce the following assumption.

(H_1) $f : J \times \mathfrak{B}_h \times \mathfrak{B}_h \to X$ is jointly continuous function and there exist positive constants L_{f1}, L_{f1} such that

$$ \|f(t, \psi, \varphi) - f(t, \xi, \chi)\|_X \leq L_{f1} \|\psi - \xi\|_{\mathfrak{B}_h} + L_{f2} \|\varphi - \chi\|_{\mathfrak{B}_h},\ \forall\ \psi, \varphi, \chi, \xi \in \mathfrak{B}_h. $$

(H_2) f is continuous and there exist positive constants M_1, M_2 such that

$$ \|f(t, \psi, \varphi)\|_X \leq M_1 \|\psi\|_{\mathfrak{B}_h} + M_2 \|\varphi\|_{\mathfrak{B}_h}, \forall\ \psi, \varphi \in \mathfrak{B}_h. $$

(H_3) The functions u, v are continuous and there are positive constants L_u, L_v such that

$$ \|u(x) - u(y)\|_X \leq L_u \|x - y\|_X;\ \|v(x) - v(y)\|_X \leq L_v \|x - y\|_X,\ \forall\ x, y \in X. $$

(H_4) The functions u, v are continuous and there are positive constants M_3, M_4 such that

$$\|u(y)\|_X \le M_3\|y\|_X;\ \|v(y)\|_X \le M_4\|y\|_X,\ \forall\ x, y \in X.$$

(H_5) The functions g_i, q_i are continuous and there are positive constants L_{g_i}, L_{q_i} such that

$$\|g_i(t,x) - g_i(t,y)\|_X \le L_{g_i}\|x-y\|_X;\ \|q_i(t,x) - q_i(t,y)\|_X \le L_{q_i}\|x-y\|_X,$$

for all $x, y \in X, t \in (t_i, s_i]$ and each $i = 1, 2, \cdots, N$.

(H_6) The functions g_i, q_i are continuous and there are positive constants M_5, M_6 such that

$$\|g_i(t,y)\|_X \le M_5\|y\|_X;\ \|q_i(t,y)\|_X \le M_6\|y\|_X,$$

for all $x, y \in X, t \in (t_i, s_i]$ and each $i = 1, 2, \cdots, N$.

Theorem 3.1.
Assume the condition $(H_1), (H_3)$ and (H_5) are satisfied and constant

$$\Delta = \max\{(L_u + TL_v + K_b\frac{T^2}{2}(L_{f_1} + L_{f_2}B^*)), L_{g_i} + TL_{q_i} + K_b\frac{T^2}{2}(L_{f1} + L_{f2}B^*)\} < 1,$$

for $i = 1, \cdots, N$. Then there exists a unique solution $y(t)$ of the problem (1)-(3) on J.

Proof. Let $\bar{\phi}, \bar{\varphi} : (-\infty, T) \to X$ be the extension of ϕ, φ to $(-\infty, T]$ such that $\bar{\phi(t)} = \phi(0), \bar{\varphi(0)} = \varphi(0)$ on J. Consider the space $\mathfrak{B}_h''' = \{y \in \mathfrak{B}_h'' : y(0) = \phi(0), y'(0) = \varphi(0)\}$ and $y(t) = \phi(t), y'(t) = \varphi(t)$ for $t \in (-\infty, 0]$ endowed with the uniform convergence topology. Let us consider an operator $P : \mathfrak{B}_h''' \to \mathfrak{B}_h'''$ defined as $Py(t) = g_i(t, \bar{y}(t))$ for $t \in (t_i, s_i]$ and

$$Py(t) = \begin{cases} \phi(0) - u(\bar{y}) + (\varphi(0) - v(\bar{y}))t + \int_0^t (t-s) f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})ds, & t \in [0, t_1], \\ g_i(s_i, \bar{y}(s_i)) + q_i(s_i, \bar{y}(s_i))t + \int_{s_i}^t (t-s) f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})ds, & t \in [s_i, t_{i+1}], \end{cases} \tag{4}$$

where $\bar{y} : (-\infty, T] \to X$ is such that $\bar{y(0)} = \phi, \bar{y'(0)} = \varphi$ and $\bar{y} = y$ on J. It is obvious that P is well defined. Now, we show that the operator P has a fixed point. Let $y(t), y^*(t) \in \mathfrak{B}_h'''$ and $t \in [0, t_1]$, we have

$$\begin{aligned} \|Py - Py^*\|_X \le\ & \|u(\bar{y}) - u(\bar{y^*})\|_X + T\|v(\bar{y}) - v(\bar{y^*})\|_X \\ & + \int_0^t (t-s)\|f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}) - f(s, \bar{y}^*_{\rho(s,\bar{y}^*_s)}, B\bar{y}^*_{\rho(s,\bar{y}^*_s)})\|_X ds \\ & \le (L_u + TL_v + K_b\frac{T^2}{2}(L_{f1} + L_{f2}B^*))\|y - y^*\|_{\mathfrak{B}_h'''}. \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|Py - Py^*\|_X &\leq \|g_i(s_i, \bar{y}(s_i)) - g_i(s_i, \bar{y}^*(s_i))\|_X + \|q_i(s_i, \bar{y}(s_i)) - q_i(s_i, \bar{y}^*(s_i))\|_X T \\ &\quad + \int_{s_i}^{t} (t-s)\|f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}) - f(s, \bar{y}^*_{\rho(s,\bar{y}^*_s)}, B\bar{y}^*_{\rho(s,\bar{y}^*_s)})\|_X ds \\ &\leq (L_{g_i} + TL_{q_i} + K_b \frac{T^2}{2}(L_{f1} + L_{f2}B^*))\|y - y^*\|_{\mathfrak{B}'''_h}. \end{aligned}$$

For $t \in (t_j, s_j]$, we get

$$\|Py - Py^*\|_X \leq L_{g_j}\|y - y^*\|_{\mathfrak{B}'''_h}, \qquad j = 1, 2, \cdots, N.$$

Gathering above results, we obtain

$$\|Py - Py^*\|_X \leq \Delta\|y - y^*\|_{\mathfrak{B}'''_h}.$$

Since $\Delta < 1$, which implies that P is a contraction map and there exists a unique fixed point which is the solution of system (1)-(3) on J. This completes the proof of the theorem. □

Theorem 3.2.
Let the assumptions $(H_2), (H_4)$ and (H_6) are satisfied. Then the problem (1)-(3) has at least one solution $y(t)$ on J.

Proof. Consider the operator $P : \mathfrak{B}'''_h \to \mathfrak{B}'''_h$, defined by (4) in theorem 3.1. We shall show P has a fixed point in $\mathfrak{B}'''_h$. First, we shall show that P is continuous, so we consider a sequence $y^n \to y$ in $\mathfrak{B}'''_h$, then for $[0, t_1]$

$$\begin{aligned} \|P(y^n) - P(y)\|_X &\leq \|u(\bar{y^n}) - u(\bar{y})\|_X + T\|v(\bar{y^n}) - v(\bar{y})\|_X \\ &\quad + \int_0^t (t-s)\|f(s, \bar{y^n}_{\rho(s,\bar{y^n}_s)}, B\bar{y^n}_{\rho(s,\bar{y^n}_s)}) - f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})\|_X ds. \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|P(y^n) - P(y)\|_X &\leq \|g_i(s_i, \bar{y^n}(s_i)) - g_i(s_i, \bar{y}(s_i))\|_X \\ &\quad + T\|q_i(s_i, \bar{y^n}(s_i)) - q_i(s_i, \bar{y}(s_i))\|_X \\ &\quad + \int_{s_i}^t (t-s)\|f(s, \bar{y^n}_{\rho(s,\bar{y^n}_s)}, B\bar{y^n}_{\rho(s,\bar{y^n}_s)}) - f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})\|_X ds. \end{aligned}$$

Since f, u, v, g_i and q_i are continuous functions, then we have

$$\|P(y^n) - P(y)\|_X \to 0 \text{ as } n \to \infty,$$

which show that P is continuous.

Let $\mathscr{B}_r = \{y \in \mathfrak{B}_h''' : \|y\|_X \leq r\}$ be a closed bounded and convex subset of $\mathfrak{B}_h'''$. Now, it is easy to prove that P maps bounded set into bounded set in $\mathscr{B}_r$. To do this we have for $[0, t_1]$

$$\begin{aligned}\|P(y)(t)\|_X &\leq \|\phi(0)\| + \|u(\bar{y})\| + T(\|\varphi(0)\| + \|v(\bar{y})\|) + \int_0^t (t-s) f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}) ds \\ &\leq \|\phi(0)\| + L_u r + T(\|\varphi(0)\| + L_v r) + \frac{T^2}{2}(M_1 + M_2 B^*) r^*.\end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned}\|P(y)(t)\|_X &\leq \|g_i(s_i, \bar{y}(s_i))\|_X + T\|q_i(s_i, \bar{y}(s_i))\|_X + \int_{s_i}^t (t-s)\|f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})\|_X ds \\ &\leq M_5 r + T M_6 r + \frac{T^2}{2}(M_1 + M_2 B^*) r^*,\end{aligned}$$

where $r^* = (M_b + J^\phi)\|\phi\|_{\mathfrak{B}_h} + K_b r$. Which implies that P maps bounded set into bounded set in B_r.

Next, we shall show that P maps bounded sets into equi-continuous sets in $\mathscr{B}_r$. Let $l_1, l_2 \in [0, t_1]$ with $l_1 < l_2$, we have

$$\begin{aligned}\|(Py)(l_2) - (Py)(l_1)\|_X &\leq (l_2 - l_1)(\|\varphi(0)\| + \|v(\bar{y})\|) \\ &\quad + \int_0^{l_1} (l_2 - l_1)\|f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})\|_X ds \\ &\quad + \int_{l_1}^{l_2} (l_2 - s)\|f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})\|_X ds \\ &\leq (l_2 - l_1)(\|\varphi(0)\| + M_4 r) + (l_2 - l_1) T(M_1 + M_2 B^*) r^* \\ &\quad + \frac{(l_2 - l_1)^2}{2}(M_1 + M_2 B^*) r^*.\end{aligned}$$

Let $l_1, l_2 \in (s_i, t_{k+1}]$ with $l_1 < l_2$, $k = 1, 2, \cdots, m$, then we have

$$\begin{aligned}\|(Py)(l_2) - (Py)(l_1)\|_X &\leq (l_2 - l_1)\|q_i(s_i, \bar{y}(s_i))\|_X \\ &\quad \int_{s_i}^{l_1} (l_2 - l_1)\|f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})\|_X ds \\ &\quad + \int_{l_1}^{l_2} (l_2 - s)\|f(s, \bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)})\|_X ds \\ &\leq (l_2 - l_1) M_6 r + (l_2 - l_1) T(M_1 + M_2 B^*) r^* \\ &\quad + \frac{(l_2 - l_1)^2}{2}(M_1 + M_2 B^*) r^*,\end{aligned}$$

as $l_2 \to l_1$, then $\|P(y)(l_2) - P(y)(l_1)\|_X \to 0$. This implies that P is equi-continuous on all $t \in J$ in $\mathscr{B}_r$. Thus, by Arzela-Ascoli Theorem, it follows that P is completely continuous. Therefore, by Schauder fixed point theorem, the operator P has a fixed point, which in turn implies that problem (1)-(3) has at least one solution on J. This is complete the proof of theorem. □

4. Example

Consider the following nonlinear impulsive fractional functional initial value problem

$$D_t^{\frac{3}{2}} y(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \Big[\int_{-\infty}^{s} e^{2(\nu-s)} \frac{y(\nu - \sigma(\|y\|))}{24} d\nu$$

$$+ \int_0^{\xi} \cos(\gamma-\xi) \frac{y(\gamma-\sigma(\|y\|))}{25} d\gamma \Big] ds,\ (t,y) \in \cup_{i=1}^{N} [s_i, t_{i+1}] \times [0,\pi], \tag{5}$$

$$y(t) + \sum_{k=1}^{r} c_k y(s_k) = \phi(t),\ t \in (-\infty, 0],\ y \in [0,\pi], \tag{6}$$

$$y'(t) + \sum_{k=1}^{r} d_k y(s_k) = \psi(t),\ t \in (-\infty, 0],\ y \in [0,\pi], \tag{7}$$

$$y(t) = G_i(t,y);\ y'(t) = H_i(t,y),\ t \in (t_i, s_i]. \tag{8}$$

For the phase space $\mathfrak{B}_h$, let $h(s) = e^{2s},\ s<0$ then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$ and define

$$\|\phi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} \|\phi(\theta)\|_{L^2} ds.$$

Hence for $(t,\phi) \in [0,1] \times \mathfrak{B}_h$,

Let $y : (-\infty, T] \to L^2[0,\pi]$ such that $y \in \mathfrak{B}_h$. Setting

$$\rho(t,\phi) = t - \sigma(\|\phi(0)\|),\ (t,\phi) \in J \times \mathfrak{B}_h,$$

we have

$$f(t,\phi,B\phi) = \int_{-\infty}^{0} e^{2(\nu)} [\frac{\phi}{24} + \int_0^{\xi} \cos(\xi-\gamma) \frac{\phi}{25} d\gamma] d\nu,$$

$$u(y) = \sum_{k=1}^{r} c_k y(s_k);\ v(y) = \sum_{k=1}^{r} d_k y(s_k),$$

$$g_i(t,y) = G_i(t,y);\ q_i(t,y) = H_i(t,y),$$

then the above equations (5)-(8) can be written in the abstract form as (1)-(3). Further more, we can see that for $(t,\phi,\xi), (t,\psi,\nu) \in J \times \mathfrak{B}_h \times \mathfrak{B}_h$, we get

$$\begin{aligned}
&\|f(t,\phi,B\phi) - f(t,\psi,B\psi)\|_{L^2} \\
&= \left[\int_0^{\pi} \left\{ \int_{-\infty}^{0} e^{2(s)} \|\frac{\phi}{24} - \frac{\psi}{24}\| ds + \int_{-\infty}^{0} e^{2(s)} \int_0^{\xi} \|\cos(\gamma-\xi)\| \frac{\phi}{25} - \frac{\psi}{25} \| d\gamma ds \right\}^2 dy \right]^{1/2} \\
&\le \left[\int_0^{\pi} \left\{ \int_{-\infty}^{0} e^{2(s)} \|\frac{\phi}{24} - \frac{\psi}{24}\| ds + \int_{-\infty}^{0} e^{2(s)} \|\frac{\phi}{25} - \frac{\psi}{25}\| ds \right\}^2 dy \right]^{1/2} \\
&\le \left[\int_0^{\pi} \left\{ \frac{1}{24} \int_{-\infty}^{0} e^{2(s)} \sup \|\phi - \psi\| ds + \frac{1}{25} \int_{-\infty}^{0} e^{2(s)} \sup \|\phi - \psi\| ds \right\}^2 dy \right]^{1/2} \\
&\le \frac{\sqrt{\pi}}{24} \|\phi - \psi\| + \frac{\sqrt{\pi}}{25} \|\phi - \psi\|.
\end{aligned}$$

Hence function f satisfies (H_1). Similarly we can show that the functions g_i, q_i, u, v satisfy $(H_3),(H_5)$. All the condition of theorem 3.1 have fulfilled so we deduced that the system (5)-(6) has a unique solution on $[0,1]$.

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