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Similarity analysis for unsteady natural convective boundary layer flow of Sisko fluid

Research Article

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Abstract: Deductive group symmetry approach is applied to the system of equations governing the unsteady natural convection boundary layer flow of the non-Newtonian Sisko fluid past non-isothermal vertical flat plate. The application of a two-parameter group reduces the number of independent variables by two, and consequently the system of governing non-linear partial differential equations with boundary conditions reduces to a system of non-linear ordinary differential equations with appropriate boundary conditions. The possible form of potential velocity for the present flow situation is derived systematically from similarity requirement.

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Keywords: Natural convection • Non-Newtonian Sisko fluid • Group theory • Similarity solution.

1. Introduction

Deductive group transformation analysis, also called symmetry analysis, is based on general group of transformation that was developed by Sophus Lie to find point transformations that map a given differential equation to itself. This method unifies almost all known exact integration techniques for both ordinary and partial differential equations [1]. Group analysis is the only rigorous mathematical method to find all symmetries of a given differential equation and no adhoc assumptions or a prior knowledge of the equation under investigation is needed. The boundary layer equations are especially interesting from a physical point of view because they have the capacity to admit a large number of invariant solutions, i.e., basically analytic solutions. In the present context, invariant solutions are meant to be a reduction to a simpler equation such as an ordinary differential equation. Prandtl's boundary layer equations admit more and different symmetry groups. Symmetry groups or simply symmetries are invariant transformations, which do not

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alter the structural form of the equation under investigation [2]. Newton's law of viscosity states that shear stress is proportional to velocity gradient. Fluids that obey this law are known as Newtonian fluids. Amongst Newtonian fluids, we can cite water, benzene, ethyl alcohol, hexane and most solutions of simple molecules. Numerous fluids violate NewtonâĂŹs law of viscosity. On the other hand, fluids that do not obey Newton's law are known as Non-Newtonian fluids.

The term non-Newtonian fluid is one of very great generality and includes all fluids for which the equations of classical hydrodynamics do not apply. As a consequence a chief difficulty in any theoretical analysis of the motion of such fluids has always been the lack of any generally acceptable equation of state between the stress tensor and the state of flow of the system. It is fortunate however that in a rather sizeable class of non-Newtonian fluids the stress-strain-velocity relations does not involve time derivatives of the stress- or the strain-velocity components and may therefore be represented under isotropic conditions by Reiner [3], Rivlin [4] invariant expressions. It was explained in the earlier paper Acrivos et al.[5] furthermore that for one-dimensional or boundary-layer types of flow these fluids may often be characterized with satisfactory accuracy by the empirical relation

$$\tau = \left(a + b\left| \frac{\partial u}{\partial y} \right|^{n-1} \right) \frac{\partial u}{\partial y} \tag{1}$$

Where a, b and n are empirical constants characteristic of the fluid, and τ and $\frac{\partial u}{\partial y}$ are the only components of the stress tensor and the deformation tensor which need to be considered under such flow conditions. Thus, although strictly speaking 1 should be replaced by the more general form of the power law which has been proposed by Mooney and Black [6] and others, it can be shown rigorously that for boundary-layer flows under either forced or free convection the term $\frac{\partial u}{\partial y}$ is so much larger than all the other elements in the deformation tensor that the one-dimensional power law shown above is in general quite adequate. The present paper will therefore be restricted to those systems which satisfy 1.

The mathematical technique used in the present paper is the two-parameter group transformation, which leads to the reduction of number of independent variables from the system of partial differential equations. A systematic formalism is presented for such a reduction of the number of independent variables in systems which consist, in general, of a set of partial differential equations and auxiliary conditions (such as boundary and/or initial conditions). In engineering, such procedures are customarily termed similarity analysis. Morgan [7] presented a theory which led to improvements over earlier similarity methods and Michal [8] extended MorganâĂŹs theory. Group methods, as a class of methods which lead to a reduction of the number of independent variables, were first introduced by Birkhoff [9, 10] The further formalism with a significant simplification of general group theory techniques developed by Moran and Gaggioli [11], which is based upon elementary group theory and earlier methods due to Birkhoff [10], Moran and Gaggioli [12] presented general systematic group formalism for similarity analysis. Similarity analysis has been applied intensively by Gabbert [13]. Abd-el-Malek and Badran [14] studied fluid flow and heat transfer characteristics for steady and unsteady, respectively by laminar free convection on a vertical circular cylinder via group method analysis. Abd-el-Malek and Badran [14] applied the group method for the analysis of unsteady free convective flow on a flat plate for the case where the velocity of flow next to the wall equal to zero.

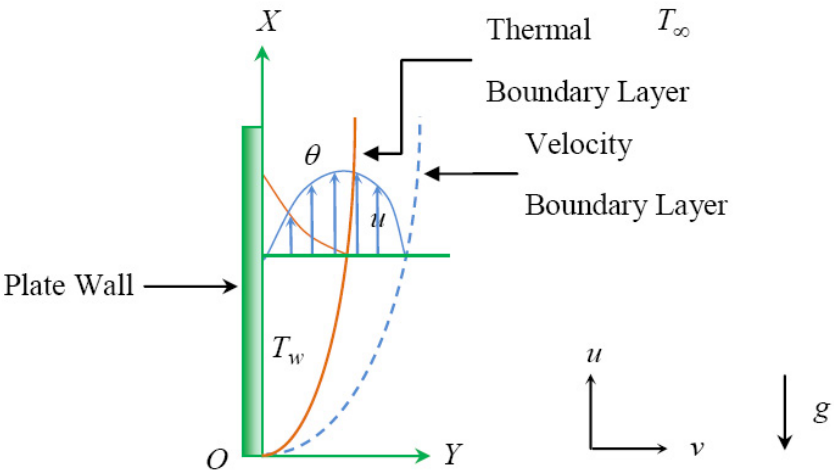

Figure 1. Schematic diagram of the problem

In the present work, we provide particular analytical solution in terms of special function for the unsteady free-convection flow non-Newtonian Power-law fluids over a continuous moving vertical plate using group methods. Under the application of two parameter group, the governing partial differential equations and boundary conditions are reduced to ordinary differential equations with the appropriate boundary conditions.

2. Governing equations

Consider an unsteady, two-dimensional, laminar, natural convection boundary layer flow of Sisko fluid past semi-infinite vertical plate We use Cartesian coordinate system *XOY*; the governing equations can be approximated by the dimensionless in the form:

Continuity Equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2}$$

Momentum Equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=T+\frac{\partial}{\partial y}\left\{\left(a+b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right)\frac{\partial u}{\partial y}\right\} \tag{3}$$

Energy Equation:

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2} \tag{4}$$

With the auxiliary boundary conditions,

$$\left.\begin{array}{l} u=0,\ v=0,\ T=T_w(x,t)\ at\ y=0 \\ u=0, T=0\ as\ y\to\infty \end{array}\right\} \tag{5}$$

Where, $x=\frac{x^*}{L}, y=\frac{y^*(G_r)^{\frac{1}{4}}}{L}, u=\frac{u^*}{U}, v=\frac{v^*(G_r)^{\frac{1}{4}}}{U}$ and $G_r=g\beta L^3\left(T^*-T^*_\infty\right)$ is the Grashof number, ν is the kinematic velocity, $P_r=\frac{\nu}{\alpha}$ is the Prandtl number, L is reference length and α is the thermal diffusivity.

From the continuity equation (2) there exist a non-dimensional stream function $\psi\left(x,y,t\right)$ such that $u=\frac{\partial\psi}{\partial y}$ and $v=-\frac{\partial\psi}{\partial x}$ which satisfies (2) identically.
If we introduce the non-dimensional temperature defines by $\theta={}^{T}/_{T_w}$ then equation (3) and (4) become

$$\frac{\partial^2\psi}{\partial y\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\theta T_w+\frac{\partial}{\partial y}\left[\left(a+b\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\right)\frac{\partial^2\psi}{\partial y^2}\right] \tag{6}$$

$$\left(T_w\frac{\partial\theta}{\partial t}+\theta\frac{\partial T_w}{\partial t}\right)+\frac{\partial\psi}{\partial y}\left(T_w\frac{\partial\theta}{\partial x}+\theta\frac{\partial T_w}{\partial x}\right)-T_w\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=\frac{1}{P_r}T_w\frac{\partial^2\theta}{\partial y^2} \tag{7}$$

With the boundary conditions,

$$\left.\begin{array}{ll} \frac{\partial\psi}{\partial y}(x,0,t)=\frac{\partial\psi}{\partial x}(x,0,t)=0, & \theta(x,0,t)=1 \\ \lim\limits_{y\to\infty}\frac{\partial\psi}{\partial y}\left(x,y,t\right)=0, & \lim\limits_{y\to\infty}\theta\left(x,y,t\right)=0 \end{array}\right\} \tag{8}$$

Here it is interesting to observe that for *a=0* and *b=1* will reduce to that of power law fluid equation. In present work we discuss the reduced power law fluid equations of Sisko fluid.

3. Solution of the Problem

The problem is solved by applying a two parameter group transformation to the partial differential equations (6 - 8). This transformation reduces the three independent variables $\left(x,y,t\right)$ to one similarity variable η and the governing equations (6 - 8) are transformed to a system of ordinary differential equations in terms of this similarity variable η.

3.1. The group systematic formulation

The procedure is initiated with the group C_G, a class of transformation of two-parameters (a_1,a_2) of the form:

$$C_G:\bar{Q}=c^Q(a_1,a_2)Q+k^Q(a_1,a_2),\ Q=x,y,t,\psi,\theta,T_w \tag{9}$$

Where *c's* and k's are real-valued and at least differentiable in each real argument (a_1,a_2)

3.2. The Invariance Analysis

To transform the differential equation, transformations of the derivatives of ψ are obtained from C_G via chain-rule operations:

$$\bar{s}_{\bar{i}} = \left(c^s / c^i\right) s_i;\ \bar{s}_{\bar{i}\,\bar{j}} = \left(c^s / c^i c^j\right) s_{ij};\ \bar{s}_{\bar{i}\,\bar{j}\,\bar{k}} = \left(c^s / c^i c^j c^k\right) s_{ijk} \tag{10}$$

Where $s = \psi, \theta, T_w;\quad i, j, k = x, y, t$

Equation (6) is said to be invariantly transformed, whenever,

$$\begin{aligned}&\frac{\partial^2\bar{\psi}}{\partial\bar{y}\partial\bar{t}} + \frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}\partial\bar{x}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2} - \bar{\theta}\,\bar{T}_{\bar{w}} - \frac{\partial\bar{\psi}}{\partial\bar{y}}\left\{\left|\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right|^{n-1}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right\}\\ &= H_1(a_1, a_2)\left[\frac{\partial^2\psi}{\partial y\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \theta\,T_w - \frac{\partial\psi}{\partial y}\left\{\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\frac{\partial^2\psi}{\partial y^2}\right\}\right]\end{aligned} \tag{11}$$

for some function $H_1(a_1, a_2)$ which may be constant.

Substituting the values from (9) and (10) in (11)

$$\begin{aligned}&\left[\frac{c^\psi}{c^y c^t}\right]\frac{\partial^2\psi}{\partial y\partial t} + \left[\frac{(c^\psi)^2}{(c^y)^2 c^x}\right]\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - \left[\frac{(c^\psi)^2}{(c^y)^2 c^x}\right]\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\\ &- c^\theta c^{T_w}\,T\theta_w - \left[\frac{(c^\psi)^n}{(c^y)^{2n+1}}\right]\left\{n\left(\frac{\partial^2\psi}{\partial y^2}\right)^{n-1}\frac{\partial^3\psi}{\partial y^3}\right\} + R_1\\ &= H_1(a_1, a_2)\left\{\frac{\partial^2\psi}{\partial y\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \theta\,T_w - \frac{\partial\psi}{\partial y}\left\{\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\frac{\partial^2\psi}{\partial y^2}\right\}\right\}\end{aligned} \tag{12}$$

Where $R_1 = c^\theta k^{T_w}\theta + c^{T_w} k^\theta T_w + k^{T_w} k^\theta$ and for the invariance of (12), $R_1 = 0$

This is satisfied by putting

$$k^{T_w} = k^\theta = 0 \tag{13}$$

And

$$\left[\frac{c^\psi}{c^y c^t}\right] = \left[\frac{(c^\psi)^2}{(c^y)^2 c^x}\right] = c^\theta c^{T_w} = \left[\frac{(c^\psi)^n}{(c^y)^{2n+1}}\right] = H_1(a_1, a_2) \tag{14}$$

Further (7) is said to be invariantly transformed, whenever,

$$\begin{aligned}&\left(\bar{T}_{\bar{w}}\frac{\partial\bar{\theta}}{\partial\bar{t}} + \bar{\theta}\frac{\partial\bar{T}_{\bar{w}}}{\partial\bar{t}}\right) + \frac{\partial\bar{\psi}}{\partial\bar{y}}\left(\bar{T}_{\bar{w}}\frac{\partial\bar{\theta}}{\partial\bar{x}} + \bar{\theta}\frac{\partial\bar{T}_{\bar{w}}}{\partial\bar{x}}\right) - \bar{T}_{\bar{w}}\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial\bar{\theta}}{\partial\bar{y}} - \frac{1}{P_r}\bar{T}_{\bar{w}}\frac{\partial^2\bar{\theta}}{\partial\bar{y}^2}\\ &= H_2(a_1, a_2)\left[\left(T_w\frac{\partial\theta}{\partial t} + \theta\frac{\partial T_w}{\partial t}\right) + \frac{\partial\psi}{\partial y}\left(T_w\frac{\partial\theta}{\partial x} + \theta\frac{\partial T_w}{\partial x}\right) - T_w\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} - \frac{1}{P_r}T_w\frac{\partial^2\theta}{\partial y^2}\right]\end{aligned} \tag{15}$$

for some function $H_2(a_1, a_2)$ which may be constant.

Substituting the values from (9) and (10) in (15)

$$\begin{aligned}&\left\{\left(c^{T_w}T_w + k^{T_w}\right)\frac{c^\theta}{c^t}\frac{\partial\theta}{\partial t} + \left(c^\theta\theta + k^\theta\right)\frac{c^{T_w}}{c^t}\frac{\partial T_w}{\partial t}\right\} + \frac{c^\psi}{c^y}\frac{\partial\psi}{\partial y}\left\{\left(c^{T_w}T_w + k^{T_w}\right)\frac{c^\theta}{c^x}\frac{\partial\theta}{\partial x} + \left(c^\theta\theta + k^\theta\right)\frac{c^{T_w}}{c^x}\frac{\partial T_w}{\partial x}\right\}\\ &\left(c^{T_w}T_w + k^{T_w}\right)\frac{c^\psi}{c^x}\cdot\frac{c^\theta}{c^y}\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} - \frac{1}{P_r}\left(c^{T_w}T_w + k^{T_w}\right)\frac{c^\theta}{(c^y)^2}\frac{\partial^2\theta}{\partial y^2}\\ &= H_2(a_1, a_2)\left[\left(T_w\frac{\partial\theta}{\partial t} + \theta\frac{\partial T_w}{\partial t}\right) + \frac{\partial\psi}{\partial y}\left(T_w\frac{\partial\theta}{\partial x} + \theta\frac{\partial T_w}{\partial x}\right) - T_w\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} - \frac{1}{P_r}T_w\frac{\partial^2\theta}{\partial y^2}\right]\end{aligned}$$

$$\left.\begin{array}{l}\left[\frac{c^{\theta}c^{T_w}}{c^t}\right]\left(T_w\frac{\partial\theta}{\partial t}+\theta\frac{\partial T_w}{\partial t}\right)+\left[\frac{c^{\psi}c^{\theta}c^{T_w}}{c^x c^y}\right]\left(T_w\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}+\theta\frac{\partial\psi}{\partial y}\frac{\partial T_w}{\partial x}-T_w\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right)\\ -\frac{1}{P_r}\left[\frac{c^{T_w}c^{\theta}}{(c^y)^2}\right]T_w\frac{\partial^2\theta}{\partial y^2}+R_2\\ =H_2(a_1,a_2)\left[\left(T_w\frac{\partial\theta}{\partial t}+\theta\frac{\partial T_w}{\partial t}\right)+\left(T_w\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}+\theta\frac{\partial\psi}{\partial y}\frac{\partial T_w}{\partial x}-T_w\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right)-\frac{1}{P_r}T_w\frac{\partial^2\theta}{\partial y^2}\right]\end{array}\right\} \tag{16}$$

Where

$$\begin{array}{l}R_1=\frac{k^{T_w}c^{\theta}}{c^t}\frac{\partial\theta}{\partial t}+\frac{k^{\theta}c^{T_w}}{c^t}\frac{\partial T_w}{\partial t}+\frac{c^{\psi}c^{\theta}k^{T_w}}{c^x c^y}\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}+\frac{c^{\psi}k^{\theta}c^{T_w}}{c^x c^y}\frac{\partial\psi}{\partial y}\frac{\partial T_w}{\partial x}\\ \quad -\frac{c^{\psi}c^{\theta}k^{T_w}}{c^x c^y}\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{1}{P_r}\frac{c^{\theta}k^{T_w}}{(c^y)^2}\frac{\partial^2\theta}{\partial y^2}\end{array}$$

and for the invariance of (16), $R_2=0$, this is evidently implies from (13) and

$$\left[\frac{c^{\theta}c^{T_w}}{c^t}\right]=\left[\frac{c^{\psi}c^{\theta}c^{T_w}}{c^x c^y}\right]=\left[\frac{c^{T_w}c^{\theta}}{(c^y)^2}\right]=H_2(a_1,a_2) \tag{17}$$

Moreover the boundary conditions (8) are also invariant in form, whenever

$$k^y=0; c^{\theta}=1 \tag{18}$$

and

$$\theta(x,0,t)=1 \quad\Rightarrow\quad \bar{\theta}(\bar{x},0,\bar{t})=1$$

Equation (14) and (17) yields

$$\left[\frac{c^{\psi}}{c^y c^t}\right]=\left[\frac{(c^{\psi})^2}{(c^y)^2 c^x}\right]=c^{T_w}=\left[\frac{(c^{\psi})^n}{(c^y)^{2n+1}}\right]=H_1(a_1,a_2) \qquad \left[\frac{c^{T_w}}{c^t}\right]=\left[\frac{c^{\psi}c^{T_w}}{c^x c^y}\right]=\left[\frac{c^{T_w}}{(c^y)^2}\right]=H_2(a_1,a_2)$$

In the virtue of above equations, we must have

$$c^{\psi}=(c^y)^2, \quad c^x=(c^y)^3, \quad c^t=(c^y)^2, \quad c^{T_w}=(c^y)^{-1}$$

Finally, we get two parameter groups which transforms invariantly the differential equations (6), (7) and the auxiliary conditions (8).
The subgroup G of the class c_G is of the form,

$$G \;:\; \left\{\begin{array}{l} G_s \;:\; \left\{\begin{array}{l}\bar{x}=(c^y)^3x+k^x\\ \bar{y}=c^y y\\ \bar{t}=(c^y)^2t+k^t\end{array}\right.\\ \bar{\psi}=(c^y)^2\psi+k^{\psi}\\ \bar{T}_{\bar{w}}=(c^y)^{-1}T_w\\ \bar{\theta}=\theta\end{array}\right.$$

3.3. The complete set of absolute invariants

If $\eta = \eta\left(x, y, t\right)$ is the absolute invariant of the independent variables then,

$$g_j\left(x, y, t, \psi, T_w, \theta\right) = F_j\left(\eta\right); \qquad j = 1,2,3 \tag{19}$$

are three absolute invariants corresponding to dependent variables ψ, T_w, θ.
The applications of basic theorem in group theory (See [15]) states that, $g\left(x, y, t, \psi, T_w, \theta\right)$ is an absolute invariant of two-parameter group G if it satisfies two partial differential equations viz,

$$\left.\begin{array}{l} \sum\limits_i (\alpha_i s_i + \alpha_{i+1}) \frac{\partial g}{\partial s_i} = 0. \\ \sum\limits_i \left(\beta_i s_i + \beta_{i+1}\right) \frac{\partial g}{\partial s_i} = 0. \end{array}\right\} \quad i = 1,3,.....11 \text{ and } s_i = x, y, t, \psi, \theta, T_w \tag{20}$$

where,

$$\alpha_1 = \left.\frac{\partial c^x}{\partial a_1}\right|_{(a_1^0, a_2^0)}, \quad \alpha_2 = \left.\frac{\partial k^x}{\partial a_1}\right|_{(a_1^0, a_2^0)}, \quad \beta_1 = \left.\frac{\partial c^x}{\partial a_2}\right|_{(a_1^0, a_2^0)}, \quad \beta_2 = \left.\frac{\partial k^x}{\partial a_2}\right|_{(a_1^0, a_2^0)} \quad etc; \tag{21}$$

and $\left(a_1^0, a_2^0\right)$ denotes the value of (a_1, a_2) which yields the identity element of the group G.

3.4. The absolute invariants of independent variables

The absolute invariant $\eta\left(x, y, t\right)$ of the independent variables $\left(x, y, t\right)$ is determined using (20) and (21) as

$$\left.\begin{array}{l} (\alpha_1 x + \alpha_2) \frac{\partial \eta}{\partial x} + \alpha_3 y \frac{\partial \eta}{\partial y} + (\alpha_5 t + \alpha_6) \frac{\partial \eta}{\partial t} = 0. \\ \left(\beta_1 x + \beta_2\right) \frac{\partial \eta}{\partial x} + \beta_3 y \frac{\partial \eta}{\partial y} + \left(\beta_5 t + \beta_6\right) \frac{\partial \eta}{\partial t} = 0. \end{array}\right\} \tag{22}$$

because $\alpha_4 = \beta_4 = 0$ as $k^y = 0$.

By definition, for each of two-parameter groups âĂŸ G_s in the class C_G there is one and only one functionally independent solution of (22), i.e. the rank of the coefficient matrix for $\left[\begin{array}{ccc} \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial t} \end{array}\right]$ is two, (the matrix has rank two whenever at least one of its two-by-two sub-matrices has a non-vanishing determinant).
According to this at least one term out of term out of

$$(\lambda_{31} x + \lambda_{32}), \ (\lambda_{35} t + \lambda_{36}) \text{ and } (\lambda_{15} x t + \lambda_{16} x + \lambda_{25} t + \lambda_{26})$$

is non zero, Where $\lambda_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$

Using the definitions of $\alpha' s$ and $\beta' s$ from (21) we have $\lambda_{31} = \lambda_{35} = \lambda_{15} = 0$.
Thus the possible three cases are as follow:
Case-I None of the coefficient vanishes, that is

$$\lambda_{32} \neq 0, \quad \lambda_{36} \neq 0, \quad \lambda_{16} x + \lambda_{25} t + \lambda_{26} \neq 0$$

In this case the system (22) becomes

$$\left.\begin{array}{l}\lambda_{32}\frac{\partial\eta}{\partial x}+\lambda_{36}\frac{\partial\eta}{\partial t}=0.\\ \lambda_{32}y\frac{\partial\eta}{\partial y}-(\lambda_{16}x+\lambda_{25}t+\lambda_{26})\frac{\partial\eta}{\partial t}=0.\end{array}\right\} \tag{23}$$

Case-II One of the coefficients vanishes.
Subcase-IIa $\lambda_{32}=0, \quad \lambda_{36}\neq 0, \quad \lambda_{16}x+\lambda_{25}t+\lambda_{26}\neq 0.$
In this case the system (22) becomes $\frac{\partial\eta}{\partial t}=0.$
This is the case requiring the steady state which may be discarded in the case of unsteady flow.

Subcase-IIb: $\lambda_{32}\neq 0, \quad \lambda_{36}=0, \quad \lambda_{16}x+\lambda_{25}t+\lambda_{26}\neq 0.$
In this case the system (22) becomes

$$\left.\begin{array}{l}\frac{\partial\eta}{\partial x}=0.\\ \lambda_{32}y\frac{\partial\eta}{\partial y}-(\lambda_{16}x+\lambda_{25}t+\lambda_{26})\frac{\partial\eta}{\partial t}=0.\end{array}\right\} \tag{24}$$

This case provides the solution which is independent of x, that is $\eta=\eta\left(y,t\right)$
Subcase-IIc $\lambda_{32}\neq 0, \quad \lambda_{36}\neq 0, \quad \lambda_{16}x+\lambda_{25}t+\lambda_{26}=0.$
In this case the system (23) becomes

$$\left.\begin{array}{l}\lambda_{32}\frac{\partial\eta}{\partial x}+\lambda_{36}\frac{\partial\eta}{\partial t}=0.\\ \lambda_{32}y\frac{\partial\eta}{\partial y}=0.\end{array}\right\} \tag{25}$$

The second equation of (25) demands $\frac{\partial\eta}{\partial y}=0$ i.e. η is independent of y which is not possible because of the auxiliary conditions.
Case-III: Any two the coefficients vanishes.

$$\begin{array}{lll}\lambda_{32}=0, & \lambda_{36}=0, & \lambda_{16}x+\lambda_{25}t+\lambda_{26}\neq 0. \quad OR\\ \lambda_{32}\neq 0, & \lambda_{36}=0, & \lambda_{16}x+\lambda_{25}t+\lambda_{26}=0. \quad OR\\ \lambda_{32}=0, & \lambda_{36}\neq 0, & \lambda_{16}x+\lambda_{25}t+\lambda_{26}=0.\end{array}$$

Observe that the first possibility yields the sub-case IIa and the later two yields sub-case IIIc. Hence this case does not allow.
Now our attention focused on those cases which are consistent with the characteristic of group and with the boundary conditions.
Case-I According to well known standard technique for linear partial differential equation the system of partial differential equation (23) has solution of the form

$$\eta=f\left(y,\xi(x,t)\right) \tag{26}$$

Where f is any arbitrary function and $\xi(x,t)$ is any function such that $\xi(x,t)=$ constant, provides a solution to

$$\frac{dx}{\lambda_{32}}=\frac{dt}{\lambda_{36}} \tag{27}$$

The solution of the equation gives

$$\xi(x,t)=\lambda_{36}x-\lambda_{32}t=\text{constant}. \tag{28}$$

Also η must satisfy second equation of (23), that is

$$y\frac{\partial f}{\partial y}-\left(\frac{\lambda_{16}x+\lambda_{25}t+\lambda_{26}}{\lambda_{32}}\cdot\frac{\partial\xi}{\partial t}\right)\frac{\partial f}{\partial\xi}=0. \tag{29}$$

Since ξ is independent of y the coefficient of $\frac{\partial f}{\partial\xi}$, i.e.

$$\left(\frac{\lambda_{16}x+\lambda_{25}t+\lambda_{26}}{\lambda_{32}}\cdot\frac{\partial\xi}{\partial t}\right) \tag{30}$$

is also independent of y. Thus f to be a function of y and ξ , it is necessary for the coefficient to depend only on ξ , that is (30) can be written as

$$y\frac{\partial f}{\partial y}-h(\xi)\frac{\partial f}{\partial\xi}=0. \tag{31}$$

where

$$h(\xi)=\left(\frac{\lambda_{16}x+\lambda_{25}t+\lambda_{26}}{\lambda_{32}}\right)\frac{\partial\xi}{\partial t} \tag{32}$$

Now we are seeking the solution of (31) and consequently of (23). From (28) we have $\frac{\partial\xi}{\partial t}=-\lambda_{32}$ and with the fact that $h(\xi)$ is completely obtained by ξ which gives $\lambda_{16}\lambda_{32}=-\lambda_{25}\lambda_{36}$ (33) yields

$$h(\xi)=-(\lambda_{16}x+\lambda_{25}t+\lambda_{26})=-\left(\frac{\lambda_{16}}{\lambda_{36}}\xi+\lambda_{26}\right)$$

Thus from (32) the solution of (23) is found as

$$\eta=f\left(y,\xi\right)=\Phi\left[yH(\xi)\right] \tag{33}$$

Here $H(\xi)$ is given by the differential equation $h(\xi)\frac{d}{d\xi}[\ln H(\xi)]=1$ and it gives

$$\begin{aligned}H(\xi)&=\exp\left(\int\frac{d\xi}{h(\xi)}\right)=\exp\left\{-\int\frac{d\xi}{\left(\frac{\lambda_{16}}{\lambda_{36}}\xi+\lambda_{26}\right)}\right\}\\&=\exp\left\{-\frac{\lambda_{36}}{\lambda_{16}}\ln\left(\frac{\lambda_{16}}{\lambda_{36}}\xi+\lambda_{26}\right)\right\}=\left(\frac{\lambda_{16}}{\lambda_{36}}\xi+\lambda_{26}\right)^{-\frac{\lambda_{36}}{\lambda_{16}}}\end{aligned}$$

$$H(\xi)\cong-(\lambda_{16}x+\lambda_{25}t+\lambda_{26})^{-\lambda_{36}/\lambda_{16}} \tag{34}$$

Hence from (33) and (34) the solution of (23) is given by

$$\eta=\Phi\left[y(Ax+Bt+C)^{m}\right]$$

where $m = -\frac{\lambda_{36}}{\lambda_{16}} = -\frac{1}{2}$, $A = \lambda_{16}$, $B = \lambda_{25}$, $C = \lambda_{26}$ provided $\lambda_{16} \neq 0$ otherwise yields the case of $\lambda_{36} = 0$ which is not permissible.
Also without loss of generality Φ can be taken to identity function. Thus

$$\eta = y\,\pi_1(x,t) \tag{35}$$

Where $\pi_1(x,t) = (Ax + Bt + C)^{-1/2}$

Case-IIb In this case we have $\frac{\partial \eta}{\partial x} = 0$, from (24) we have

$$\lambda_{32} y \frac{\partial \eta}{\partial y} - (\lambda_{16} x + \lambda_{25} t + \lambda_{26}) \frac{\partial \eta}{\partial t} = 0.$$

Since η is independent of x we must have $\lambda_{16} = 0$, this gives

$$\lambda_{32} y \frac{\partial \eta}{\partial y} - (\lambda_{25} t + \lambda_{26}) \frac{\partial \eta}{\partial t} = 0.$$

According to well known standard technique for linear partial differential equation, we get

$$\frac{dy}{\lambda_{32} y} = -\frac{dt}{\lambda_{25} t + \lambda_{26}} = \frac{d\eta}{0}$$

This equation yields,

$$\eta = \eta(y,t) = y\left(t + \frac{\lambda_{26}}{\lambda_{25}}\right)^{\frac{\lambda_{32}}{\lambda_{25}}}$$

That is

$$\eta = y\,\pi_2(t) \tag{36}$$

Where $\pi_2(t) = (t + A)^{-1/2}$, $\quad A = \frac{\lambda_{26}}{\lambda_{25}}$, $\quad \frac{\lambda_{32}}{\lambda_{25}} = -\frac{1}{2}$

In the next step we obtain the absolute invariant corresponding to the dependent variable viz θ, ψ and T_w .
From (18) is an absolute invariant of itself, therefore

$$g_1(x,y,t,\theta) = \theta(\eta) \tag{37}$$

A function $g_2(x,t,\psi)$ is an absolute invariant of $\psi(x,y,t)$ for two parameter group G, if it satisfies two linear partial differential equations

$$\left.\begin{aligned} (\alpha_1 x + \alpha_2) \tfrac{\partial g_2}{\partial x} + (\alpha_5 t + \alpha_6) \tfrac{\partial g_2}{\partial t} + (\alpha_7 \psi + \alpha_8) \tfrac{\partial g_2}{\partial \psi} = 0. \\ (\beta_1 x + \beta_2) \tfrac{\partial g_2}{\partial x} + (\beta_5 t + \beta_6) \tfrac{\partial g_2}{\partial t} + (\beta_7 \psi + \beta_8) \tfrac{\partial g_2}{\partial \psi} = 0. \end{aligned}\right\} \tag{38}$$

The solutions of these equations give

$$g_2\left(x,t,\psi\right)=\Phi_1\left\{\frac{\psi}{\Gamma(x,t)}\right\}=F\left(\eta\right) \tag{39}$$

In similar manner we can obtain

$$g_3\left(x,t,T_w\right)=\Phi_2\left\{\frac{T_w}{\omega(x,t)}\right\}=E\left(\eta\right) \tag{40}$$

Where $\Gamma(x,t)$ and $\omega(x,t)$ are functions to be determined and without loss of generality the arbitrary functions $\Phi's$ are assumed to be identity functions in (39) and (40), whence

$$\psi\left(x,y,t\right)=\Gamma(x,t)F\left(\eta\right) \tag{41}$$

$$T_w(x,t)=\omega(x,t)E\left(\eta\right) \tag{42}$$

As $\Gamma(x,t)$ and $\omega(x,t)$ are independent of y where as η depends on y (42) follows that $E\left(\eta\right)$ must be constant say T_0, that is

$$T_w(x,t)=T_0\,\omega(x,t) \tag{43}$$

The forms of $\Gamma(x,t)$ and $\omega(x,t)$ are those for which equations (6) and (7) reduces to the system of ordinary differential equations.

3.5. The reduction to ordinary differential equations

In above analysis the absolute invariant $\eta\left(x,y,t\right)$ of independent variables is generally of the form

$$\eta\left(x,y,t\right)=y\,\pi(x,t) \tag{44}$$

Where $\pi(x,t)$ assigned its own form according to different cases.
Substituting the values in (6) from (41), (43), and (44), we get

$$\begin{aligned}&\left(-n\Gamma^n\pi^{2n+1}\right)(F'')^{n-1}F'''+\eta\Gamma\frac{\partial\pi}{\partial t}F''+\pi\frac{\partial\Gamma}{\partial t}F'+\Gamma\frac{\partial\pi}{\partial t}F'\\&+\pi\Gamma^2\frac{\partial\pi}{\partial x}(F')^2+\Gamma\pi^2\frac{\partial\Gamma}{\partial x}(F')^2-\Gamma\pi^2\frac{\partial\Gamma}{\partial x}FF''-\theta\,T_0\omega=0\end{aligned}$$

Where prime denotes the ordinary derivative with respect to η.

Dividing the above equation by leading coefficient,

$$\begin{aligned}&(F'')^{n-1}F'''-\frac{\left(\frac{\partial\pi}{\partial t}\right)}{n\Gamma^{n-1}\pi^{2n+1}}\eta F''-\frac{\left(\frac{\partial\Gamma}{\partial t}\right)}{n\Gamma^{n}\pi^{2n}}F'-\frac{\left(\frac{\partial\pi}{\partial t}\right)}{n\Gamma^{n-1}\pi^{2n+1}}F'\\&-\frac{\left(\frac{\partial\pi}{\partial x}\right)}{n\Gamma^{n-2}\pi^{2n}}(F')^2-\frac{\left(\frac{\partial\Gamma}{\partial x}\right)}{n\Gamma^{n-1}\pi^{2n-1}}\left\{(F')^2-FF''\right\}+\frac{T_0\omega}{n\Gamma^{n}\pi^{2n+1}}\theta=0\end{aligned} \tag{45}$$

Similarly on substituting the values from (41)-(44) in (7), we get

$$\frac{1}{P_r}\theta'' - \frac{\left(\frac{\partial \pi}{\partial t}\right)}{\pi^3}\eta\theta' + \frac{\left(\frac{\partial \Gamma}{\partial x}\right)}{\pi}F\theta' - \frac{\Gamma\left(\frac{\partial \omega}{\partial x}\right)}{\pi\omega}F'\theta - \frac{\left(\frac{\partial \omega}{\partial t}\right)}{\omega\pi^2}\theta = 0 \tag{46}$$

To reduce (45) and (46) to a the system of ODE in a single variable η , it is necessary that the coefficients of the functions $F\left(\eta\right)$, $\theta\left(\eta\right)$ and their derivatives must be either constants or functions of η only. These coefficients are:

$$\left.\begin{array}{lll} C_1 = \frac{\left(\frac{\partial \pi}{\partial t}\right)}{n\Gamma^{n-1}\pi^{2n+1}}, & C_2 = \frac{\left(\frac{\partial \Gamma}{\partial t}\right)}{n\Gamma^{n}\pi^{2n}}, & C_3 = \frac{\left(\frac{\partial \pi}{\partial x}\right)}{n\Gamma^{n-2}\pi^{2n}} \\ C_4 = \frac{\left(\frac{\partial \Gamma}{\partial x}\right)}{n\Gamma^{n-1}\pi^{2n-1}}, & C_5 = \frac{T_0\omega}{n\Gamma^{n}\pi^{2n+1}}, & C_6 = \frac{\left(\frac{\partial \pi}{\partial t}\right)}{\pi^3} \\ C_7 = \frac{\left(\frac{\partial \Gamma}{\partial x}\right)}{\pi}, & C_8 = \frac{\Gamma\left(\frac{\partial \omega}{\partial x}\right)}{\pi\omega}, & C_9 = \frac{\left(\frac{\partial \omega}{\partial t}\right)}{\omega\pi^2} \end{array}\right\} \tag{47}$$

With this notations (45) and (46) reduces to,

$$\left(F''\right)^{n-1}F''' + \left(C_4F - C_1\eta\right)F'' - (C_3 + C_4)\left(F'\right)^2 - (C_1 + C_2)F' + C_5\theta = 0 \tag{48}$$

$$\frac{1}{P_r}\theta'' + \left(C_7F - C_6\eta\right)\theta' - \left(C_8F' + C_9\right)\theta = 0 \tag{49}$$

Together with the reduced boundary conditions

$$\left.\begin{array}{ll} F(0) = F'(0) = 0, & \theta(0) = 1 \\ F'(\infty) = 0, & \theta(\infty) = 1 \end{array}\right\} \tag{50}$$

3.6. To find the unknown functions

Case-I In this case $\eta = y\,\pi_1(x,t) = y(Ax + Bt + C)^{-1/2}$

It follows that, $\frac{\partial \pi_1}{\partial x} = -\frac{A}{2}\pi_1^3$ and $\frac{\partial \pi_1}{\partial t} = -\frac{B}{2}\pi_2^3$
Substituting these values in (47) and using the mathematical analysis one can obtain

$$\left.\begin{array}{l} \Gamma(x,t) = K(Ax + Bt + C)^{1/2}, \quad K \text{ is some constant.} \\ C_1 = \frac{B}{KA}C_3; \quad C_2 = \frac{B}{KA}C_4; \quad C_1 = -\frac{B}{KA}C_4 \end{array}\right\} \tag{51}$$

The constant C_5 is taken to be unity, this cane achieved without restricting the expression of T_w, thus

$$T_w = T_0\,\omega(x,t) = nK^n\pi_1^{n+1}$$

Therefore, $\frac{\partial\omega}{\partial x}=-\frac{(n+1)A}{2}\,\omega\pi_1^2$ and $\frac{\partial\omega}{\partial t}=-\frac{(n+1)B}{2}\,\omega\pi_1^2$
From (47) we get,

$$C_5=1;\quad C_6=-\frac{B}{2};\quad C_7=\frac{KA}{2};\quad C_8=-\frac{KA(n+1)}{2};\quad C_9=-\frac{B(n+1)}{2} \tag{52}$$

From (51) we obtain the relative constants as

$$C_1=-C_2=\frac{B}{KA}C_3=-\frac{B}{KA}C_4;\quad C_5=1 \tag{53}$$

Substituting the obtained constants in equations (48) and (49), yields

$$\left(F''\right)^{n-1}F'''-C_1\left(\frac{KA}{B}F+\eta\right)F''+\theta=0 \tag{54}$$

$$\frac{1}{P_r}\theta''+\frac{1}{2}\left(KAF+B\eta\right)\theta'+\frac{(n+1)}{2}\left(KAF'+B\right)\theta=0 \tag{55}$$

Together with boundary conditions

$$\left.\begin{array}{ll} F(0)=F'(0)=0, & \theta(0)=1 \\ F'(\infty)=0; & \theta(\infty)=1 \end{array}\right\} \tag{56}$$

The stream function is given by

$$\psi\left(x,y,t\right)=K(Ax+Bt+C)^{1/2}F\left(\eta\right);\ \text{where}\ \eta=y(Ax+Bt+C)^{-1/2} \tag{57}$$

<u>Subcase-IIb</u>

In this case we have $\eta=y\,\pi_2(x,t)=y(t+A)^{-1/2}$
Preceding the similar analysis as in case I we obtain

$$\left.\begin{array}{l} \Gamma(x,t)=K(t+A)^{1/2},\quad K\ \text{is some constant.} \\ C_1=1;\quad C_2=-1;\quad C_3=0;\quad C_4=0;\quad C_5=1 \\ C_6=-\frac{1}{2};\quad C_7=0;\quad C_8=0;\quad C_9=\frac{1}{2} \end{array}\right\} \tag{58}$$

Substituting the obtained constants (48) and (49), yields, together with boundary conditions (57) as

$$\left(F''\right)^{n-1}F'''-\eta F''+\theta=0 \tag{59}$$

$$\frac{1}{P_r}\theta''+\frac{1}{2}\eta\theta'+\frac{1}{2}\theta=0 \tag{60}$$

The stream function is given by

$$\psi\left(x,y,t\right)=K(t+A)^{1/2}F\left(\eta\right);\text{where, }\eta=y(t+A)^{-1/2} \tag{61}$$

Equation (60) gives the surface-temperature distribution corresponding to the case, which is independent of x i.e. uniform. It is a function of time t. As a special case, for $P_r=1$ (60) takes the form

$$\theta''+\frac{1}{2}\eta\theta'+\frac{1}{2}\theta=0 \tag{62}$$

It is worth to observe that equation (62) has analytic solution in term of the confluent hyper-geometric function (see [16]) is

$$\theta\left(\eta\right)=e^{-\eta^2/4}U\left(-\frac{1}{2};\frac{1}{2},\frac{\eta^2}{4}\right) \tag{63}$$

The boundary layer characteristics are:

(1) The vertical velocity

$$u=\frac{\partial\psi}{\partial y}=K\,F'\left(\eta\right) \tag{64}$$

(2) The horizontal velocity

$$v=\frac{\partial\psi}{\partial x}=0 \tag{65}$$

Which are evidently form the continuity equation.

(3) The surface heat flux

$$q=K(t+A)^{-3/4}\left[-\theta'(0)\right] \tag{66}$$

4. Conclusion

Using two-parameter deductive group theoretic technique, similarity transformations for two dimensional unsteady laminar boundary layer flow of non-Newtonian power law fluid past a non-isothermal vertical plate are systematically derived. The possible form of surface temperature variations with position and time are also systematically derived from the similarity requirement. The governing highly non-linear system of partial differential equations with multi variables reduce to the non-linear system of ordinary differential equations with appropriate boundary conditions. Controlling the parameters one solution of the one particular case in terms of confluent hypergeometric function is derived. Present similarity solution is one of the quite a few solutions for power law non-Newtonian fluid obtained by applications of two parameter deductive group transformations technique.

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