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Diffusion-reaction problem for the Bingham fluid with Lipschitz Source

Research Article

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Abstract: The paper is devoted to the study of the diffusion-reaction problem for the Bingham fluid in three dimensional bounded domain, whose viscosity, yield limit and diffusivity depend on the concentration. We prove the existence and uniqueness of local strong solution. It is also shown that, under some additional hypotheses, the concentration becomes a positive function.

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1. Introduction

The model of Bingham is a rigid viscoplastic and incompressible fluid with yield limit. It has been studied by mathematicians, physicists and engineers as intensively as the Navier-Stokes. While this model describes adequately a large class of flows. Due to existence of yield limit, the model can capture phenomena connected with the development of discontinuous stresses. A particularity of the Bingham model lies in the presence of rigid zones located in the interior of the flow. As the yield limit increases, these rigid zones become larger and may completely block the flow. This property is called the blocking phenomenon. The Bingham fluid has the following further physical property: if the yield limit is not reached, the fluid deforms like a rigid medium and if the stresse tensor reaches the yield limit, it behaves like a viscous fluid. One of the main features of the model consists of the existence of free boundary which separates the rigid behaviour and the viscoplastic behaviour of the material.

These phenomena can be physically explained by the fact that these fluids are mostly suspensions of quasi-spherical particles in a solvent. When the particles are low concentred, the only effect of their presence is to increase the viscosity

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proportionally to the concentration of the particles. If we still increase the concentration, particles end up touching. The fluid becomes pasty due to the forces between the particles in contact. To make it flow, it is necessary to destroy all these forces.

The Bingham fluid has been used in various publications in order to model the flow of metals, plastic solids and a variety of polymers. Such situations abound in industry, for example the wire-drawing process. More recently, the model was considered in blood modelling due to the yield limit and blockage phenomena, which allow us to describe the coagulation and the formation and lysis of blood clots. The literature concerning this topic is extensive; see e.g. [3, 5, 6] and references therein.

An early attempt in the study of existence of weak solution for the Bingham model can be found in [8] and in [7] for the thermal flow.

In three dimensional domain, uniqueness of local strong solutions as well as the existence and uniqueness of global strong solution were considered in the references [11–13]. The numerical analysis of the corresponding problems was provided in [9, 10].

The purpose of this paper is to study the local existence and uniqueness of strong solution, in three dimensional domain, to a physical problem descibing the reaction (interaction and collision) between the particles in a Bingham fluid in motion, endowed with viscosity, yield limit and diffusivity depending on the concentration of the particles.

The paper is organized as follows. In Section 2 we present the mechanical problem which describes the diffusion-reaction phenomenon whith a Lipschitz source for the unsteady flow of Bingham fluid in a three dimensional bounded domain . In addition, we introduce some notations and preliminaries, and we derive the variational formulation of the problem. We prove in Section 3 the existence and uniqueness of local strong solution as well as a maximum principle for the concentration, under some hypotheses.

2. Problem Statement

Let $T>0, \Omega\subset\mathbb{R}^n$ be a bounded domain with strong Lipschitz boundary Γ. We denote by $\mathbb{S}_3$ the space of symmetric tensors on $\mathbb{R}^3$. We define the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}_3$, respectively, by

$$\mathbf{u}\cdot\mathbf{v}=u_i v_i \quad \forall\,\mathbf{u},\,\mathbf{v}\in\mathbb{R}^3 \ \text{ and } \ \sigma\cdot\tau=\sigma_{ij}\tau_{ij} \quad \forall\,\sigma,\,\tau\in\mathbb{S}_3.$$

$$|\mathbf{u}|=(\mathbf{u}\cdot\mathbf{u})^{\frac{1}{2}} \quad \forall\mathbf{u}\in\mathbb{R}^3 \ \text{ and } \ |\sigma|=(\sigma\cdot\sigma)^{\frac{1}{2}} \quad \forall\sigma\in\mathbb{S}_3.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used. We denote by σ the deviator of $\sigma=\left(\sigma_{ij}\right)$ given by

$$\sigma^D=\left(\sigma^D_{ij}\right),\ \ \sigma^D_{ij}=\sigma_{ij}-\frac{\sigma_{kk}}{n}\delta_{ij},$$

We consider the rate of deformation operator defined for every $\mathbf{u}\in H^1(\Omega)^3$ by

$$D(\mathbf{u})=\left(D_{ij}(\mathbf{u})\right),\quad D_{ij}(\mathbf{u})=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right).$$

The diffusion-reaction problem for the unsteady flow of Bingham fluid may be formulated as follows.

Problem P1. Find the velocity field $\mathbf{u}=(u_i):\Omega\times(0,T)\longrightarrow\mathbb{R}^3$, the stress field $\sigma=(\sigma_{ij}):\Omega\times(0,T)\longrightarrow\mathbb{S}_3$ and the concentration $C:\Omega\times(0,T)\longrightarrow\mathbb{R}$ such that

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=div(\sigma)+\mathbf{f}\text{ in }\Omega\times(0,T), \tag{1}$$

$$\begin{cases}\sigma^D=2\mu(C)D(\mathbf{u})+g(C)\dfrac{D(\mathbf{u})}{|D(\mathbf{u})|} & \text{if } |D(\mathbf{u})|\neq 0\\ \left|\sigma^D\right|\leq g(C) & \text{if } |D(\mathbf{u})|=0\end{cases}\quad\text{in }\Omega\times(0,T), \tag{2}$$

$$div(\mathbf{u})=0\text{ in }\Omega\times(0,T), \tag{3}$$

$$\frac{\partial C}{\partial t}+\mathbf{u}\cdot\nabla C-div\left(\eta(C)\nabla C\right)=R(C)\text{ in }\Omega\times(0,T), \tag{4}$$

$$\mathbf{u}=0\text{ on }\Gamma\times(0,T), \tag{5}$$

$$C=0\text{ on }\Gamma\times(0,T), \tag{6}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\ C(0)=C_0\text{ in }\Omega. \tag{7}$$

Here, the flow is given by the equation (1) where the density is assumed equal to one and $\mathbf{f}$ is the density of volumes forces. Relation (2) is the constitutive law of a Bingham fluid whose the viscosity μ and the yield limit g depend on the concentration C. (3) represents the incompressibility condition. Equation (4) represents the diffusion-reaction equation where the diffusivity η is supposed depends also on the concentration and R is a non linear function which represents the reaction source. (5) gives the velocity on Γ. Condition (6) is an homogeneous Dirichlet boundary condition on Γ for the concentration. Finally, in (7), the functions $\mathbf{u}_0$ and C_0 are the initial data.

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem. Let $s\geq 0$ and $0\leq\tau\leq T$. We define the function spaces

$$\mathcal{V}_s=\left\{\mathbf{v}\in H_0^s(\Omega)^3\mid div(\mathbf{v})=0\text{ in }\Omega\right\},$$

$$\mathcal{W}(0,\tau)=\left\{\mathbf{u}\in L^\infty\left(0,\tau;\mathcal{V}_1\cap W_0^{1,6}(\Omega)^3\right)\mid\frac{\partial\mathbf{u}}{\partial t}\in L^2(0,\tau;\mathcal{V}_1)\cap L^\infty(0,\tau;\mathcal{V}_0)\right\}$$

$$\mathcal{X}(0,\tau)=\left\{\phi\in L^2\left(0,\tau;H_0^1(\Omega)\right)\cap L^\infty\left(0,\tau;L^2(\Omega)\right)\mid\frac{\partial\phi}{\partial t}\in L^2\left(0,\tau;L^2(\Omega)\right)\right\}.$$

$\mathcal{V}_s$ is an Hilbert space equipped with the inner product and the induced norm, respectively,

$$(\mathbf{u},\mathbf{v})_{\mathcal{V}_s}=(u_i,v_i)_{H^s(\Omega)},\quad\|\mathbf{v}\|_{\mathcal{V}_s}=(\mathbf{u},\mathbf{u})_{\mathcal{V}_s}^{\frac{1}{2}}\quad\forall\mathbf{u},\ \mathbf{v}\in\mathcal{V}_s.$$

$\mathscr{W}(0,\tau)$ and $\mathscr{X}(0,\tau)$ are Banach spaces equipped with the norms, respectively,

$$\|\mathbf{v}\|_{\mathscr{W}(0,\tau)} = \|\mathbf{v}\|_{L^\infty(0,\tau;\mathscr{V}_1)} + \|\mathbf{v}\|_{L^\infty\left(0,\tau;W_0^{1,6}(\Omega)^3\right)} + \left\|\frac{\partial \mathbf{v}}{\partial t}\right\|_{L^2(0,\tau;\mathscr{V}_1)} + \left\|\frac{\partial \mathbf{v}}{\partial t}\right\|_{L^\infty(0,\tau;\mathscr{V}_0)}$$

$$\forall \mathbf{v} \in \mathscr{W}_s(0,\tau).$$

$$\|\phi\|_{\mathscr{X}(0,\tau)} = \|\phi\|_{L^2\left(0,\tau;H_0^1(\Omega)\right)} + \|\phi\|_{L^\infty(0,\tau;L^2(\Omega))} + \left\|\frac{\partial \phi}{\partial t}\right\|_{L^2(0,\tau;L^2(\Omega))} \quad \forall \phi \in \mathscr{X}(0,\tau).$$

We introduce the functionals

$$\left\{\begin{array}{c} B : \mathscr{V}_1 \times \mathscr{V}_1 \times \mathscr{V}_1 \to \mathbb{R}, \quad B(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\, dx \\ E : H_0^1(\Omega) \times H_0^1(\Omega) \times \left(\mathscr{V}_1 \cap W_0^{1,6}(\Omega)^3\right) \longrightarrow \mathbb{R}, \\ E\left(\psi,\phi,\mathbf{v}\right) =_\Omega \left(\nabla\psi\right)\phi\cdot\mathbf{v}\, dx. \end{array}\right. \tag{8}$$

In the study of the mechanical problem (*P1*), we consider the following hypotheses

$\mu, g, \eta : \Omega \times (0,T) \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\left\{\begin{array}{c} \mu\left(.,.,\phi\right), g\left(.,.,\phi\right), \eta\left(.,.,\phi\right) \text{ is measurables } \forall\phi \in \mathbb{R}, \\ \mu(x,t,.), g(x,t,.), \eta(x,t,.) \in \mathscr{C}^0(\mathbb{R}) \text{ a.e. } (x,t) \in \Omega \times (0,T), \\ \exists \mu_* > 0 : \mu\left(x,t,\phi\right) \geq \mu_* \text{ a.e. } (x,t) \in \Omega\times(0,T), \forall\phi \in \mathbb{R}, \\ \exists g^* > 0 : 0 \leq g\left(x,t,\phi\right) \leq g^* \text{ a.e. } (x,t) \in \Omega\times(0,T), \forall\phi \in \mathbb{R}, \\ \exists \eta_* : \eta\left(x,t,\phi\right) \geq \eta_* \text{ a.e. } (x,t) \in \Omega\times(0,T), \forall\phi \in \mathbb{R}, \end{array}\right. \tag{9}$$

$$\mathbf{f} \in L^2\left(0,T;L^2(\Omega)^3\right) \text{ and } \frac{\partial \mathbf{f}}{\partial t} \in L^2\left(0,T;H^{-1}(\Omega)^3\right). \tag{10}$$

$R : \Omega \times (0,T) \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\left\{\begin{array}{c} R\left(.,.,\phi\right) \text{ is measurable } \forall\phi \in \mathbb{R}, \\ R(x,t,0) \in L^2\left(0,T;L^2(\Omega)\right) \text{ a.e. } (x,t) \in \Omega\times(0,T), \\ \exists L \in L^2(\Omega) \mid \left|R\left(x,t,\phi_1\right) - R\left(x,t,\phi_2\right)\right| \leq L\left|\phi_1 - \phi_2\right| \\ \text{a.e. } (x,t) \in \Omega\times(0,T), \forall\phi_1,\phi_2 \in \mathbb{R}. \end{array}\right. \tag{11}$$

Remark 2.1.
In the constitutive law (2) of Bingham fluid
1. The pressure is given by the function

$$P = -\frac{1}{3}\mathrm{tr}(\sigma). \tag{12}$$

2. If $C = 0$ the fluid becomes Newtonian, which allows us to impose the following physical condition.

$$g(x,t,0) = 0 \text{ a.e. } (x,t) \in \Omega\times(0,T). \tag{13}$$

The following lemma, see the references [14, 16, 17], gives some properties of the functionals B and E.

Lemma 2.1.
1. *B is trilinear and continuous on $\mathscr{V}_1 \times \mathscr{V}_1 \times \mathscr{V}_1$. Moreover, $B(\mathbf{u},\mathbf{v},\mathbf{w}) = -B(\mathbf{u},\mathbf{w},\mathbf{v})\ \ \forall(\mathbf{u},\mathbf{v},\mathbf{w}) \in \mathscr{V}_1 \times \mathscr{V}_1 \times \mathscr{V}_1$.*
2. *E is trilinear and continuous on $H_0^1(\Omega) \times H_0^1(\Omega) \times \left(\mathscr{V}_1 \cap W_0^{1,6}(\Omega)^3\right)$. Moreover, $E(\psi,\phi,\mathbf{v}) = -E(\phi,\psi,\mathbf{v})\ \ \forall(\psi,\phi,\mathbf{v}) \in H_0^1(\Omega) \times H_0^1(\Omega) \times \left(\mathscr{V}_1 \cap W_0^{1,6}(\Omega)^3\right)$.*

We also need the following lemma, made in [14].

Lemma 2.2.
Let Ω be a bounded domain of $\mathbb{R}^3$ with a Lipschitz boundary and let $v \in L^2\left(0,T;H_0^1(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right)$, then

$$v \in L^4\left(0,T;L^6(\Omega)\right), \tag{14}$$

with continuous injection.

The use of Green's formula permits us to derive the following variational formulation of mechanical problem (*P1*).

Problem P2. Find the velocity field $\mathbf{u} = (u_i) : \Omega \times (0,T) \longrightarrow \mathbb{R}^3$ and the concentration $C : \Omega \times (0,T) \longrightarrow \mathbb{R}$ satisfying the variational system

$$\begin{aligned} &\int_\Omega \frac{\partial \mathbf{u}}{\partial t}(\mathbf{v}-\mathbf{u})\,dx + B(\mathbf{u},\mathbf{u},\mathbf{v}) + 2\int_\Omega \mu(C)D(\mathbf{u})\cdot D(\mathbf{v}-\mathbf{u})\,dx + \\ &\int_\Omega g(C)|D(\mathbf{v})|\,dx - \int_\Omega g(C)|D(\mathbf{u})|\,dx \geq_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\,dx \quad \forall \mathbf{v} \in \mathscr{V}_1, \end{aligned} \tag{15}$$

$$\int_\Omega \frac{\partial C}{\partial t}\phi\,dx + E\left(C,\phi,\mathbf{u}\right) +_\Omega \eta(C)\nabla C\cdot\nabla\phi\,dx =_\Omega R(C)\phi\,dx \quad \forall \phi \in H_0^1(\Omega). \tag{16}$$

3. Main Results

We begin by establishing an existence and uniqueness theorems to the problem (P2) as well as a maximum principle for the concentration.

Theorem 3.1.
Under the assumptions (9)-(12), there exists $T_0 \in \left]0,T\right]$ such that the problem (P2) admits at least one local solution $(\mathbf{u},C)$ having the regularity

$$\begin{cases} \mathbf{u} \in \mathscr{W}(0,T_0), \\ C \in \mathscr{X}(0,T_0). \end{cases} \tag{17}$$

It is easy to check that regularity (17) implies in particular, using the theorem of intermediary derivatives and Sobolev embeddings, see for instance [1, 15], that, after a possible modification on a set of measure zero, the solution is such that

$$\begin{cases} \mathbf{u} \in \mathscr{C}^0([0,T_0];\mathscr{V}_1) \cap L^\infty\left(0,T_0;\mathscr{C}^{0,\frac{1}{2}}(\Omega)^3\right), \\ C \in \mathscr{C}^0\left([0,T_0];L^2(\Omega)\right). \end{cases} \tag{18}$$

The proof of Theorem 3.1 is based on the application of Schauder's fixed point theorem, using two auxiliary existence results. The first one has been obtained by two authors, see [12, 13]. The second one results from the theory of parabolic equations and Banach's fixed point theorem, see [4, 14]. Finally, some arguments of non linear analysis are used to conclude the proof.

The first existence auxiliary result is given by the following proposition.

Proposition 3.1.
For every $\lambda \in \mathscr{X}(0,T)$, there exists a unique local solution $\mathbf{u}_\lambda \in \mathscr{W}(0,T_0)$ to the problem

$$\int_\Omega \frac{\partial \mathbf{u}_\lambda}{\partial t}(\mathbf{v}-\mathbf{u}_\lambda)\,dx + B(\mathbf{u}_\lambda,\mathbf{u}_\lambda,\mathbf{v}) + 2\int_\Omega \mu(\lambda) D(\mathbf{u}_\lambda)\cdot D(\mathbf{v}-\mathbf{u}_\lambda)\,dx + \int_\Omega g(\lambda)|D(\mathbf{v})|\,dx - \int_\Omega g(\lambda)|D(\mathbf{u}_\lambda)|\,dx \geq_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u}_\lambda)\,dx \quad \forall \mathbf{v}\in \mathscr{V}_1, \tag{19}$$

and it satisfies the estimate

$$\|\mathbf{u}_\lambda\|_{\mathscr{W}_1(0,T_0)} \leq d_1, \tag{20}$$

where d_1 is a positive constant independs on λ.

For more details about the proof of this proposition, see [12, 13].

The second auxiliary existence result is given by the proposition below.

Proposition 3.2.
Let $\mathbf{u}_\lambda \in \mathscr{W}(0,T_0)$ be the solution of problem (19) given by Proposition 3.1. Then, there exists a unique solution $C_\lambda \in \mathscr{X}(0,T_0)$ to the problem

$$\int_\Omega \frac{\partial C_\lambda}{\partial t}\phi\,dx + E\left(C_\lambda,\phi,\mathbf{u}_\lambda\right) +_\Omega \eta(\lambda)\nabla C_\lambda\cdot\nabla\phi\,dx =_\Omega R(C_\lambda)\phi\,dx \quad \forall \phi\in H_0^1(\Omega). \tag{21}$$

and it satisfies the estimate

$$\|C_\lambda\|_{\mathscr{X}(0,T_0)} \leq d_2, \tag{22}$$

where d_2 is a positive constant independs on λ.

We remark that the solutions $\mathbf{u}_\lambda$ and C_λ verify also the regularity (18).

Proof. The proof will be done in several steps. Based on classical arguments of functional analysis concerning parabolic problems and Banach fixed point theorem.

First, remark that the continuity of E on $H_0^1(\Omega)\times H_0^1(\Omega)\times\left(\mathscr{V}_1\cap W_0^{1,6}(\Omega)^3\right)$ leads, using Hölder's inequality with respect to the time variable, to

$$\left|_0^{T_0} E\left(\psi,\phi,\mathbf{v}\right)dt\right| \leq \left\|\psi\right\|_{L^2\left(0,T_0;H_0^1(\Omega)\right)}\left\|\phi\right\|_{L^2\left(0,T_0;H_0^1(\Omega)\right)}\|\mathbf{v}\|_{L^\infty\left(0,T_0;W_0^{1,6}(\Omega)\cap\mathscr{V}_1\right)}. \tag{23}$$

Which means that E is continuous on $L^2\left(0,T_0;H_0^1(\Omega)\right)^2\times L^\infty\left(0,T_0;\mathscr{V}_1\cap W_0^{1,6}(\Omega)^3\right)$.

Now, denoting by r an arbitrary element of $L^2\left(0,T_0;L^2(\Omega)\right)$ and introducing the linear problem

$$\int_\Omega \frac{\partial C_{\lambda r}}{\partial t}\phi\,dx + E\left(C_{\lambda r},\phi,\mathbf{u}_\lambda\right) +_\Omega \eta(\lambda)\nabla C_{\lambda r}\cdot\nabla\phi\,dx \tag{24}$$

$$=_\Omega r\phi\,dx \quad \forall\phi\in H_0^1(\Omega).$$

Consider the bilinear form

$$G: H_0^1(\Omega)\times H_0^1(\Omega)\longrightarrow\mathbb{R},$$

$$G\left(\psi,\phi\right) = E\left(\psi,\phi,\mathbf{u}_\lambda\right) +_\Omega \eta(\lambda)\nabla\psi\cdot\nabla\phi\,dx + \int_\Omega \psi\phi\,dx. \tag{25}$$

The linear problem (24) can be rewritten, making use G

$$\int_\Omega \frac{\partial C_{\lambda r}}{\partial t}\phi\,dx + G\left(C_{\lambda r},\phi\right) =_\Omega r\phi\,dx \quad \forall\phi\in H_0^1(\Omega), \tag{26}$$

We can easily prove, via Lemma 2.1 and hypothesis (9), that the bilinear form G is continuous and coercive on $H^1(\Omega)\times H^1(\Omega)$.

Consequently, we deduce, via classical arguments of functional analysis concerning linear parabolic equations, see [4, 14], that equation (26) admits a unique solution $C_{\lambda r}\in\mathscr{X}(0,T_0)$.

To obtain the solution of the auxiliary problem (21), we use the Banach fixed point theorem. To this aim, we introduce the operator

$$\begin{cases} \mathscr{L}: L^2\left(0,T_0;L^2(\Omega)\right)\longrightarrow L^2\left(0,T_0;L^2(\Omega)\right), \\ \mathscr{L}r = R(C_{\lambda r}). \end{cases} \tag{27}$$

For any $r_1, r_2\in L^2\left(0,T_0;L^2(\Omega)\right)$ the Lipschitzianity of R affirms that

$$|\mathscr{L}r_1 - \mathscr{L}r_2| \le L\left|C_{\lambda r_1} - C_{\lambda r_2}\right|. \tag{28}$$

Remembering that $C_{\lambda r_1}$ and $C_{\lambda r_2}$ are, respectively, the solutions of problems

$$\int_\Omega \frac{\partial C_{\lambda r_1}}{\partial t}\phi\,dx + G\left(C_{\lambda r_1},\phi\right) =_\Omega r_1\phi\,dx \quad \forall\phi\in H_0^1(\Omega), \tag{29}$$

$$\int_\Omega \frac{\partial C_{\lambda r_2}}{\partial t}\phi\,dx + G\left(C_{\lambda r_2},\phi\right) =_\Omega r_2\phi\,dx \quad \forall\phi\in H_0^1(\Omega). \tag{30}$$

Subtracting the equation (29) and (30), using $\phi = C_{\lambda r_1} - C_{\lambda r_2}$ as test function in the obtained equation and integrating over the interval time $(0,t)$, $t\le T_0$, it follows

$$\frac{1}{2}\left\|C_{\lambda r_1}(t) - C_{\lambda r_2}(t)\right\|_{L^2(\Omega)}^2 + c_1{}_0^t\left\|C_{\lambda r_1}(s) - C_{\lambda r_2}(s)\right\|_{H_0^1(\Omega)}^2 ds$$

$$\le_0^t \|r_1(s) - r_2(s)\|_{L^2(\Omega)}\left\|C_{\lambda r_1}(s) - C_{\lambda r_2}(s)\right\|_{H_0^1(\Omega)} ds \text{ a.e. } t\in(0,T_0).$$

Such that $c_1 = c_0^2 \eta_*$, where c_0 represents the Poincaré constant.

By Young's inequality, we get

$$\left\| C_{\lambda r_1}(t) - C_{\lambda r_2}(t) \right\|_{L^2(\Omega)}^2 + c_1 \int_0^t \left\| C_{\lambda r_1}(s) - C_{\lambda r_2}(s) \right\|_{H_0^1(\Omega)}^2 ds$$
$$\leq \frac{1}{c_1} \int_0^t \| r_1(s) - r_2(s) \|_{L^2(\Omega)}^2 ds. \text{ a.e. } t \in (0, T_0)$$

This leads, combining with (28)

$$\| \mathscr{L} r_1(t) - \mathscr{L} r_2(t) \|_{L^2(\Omega)}^2 \leq \frac{L^2}{c_1} \int_0^t \| r_1(s) - r_2(s) \|_{L^2(\Omega)}^2 ds \text{ a.e. } t \in (0, T_0).$$

Applying another time the operator $\mathscr{L}$, we find

$$\left\| \mathscr{L}^2 r_1(t) - \mathscr{L}^2 r_2(t) \right\|_{L^2(\Omega)}^2 \leq \frac{T_0 L^2}{c_1} \int_0^t \| r_1(s) - r_2(s) \|_{L^2(\Omega)}^2 ds \text{ a.e. } t \in (0, T_0).$$

Generalizing the above processes, by recurrence on $m \geq 1$, we get, using some algebraic calculations

$$\| \mathscr{L}^m r_1(t) - \mathscr{L}^m r_2(t) \|_{L^2(\Omega)}^2 \leq \frac{L^2 T_0^{m-1}}{c_1 (m-1)!} \int_0^t \| r_1(s) - r_2(s) \|_{L^2(\Omega)}^2 ds \text{ a.e. } t \in (0, T_0).$$

Thus,

$$\int_0^{T_0} \| \mathscr{L}^m r_1(t) - \mathscr{L}^m r_2(t) \|_{L^2(\Omega)}^2 dt$$
$$\leq \frac{L^2 T_0^m}{c_1 (m-1)!} \int_0^{T_0} \| r_1(t) - r_2(t) \|_{L^2(\Omega)}^2 dt. \tag{31}$$

We deduce from (31) that for m large enough, a power m of the operator $\mathscr{L}$ is a contraction on the space $L^2\big(0, T_0; L^2(\Omega)\big)$ and, therefore, Banach's fixed point theorem proves that $\mathscr{L}^m$ admits a fixed point

$$\bar{r} \in L^2\big(0, T_0; L^2(\Omega)\big).$$

Then, it follows

$$\mathscr{L}^{m+1} \bar{r} = \mathscr{L} \bar{r}.$$

Hence, the uniqueness of fixed point shows that $\mathscr{L} \bar{r} = \bar{r}$, this gives $R(C_{\lambda \bar{r}}) = \bar{r}$. Which implies that equation (21) admits a unique solution $C_\lambda \in \mathscr{X}(0, T_0)$.

Our goal now is to prove the estimation (22). To do this, we choose C_λ as test function in equation (21), to find, by integrating over the interval time $(0, t)$, $t \leq T_0$ and using Holder's and Young's inequalities

$$\frac{1}{2} \| C_\lambda(t) \|_{L^2(\Omega)}^2 + c_1 \int_0^t \| C_\lambda(s) \|_{H_0^1(\Omega)}^2 ds \leq \left(L + \frac{1}{2} \right) \int_0^t \| C_\lambda(s) \|_{L^2(\Omega)}^2 ds$$
$$+ \frac{1}{2} \int_0^t \| R(s, 0) \|_{L^2(\Omega)}^2 ds + \frac{1}{2} \| C_0 \|_{L^2(\Omega)}^2 \text{ a.e. } t \in (0, T_0). \tag{32}$$

Particularly,

$$\|C_\lambda(t)\|^2_{L^2(\Omega)} \le (2L+1)\int_0^t \|C_\lambda(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \|R(s,0)\|^2_{L^2(\Omega)}\,ds$$
$$+\|C_0\|^2_{L^2(\Omega)} \text{ a.e. } t\in(0,T_0).$$

This yields, employing Gronwall's lemma

$$\|C_\lambda(t)\|_{L^2(\Omega)} \le \left(\|R(0)\|_{L^2(0,T_0;L^2(\Omega))} + \|C_0\|_{L^2(\Omega)}\right)e^{\left(L+\frac{1}{2}\right)T_0}.$$

Then, taking into account (32), the following estimate holds

$$\|C_\lambda\|_{L^2\left(0,T_0;H_0^1(\Omega)\right)\cap L^\infty((0,T_0);L^2(\Omega))}$$
$$\le \left(\|R(0)\|_{L^2(0,T_0;L^2(\Omega))} + \|C_0\|_{L^2(\Omega)}\right)\times$$
$$\max\left(e^{\left(L+\frac{1}{2}\right)T_0}, \sqrt{\frac{1}{2c_1}\left((2L+1)\,T_0 e^{(2L+1)T_0}+1\right)}\right). \tag{33}$$

Our object now is to obtain an estimate of the derivative $\frac{\partial C_\lambda}{\partial t}$. To this aim, let us remember that

$$\frac{\partial C_\lambda}{\partial t} = -\mathbf{u}_\lambda\cdot\nabla C_\lambda + div\left(\eta(C_\lambda)\nabla C_\lambda\right) + R(C_\lambda) \text{ in } \Omega\times(0,T_0). \tag{34}$$

Hence, the problem consists in finding an estimate of the term $\mathbf{u}_\lambda\cdot\nabla C_\lambda$.

We obtain, exploiting (18)

$$\int_0^{T_0} \|\mathbf{u}_\lambda(t)\cdot\nabla C_\lambda(t)\|^2_{L^2(\Omega)}\,dt \le \|\mathbf{u}_\lambda(x,t)\|^2_{L^\infty(0,T_0;\mathscr{C}^0(\Omega))}\|C_\lambda\|^2_{L^2(0,T_0;H^1(\Omega))}.$$

Thus, via the estimates (20) and (33),

$$\|\mathbf{u}_\lambda\cdot\nabla C_\lambda\|_{L^2(0,T_0;L^2(\Omega))} \le c. \tag{35}$$

It follows

$$\left\|\frac{\partial C_\lambda}{\partial t}\right\|_{L^2(0,T_0;L^2(\Omega))} \le c. \tag{36}$$

Combining (36) and ((33)) the estimate (22) immediately follows. □

Proof of Theorem 3.1. In order to apply the Schauder fixe point theorem, we consider the closed convex ball

$$\mathscr{B} = \left\{\lambda\in\mathscr{X}(0,T_0),\ \|\lambda\|_{\mathscr{X}(0,T_0)} \le d_2\right\}, \tag{37}$$

Let us built the mapping Λ as follows

$$\begin{cases} \Lambda:\mathscr{B}\longrightarrow\mathscr{B}, \\ \lambda\longmapsto\Lambda(\lambda) = C_\lambda\in\mathscr{B}. \end{cases} \tag{38}$$

Our interest is to verify the compactness of the mapping Λ. To do so, let us consider a sequences $\lambda_n \in \mathscr{B}$. We can then extract a subsequences, still denoted by λ_n and $C_n = C_{\lambda_n}$ such that

$$\lambda_n \longrightarrow \lambda \text{ in } L^2\left(0, T_0; H^1(\Omega)\right) \text{ weakly,} \tag{39}$$

$$\lambda_n \longrightarrow \lambda \text{ in } L^\infty\left(0, T_0; L^2(\Omega)\right) \text{ weakly*,} \tag{40}$$

$$\frac{\partial \lambda_n}{\partial t} \longrightarrow \frac{\partial \lambda}{\partial t} \text{ in } L^2\left(0, T_0; L^2(\Omega)\right) \text{ weakly,} \tag{41}$$

$$C_n \longrightarrow C_\lambda \text{ in } L^2\left(0, T_0; H^1(\Omega)\right) \text{ weakly,} \tag{42}$$

$$C_n \longrightarrow C_\lambda \text{ in } L^\infty\left(0, T_0; L^2(\Omega)\right) \text{ weakly*,} \tag{43}$$

$$\frac{\partial C_n}{\partial t} \longrightarrow \frac{\partial C_\lambda}{\partial t} \text{ in } L^2\left(0, T_0; L^2(\Omega)\right) \text{ weakly.} \tag{44}$$

Moreover, it is known that C_n is solution of the equation

$$\int_\Omega \frac{\partial C_n}{\partial t} \phi \, dx + E\left(C_n, \phi, \mathbf{u}_n\right) +_\Omega \eta(\lambda_n) \nabla C_n \cdot \nabla \phi \, dx$$

$$=_\Omega R(C_n) \phi \, dx \quad \forall \phi \in H^1(\Omega), \tag{45}$$

where $\mathbf{u}_n$ verifies the inequality

$$\int_\Omega \frac{\partial \mathbf{u}_n}{\partial t} (\mathbf{v} - \mathbf{u}_n) \, dx + B(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}) + 2 \int_\Omega \mu(\lambda_n) D(\mathbf{u}_n) \cdot D(\mathbf{v} - \mathbf{u}_n) \, dx +$$
$$\int_\Omega g(\lambda_n) |D(\mathbf{v})| \, dx - \int_\Omega g(\lambda_n) |D(\mathbf{u}_n)| \, dx \geq_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}_n) \, dx \quad \forall \mathbf{v} \in \mathscr{V}_1. \tag{46}$$

Proposition 3.1 affirms that this inequality admits a unique local strong solution $\mathbf{u}_n \in \mathscr{W}(0, T_0)$. Then, we can extract a subsequence, still denoted by $\mathbf{u}_n$ satisfying

$$\mathbf{u}_n \longrightarrow \mathbf{u}_\lambda \text{ in } L^\infty\left(0, T_0; \mathscr{V}_1 \cap W_0^{1,6}(\Omega)^3\right) \text{ weakly*,} \tag{47}$$

$$\frac{\partial \mathbf{u}_n}{\partial t} \longrightarrow \frac{\partial \mathbf{u}_\lambda}{\partial t} \text{ in } L^2(0, T_0; \mathscr{V}_1) \text{ weakly,} \tag{48}$$

$$\frac{\partial \mathbf{u}_n}{\partial t} \longrightarrow \frac{\partial \mathbf{u}_\lambda}{\partial t} \text{ in } L^\infty(0, T_0; \mathscr{V}_0) \text{ weakly*.} \tag{49}$$

Since the injection $W_0^{1,6}(\Omega) \subset \mathscr{C}^0\left(\bar{\Omega}\right)$ is compact, see, [1], it follows, making use Aubin's compactness theorem, see for more details the reference [19], that we can extract a subsequences λ_n, C_n and $\mathbf{u}_n$ such that

$$\lambda_n \longrightarrow \lambda \text{ in } L^2\left(0, T_0; L^2(\Omega)\right) \text{ strongly and a.e. in } \Omega \times (0, T_0), \tag{50}$$

$$C_n \longrightarrow C_\lambda \text{ in } L^2\left(0, T_0, L^2(\Omega)\right) \text{ strongly and a.e. in } \Omega \times (0, T_0), \tag{51}$$

$$\mathbf{u}_n \longrightarrow \mathbf{u}_\lambda \text{ in } L^\infty\left(0, T_0; \mathscr{C}^0\left(\bar{\Omega}\right)^3\right) \text{ strongly and a.e. in } \Omega \times (0, T_0), \tag{52}$$

Furthermore, inequality (46) can be rewritten

$$\int_\Omega \frac{\partial \mathbf{u}_n}{\partial t}\mathbf{v}dx + B(\mathbf{u}_n,\mathbf{u}_n,\mathbf{v}) + 2\int_\Omega \mu(\lambda_n)D(\mathbf{u}_n)\cdot D(\mathbf{v})\,dx$$
$$+\int_\Omega g(\lambda_n)|D(\mathbf{v})|\,dx - {}_\Omega \mathbf{f}\cdot\mathbf{v}dx \geq \frac{1}{2}\frac{d}{dt}\|\mathbf{u}_n\|^2_{\mathscr{V}_0}$$
$$+2\int_\Omega \mu(\lambda_n)|D(\mathbf{u}_n)|^2\,dx + \int_\Omega g(\lambda_n)|D(\mathbf{u}_n)|\,dx - {}_\Omega \mathbf{f}\cdot\mathbf{u}_n dx \quad \forall \mathbf{v}\in\mathscr{V}_1. \tag{53}$$

Choosing $\mathbf{v}\in L^2(0,T_0;\mathscr{V}_1)$ as test function in the inequality above and integrating over the interval time $(0,T_0)$, if follows

$$\int_{\Omega\times(0,T_0)} \frac{\partial \mathbf{u}_n}{\partial t}\cdot\mathbf{v}dxdt + \int_0^{T_0} B(\mathbf{u}_n,\mathbf{u}_n,\mathbf{v})dt + 2\int_{\Omega\times(0,T_0)} \mu(\lambda_n)D(\mathbf{u}_n)\cdot D(\mathbf{v})\,dxdt$$
$$+\int_{\Omega\times(0,T_0)} g(\lambda_n)|D(\mathbf{v})|\,dxdt - \int_{\Omega\times(0,T_0)} \mathbf{f}\cdot\mathbf{v}dxdt + \frac{1}{2}\|\mathbf{u}_0\|^2_{\mathscr{V}_0} \geq \frac{1}{2}\|\mathbf{u}_n(T_0)\|^2_{\mathscr{V}_0}$$
$$+2\int_{\Omega\times(0,T_0)} \mu(\lambda_n)|D(\mathbf{u}_n)|^2\,dxdt + \int_{\Omega\times(0,T_0)} g(\lambda_n)|D(\mathbf{u}_n)|\,dxdt$$
$$-\int_{\Omega\times(0,T_0)} \mathbf{f}\cdot\mathbf{u}_n dxdt. \tag{54}$$

It is well-known, see [8, 14, 16], that the term $B(\mathbf{u}_n,\mathbf{u}_n,\mathbf{v})$ can easily pass to the limit. Which means that

$$\int_0^{T_0} B(\mathbf{u}_n,\mathbf{u}_n,\mathbf{v})dt \longrightarrow \int_0^{T_0} B(\mathbf{u}_\lambda,\mathbf{u}_\lambda,\mathbf{v})dt \quad \forall \mathbf{v}\in L^2(0,T_0;\mathscr{V}_1). \tag{55}$$

Using now the convergence results (47) and (48), the fact that $\lambda_n \longrightarrow \lambda$ in a.e. in $\Omega\times(0,T_0)$, the hypothesis (9) and Lebesgue's dominated convergence theorem, we obtain, for all $\mathbf{v}\in L^2(0,T_0;\mathscr{V}_1)$, the following convergence results

$$\begin{cases} \int_{\Omega\times(0,T_0)} \mu(\lambda_n)D(\mathbf{u}_n)\cdot D(\mathbf{v})\,dxdt \longrightarrow \int_{\Omega\times(0,T_0)} \mu(\lambda)D(\mathbf{u}_\lambda)\cdot D(\mathbf{v})\,dxdt, \\ \int_{\Omega\times(0,T_0)} \frac{\partial \mathbf{u}_n}{\partial t}\cdot\mathbf{v}dxdt \longrightarrow \int_{\Omega\times(0,T_0)} \frac{\partial \mathbf{u}_\lambda}{\partial t}\cdot\mathbf{v}dxdt, \\ \int_{\Omega\times(0,T_0)} \mathbf{f}\cdot\mathbf{u}_n dxdt \longrightarrow \int_{\Omega\times(0,T_0)} \mathbf{f}\cdot\mathbf{u}_\lambda dxdt. \end{cases} \tag{56}$$

Moreover, we have

$$2\int_{\Omega\times(0,T_0)} \mu(\lambda_n)|D(\mathbf{u}_n)|^2\,dxdt + \int_{\Omega\times(0,T_0)} g(\lambda_n)|D(\mathbf{u}_n)|\,dxdt$$
$$= 2\int_{\Omega\times(0,T_0)} \mu(\lambda)|D(\mathbf{u}_n)|^2\,dxdt + 2\int_{\Omega\times(0,T_0)} \big(\mu(\lambda_n)-\mu(\lambda)\big)|D(\mathbf{u}_n)|^2\,dxdt$$
$$+\int_{\Omega\times(0,T_0)} g(\lambda)|D(\mathbf{u}_n)|\,dxdt + \int_{\Omega\times(0,T_0)} \big(g(\lambda_n)-g(\lambda)\big)|D(\mathbf{u}_n)|\,dxdt.$$

Since $\lambda_n \longrightarrow \lambda$ a.e. in $\Omega\times(0,T_0)$, the functions μ and g are continuous and due to the weak lower semicontinuity, in the space $L^2(0,T_0;\mathscr{V}_1)$, of the continuous and convex functionals $\int_{\Omega\times(0,T_0)} \mu(\lambda)|D(\mathbf{v})|^2\,dxdt$ and $\int_{\Omega\times(0,T_0)} g(\lambda)|D(\mathbf{v})|\,dxdt$, we deduce via Fatou's lemma

$$
\begin{aligned}
&2\liminf \int_{\Omega\times(0,T_0)} \mu(\lambda_n)|D(\mathbf{u}_n)|^2\,dx\,dt + \liminf \int_{\Omega\times(0,T_0)} g(\lambda_n)|D(\mathbf{u}_n)|\,dx\,dt \\
&\geq 2\int_{\Omega\times(0,T_0)} \mu(\lambda)|D(\mathbf{u}_\lambda)|^2\,dx\,dt + \int_{\Omega\times(0,T_0)} g(\lambda)|D(\mathbf{u}_\lambda)|\,dx\,dt.
\end{aligned}
\tag{57}
$$

On the other hand, the fact that

$$
\mathbf{u}_n,\ \mathbf{u}_\lambda \in \mathscr{C}^0([0,T_0];\mathscr{V}_1),
$$

leads, making use (52), to

$$
\liminf \|\mathbf{u}_n(T_0)\|_{\mathscr{V}_0} \geq \|\mathbf{u}_\lambda(T_0)\|_{\mathscr{V}_0}. \tag{58}
$$

Hence, (54)-(58) permit us to conclude that $\mathbf{u}$ solves the inequality (19).

Furthermore, it is necessary to pass to the limit in equation (45). To do this, for each $\phi \in L^2\left(0,T_0;H^1(\Omega)\right)$ we have

$$
\begin{aligned}
&\left|\int_0^{T_0} \left(E\left(C_n,\phi,\mathbf{u}_n\right) - E\left(C_\lambda,\phi,\mathbf{u}_\lambda\right)\right)dt\right| \\
&= \left|\int_{\Omega\times(0,T_0)} \left(\mathbf{u}_n\cdot\nabla(C_n - C_\lambda)\phi + (\mathbf{u}_n - \mathbf{u}_\lambda)\cdot(\nabla C_\lambda)\phi\right)dx\,dt\right|.
\end{aligned}
$$

Lemma 2.1 leads to

$$
\begin{aligned}
\left|\int_0^{T_0} \left(E\left(C_n,\phi,\mathbf{u}_n\right) - E\left(C_\lambda,\phi,\mathbf{u}_\lambda\right)\right)dt\right| &\leq \left|\int_{\Omega\times(0,T_0)} (C_n - C_\lambda)\mathbf{u}_n\cdot\nabla\phi\,dx\,dt\right| \\
&+ \left|\int_{\Omega\times(0,T_0)} C_\lambda(\mathbf{u}_n - \mathbf{u}_\lambda)\cdot\nabla\phi\,dx\,dt\right|.
\end{aligned}
\tag{59}
$$

Elsewhere, via (18), the following inequality holds

$$
\int_{\Omega\times(0,T_0)} \left|\mathbf{u}_n\cdot\nabla\phi\right|^2 dx\,dt \leq \|\mathbf{u}_n(x,t)\|^2_{L^\infty(0,T_0;\mathscr{C}^0(\Omega))} \left\|\phi\right\|^2_{L^2(0,T_0;H^1(\Omega))}
$$

$$
\forall\phi \in L^2\left(0,T_0;H^1(\Omega)\right).
$$

Which implies, using the convergence result (42), that the first integral in the left hand side of (59) tends to zero.

Moreover, since $C_\lambda \in \mathscr{X}(0,T_0)$, Lemma 2.2 shows in particular that

$$
C_\lambda \in L^4\left(0,T_0;L^4(\Omega)\right).
$$

This implies

$$
C_\lambda\nabla\phi \in L^{\frac{4}{3}}\left(0,T_0;L^{\frac{4}{3}}(\Omega)^3\right)\ \ \forall\phi \in L^2\left(0,T_0;H^1(\Omega)\right). \tag{60}
$$

Furthermore, from (52), we can infer

$$\mathbf{u}_n \longrightarrow \mathbf{u}_\lambda \text{ in } L^4\left(0,T_0;L^4(\Omega)^3\right) \text{ strongly.} \tag{61}$$

Combining (60) with (61), we deduce that the second integral in the left hand side of (59) tends also to zero.

Thus, we get

$$\int_0^{T_0} \left(E\left(C_n,\phi,\mathbf{u}_n\right) - E\left(C_\lambda,\phi,\mathbf{u}_\lambda\right)\right) dt \longrightarrow 0 \;\; \forall \phi \in L^2\left(0,T_0;H^1(\Omega)\right). \tag{62}$$

Furthermore, sinse $\lambda_n \longrightarrow \lambda$ a.e. in $\Omega \times (0,T_0)$, the limit below can be easily obtained, via (42), the hypothesis (9) and Lebesgue's dominated convergence theorem

$$\int_{\Omega\times(0,T_0)} \eta(\lambda_n)\nabla C_n \cdot \nabla\phi \, dx\, dt \longrightarrow \int_{\Omega\times(0,T_0)} \eta(\lambda)\nabla C_\lambda \cdot \nabla\phi \, dx\, dt$$
$$\forall \phi \in L^2\left(0,T_0;H^1(\Omega)\right). \tag{63}$$

So that to achieve the passage to the limit in equation (45) it remains to verify that

$$\int_{\Omega\times(0,T_0)} R(C_n)\phi \, dx\, dt \longrightarrow \int_{\Omega\times(0,T_0)} R(C_\lambda)\phi \, dx\, dt \;\; \forall \phi \in L^2\left(0,T_0;H^1(\Omega)\right). \tag{64}$$

Which can be easily obtained taking into account the Lipschitzianity of R and (51).

Our goal now is to prove that

$$C_n \longrightarrow C_\lambda \text{ in } \mathscr{X}(0,T_0) \text{ strongly.} \tag{65}$$

To do this, subtracting the two equations (21) and (45), using $\phi = C_n - C_\lambda$ as test function in the obtained equation and integration over the interval time $(0,T_0)$, we find

$$\frac{1}{2}\|C_n - C_\lambda\|^2_{H^1(\Omega)} + \int_0^{T_0} \left(E(C_n, C_n - C_\lambda, \mathbf{u}_n) - E(C_\lambda, C_n - C_\lambda, \mathbf{u}_\lambda)\right) dt +$$
$$\int_{\Omega\times(0,T_0)} \left(\eta(\lambda_n)\nabla C_n - \eta(\lambda)\nabla C_\lambda\right)\cdot\nabla(C_n - C_\lambda)\, dx\, dt$$
$$= \int_{\Omega\times(0,T_0)} (R(C_n) - R(C_\lambda))(C_n - C_\lambda)\, dx\, dt.$$

Hence, it follows making use the hypothesis (9)

$$\frac{1}{2}\|C_n - C_\lambda\|^2_{H^1(\Omega)} + \int_{\Omega\times(0,T_0)} \eta(\lambda_n)|\nabla(C_n - C_\lambda)|^2\, dx\, dt$$
$$\leq \int_{\Omega\times(0,T_0)} \left|\eta(\lambda_n) - \eta(\lambda)\right| |\nabla C_\lambda| |\nabla(C_n - C_\lambda)|\, dx\, dt +$$
$$\int_0^{T_0} |E(C_n, C_n - C_\lambda, \mathbf{u}_n) - E(C_\lambda, C_n - C_\lambda, \mathbf{u}_\lambda)|\, dt$$
$$= L\int_{\Omega\times(0,T_0)} |C_n - C_\lambda|^2\, dx\, dt. \tag{66}$$

On the other hand, (62) shows, in particular, that

$$\int_0^{T_0} E(C_n, C_n - C_\lambda, \mathbf{u}_n) - E(C_\lambda, C_n - C_\lambda, \mathbf{u}_\lambda)\, dt \longrightarrow 0. \tag{67}$$

Consequently, we get, keeping in mind equation (66), hypothesis (9), the convergence results (50), (51) and the fact that $|\nabla C_\lambda||\nabla(C_n - C_\lambda)|$ belongs in a bounded of $L^1\left(0, T_0; L^1(\Omega)\right)$, via Lebesgue dominated convergence theorem

$$C_n \longrightarrow C_\lambda \text{ in } L^2\left(0, T_0; H^1(\Omega)\right) \cap L^\infty\left(0, T_0; L^2(\Omega)\right) \text{ strongly.} \tag{68}$$

The final object is prove that

$$\frac{\partial C_n}{\partial t} \longrightarrow \frac{\partial C_\lambda}{\partial t} \text{ in } L^2\left(0, T_0; L^2(\Omega)\right) \text{ strongly.} \tag{69}$$

To this end, we can infer by subtraction

$$\begin{aligned}\int_{\Omega\times(0,T_0)} |\mathbf{u}_n \cdot \nabla C_n - \mathbf{u}_\lambda \cdot \nabla C_\lambda|^2\, dx\, dt \;\leq\; & c \int_{\Omega\times(0,T_0)} |\mathbf{u}_n|^2 |\nabla(C_n - C_\lambda)|^2\, dx\, dt + \\ & c \int_{\Omega\times(0,T_0)} |\mathbf{u}_n - \mathbf{u}_\lambda|^2 |\nabla C_\lambda|^2\, dx\, dt\end{aligned}$$

Hence, the use of Hölder's inequality gives, via (18),

$$\begin{aligned}\int_{\Omega\times(0,T_0)} |\mathbf{u}_n \cdot \nabla C_n - \mathbf{u}_\lambda \cdot \nabla C_\lambda|^2\, dx\, dt \leq\; & c\, \|\mathbf{u}_n\|^2_{L^\infty\left(0,T_0;\mathscr{C}^0\left(\bar{\Omega}\right)\right)} \|C_n - C_\lambda\|^2_{H^1(\Omega)} + \\ & c\, \|\mathbf{u}_n - \mathbf{u}_\lambda\|^2_{L^\infty\left(0,T_0;\mathscr{C}^0\left(\bar{\Omega}\right)\right)} \|C_\lambda\|^2_{H^1(\Omega)}\end{aligned} \tag{70}$$

Then, (69) can be deduced making use (52), (68) and (70).

Consequently, C_n converges strongly to C_λ in the space $\mathscr{X}(0, T_0)$. Thus the mapping Λ is compact. By virtue of Schauder's fixed point theorem, the mapping has at least one fixed point $\bar{\lambda} = C_{\bar{\lambda}} \in \mathscr{X}(0, T_0)$.

We conclude finally that the function $(\mathbf{u}_{\bar{\lambda}}, C_{\bar{\lambda}}) \in \mathscr{W}(0, T_0) \times \mathscr{X}(0, T_0)$ solves the problem (P3). □

We prove in the following result that if η depends only on (x, t) and the functions μ and g are Lipschitzian then the solution is unique.

Theorem 3.2.

If we suppose in addition that:
1. η dependsonly on the variable $(x, t) \in \Omega \times (0, T)$.
2. There exists $L_\mu, L_g > 0$ such that for a.e. $(x, t) \in \Omega \times (0, T)$ and every $\phi_1, \phi_2 \in \mathbb{R}$, we have

$$\begin{cases} \left|\mu\left(x, t, \phi_1\right) - \mu\left(x, t, \phi_2\right)\right| \leq L_\mu \left|\phi_1 - \phi_2\right|, \\ \left|g\left(x, t, \phi_1\right) - g\left(x, t, \phi_2\right)\right| \leq L_g \left|\phi_1 - \phi_2\right|. \end{cases} \tag{71}$$

Then, the solution $(\mathbf{u}, C)$ is unique in the space $\mathscr{W}(0, T_0) \times \mathscr{X}(0, T_0)$.

Proof. Suppose that the problem *(P3)* admits two solutions $(\mathbf{u}_1, C_1), (\mathbf{u}_2, C_2) \in \mathscr{W}(0, T_0) \times \mathscr{X}(0, T_0)$.

Then, we get

$$\int_\Omega \frac{\partial \mathbf{u}_1}{\partial t}(\mathbf{v}-\mathbf{u}_1)\,dx + B(\mathbf{u}_1,\mathbf{u}_1,\mathbf{v}) + 2\int_\Omega \mu(C_1)D(\mathbf{u}_1)\cdot D(\mathbf{v}-\mathbf{u}_1)\,dx +$$
$$\int_\Omega g(C_1)|D(\mathbf{v})|\,dx - \int_\Omega g(C_1)|D(\mathbf{u}_1)|\,dx \geq_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u}_1)\,dx \quad \forall \mathbf{v}\in\mathscr{V}_1, \tag{72}$$

$$\int_\Omega \frac{\partial C_1}{\partial t}\phi\,dx + E\left(C_1,\phi,\mathbf{u}_1\right) +_\Omega \eta\nabla C_1\cdot\nabla\phi\,dx$$
$$=_\Omega R(C_1)\phi\,dx \quad \forall\phi\in H^1(\Omega). \tag{73}$$

and

$$\int_\Omega \frac{\partial \mathbf{u}_2}{\partial t}(\mathbf{v}-\mathbf{u}_2)\,dx + B(\mathbf{u}_2,\mathbf{u}_2,\mathbf{v}) + 2\int_\Omega \mu(C_2)D(\mathbf{u}_2)\cdot D(\mathbf{v}-\mathbf{u}_2)\,dx +$$
$$\int_\Omega g(C_2)|D(\mathbf{v})|\,dx - \int_\Omega g(C_2)|D(\mathbf{u}_2)|\,dx \geq_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u}_2)\,dx \quad \forall \mathbf{v}\in\mathscr{V}_1, \tag{74}$$

$$\int_\Omega \frac{\partial C_2}{\partial t}\phi\,dx + E\left(C_2,\phi,\mathbf{u}_2\right) +_\Omega \eta\nabla C_2\cdot\nabla\phi\,dx$$
$$=_\Omega R(C_2)\phi\,dx \quad \forall\phi\in H^1(\Omega). \tag{75}$$

We find by subtracting the two equation (73) and (75), setting $C_2 - C_1$ as test function in the obtained equation, integrating from 0 to $t\in(0,T_0)$, employing the Lipschitzianity of R and Lemma 2.1

$$\frac{1}{2}\|C_2-C_1\|^2_{L^2(\Omega)} + \int_0^t E(C_1, C_2-C_1, \mathbf{u}_2-\mathbf{u}_1)\,ds + c_1\int_0^t \|C_2-C_1\|^2_{H^1(\Omega)}\,ds$$
$$\leq L\int_0^t \|C_2-C_1\|^2_{L^2(\Omega)}\,ds \;\text{ a.e. } t\in(0,T_0),$$

which gives by Hôlder's inequality

$$\frac{1}{2}\|C_2-C_1\|^2_{L^2(\Omega)} + c_1\int_0^t \|C_2-C_1\|^2_{H^1(\Omega)}\,ds \leq L(T_0|\Omega|)^{\frac{1}{2}}\left(\int_0^t \|C_2-C_1\|^4_{L^4(\Omega)}\,ds\right)^{\frac{1}{2}} +$$
$$c\left(\int_0^t \|\nabla C_1\|^2_{L^2(\Omega)^3}\,ds\right)^{\frac{1}{2}}\left(\int_0^t \|C_2-C_1\|^4_{L^4(\Omega)}\,ds\right)^{\frac{1}{4}}\left(\int_0^t \|\mathbf{u}_2-\mathbf{u}_1\|^4_{L^4(\Omega)^3}\,ds\right)^{\frac{1}{4}} \;\text{ a.e. } t\in(0,T_0).$$

Hence, since $C_1 \in \mathscr{X}(0,T_0)$, we get

$$\frac{1}{2}\|C_2-C_1\|^2_{L^2(\Omega)}+c_1\int_0^t \|C_2-C_1\|^2_{H^1(\Omega)}\,ds \le c\left(\int_0^t \|C_2-C_1\|^4_{L^4(\Omega)}\,ds\right)^{\frac{1}{2}}+$$
$$c\left(\int_0^t \|C_2-C_1\|^4_{L^4(\Omega)}\,ds\right)^{\frac{1}{4}}\left(\int_0^t \|\mathbf{u}_2-\mathbf{u}_1\|^4_{L^4(\Omega)^3}\,ds\right)^{\frac{1}{4}}\quad \text{a.e. } t\in(0,T_0). \tag{76}$$

On the other hand, choosing $\mathbf{v}=\mathbf{u}_2$ and $\mathbf{v}=\mathbf{u}_1$ as test function in the inequalities (72) and (74), respectively, subtracting the two obtained inequalities and integrating over the interval time $(0,T_0)$, we obtain after some calculations

$$\frac{1}{2}\|\mathbf{u}_2-\mathbf{u}_1\|^2_{\mathscr{V}_0}+2\mu_*\int_0^t \|D(\mathbf{u}_2-\mathbf{u}_1)\|^2_{L^2(\Omega)^9}\,ds$$
$$\le \int_0^t \left(|B(\mathbf{u}_2,\mathbf{u}_2,\mathbf{u}_2-\mathbf{u}_1)-B(\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_2-\mathbf{u}_1)|\right)ds+$$
$$\int_0^t \left\|\mu(C_2)-\mu(C_1)\right\|_{L^4(\Omega)}\|D(\mathbf{u}_2)\|_{L^4(\Omega)^9}\|D(\mathbf{u}_2-\mathbf{u}_1)\|_{L^2(\Omega)^9}\,ds+$$
$$\int_0^t \left\|g(C_2)-g(C_1)\right\|_{L^2(\Omega)}\|D(\mathbf{u}_2-\mathbf{u}_1)\|_{L^2(\Omega)^9}\,ds\quad \text{a.e. } t\in(0,T_0). \tag{77}$$

Moreover, Lemma 2.1 shows that

$$\int_0^t \left(|B(\mathbf{u}_2,\mathbf{u}_2,\mathbf{u}_2-\mathbf{u}_1)-B(\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_2-\mathbf{u}_1)|\right)ds=\int_0^t |B(\mathbf{u}_2-\mathbf{u}_1,\mathbf{u}_2-\mathbf{u}_1,\mathbf{u}_2)|\,ds.$$

Thus, thinks to Hölder's and Young's inequalities, one can find, taking into account the fact that $\mathbf{u}_2 \in \mathscr{W}(0,T_0)$

$$\int_0^t \left(|B(\mathbf{u}_2,\mathbf{u}_2,\mathbf{u}_2-\mathbf{u}_1)-B(\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_2-\mathbf{u}_1)|\right)ds$$
$$\int_0^t \|\mathbf{u}_2-\mathbf{u}_1\|_{L^3(\Omega)^3}\|\nabla(\mathbf{u}_2-\mathbf{u}_1)\|_{L^2(\Omega)^9}\|\mathbf{u}_2\|_{L^6(\Omega)^3}\,ds$$
$$\le\ c\int_0^t \left(\|\mathbf{u}_2-\mathbf{u}_1\|^2_{L^3(\Omega)^3}+\|\nabla(\mathbf{u}_2-\mathbf{u}_1)\|^2_{L^2(\Omega)^9}\right)ds.$$

Exploiting now the following interpolation inequality, see [8]

$$\|v\|_{L^3(\Omega)}\le c\,\|v\|^{\frac{1}{2}}_{L^2(\Omega)}\|v\|^{\frac{1}{2}}_{H^1(\Omega)}\quad \forall v\in H^1_0(\Omega), \tag{78}$$

to get, using Poincaré's inequality

$$\int_0^t (|B(\mathbf{u}_2,\mathbf{u}_2,\mathbf{u}_2-\mathbf{u}_1)-B(\mathbf{u}_1,\mathbf{u}_1,\mathbf{u}_2-\mathbf{u}_1)|)\,ds$$
$$\leq c\int_0^t \left(\|\mathbf{u}_2-\mathbf{u}_1\|^2_{\mathscr{V}_0}+\|\nabla(\mathbf{u}_2-\mathbf{u}_1)\|^2_{L^2(\Omega)^9}\right)ds. \tag{79}$$

Combined (79) with (77), we deduce, using Young's and Korn's inequalities, as well as the fact that $\mathbf{u}_2\in\mathscr{W}(0,T_0)$, that there exists a constant $c>0$ such that for a.e. $t\in(0,T_0)$

$$\|\mathbf{u}_2-\mathbf{u}_1\|^2_{\mathscr{V}_0}+\int_0^t \|D(\mathbf{u}_2-\mathbf{u}_1)\|^2_{L^2(\Omega)^9}\,ds$$
$$\leq c\int_0^t \left(\|\mathbf{u}_2-\mathbf{u}_1\|^2_{\mathscr{V}_0}+\left\|\mu(C_2)-\mu(C_1)\right\|^2_{L^4(\Omega)}+\left\|g(C_2)-g(C_1)\right\|^2_{L^2(\Omega)}\right)ds. \tag{80}$$

In particular, Gronwall's lemma leads to

$$\|\mathbf{u}_2-\mathbf{u}_1\|^2_{\mathscr{V}_0}\leq c\int_0^t \left(\left\|\mu(C_2)-\mu(C_1)\right\|^2_{L^4(\Omega)}+\left\|g(C_2)-g(C_1)\right\|^2_{L^2(\Omega)}\right)ds \text{ a.e. } t\in(0,T_0).$$

This yields, making use again (80), the Lipschitzianity of the functions μ and g and Hölder's inequality

$$\|\mathbf{u}_2-\mathbf{u}_1\|^2_{\mathscr{V}_0}+\int_0^t \|D(\mathbf{u}_2-\mathbf{u}_1)\|^2_{L^2(\Omega)^9}\,ds\leq c\left(\int_0^t \|C_2-C_1\|^4_{L^4(\Omega)}\,ds\right)^{\frac{1}{2}} \text{ a.e. } t\in(0,T_0).$$

Hence, by Lemma 2.2, the following estimate holds

$$\int_0^t \|\mathbf{u}_2-\mathbf{u}_1\|^4_{L^4(\Omega)^3}\,ds\leq c\int_0^t \|C_2-C_1\|^4_{L^4(\Omega)}\,ds \text{ a.e. } t\in(0,T_0). \tag{81}$$

Substituting (81) in the inequality (76), we derive the estimate

$$\frac{1}{2}\|C_2-C_1\|^2_{L^2(\Omega)}+c_1\int_0^t \|C_2-C_1\|^2_{H^1(\Omega)}\,ds\leq c\left(\int_0^t \|C_2-C_1\|^4_{L^4(\Omega)}\,ds\right)^{\frac{1}{2}} \text{ a.e. } t\in(0,T_0).$$

Then, by Hölder's inequality, we can infer

$$\frac{1}{2}\|C_2-C_1\|^4_{L^2(\Omega)}+c_1^2\left(\int_0^t \|C_2-C_1\|^2_{H^1(\Omega)}\,ds\right)^2$$
$$\leq c\int_0^t \|C_2-C_1\|_{L^2(\Omega)}\|C_2-C_1\|^3_{L^6(\Omega)}\,ds \text{ a.e. } t\in(0,T_0).$$

Thus, by application of Young's inequality and Lemma 2.2, it follows

$$\|C_2 - C_1\|^4_{L^2(\Omega)} + \|C_2 - C_1\|^4_{L^2(0,t;H^1(\Omega))} \leq c \int_0^t \|C_2 - C_1\|^4_{L^2(\Omega)}\, ds \text{ a.e. } t \in (0, T_0). \tag{82}$$

We deduce finally, using Gronwall's lemma, that $C_1 = C_2$ and, taking into account Proposition 3.1, we find $\mathbf{u}_1 = \mathbf{u}_2$. Which permits us to conclude the proof. □

We prove now, under some supplementary hypothesis, a maximum principle for the concentration function.

Theorem 3.3 (Positivity of the concentration).
Let the hypotheses of Theorem 3.1 hold and suppose in addition that

$$R(x, t, \phi) \geq 0 \text{ a.e. in } \Omega \times (0, T) \text{ and } \forall \phi \in \mathbb{R}, \tag{83}$$

$$C_0 \geq 0 \text{ a.e. in } \Omega \times (0, T). \tag{84}$$

Then, the solution $\{\mathbf{u}, C\}$ *to problem (P2) is such that*

$$C(x, t) \geq 0 \text{ for a.e. } (x, t) \in \Omega \times (0, T_0). \tag{85}$$

Proof. We use a maximum principle argument. Thus, we test the equation (16) by the function $-C^-$, where ϕ^- denoting the so-called negative part of a function ϕ i.e. $\phi^- = \max\{0, -\phi\}$, and integrate over $(0, t)$. We can infer, using Lemma 2.1

$$\frac{1}{2}\left\|C^-\right\|^2_{L^\infty(0,t;L^2(\Omega))} - \frac{1}{2}\left\|C_0^-\right\|^2_{L^2(\Omega)} + c_1\left\|C^-\right\|^2_{L^2(0,t;H^1(\Omega))} \leq -{}_0^t{}_\Omega R\left(C^-\right) C^- \,dx\,ds \text{ a.e. } t \in (0, T_0).$$

Consequently, hypothesis (83) and (84) permit us to deduce that

$$\left\|C^-\right\|_{L^2(0,T_0;H^1(\Omega)) \cap L^\infty(0,T_0;L^2(\Omega))} \leq 0,$$

which eventually gives (85). □

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