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Existence of fractional order mixed type functional integro-differential equations with nonlocal conditions

Research Article

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Abstract: In this paper, we prove the existence of mild solutions for the semilinear fractional order functional of Volterra-Fredholm type differential equations with nonlocal conditions in a Banach space. The results are obtained by using the theory of fractional calculus, the analytic semigroup theory of linear operators and the fixed point techniques.

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1. Introduction

During the last few years, the theory of fractional calculus and fractional differential differential systems have been received much attention due to their applications in various fields of technical sciences, engineering and technolgy (see [20, 28, 30, 36, 41]). Differential and integral operators of fractional -order do share some of the characterstics and properties exhibited by the processes associated with complex syetems having long-memory in time. This feautre has contributed siginificantly to the popularity of the subject and has motivated many researchers and scientists to focus on fractional order models. For more details (see [7, 14, 27, 31, 32, 42, 50, 51]).

Impulsive differential equations have become more important in recent years in some mathematical models of processes and phenomena studied in physics, optimal control, chemotherapy, biotechnology, population dynamics and ecology, we refer the monographs (see [6, 13, 29, 46]) and the papers (see [1–4, 17, 19, 23, 24, 33, 34, 44, 47]) and references therein.

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Nowadays many authors have been studied the existence results combined with fractional derivative and impulsive conditions (see [11, 12, 18, 25, 37, 43, 48]).

The nonlocal Cauchy problem for abstract evolution differential equation was first considered by Byszewski (see [16]). Subsequently, several authors have investigated the problem for different types of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces (see [8–10, 15, 21])

Recently Mophou et al. (see [38–40]) studied some semilinear fractional differential equations with nonlocal conditions and neutral fractional functional evolution equations with infinite delay in Banach spaces, and Matar (see [35]) studied the existence and uniqueness for fractional semilinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions in Banach Spaces, and Anguraj et al. (see [5]) discussed the nonlocal impulsive fractional semilinear differential equations with almost sectorial operators. Using the concepts of above mentioned papers, we proved the existence of mild solutions for the semilinear fractional order functional of integrodifferential equations with nonlocal conditions in a Banach space.

Here we consider the existence of mild solutions for the fractional order semilinear functional integrodifferential equations with nonlocal conditions as follows,

$$D^{\alpha}x(t)=Ax(t)+f(t,x(t),Kx(t),Hx(t)),\quad t\in J=[0,T],\ t\neq t_k, \tag{1}$$

$$\Delta x|_{t=t_k}=I_k(x(t_k^-)),\ k=1,2,\cdots,m, \tag{2}$$

$$x(0)=x_0+g(x), \tag{3}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $\{T(t),t\geq 0\}$ on a Banach space $\mathbb{X}$, $f : J\times\mathbb{X}\times\mathbb{X}\times\mathbb{X}\to\mathbb{X}$, is continuous.

$$Kx(t)=\int_0^t K(t,s)x(s)ds,\quad K\in C[D,R^+], Hx(t)=\int_0^{T_0} H(t,s)x(s)ds,\quad H\in C[D_0,R^+],$$

$D=\{(t,s)\in R^2 ; 0\leq s\leq t\leq T_0\}$, $D_0=\{(t,s)\in R^2 : 0\leq t,\ s\leq T_0\}$, and $0=t_0<t_1<,\cdots,<t_m<t_{m+1}=T_0$, $I_k\in C(X,X)$ $(k=1,...,m)$ are bounded functions. $\Delta x|_{t=t_k}=x(t_k^+)-x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ represent the left and right limits of x(t) at $t=t_k$ respectively. The results are obtained by using Banach contraction mapping principle and Kraosnoselskii fixed point theorem.

In this paper, we use the analytic semigroup theory of linear operators and fixed point method to prove the existence and uniqueness of mild solution. In Section 2, we present some definition and preliminary facts. In Sections 3, we prove the existence results of PC-mild solutions for the fractional impulsive mixed nonlocal Cauchy problem (1-3).

2. Preliminaries

In this section, we give some notations about sectorial operators, solution operators, analytic solution operators and then present the definition of a mild solution of (1-3) by investigating the classical solutions of (1-3).

An operator A is said to be sectorial if there are constants $\omega \in R$, $\theta \in [\pi/2, \pi]$, $M > 0$ such that the following two conditions are satisfied:

$$\begin{cases} (1) \quad \rho(A) \subset \sum_{\theta,\omega} = \{\lambda \in C : \lambda \neq \omega, \ |arg(\lambda-\omega)| < \theta\}, \\ (2) \quad ||R(\lambda, A)||_{L(X)} \leq \frac{M}{|\lambda-\omega|}, \ \lambda \in \sum_{\theta,\omega}. \end{cases}$$

Consider the following Cauchy problem for the Caputo fractional derivative evolution equation of order $\alpha(m-1 < \alpha < m, \ m > 0$ is an integer):

$$\begin{cases} D^{\alpha}x(t) = Ax(t), \\ x(0) = x, \ x^{(k)}(0) = 0, \ k = 1, 2, \cdots, m-1 \end{cases} \tag{4}$$

where A is a sectorial operator. The solution operators $S_\alpha(t)$ of (4) is defined by (see [26])

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda,$$

where Γ is a suitable path lying on $\sum_{\theta,\omega}$.

An operator A is said to belong to $C^\alpha(X; M, \omega)$, if problem (4) has a solution operator $S_\alpha(t)$ satisfying $||S_\alpha(t)|| \leq Me^{\omega t}$, $t \geq 0$. Denote $C^\alpha(\omega) := \{C^\alpha(X; M, \omega) : M \geq 1\}$, and $C^\alpha := \{C^\alpha(\omega) : \omega \geq 0\}$.

Definition 2.1 (see [49]).
A solution operator $S_\alpha(t)$ of (4) is called analytic if $S_\alpha(t)$ admits an analytic extension to a sector $\sum_{\theta_0} := \{\lambda \in C\backslash\{0\} : |arg\lambda| < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type, (θ_0, ω_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is an $M = M(\theta, \omega)$ such that $||S_\alpha(t)|| \leq Me^{\omega Ret}$, $\sum_\theta := \{t \in C\backslash\{0\} : |arg t| < \theta\}$. Denote $A^\alpha(\theta_0, \omega_0) :=$
$\{A \in C^\alpha := A \ generates \ analytic \ solution \ operators S_\alpha(t) \ of \ type \ (\theta_0, \omega_0)\}$.

Lemma 2.1 (see [49]).
Let $\alpha \in (0, 2)$, a linear closed densely defined operator A belong to $A^\alpha(\theta_0, \omega_0)$ iff $\lambda^\alpha \in \rho(A)$ for each $\lambda \in \sum_{\theta_0+\frac{\pi}{2}}$, and for any $\theta < \theta_0$, $\omega > \omega_0$, there is a constant $C = C(\theta, \omega)$ such that

$$\left\|\lambda^{\alpha-1}R(\lambda^\alpha, A)\right\| \leq \frac{C}{|\lambda-\omega|}, \ \lambda \in \sum_{\theta+\frac{\pi}{2}}(\omega).$$

Next, we consider the definition of the mild solution of (1-3). Consider the following Cauchy problem

$$\begin{cases} D_*^\alpha x(t) = Ax(t) + f(t), \ \ 0 < \alpha < 1, \\ x(t) = x_0 \in \mathbb{X}. \end{cases} \tag{5}$$

where f is an abstract function defined on $[0, \infty)$ and with values in $\mathbb{X}$

Theorem 2.1 (see [49]).
If f satisfies the uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem (5) is given by

$$x(t) = S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f(s)ds, \tag{6}$$

where

$$S_\alpha(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda, \;\; T_\alpha(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^\alpha, A)d\lambda$$

and Γ is suitable path lying on $\sum_{\theta,\omega}$.

Lemma 2.2 (see [49]).
If $\alpha \in (0,1)$ and $A \in A^\alpha(\theta_0, \omega_0)$, then for any $x \in X$ and $t > 0$, we have

$$||T_\alpha(t)|| \leq Ce^{\omega t}(1 + t^{\alpha-1}), \;\; t > 0, \; \omega > \omega_0.$$

Lemma 2.3 (see [45]).
For $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are continuous in the uniform operator topology. Moreover, for every $r > 0$, the continuity is uniform on $[r, \infty)$.

Theorem 2.2.
If f satisfies a uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator, then any solution of the Cauchy problem (1-3) is a fixed point of the operator given below

$$\Gamma x(t) = \begin{cases} S_\alpha(t)(x_0 + g(x)) \\ \quad + \int_0^t T_\alpha(t-s)f(s, x(s), Kx(s), Hx(s))ds, \; t \in [0, t_1]; \\ S_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ \quad + \int_{t_1}^t T_\alpha(t-s)f(s, x(s), Kx(s), Hx(s))ds, \; t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t-t_m)[x(t_m^-) + I_m(x(t_m^-))] \\ \quad + \int_{t_m}^t T_\alpha(t-s)f(s, x(s), Kx(s), Hx(s))ds, \; t \in [t_m, T_0]; \end{cases}$$

In fact, from (6) it is easy to see that Theorem 2.2 holds.

In order to define the concept of mild solution, we need to define the following spaces. $PC(J, \mathbb{X}) = \{x : J \to \mathbb{X} : x \in C((t_k, t_{k+1}], \mathbb{X}), \mathbb{X}), \; k = 0, 1, 2, \cdots, m \; and \; there \; exist$
$x(t_k^-) \; and \; x(t_k^+), \; k = 1, 2, \cdots, m \; with \; x(t_k^-) = x(t_k)\}$.

Endowed with the norm $||x||_{PC} = \sup_{t \in J} ||x(t)||$, $(PC(J, \mathbb{X}), ||.||)$ is a Banach space. From Theorem 2.2, we can define the mild solution of (1-3)).

Definition 2.2.
A function $x : J \to \mathbb{X}$ is called a solution of (1-3) if $x \in PC(J, \mathbb{X})$ and satisfies the following equation

$$x(t) = \begin{cases} S_\alpha(t)(x_0 + g(x)) \\ \quad + \int_0^t T_\alpha(t-s)f(s, x(s), Kx(s), Hx(s))ds, \quad t \in [0, t_1]; \\ S_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ \quad + \int_{t_1}^t T_\alpha(t-s)f(s, x(s), Kx(s), Hx(s))ds, \quad t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t-t_m)[x(t_m^-) + I_m(x(t_m^-))] \\ \quad + \int_{t_m}^t T_\alpha(t-s)f(s, x(s), Kx(s), Hx(s))ds, \quad t \in [t_m, T_0]; \end{cases}$$

Now to prove the existence of mild solution of IVP (1-3).

3. Existence results

In this section, we study the existence of mild solutions for the system (1-3). To establish our results, we introduce the following conditions;

(H1) $f:[0,T_0]\times\mathbb{X}\times\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ is continuous, and there exists $L_f>0$ such that

$$||f(t,\varphi_1,\varphi_2,\varphi_3)-f(t,\Psi_1,\Psi_2,\Psi_3)||\le L_f\Big[||\varphi_1-\Psi_1||+||\varphi_2-\Psi_2||+||\varphi_3-\Psi_3||\Big],$$
$$t\in J,\ \varphi_i,\ \Psi_i\in\mathbb{X},\ i=1,2,3\ \text{ with}$$
$$||f(t,0,0,0)||\le m_1,$$

(H2) $K^*=\sup_{t\in[0,T_0]}\int_0^t |K(t,s|dt<\infty,\ H^*=\sup_{t\in[0,T_0]}\int_0^{T_0}|H(t,s)|dt<\infty,\ N_s=\sup_{0<t<T_0}||S_\alpha(t)||$ and $N_T=\sup_{0<t<T_0}Ce^{\omega t}(1+t^{1-\alpha})$.

(H3) $g:PC([0,T_0],\mathbb{X})$ is continuous and there exists a constant $L_g>0$ such that

$$||g(x)-g(y)||\le L_g||x-y||\ \forall\ x,y\in PC([0,T_0],\mathbb{X})\ \text{ with}$$
$$||g(0)||\le m_2.$$

(H4) For each $k=1,2,\cdots,m$, there exist a $\rho_k>0$ such that

$$||I_k(x)-I_k(y)||\le\ \rho_k||x-y||,\quad \forall x,y\in\mathbb{X}\ \text{ with}$$
$$||I_k(0)||\le m_3.$$

(H5) For each $x_0\in\mathbb{X}$, there exists a constant $r>0$ such that

$$r\ge\max_{1\le i\le m}\Big\{N_s[||x_0||+r(1+\rho_i+L_g)+m_2+m_3] \\ +\frac{N_T T^\alpha}{\alpha}\big[L_f T_0(1+K^*+H^*)r+m_1\big]\Big\}.$$

We have the following theorem regarding the existence and uniqueness of mild solution for the IVP (1-3).

Theorem 3.1.
Assume conditions (H1)-(H4) are satisfied, then the problem (1-3) has a unique mild solution provided that

$$\Gamma:=N_s(\rho_i+1)+\frac{1}{\alpha}N_T L_f(1+K^*+H^*)T_0T^\alpha<1. \tag{7}$$

Proof. Transform the problem (1-3) into a fixed point problem. Consider the operator $F : PC(J,\mathbb{X}) \to PC(J,\mathbb{X})$ defined by

$$Fx(t) = \begin{cases} S_\alpha(t)(x_0 + g(x)) \\ \quad + \int_0^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s))\, ds, \ t \in [0, t_1]; \\ S_\alpha(t - t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ \quad + \int_{t_1}^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s))\, ds, \ t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t - t_m)[x(t_m^-) + I_m(x(t_m^-))] \\ \quad + \int_{t_m}^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s))\, ds, \ t \in [t_m, T_0]; \end{cases}$$

Let $x, y \in PC(J,\mathbb{X})$, then for each $t \in (0, t_1]$, we have

$$\begin{aligned} \|Fx(t) - Fy(t)\| &\le N_s L_g \|x(s) - y(s)\|_{PC} + N_T \int_0^t (t-s)^{\alpha-1} \\ &\times \big[\|f(s, x(s), Kx(s), Hx(s)) - f(s, y(s), Ky(s), Hy(s))\|\big] \\ &\le N_s L_g \|x(s) - y(s)\|_{PC} + N_T \int_0^t (t-s)^{\alpha-1} L_f \\ &\times \big[\|x(s) - y(s)\|_{PC} + \|Kx(s) - Ky(s)\| + \|Hx(s) - Hy(s)\|\big]\, ds. \end{aligned} \tag{8}$$

Now,

$$\begin{aligned} \int_0^t \|Kx(s) - Ky(s)\| ds &\le \int_0^t \int_0^s \|K(s,\tau)\|\, \|x(\tau) - y(\tau)\| d\tau\, ds \\ &\le \int_0^t \|x(s) - y(s)\| \int_0^s \|K(s,\tau)\| d\tau\, ds \\ &\le \|x(t) - y(t)\| \int_0^t K^* ds \\ &\le \|x - y\|_{PC} K^* T_0. \end{aligned} \tag{9}$$

Similarly,

$$\int_0^t \|Hx(s) - Hy(s)\| ds \le \|x - y\|_{PC} H^* T_0. \tag{10}$$

Substitute (9) and (10) in (8) we get

$$\begin{aligned} \|Fx(t) - Fy(t)\| &\le N_s L_g \|x - y\|_{PC} \\ + \frac{1}{\alpha} N_T T^\alpha L_f &\big[T_0 \|x - y\|_{PC} + K^* T_0 \|x - y\|_{PC} + H^* T_0 \|x - y\|_{PC}\big] \\ &\le \Big[N_s L_g + \frac{1}{\alpha} N_T T^\alpha L_f (1 + K^* + H^*) T_0\Big] \|x - y\|_{PC}. \end{aligned}$$

For $t \in (t_1, t_2]$, we have

$$\begin{aligned} \|Fx(t) - Fy(t)\| &\le N_s[\|x(t_1^-) - y(t_1^-)\| + \rho\|x(t_1^-) - y(t_1^-)\|] \\ &\quad + N_T \int_0^t (t-s)^{\alpha-1} L_f[1 + K^* + H^*]\|x(s) - y(s)\| ds \\ &\le \left[N_s(\rho_1 + 1) + \frac{1}{\alpha} N_T L_f(1 + K^* + H^*) T_0 T^\alpha\right] \|x - y\|_{PC}. \end{aligned}$$

Similarly, we have

$$\|Fx(t) - Fy(t)\| \le \left[N_s(\rho_i + 1) + \frac{1}{\alpha} N_T L_f(1 + K^* + H^*) T_0 T^\alpha\right] \|x - y\|_{PC}, \quad t \in (t_i, t_{i+1}],$$

and

$$\|Fx(t) - Fy(t)\| \le \left[N_s(\rho_m + 1) + \frac{1}{\alpha} N_T L_f(1 + K^* + H^*) T_0 T^\alpha\right] \|x - y\|_{PC}, \quad t \in (t_m, T_0].$$

Then, for each $t \in [0, T_0]$, we have

$$\begin{aligned} \|Fx(t) - Fy(t)\| &\le \max_{1 \le i \le m} \left[N_s(\rho_i + 1) + \frac{1}{\alpha} N_T L_f(1 + K^* + H^*) T_0 T^\alpha\right] \|x - y\|_{PC}, \\ &\le \Gamma \|x - y\|_{PC} \end{aligned}$$

Since $0 < \Gamma < 1$, therefore F is a contraction operator, hence F has a unique fixed point by the Banach contraction principle. That is problem (1-3) has a unique mild solution. □

Our next result is based on Krasnoselskii's fixed point theroem.

Lemma 3.1 (Krasnoselskii's Fixed point theorem [22]).
Let E be a bounded closed convex and nonempty subset of a Banach space $\mathbb{X}$. Let F_1 and F_2 be two operators such that $(i) F_1 x + F_2 y \in E$ whenever $x, y \in E$, (ii) F_1 is a contraction, $(iii) F_2$ is completely continuous. Then there exists $z \in E$ such that $z = F_1 z + F_2 z$.

Theorem 3.2.
Assume that the hypothesis (H1)-(H5) are satisfied, then the system has atleast one mild solution on J.

Proof. Transform the problem (1-3) into a fixed point problem, consider the operator $F : PC(J, \mathbb{X}) \to PC(J, \mathbb{X})$ define by

$$Fx(t) = \begin{cases} S_\alpha(t)(x_0 + g(x)) \\ \quad + \int_0^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s)) ds, \ t \in [0, t_1]; \\ S_\alpha(t - t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ \quad + \int_{t_1}^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s)) ds, \ t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t - t_m)[x(t_m^-) + I_m(x(t_m^-))] \\ \quad + \int_{t_m}^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s)) ds, \ t \in [t_m, T_0]; \end{cases}$$

Define B_r as $B_r = \{x \in PC(J,\mathbb{X}) : ||x||_{PC} \leq r\}$. Then B_r is a closed, bounded and convex subset of $PC(J,\mathbb{X})$. On B_r, we define the operators F_1 and F_2 as follows.

$$F_1 x(t) = \begin{cases} S_\alpha(t)(x_0 + g(x)), & t \in [0, t_1]; \\ S_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))], & t \in (t_1, t_2]; \\ \vdots & \\ S_\alpha(t-t_m)[x(t_m^-) + I_m(x(t_m^-))], & t \in [t_m, T_0]; \end{cases}$$

and

$$F_2 x(t) = \begin{cases} \int_0^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s))\, ds, & t \in [0, t_1]; \\ \int_{t_1}^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s))\, ds, & t \in (t_1, t_2]; \\ \vdots & \\ \int_{t_m}^t T_\alpha(t-s) f(s, x(s), Kx(s), Hx(s))\, ds, & t \in [t_m, T_0]; \end{cases}$$

Now we show that $F_1 + F_2$ has a fixed point in B_r. The proof is divided into three steps.

Step 1 $F_1 x + F_2 y \in B_r$, for every pair $x, y \in B_r$, consider for any $x, y \in B_r$ and for $t \in [0, t_1]$ we have

$$||F_1 x(t) + F_2 y(t)|| \leq ||S_\alpha(t)|| \Big[||x_0|| + ||g(x) - g(0)|| + ||g(0)|| \Big] + \int_0^t ||T_\alpha(t-s)|| \Big[||f(s, y(s), Ky(s), Hy(s)) - f(s,0,0,0)|| + ||f(s,0,0,0)|| \Big] ds$$

$$\leq N_s \Big[||x_0|| + L_g ||x|| + m_2 \Big] + N_T L_f \Big[||y|| (1 + K^* + H^*) T_0 + m_1 \Big] \int_0^t (t-s)^{\alpha-1} ds \quad \leq N_s \Big[||x_0|| + L_g r + m_2 \Big] + \frac{N_T T^\alpha}{\alpha} L_f \Big[r(1 + K^* + H^*) T_0 + m_1 \Big].$$

For $t \in (t_1, t_2]$, we have

$$\begin{aligned} ||F_1 x(t) + F_2 y(t)|| &\leq ||S_\alpha(t-t_1)|| \, [||x(t_1^-)|| + ||I_1(x(t_1^-)) - I_1(0)|| + ||I_1(0)|| \\ &+ \int_{t_1}^t ||T_\alpha(t-s)|| \, [||f(s, y(s), Ky(s), Hy(s)) - f(s,0,0,0)|| \\ &+ ||f(s,0,0,0)||] ds \\ &\leq N_s [r + \rho_1 r + m_3] + \frac{N_T T^\alpha}{\alpha} \big[L_f T_0 (1 + K^* + H^*) r + m_1 \big] \\ &\leq N_s [r(1+\rho_1) + m_3] + \frac{N_T T^\alpha}{\alpha} \big[L_f T_0 (1 + K^* + H^*) r + m_1 \big] \end{aligned}$$

Similarly, we have

$$||F_1 x(t) + F_2 y(t)|| \leq N_s [r(1+\rho_i) + m_3] + \frac{N_T T^\alpha}{\alpha} \big[L_f T_0 (1 + K^* + H^*) r + m_1 \big], \quad \forall t \in (t_i, t_{i+1}],$$

and

$$||F_1 x(t) + F_2 y(t)|| \leq N_s [r(1+\rho_m) + m_3] + \frac{N_T T^\alpha}{\alpha} \big[L_f T_0 (1 + K^* + H^*) r + m_1 \big], \quad \forall t \in (t_m, T_0].$$

Thus, for all $t \in [0, T_0]$ and by (H_5), we have

$$\begin{aligned}||F_1 x(t) + F_2 y(t)|| \leq \max_{1\leq i\leq m} \Big\{ N_s[||x_0|| + r(1+\rho_i + L_g) + m_2 + m_3] \\ + \frac{N_T T^\alpha}{\alpha}\left[L_f T_0(1+K^* + H^*)r + m_1\right]\Big\} \\ \leq r.\end{aligned}$$

which means that $F_1 x + F_2 y \in B_r$, for any $x, y \in B_r$.

Step 2: F_1 is contraction on B_r.

Let $x, y \in B_r$. By (H2) - (H4) for each $t \in [0, t_1]$,

$$\begin{aligned}||F_1 x(t) - F_1 y(t)|| &\leq ||S_\alpha(t)||\, ||g(x) - g(y)|| \\ &\leq N_s L_g ||x - y|| \\ &\leq L_g N_s ||x - y||\end{aligned}$$

For $t \in (t_1, t_2]$,

$$\begin{aligned}||F_1 x(t) - F_1 y(t)|| &\leq ||S_\alpha(t - t_1)|| \left[||x(t_1^-) - y(t_1^-)|| + ||I_1(x(t_1^-)) - I_1(y(t_1^-))||\right] \\ &\leq N_s[1+\rho_1]||x - y||\end{aligned}$$

Similarly, for all $t \in (t_i, t_{i+1}]$,

$$||F_1 x(t) - F_1 y(t)|| \leq N_s(\rho_i + 1)||x - y||$$

and therefore for all $t \in (t_m, T_0]$,

$$||F_1 x(t) - F_1 y(t)|| \leq N_s(\rho_m + 1)||x - y||$$

Thus, for all $t \in [0, T_0]$,

$$\begin{aligned}||F_1 x(t) - F_1 y(t)|| &\leq \max_{1\leq i\leq m}\{N_s(L_g + \rho_i + 1)\}||x - y|| \\ &\leq \Gamma\, ||x - y||\end{aligned}$$

Thus, from equation (7) , F_1 is contraction on B_r.

Step 3: Now, we show that F_2 is completely continuous operator.

For that consider, for any $t \in [0, t_1]$, we have

$$\begin{aligned}||F_2 x(t)|| &\leq \int_0^t ||T_\alpha(t-s)||\, ||f(s, x(s), Kx(s), Hx(s)||ds \\ &\leq \frac{N_T T^\alpha}{\alpha}\left[L_f T_0(1+K^* + H^*)r + m_1\right]\end{aligned}$$

Similarly, for all $t \in (t_i, t_{i+1}]$,

$$||F_2 x(t)|| \leq \frac{N_T T^\alpha}{\alpha}\left[L_f T_0(1+K^*+H^*)r + m_1\right]$$

Thus, from the above inequalities, $\{F_2 x \ : \ x \in B_r\}$ is uniformly bounded for every $t \in [0, T_0]$.

Next, we will prove that $\{F_2 x \ : \ x \in B_r\}$ is equicontinuous.

Let, $s_1, \ s_2 \in [0, t_1]$, with $s_1 < s_2$, then $\forall s_1, \ s_2$, we have

$$\begin{aligned}
||(F_2 x)(s_2) - (F_2 x)(s_1)|| &\leq \int_0^{s_2} ||T_\alpha(s_2 - s)|| \, ||f(s, x(s), K x(s), H(s))|| ds \\
&\quad - \int_0^{s_1} ||T_\alpha(s_1 - s)|| \, ||f(s, x(s), K x(s), H x(s))|| ds \\
&\leq N_T \left[\int_0^{s_1} \left[(s_2 - s)^{\alpha-1} - (s_1 - s)^{\alpha-1}\right] ||f(s, x(s), K x(s), H x(s))|| ds \right. \\
&\quad \left. + \int_{s_1}^{s_2} (s_2 - s)^{\alpha-1} ||f(s, x(s), K x(s), H x(s))|| ds \right] \\
&\leq \frac{N_T \left(L_f T_0(1+K^*+H^*)r + m_1\right)}{\alpha}\left[s_2^\alpha - s_1^\alpha\right]
\end{aligned}$$

Similarly, $\forall s_1, s_2 \in (t_i, t_{i+1})$, with $s_1 < s_2, \ i = 1, 2, \cdots, m$, we have

$$||(F_2 x)(s_2) - (F_1 x)(s_1)|| \leq \frac{N_T \left(L_f T_0(1+K^*+H^*)r + k_1\right)}{\alpha}[(s_2 - t_i)^\alpha - (s_1 - t_i)^\alpha]$$

Thus, from the above inequalities, we have

$$\lim_{s_2 \to s_1} ||(F_2 x)(s_2) - (F_2 x)(s_1)|| = 0$$

So, F_2 is equicontinuous. Moreover, it is clear that from the lemma 2.3 F_2 is continuous. So, F_2 is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $F = F_1 + F_2$ has a fixed point on B_r and hence the system (1-3) has a solution on J. □

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