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A numerical scheme for solving Space-Fractional equation by finite differences theta-method

Research Article

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Abstract: This paper aims to present a general framework of the $\theta-$ and spatial extrapolation method. We examine $\theta-$method to solve Fractional Diffusion Differential Equations and we use spatial extrapolation for improving results ($0 \le \theta \le 1$). We use Riemann-Liouville derivative based on Shifted Grunwald stimates for fractional derivative. Consistency, stability and convergence analysis of the method is discussed. At the end, two illustrative examples have been presented. The obtained results reveal that the proposed technique is very effective, convenient and quite accurate to such considered problems.

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Keywords: Fractional PDE (FPDE) • Finite differences $\theta-$ method • Riemann-Liouville derivative • Shifted Grunwald formula

1. Introduction

In the recent years, fractional calculus is one of the interest issues that attract many scientists, specially mathematicians and engineer scientists. Many natural phenomena can be present by fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, viscoelasticity, try to modeling of these phenomena by fractional differential equations. Fractional calculus applied to model many meaningful things, such as fractional differential equation can model price volatility in finance [1], [2], model fast spreading of pollutants in hydrology [3], model the particle motions in a heterogeneous environment and long particle jumps of the anomalous diffusion in physics [4], [5]. The most common hydrologic and physics application of fractional calculus is the generation of fractional Brownian motion as a representation of aquifer material with long-range correlation structure [6] [7]. Other exact description of the applications of engineering, mechanics and mathematics, the literature is made to [8], [9], [10], [11]. Many cases of the real physical processes could be modeled in a reliable manner using fractional-order differential equations [12]. Most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used.
Fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle plume spreads at a rate inconsistent with the classical Brownian motion model. When a fractional derivative replaces the second derivative in a diffusion or dispersion model, it leads to enhanced diffusion (also called superdiffusion).
In this paper, we develop the basic theory of numerical solution for the one-dimensional fractional diffusion differential equation

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + q(x,t), \qquad t \in [0,T]. \tag{1}$$

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on a finite domain $x_L < x < x_R$. Here, we assume that $1 < \alpha \leq 2$ as the fractional order of the spatial derivative. The function $d(x) > 0$ is a given as the diffusion coefficient and $q(x,t)$ is a known function. Initial condition $u(x,0) = s(x)$ for $x_L < x < x_R$ and Dirichlet boundary conditions are as follows:
$u(x_L, t) = 0$ and $u(x_R, t) = 0$.
Published papers on the numerical solution of fractional partial differential equations are scarce. A different method for solving the fractional partial differential equation (1.1) is pursued in the recent papers [13], [14]. The theta-method is generalization of implicit , explicit and Crank-Nicholson methods.

2. Preliminaries

For implementation of this method we need to the following definitions.See [15], [16]

Definition 2.1 (Riemann-Lioville Fractional derivative).
If f be a real function and has continues derivatives of integer order n, then

$$D_x^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(\xi)}{(x-\xi)^{\alpha+1-n}} d\xi,$$

is Riemann-Lioville fractional derivative of order α that $n-1 < \alpha \leq n$.

Definition 2.2 (Shifted Grunwald formula).
For $1 < \alpha \leq 2$ shifted Grunwald formula defines:

$$\frac{d^{\alpha} f}{dx^{\alpha}} = \lim_{M\to\infty} \frac{1}{h^{\alpha}} \sum_{k=0}^{M} g_{\alpha,k} . f(x-(k-1)h),$$

such that shifted Grunwald estimates for fractional derivative defines :

$$\frac{d^{\alpha} f}{dx^{\alpha}} = \frac{1}{h^{\alpha}} \sum_{k=0}^{M} g_{\alpha,k} . f(x-(k-1)h) + \mathcal{O}(h^{\alpha}),$$

and M is a positive integers and $h = \frac{x_R - x_L}{M}$. Moreover, normalized Grunwald Weights are defined by :

$$g_{\alpha,k} = (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \qquad k = 0,1,2,\ldots,$$

Definition 2.3 (Taylor expansion of fractional order).
If f be a continous function that for any positive integer k and any $1 \leq \alpha \leq 2$ has fractional derivative of order $k\alpha$ then fractional Taylor expansion is as follows:

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha}}{\Gamma(1+k\alpha)} f^{(k\alpha)}(x),$$

such that

$$f(x+h) = E_{\alpha}(h^{\alpha} D_x^{\alpha}) f(x),$$

that $E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)}$ is Mittag-Leffler function.

3. Formulation of problem by $\theta-$Method

Here, we assume $h = \Delta x = \frac{x_R - x_L}{M}$ for x-axis and $k = \Delta t = \frac{T}{K}$ for t-axis as grid size therefore we have:
$x_i = x_L + ih; i = 1,2,\ldots,M$ and $t_j = jk; j = 1,2,\ldots,K$.
Now, if $U_{i,j} = U(x_i, t_j)$ represent the numerical approximation solution with $\theta-$ method, we have:

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} = d_i \left(\theta \delta_{\alpha,x} U_{i,j+1} + (1-\theta)\delta_{\alpha,x} U_{i,j}\right) + q_i^{j+1/2}, \qquad (2)$$

such that we define $\delta_{\alpha,x} U_{i,j} = \frac{1}{(\Delta x)^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k}.U_{i-k+1,j}, \quad and \quad q_i^{j+1/2} = \frac{q_{i,j}+q_{i,j+1}}{2}.$
In other words, we can rewrite (2) as follows:

$$\left(1 - d_i \theta \Delta t \delta_{\alpha,x}\right) U_{i,j+1} = \left(1 + d_i (1-\theta) \Delta t \delta_{\alpha,x}\right) U_{i,j} + \Delta t . q_i^{j+1/2}. \tag{3}$$

Also, if we consider $r = \frac{\Delta t}{(\Delta x)^\alpha}$ then we have:

$$U_{i,j+1} = U_{i,j} + r d_i \theta \sum_{p=0}^{i+1} g_{\alpha,p}.U_{i-p+1,j+1} + r d_i (1-\theta) \sum_{p=0}^{i+1} g_{\alpha,p}.U_{i-p+1,j} + \Delta t . q_i^{j+1/2}, \tag{4}$$

that it can define as a linear system :

$$(I - \theta A)\underline{U}^{j+1} = (I + (1-\theta)A)\underline{U}^j + \underline{Q}\Delta t, \tag{5}$$

such that

$$\underline{U}^j = [U_{0,j}, U_{1,j}, \ldots, U_{M,j}]^T,$$

and

$$\underline{Q}^{n+1/2} = [0, \frac{q_{1,j}+q_{1,j+1}}{2}, \frac{q_{2,j}+q_{2,j+1}}{2}, \ldots, \frac{q_{M-1,j}+q_{M-1,j+1}}{2}, 0]^T.$$

Entries of matrix A for $i, j = 1, 2, \ldots, M-1$ are as follows:

$$A_{i,j} = \begin{cases} 0 \quad , & j > i+1 \\ r d_i g_{\alpha,1} \quad , & j = i \\ r d_i g_{\alpha,i-j+1} \quad , & \text{otherwise} \end{cases}$$

and for $i, j = 0, 1, 2, ..., M$ we have:

$$A_{0,j} = A_{K,j} = A_{j,0} = A_{j,M} = 0.$$

Therefore, A is a tridiagonal symmetric matrix and :

$$\|A\|_2 = \rho(A)$$

.

4. Analysis of stability, consistency and convergence

If U be an approximated solution and u be exact solution and $F_{i,j}(U) = 0$ represent approximated difference equation of FPDE at mesh point (x_i, t_j). By substitution U with u value $F_{i,j}(u) = T_{i,j}$ represented local truncation error (LTE) at mesh point (x_i, t_j).

Theorem 4.1.
The LTE of the fractional $\theta-$ method $(1 \le \alpha \le 2)$ for FPDE (1.1) at the point $(ih, j\Delta t)$ is:

$$T_{i,j} = \mathcal{O}(|\theta - \frac{1}{2}|\Delta t) + \mathcal{O}((\Delta t)^2) + \mathcal{O}(h^\alpha) + \mathcal{O}(h^\alpha \Delta t).$$

Proof. According to the discretization, we have:

$$F_{i,j}(U) = \frac{U_{i,j+1} - U_{i,j}}{\Delta t} - d_i(\theta \delta_{\alpha,x} U_{i,j+1} + (1-\theta)\delta_{\alpha,x} U_{i,j}) - \frac{q_{i,j}+q_{i,j+1}}{2} = 0, \tag{6}$$

In other words, we can write

$$F_{i,j}(U) = \frac{U_{i,j+1} - U_{i,j}}{\Delta t} - \frac{d_i}{h^\alpha}(\theta \sum_{k=0}^{i+1} g_{\alpha,k} U_{i,j+1} + (1-\theta) \sum_{k=0}^{i+1} g_{\alpha,k} U_{i,j}) - \frac{q_{i,j}+q_{i,j+1}}{2} = 0. \tag{7}$$

Therefore, for LTE we conclude that

$$T_{i,j} = F_{i,j}(u) = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} - \frac{d_i}{h^\alpha}(\theta \sum_{k=0}^{i+1} g_{\alpha,k} u_{i,j+1} + (1-\theta) \sum_{k=0}^{i+1} g_{\alpha,k} u_{i,j}) - \frac{q_{i,j}+q_{i,j+1}}{2} = 0. \tag{8}$$

On the other hand, by standard Taylor expansion, we have:

$$u_{i,j+1} = u(x_i, t_{j+1}) = u_{i,j} + \Delta t \left(\frac{\partial u}{\partial t}\right)_{i,j} + \frac{1}{2}(\Delta t)^2 \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} + \frac{1}{6}(\Delta t)^3 \left(\frac{\partial^3 u}{\partial t^3}\right)_{i,j} + \dots, \tag{9}$$

and by the fractional Taylor expansion:

$$u_{i-k+1,j} = u(x_{i-k+1}, t_j) = u_{i,j} + \frac{[(1-k)h]^\alpha}{\alpha!} \left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)_{i,j} + \frac{[(1-k)h]^{2\alpha}}{(2\alpha)!} \left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right)_{i,j} + \dots, \tag{10}$$

Also ,we can write:

$$\begin{aligned} u_{i-k+1,j+1} = u(x_{i-k+1}, t_{j+1}) &= u_{i,j} + \Delta t \left(\frac{\partial u}{\partial t}\right)_{i,j} + \frac{1}{2}(\Delta t)^2 \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} + \dots \\ &+ \frac{[(1-k)h]^\alpha}{\alpha!} \left[\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)_{i,j} + \Delta t \left(\frac{\partial^{\alpha+1} u}{\partial x^\alpha \partial t}\right)_{i,j} + \frac{1}{2}(\Delta t)^2 \left(\frac{\partial^{\alpha+2} u}{\partial x^\alpha \partial t^2}\right)_{i,j} + \dots\right] \\ &+ \frac{[(1-k)h]^{2\alpha}}{(2\alpha)!} \left[\left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right)_{i,j} + \Delta t \left(\frac{\partial^{2\alpha+1} u}{\partial x^{2\alpha} \partial t}\right)_{i,j} + \frac{1}{2}(\Delta t)^2 \left(\frac{\partial^{2\alpha+2} u}{\partial x^{2\alpha} \partial t^2}\right)_{i,j} + \dots\right] + \dots, \end{aligned} \tag{11}$$

and by standard Taylor expansion for $q_{i,j+1}$:

$$q_{i,j+1} = q(x_i, t_{j+1}) = q_{i,j} + \Delta t \left(\frac{\partial q}{\partial t}\right)_{i,j} + \frac{1}{2}(\Delta t)^2 \left(\frac{\partial^2 q}{\partial t^2}\right)_{i,j} + \frac{1}{6}(\Delta t)^3 \left(\frac{\partial^3 q}{\partial t^3}\right)_{i,j} + \dots. \tag{12}$$

By substitution (9), (10), (11) and (12) in equation (8), we have:

$$\begin{aligned} T_{i,j} = F_{i,j}(u) &= \left(\frac{\partial u}{\partial t}\right)_{i,j} + \frac{1}{2}\Delta t \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} + \frac{1}{6}(\Delta t)^2 \left(\frac{\partial^3 u}{\partial t^3}\right)_{i,j} + \dots \\ &- \frac{d_i}{h^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k} \cdot \Bigg\{ \theta \left[u_{i,j} + \frac{[(1-k)h]^\alpha}{\alpha!} \left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)_{i,j} + \frac{[(1-k)h]^{2\alpha}}{(2\alpha)!} \left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right)_{i,j} + \dots \right] \\ &\qquad + (1-\theta) \Bigg[u_{i,j} + \Delta t \left(\frac{\partial u}{\partial t}\right)_{i,j} + \frac{1}{2}(\Delta t)^2 \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} + \dots \\ &+ \frac{[(1-k)h]^\alpha}{\alpha!} \left(\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)_{i,j} + \Delta t \left(\frac{\partial^{\alpha+1} u}{\partial x^\alpha \partial t}\right)_{i,j} + \frac{1}{2}(\Delta t)^2 \left(\frac{\partial^{\alpha+2} u}{\partial x^\alpha \partial t^2}\right)_{i,j} + \dots \right) \\ &\qquad + \frac{[(1-k)h]^{2\alpha}}{(2\alpha)!} \left(\left(\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right)_{i,j} + \Delta t \left(\frac{\partial^{2\alpha+1} u}{\partial x^{2\alpha} \partial t}\right)_{i,j} + \frac{(\Delta t)^2}{2} \left(\frac{\partial^{2\alpha+2} u}{\partial x^{2\alpha} \partial t^2}\right)_{i,j} + \dots \right) \Bigg] \Bigg\} \\ &- \left(q_{i,j} + \frac{\Delta t}{2} \left(\frac{\partial q}{\partial t}\right)_{i,j} + \frac{1}{4}(\Delta t)^2 \left(\frac{\partial^2 q}{\partial t^2}\right)_{i,j} + \dots \right), \end{aligned} \tag{13}$$

From [14] :

$$\sum_{k=0}^{\infty} g_{\alpha,k} = 0, \tag{14}$$

and also we can prove that:

$$\sum_{k=0}^{\infty} g_{\alpha,k} . (1-k)^\alpha = \alpha!, \tag{15}$$

From (13) , (14) and (15) :

$$T_{i,j} = \left(\frac{\partial U}{\partial t} - d \frac{\partial^\alpha U}{\partial x^\alpha} - q\right)_{i,j} + \frac{1}{2}\Delta t \left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} + \frac{1}{6}(\Delta t)^2 \left(\frac{\partial^3 u}{\partial t^3}\right)_{i,j} + \dots$$

$$-d_i\theta\left(h^{\alpha}\left(\frac{\partial^{2\alpha}U}{\partial x^{2\alpha}}\right)_{i,j}+h^{2\alpha}\left(\frac{\partial^{3\alpha}U}{\partial x^{3\alpha}}\right)_{i,j}+\dots\right)$$

$$-\frac{d_i(1-\theta)}{h^{\alpha}}\left[u_{i,j}+\Delta t\left(\frac{\partial u}{\partial t}\right)_{i,j}+\frac{1}{2}(\Delta t)^2\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}+\dots\right]$$

$$-d_i(1-\theta)\left(\Delta t\left(\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)_{i,j}+\frac{1}{2}(\Delta t)^2\left(\frac{\partial^{\alpha+2}u}{\partial x^{\alpha}\partial t^2}\right)+\dots\right)$$

$$-d_i(1-\theta)h^{\alpha}\left(\Delta t\left(\frac{\partial^{2\alpha+1}u}{\partial x^{2\alpha}\partial t}\right)_{i,j}+\frac{1}{2}(\Delta t)^2\left(\frac{\partial^{2\alpha+2}u}{\partial x^{2\alpha}\partial t^2}\right)+\dots\right)$$

$$-\left(\frac{\Delta t}{2}\left(\frac{\partial q}{\partial t}\right)_{i,j}+\frac{1}{4}(\Delta t)^2\left(\frac{\partial^2 q}{\partial t^2}\right)_{i,j}+\dots\right) \tag{16}$$

We know : $\left(\frac{\partial u}{\partial t}-d\frac{\partial^{\alpha}u}{\partial x^{\alpha}}-q\right)_{i,j}=0$, because u is exact solution of equation (1), therefore :

$$\left(\frac{\partial^2 u}{\partial t^2}-d\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}-\frac{\partial q}{\partial t}\right)_{i,j}=0,$$

By adding: $\pm\frac{d_i}{2}\left(\Delta t\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)$, Finally the principal part of LTE is:

$$\left[(\theta-\frac{1}{2})d.\Delta t\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}+\left(\frac{1}{6}.\frac{\partial^3 u}{\partial t^3}-\frac{1}{4}.\frac{\partial^2 q}{\partial t^2}\right)(\Delta t)^2-d\,h^{\alpha}\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}-\frac{d}{2}.h^{\alpha}\Delta t\frac{\partial^{2\alpha+1}u}{\partial x^{2\alpha}\partial t}\right]_{i,j},$$

therefore:

$$T_{i,j}=\mathcal{O}(|\theta-\frac{1}{2}|\Delta t)+\mathcal{O}((\Delta t)^2)+\mathcal{O}(h^{\alpha})+\mathcal{O}(h^{\alpha}\Delta t).$$

□

Corollary 4.1.
This theorem shows that this method is consistent , because for $h\to 0$ and $\Delta t\to 0$, LTE tend to zero.

Corollary 4.2.
For Crank-Nicholson method ($\theta=\frac{1}{2}$) LTE is:

$$T_{i,j}=\mathcal{O}((\Delta t)^2)+\mathcal{O}(h^{\alpha})+\mathcal{O}(h^{\alpha}\Delta t).$$

Corollary 4.3.
For matrix in (5), we have:

$$\underline{U}^{j+1}=(I-\theta A)^{-1}(I+(1-\theta)A)\underline{U}^{j}+(I-\theta A)^{-1}\underline{Q}\Delta t,$$

If μ be eigenvalue of matrix A then eigenvalues of matrix $B=(I-\theta A)^{-1}(I+(1-\theta)A)$ is : $\lambda=\frac{1+(1-\theta)\mu}{1-\theta\mu}$. Now, if apply the condition $\rho(B)\le 1$ we can show that θ-method for $\frac{1}{2}\le\theta\le 1$ is unconditionally stable and for $0\le\theta\le\frac{1}{2}$ the condition of stability is $r\le\frac{2}{1-2\theta}$.

Corollary 4.4.
According to stability analysis and consistency analysis of this method, now from Lax-Richtmyer's equivalence theorem this method is convergence.

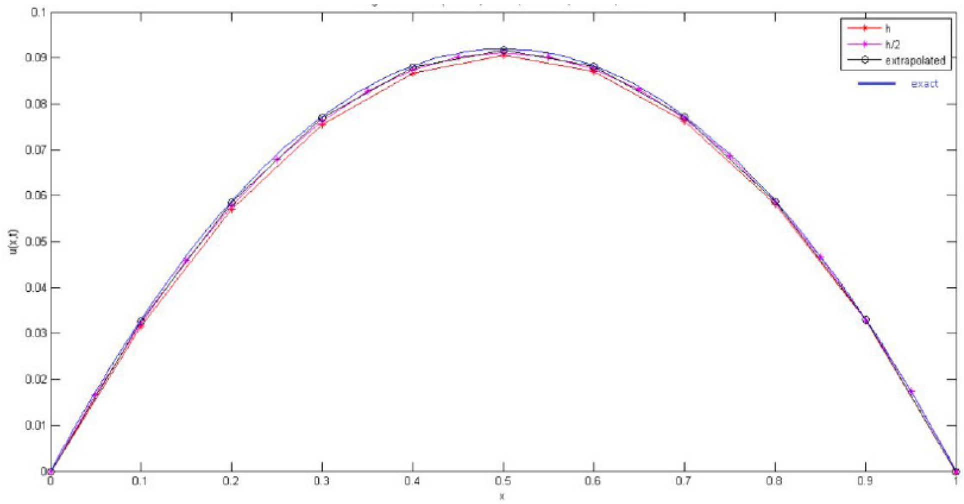

Fig. 1. Comparison between exact and numerical solutions for the Example 5.1 with $\theta = 0.5$, (Crank-Nicholson,) $\Delta x = 0.1, \Delta t = 0.1$

5. Numerical examples

Example 5.1.
Consider following equation (1) on $\Omega = \{(x,t)|0 \le x \le 1, t > 0\}$:

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t),$$

where :

$$d(x) = \Gamma(0.2)x^{1.8} \quad and \quad q(x,t) = -(2x - 11x^2)e^{-t},$$

with initial and boundary conditions :

$$u(x,0) = x(1-x), \quad u(0,t) = 0, \quad u(1,t) = 0,$$

We can show that exact solution is:

$$u(x,t) = x(1-x)e^{-t}.$$

We use spatial extrapolation as $U_{ext.} = 2U(\frac{h}{2}) - U(h)$.
The values of Maximum absolute-error are shown in Table 1 for the above example problem for different values of h and θ. Figs. 1 and 2 having different values of h and θ verify the efficiency of the proposed scheme.

Table 1. Max. absolute Error of Example 5.1 for different θ at time $T = 1$. (N.C= Not Convergence.)

$\theta=$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
h=0.1	N.C	0.003845	0.002289	0.001396	0.001967	0.003518	0.005079
h=0.05	N.C	N.C	0.001288	9.116×10^{-4}	9.501×10^{-4}	0.001742	0.002548
Ext.	N.C	N.C	4.583×10^{-4}	4.764×10^{-4}	4.908×10^{-4}	4.996×10^{-4}	5.0240×10^{-4}

Example 5.2.
Consider following equation (1) on $\Omega = \{(x,t)|0 \le x \le 2, t > 0\}$:

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t),$$

where :

$$d(x) = \Gamma(1.2)x^{1.8} \quad and \quad q(x,t) = e^{-t}\{x^2(2-x)^2 + 8(x^2 - \frac{5}{2}x^3 + \frac{25}{22}x^4)\},$$

with initial and boundary conditions :

$$u(x,0) = x^2(2-x)^2, \quad u(0,t) = 0, \quad u(2,t) = 0,$$

We can show that exact solution is:

$$u(x,t) = e^{-t}x^2(2-x)^2.$$

We use spatial extrapolation as $U_{ext.} = 2U(\frac{h}{2}) - U(h)$.
The values of Maximum absolute-error are shown in Table 2 for the above example problem for different values of h and θ. Figs. 3 and 4 having different values of h and θ verify the efficiency of the proposed scheme.

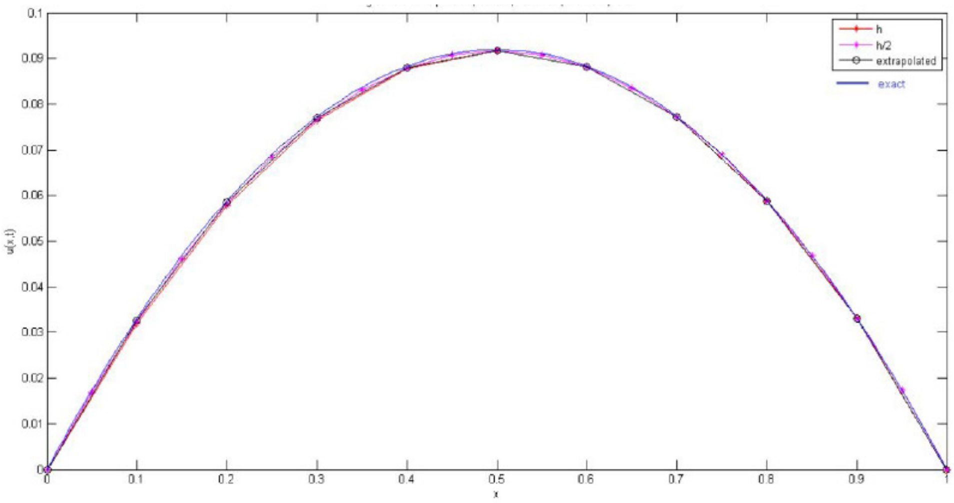

Fig. 2. Comparison between exact and numerical solutions for the Example 5.1 with $\theta = 0.6$, $\Delta x = 0.1, \Delta t = 0.1$

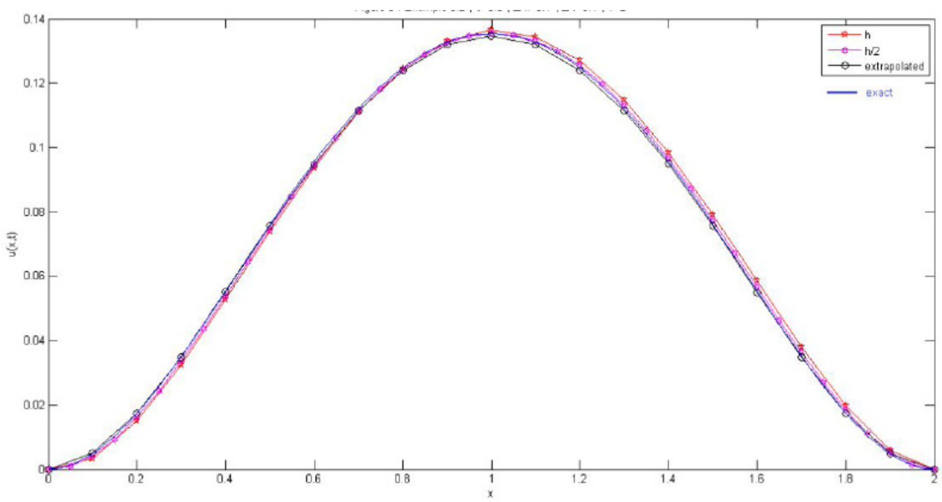

Fig. 3. Comparison between exact and numerical solutions for the Example 5.2 with $\theta = 0.5$, (Crank-Nicholson,) $\Delta x = 0.1, \Delta t = 0.1$

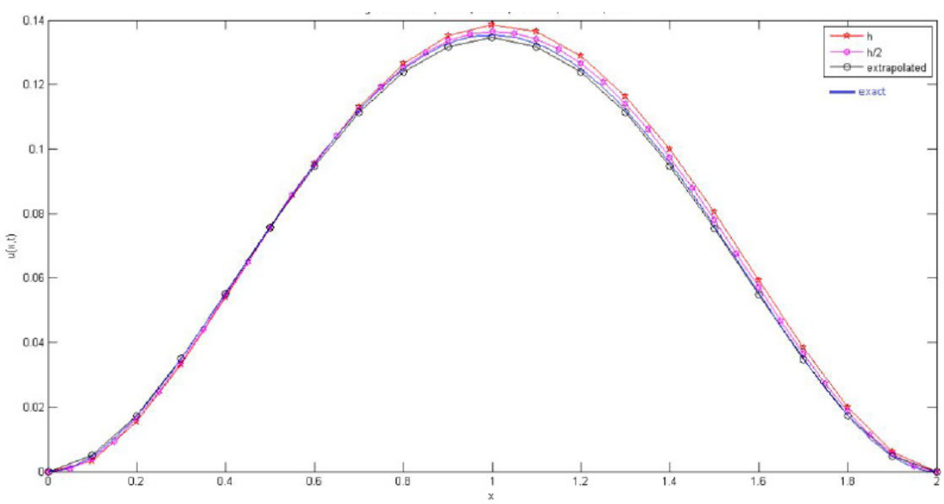

Fig. 4. Comparison between exact and numerical solutions for the Example 5.2 with $\theta = 0.6$, $\Delta x = 0.1, \Delta t = 0.1$

Table 2. Max. absolute Error of Example 5.2 for different θ at time $T = 2$. (N.C= Not Convergence.)

$\theta=$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
h=0.1	N.C	0.318188	0.016448	0.018421	0.021667	0.025419	0.029206
h=0.05	N.C	N.C	0.007325	0.008511	0.010105	0.011992	0.014020
Ext.	N.C	N.C	0.004604	0.002509	0.001693	0.001657	0.001638

6. Conclusion

In this paper we presented a numerical scheme for solving one-dimensional fractional diffusion differential equation. The method employed to find the numerical solutions of these equations is based on the Grunwald estimates for Rieman-Liouvile fractional derivative. The computational results are found to be in good agreement with the exact solutions.

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